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Abstract

We consider a class of (very) weak solutions to the Navier-Stokes-Fourier system describing the time evolution of the density ϱ, the absolute temperature ϑ, and the macroscopic velocity $\mathbf{u}$. It is shown that a weak solution emanating from smooth initial data is regular as long as $\mathbf{u}$ and ϑ are bounded and $\|\mathrm{div}_x \mathbf{u}\|_{L^\infty}$ integrable in the existence interval $(0,T)$. Using the method of relative energy we first show that any weak solution enjoying the above mentioned regularity coincides with a strong one as long as the latter exists. In such a way, the proof reduces to showing that the life span of the strong solution can be extended to the desired existence interval $(0,T)$.

Key words: Navier-Stokes-Fourier system, weak solution, regularity criteria

Contents

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1 Introduction

The time evolution of the mass density $\varrho = \varrho(t,x)$, the absolute temperature $\vartheta = \vartheta(t,x)$, and the velocity field $\mathbf{u} = \mathbf{u}(t,x)$ of a compressible, viscous and heat conducting fluid can be described by the *Navier-Stokes-Fourier system* of partial differential equations:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{1.1}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}), \tag{1.2}$$

$$c_v \Big[\partial_t(\varrho \vartheta) + \mathrm{div}_x(\varrho \vartheta \mathbf{u}) \Big] + \mathrm{div}_x \mathbf{q} = \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \vartheta \frac{\partial p(\varrho, \vartheta)}{\partial \vartheta} \mathrm{div}_x \mathbf{u}, \tag{1.3}$$

where $p = p(\varrho, \vartheta)$ is the pressure, $\mathbb{S}(\nabla_x \mathbf{u})$ the viscous stress tensor, c_v denotes the specific heat at constant volume, and $\mathbf{q}$ the heat flux. In addition, we suppose that $\mathbb{S}$ is given by *Newton's law*

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \eta \mathrm{div}_x \mathbf{u} \mathbb{I}, \ \mu > 0, \ \eta \geq 0, \tag{1.4}$$

while $\mathbf{q}$ obeys *Fourier's law*

$$\mathbf{q} = -\kappa(\vartheta) \nabla_x \vartheta. \tag{1.5}$$

The fluid is confined to a bounded physical domain $\Omega \subset R^3$, on the boundary of which

$$\mathbf{u} \cdot \mathbf{n}|_{\partial \Omega} = 0, \ \mathbf{q} \cdot \mathbf{n}|_{\partial \Omega} = 0. \tag{1.6}$$

Furthermore, the velocity field $\mathbf{u}$ satisfies either the no-slip

$$\mathbf{u} \times \mathbf{n}|_{\partial \Omega} = 0, \tag{1.7}$$

or the complete slip

$$(\mathbb{S}(\nabla_x \mathbf{u}) \cdot \mathbf{n}) \times \mathbf{n}|_{\partial\Omega} = 0 \tag{1.8}$$

boundary conditions.

The problem is completed by prescribing the initial conditions

$$\varrho(0,\cdot) = \varrho_0,\ \vartheta(0,\cdot) = \vartheta_0,\ \mathbf{u}(0,\cdot) = \mathbf{u}_0. \tag{1.9}$$

Our goal is to study solvability of the Navier-Stokes-Fourier system (1.1-1.9) for sufficiently regular initial data satisfying the physically relevant restriction

$$\varrho_0 > 0,\ \vartheta_0 > 0 \text{ in } \overline{\Omega}. \tag{1.10}$$

Our principal working hypothesis is that the specific heat at constant volume $c_v > 0$ is constant, therefore the associated specific internal energy reads

$$e(\varrho,\vartheta) = c_v\vartheta + P(\varrho). \tag{1.11}$$

Moreover, we suppose that the heat conductivity $\kappa(\vartheta)$ is an increasing function of the absolute temperature, while the viscosity coefficients $\mu > 0$ and $\eta \geq 0$ remain constant.

