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Local error indicators and guaranteed upper bounds

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Numerical solutions of partial differential equations are widely used in engineering and science as computational models of various phenomena. However, there is a general agreement that the numerical solution as a single output of the computations is not sufficient. An information about the approximation error is required in order to asses the reliability of the computations. Such an information is provided by *a posteriori error estimates.*

It is advantageous to use the a posteriori error estimates in conjunction with the adaptive algorithm. This enables to prescribe the desired error tolerance and the algorithm is then automatically capable to deliver an approximate solution within the specified error tolerance. In the context of the finite element method the standard adaptive algorithm follows these general steps:

1. *Initialize*: Construct the initial mesh $\mathcal{T}_h$.
2. *Solve*: Find the approximate solution u_h on $\mathcal{T}_h$.
3. *Indicators*: Compute error indicators η_K for all elements $K \in \mathcal{T}_h$.
4. *Estimator*: Compute the error estimator η.
5. *Stop*: If $\eta \leq \mathrm{TOL}$ then STOP.
6. *Mark*: If $\eta_K \geq \Theta \max\{\eta_K : K \in \mathcal{T}_h\}$ then mark K.
7. *Refine*: Refine the marked elements and build the new mesh $\mathcal{T}_h$.
8. Go to 2.

If a function u defined in a domain Ω is the exact solution of the underlined partial differential problem and if u_h is its approximation then $e = u - u_h$ is the error and the error indicators η_K (see Step 3) estimate a suitable norm of the error restricted to the element K, i.e., $\eta_K \approx \|e\|_K$. Similarly, the error estimator η (see Step 4) estimates the norm of the entire domain Ω, i.e, $\eta \approx \|e\|$. In Step 5 we assume that TOL is the user prescribed absolute tolerance on the error and we test wheather the error estimator η is below it. In Step 6 we mark the elements where large error is indicated. The parammeter Θ is used to control the amount of elements to be refined within each adaptive step. The value $\Theta = 1/2$ is often chosen in practice. In Step 7 the marked elements are refined and the new finite element mesh is produced. The new mesh enters the next adaptive cycle in Step 2.

Let us point out two roles of a posteriori error estimators in this adaptive algorithm. The global estimator η is used in the stopping criterion. The optimal property of η is to provide a *guaranteed upper bound* on error, i.e. $\|e\| \leq \eta$. If this is the case then the stopping criterion in Step 5 guarantees the approximation error to be below the tolerance, i.e. $\|e\| \leq \eta \leq \mathrm{TOL}$.

On the other hand, the local error indicators η_K serve for the location of places where the error is higher than elsewhere and where it is suitable to refine the mesh. They do not need to estimate the actual size of the error in the element K. It suffices if they are equivalent to the local error in the sense of inequalities $C_1\eta_K \leq \|e\|_K \leq C_2\eta_K$, where the actual values of constants C_1 and C_2 can be unknown. These inequalities are often referred as *efficiency* and *reliability*, respectively.

There are several known methos for computation of efficient and reliable local error indicators η_K [1, 3, 6]. There are explicit and implicit residual indicators, hierarchical estimates and estimates based on gradient recovery. These methods are theoretically quite well understood, they can be implemented in a fast way, and their numerical behavior in the adaptive algorithms is satisfactory.

On the other hand, as far as the author is aware, the only technique leading to the guaranteed upper bounds of the energy norm of the error is based on the *complementarity* [4, 5]. The idea is to construct a complementary problem whose approximate solution is used for evaluation of the guaranteed error estimate η. This approach is often expensive in terms of the computational time but for simple problems (like Poisson problem) fast implementations of this technique exist [2].

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References

[1] Ainsworth, M. and Oden, J.T.: *A posteriori error estimation in finite element analysis*. John Wiley & Sons, New York, 2000.

[2] Ainsworth, M. and Vejchodský, T.: Fully computable robust a posteriori error bounds for singularly perturbed reaction–diffusion problems. Preprint of the Institute of Mathematics AS CR, 2010.

[3] Babuška, I. and Strouboulis, T.: *The finite element method and its reliability*, The Clarendon Press, Oxford University Press, New York, 2001.

[4] Neittaanmäki, P. and Repin, S.: *Reliable methods for computer simulation*, Elsevier, Amsterdam, 2004.

[5] Repin, S.: *A posteriori estimates for partial differential equations*, de Gruyter, Berlin, 2008.

[6] Verfürth, R.: *A review of a posteriori error estimation and adaptive mesh-refinement techniques*, Wiley-Teubner, Stuttgart, 1996.