We consider the class of (very) weak solutions introduced in the monograph [7], see Section 2 below. Roughly speaking, these are distributional solutions of the equations (1.1), (1.2), while the thermal energy balance (1.3) is replaced by two inequalities:

$$c_v\Big[\partial_t(\varrho\vartheta) + \mathrm{div}_x(\varrho\mathbf{u}\otimes\mathbf{u})\Big] - \Delta K(\vartheta) \geq \mathbb{S}(\nabla_x\mathbf{u}) : \nabla_x\mathbf{u} - \vartheta\frac{\partial p(\varrho,\vartheta)}{\partial\vartheta}\mathrm{div}_x\mathbf{u}, \tag{1.12}$$

with

$$K(\vartheta) = \int_0^\vartheta \kappa(z)\ \mathrm{dz},$$

and

$$\int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta)\right](\tau,\cdot)\ \mathrm{d}x \leq \int_\Omega \left[\frac{1}{2}\varrho_0|\mathbf{u}_0|^2 + \varrho_0 e(\varrho_0,\vartheta_0)\right]\ \mathrm{d}x. \tag{1.13}$$

As shown in [7, Chapter 4, Section 4.3.2], both (1.12) and (1.13) reduce to equalities as soon as the corresponding weak solution is smooth enough.

Recently [9], an alternative approach has been developed based on the weak formulation of the Navier-Stokes-Fourier system, where the thermal energy balance (1.3) is replaced by the entropy inequality

$$\partial_t(\varrho s(\varrho,\vartheta)) + \mathrm{div}_x(\varrho s(\varrho,\vartheta)\mathbf{u}) + \mathrm{div}_x\left(\frac{\mathbf{q}}{\vartheta}\right) = \sigma,\ \ \sigma \geq \frac{1}{\vartheta}\left(\mathbb{S} : \nabla_x\mathbf{u} - \frac{\mathbf{q}\cdot\nabla_x\vartheta}{\vartheta}\right), \tag{1.14}$$

where $s = s(\varrho,\vartheta)$ is the specific entropy determined through Gibbs' relation

$$\vartheta Ds(\varrho,\vartheta) = De(\varrho,\vartheta) + p(\varrho,\vartheta)D\left(\frac{1}{\varrho}\right). \tag{1.15}$$

The relevant existence theory based on the entropy formulation (1.14), developed in [9, Chapter 3], requires *all* transport coefficients μ, η, and κ to depend effectively on the temperature, whereas the pressure as well as the internal energy have to be augmented by a radiation component proportional to ϑ^4 to avoid the problem of temperature oscillations on the hypothetical vacuum zones. We note that yet another rather different approach to weak solutions to the Navier-Stokes-Fourier system has been proposed by Bresch and Desjardins [2], where the viscosity coefficients μ and λ are functions of the density ϱ interrelated in a specific way, and the pressure contains a singular component unbounded when ϱ approaches zero.

Although the weak formulation based on the entropy balance (1.14) rather than (1.3) is very convenient from the purely theoretical point of view, giving rise to a number of interesting properties of the weak solutions including the *weak-strong uniquness* (see [10]) and *conditional regularity* results (see [11]), the presence of the radiation terms as well as the explicit dependence of the transport coefficients on the temperature makes it too complicated for possible numerical implementations.

Our interest in the weak formulation based on the inequalities (1.12), (1.13) is motivated by the recent development of relevant *numerical schemes* based on a mixed discontinuous Galerkin method, see Karper [15] and [8]. As shown in [8], such a numerical scheme *converges* to a weak solution of the Navier-Stokes-Fourier system specified through (1.12), (1.13). In particular, the conditional regularity based on mere *boundedness* of the numerical solutions may lead to rigorous *error estimates* for the latter. Last but not least, we point out that although the class of weak solutions is apparently larger, the regularity criterion we obtain is considerably stronger than in [11].

Our goal is to show that any weak solution of the Navier-Stokes-Fourier system (1.1-1.8), originating from sufficiently smooth initial data (1.9), (1.10), is regular in $(0,T) \times \Omega$ as soon as

$$\vartheta(t,x) \leq \overline{\vartheta}, |\mathbf{u}(t,x)| \leq \overline{u} \text{ for a.a. } (t,x) \in (0,T) \times \Omega, \ \int_0^T \|\mathrm{div}_x \mathbf{u}(t,\cdot)\|_{L^\infty(\Omega)} \ \mathrm{d}t \leq L. \tag{1.16}$$

In comparison with the standard blow up criteria applicable to *strong solutions* of the Navier-Stokes-Fourier system (see [6], [14], among others), the problem of conditional regularity for *weak solutions* features an additional difficulty that consists in showing that a weak and strong solution, starting from the same initial data, coincide on their common existence interval. To this end, we adapt the method based on the relative energy functional developed in [10]. As this approach relies on the entropy inequality (1.14) rather than the thermal energy balance (1.12), we have to show that any weak solution that complies with (1.12), (1.13), together with the extra regularity properties (1.16), necessarily satisfies also (1.14). The remaining part of the proof of conditional regularity is then reduced to the class of strong solutions.

The paper is organized as follows. In Section 2, we review the basic results of the existence theory developed in [7] and state our main result. In Section 3, we show that any weak solution enjoying the extra regularity (1.16) satisfies also the entropy inequality (1.14) - a result that may be of independent interest. To this end, we use a variant of the technique developed by Freshe et. al. [12] in the context of non-homogeneous fluids. Then we show the weak-uniqueness properties by means of a straightforward application of the relative energy method. Final, we complete the proof by showing a blow-up criterion for strong solutions in Section 4.

2 Weak solutions, main results

We start by specifying the structural restrictions imposed on the thermodynamics functions. It follows from our basic hypothesis (1.11) combined with the Gibbs' relation (1.15) that the pressure p can be written in the form

$$p(\varrho, \vartheta) = p_e(\varrho) + \vartheta p_{th}(\varrho), \tag{2.1}$$

where p_e denotes the elastic (cold) pressure and p_{th} the thermal pressure component. To simplify the presentation, we shall assume that

$$p(\varrho, \vartheta) = \varrho^\gamma + \varrho\vartheta. \tag{2.2}$$

In addition, the heat conductivity coefficient κ is taken

$$\kappa(\vartheta) = \overline{\kappa}(1 + \vartheta^2). \tag{2.3}$$

As a matter of fact, these assumptions are relevant only to establish the *existence* of weak solutions, whereas the specific form of p and κ does not play any role in the regularity criterion as (1.16) implies that both ϱ and ϑ must be bounded.

2.1 Weak solutions

Definition 2.1. *We say that* $[\varrho, \vartheta, \mathbf{u}]$ *is a* weak solution *of the Navier-Stokes-Fourier system (1.1 - 1.9) if:*

- *The functions* $[\varrho, \vartheta, \mathbf{u}]$ *belong to the regularity class:*

$$\varrho \geq 0, \ \varrho \in C_{\rm weak}([0,T]; L^\gamma(\Omega)), \ \gamma > 1,$$

$$\mathbf{u} \in L^2(0,T; W^{1,2}(\Omega)), \ (\varrho\mathbf{u}) \in C_{\rm weak}([0,T]; L^{2\gamma/(\gamma+1)(\Omega;R^3)}, \ \varrho|\mathbf{u}|^2 \in L^\infty(0,T; L^1(\Omega)),$$

$$\mathbf{u}|_{\partial\Omega} = 0 \ \textit{in the case of no-slip (1.7)}, \ \mathbf{u}\cdot\mathbf{n}|_{\partial\Omega} \ \textit{in the case of complete slip (1.8)}$$

$$\vartheta > 0 \ \textit{a.a. in} \ (0,T)\times\Omega, K(\vartheta) \in L^1((0,T)\times\Omega), \ \varrho\vartheta \in L^2((0,T)\times\Omega).$$

- *The equations (1.1-1.3 are replaced by the following integral identities:*

$$\int_\Omega \varrho\varphi \ {\rm d}x\Big|_{t=\tau_1}^{t=\tau_2} = \int_{\tau_1}^{\tau_2}\int_\Omega [\varrho\partial_t\varphi + \varrho\mathbf{u}\cdot\nabla_x\varphi] \ {\rm d}x{\rm d}t, \ \varrho(0,\cdot) = \varrho_0, \tag{2.4}$$

for any $0 \leq \tau_1 < \tau_2 \leq T$ *and any test function* $\varphi \in C_c^\infty([0,T]\times\overline{\Omega})$*;*

$$\int_\Omega \varrho\mathbf{u}\cdot\varphi \ {\rm d}x\Big|_{t=\tau_1}^{t=\tau_2} = \tag{2.5}$$

$$\int_{\tau_1}^{\tau_2}\int_\Omega [\varrho\mathbf{u}\cdot\partial_t\varphi + \varrho\mathbf{u}\otimes\mathbf{u} : \nabla_x\varphi + p(\varrho,\vartheta){\rm div}_x\varphi - \mathbb{S}(\nabla_x\varphi) : \nabla_x\varphi] \ {\rm d}x \ {\rm d}t, \ \varrho\mathbf{u}(0,\cdot) = \varrho_0\mathbf{u}_0,$$

for any $0 \leq \tau_1 < \tau_2 \leq T$*, and any test function* $\varphi \in C^\infty_c([0,T] \times \Omega; R^3)$ *in the case of the no-slip (1.7),* $\varphi \in C^\infty_c([0,T] \times \overline{\Omega}; R^3)$*,* $\varphi \cdot \mathbf{n}|_{\partial\Omega} = 0$ *in the case of the complete slip (1.8);*

$$c_v \int_\Omega \varrho\vartheta\varphi \ \mathrm{d}x \Big|_{t=\tau_1}^{t=\tau_2} \tag{2.6}$$

$$\geq \int_{\tau_1}^{\tau_2} \int_\Omega \left[c_v \left(\varrho\vartheta\partial_t\varphi + \varrho\vartheta\mathbf{u} \cdot \nabla_x\varphi \right) - K(\vartheta)\Delta\varphi + \mathbb{S}(\nabla_x\mathbf{u}) : \nabla_x\mathbf{u}\varphi - \vartheta \frac{\partial_\vartheta p(\varrho,\vartheta)}{\partial\vartheta} \mathrm{div}_x\mathbf{u}\varphi \right] \ \mathrm{d}x \ \mathrm{d}t$$

for a.a. $0 \leq \tau_1 < \tau_2 \leq T$ *including* $\tau_1 = 0$*, where* $\varrho\vartheta(0,\cdot) = \varrho_0\vartheta_0$*, and for any test function* $\varphi \in C^\infty_c([0,T] \times \overline{\Omega})$*,* $\varphi \geq 0$*,* $\nabla_x\varphi \cdot \mathbf{n}|_{\partial\Omega} = 0$*;*

$$\int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta) \right](\tau,\cdot) \ \mathrm{d}x \leq \int_\Omega \left[\frac{1}{2}\varrho_0|\mathbf{u}_0|^2 + \varrho_0 e(\varrho_0,\vartheta_0) \right] \ \mathrm{d}x \text{ for a.a. } \tau \in [0,T]. \tag{2.7}$$

The *existence* of global-in-time weak solutions was established in the monograph [7, Chapter 7, Theorem 7.1] under the hypotheses (2.2), (2.3) with

$$\gamma > 3, \ \Omega \subset R^3 \text{ -a bounded domain of class } C^{2+\nu}.$$

In addition, the temperature ϑ constructed in [7] enjoys higher regularity outside the hypothetical vacuum zone, namely

$$\varrho\vartheta = \varrho\tilde{\vartheta}, \text{ where } \tilde{\vartheta}, \ \log(\tilde{\vartheta}) \in L^2(0,T; W^{1,2}(\Omega)). \tag{2.8}$$

Finally, the initial value of the temperature is attained in the following sense:

$$\left. \begin{array}{r} \mathrm{ess}\lim_{t\to 0+} \int_\Omega \varrho\vartheta(t,\cdot)\varphi \ \mathrm{d}x = \int_\Omega \varrho_0\vartheta_0\varphi \ \mathrm{d}x, \\ \mathrm{ess}\liminf_{t\to 0+} \int_\Omega \varrho\vartheta^\alpha(t,\cdot)\varphi \ \mathrm{d}x \geq \int_\Omega \varrho_0\vartheta_0^\alpha\varphi \ \mathrm{d}x \end{array} \right\} \text{for any } 0 \leq \alpha < 1, \text{ and any } \varphi \in C^\infty_c(\overline{\Omega}), \ \varphi \geq 0, \tag{2.9}$$

see [7, Chapter 4].

2.2 Main result

We are ready to state the main result of the present paper.

Theorem 2.1. *Let $\Omega \subset R^3$ be a bounded domain of class $C^{2+\nu}$. Suppose that p and κ are given by (2.2), (2.3), with*

$$\gamma > 3. \tag{2.10}$$

Let the initial data $[\varrho_0, \vartheta_0, \mathbf{u}_0]$ satisfy (1.10),

$$\varrho_0, \ \vartheta_0 \in W^{3,2}(\Omega), \ \mathbf{u}_0 \in W^{3,2}(\Omega; R^3), \tag{2.11}$$

with the relevant compatibility conditions. Let $[\varrho, \vartheta, \mathbf{u}]$ be a weak solution of the Navier-Stokes-Fourier system (1.1 - 1.9) in $(0, T) \times \Omega$ satisfying (2.8) and enjoying the extra regularity (1.16).

Then $[\varrho, \vartheta, \mathbf{u}]$ is a regular (classical) solution belonging to the class

$$\varrho, \vartheta \in C([0, T]; W^{3,2}(\Omega)), \ \mathbf{u} \in C([0, T]; W^{3,2}(\Omega; R^3)), \tag{2.12}$$

$$\vartheta \in L^2(0, T; W^{4,2}(\Omega)), \ \mathbf{u} \in L^2(0, T; W^{4,2}(\Omega; R^3)), \tag{2.13}$$

$$\partial_t \vartheta \in L^2(0, T; W^{2,2}(\Omega)), \ \partial_t \mathbf{u} \in L^2(0, T; W^{2,2}(\Omega; R^3)). \tag{2.14}$$

The rest of the paper is devoted to the proof of Theorem 2.1. We note that the local-in-time strong solutions to the Navier-Stokes-Fourier system in the class (2.12 - 2.14) were constructed by Valli [20], [21], see also Valli and Zajaczkowski [22]. Global-in-time solutions with the data close to an equilibrium were obtained in the seminal papers by Matsumura and Nishida [19], [18]. Hoff [13] considered a simplified barotropic system supplemented by the complete slip conditions in a slightly weaker framework than (2.12-2.14). Similar results for the full system were obtained by Cho et al. [3]. Finally, in view of the standard parabolic regularity, it is easy to observe that all relevant derivatives of $[\varrho, \vartheta, \mathbf{u}]$ are continuous in the open set $(0, T) \times \Omega$; whence the solution belonging to the regularity class (2.12-2.14) is classical smooth.

2.3 Additional regularity of weak solutions

We derive certain additional regularity properties of the weak solutions satisfying (1.16) that may be of independent interest.

2.3.1 Uniform bounds on the density

Following DiPerna and Lions [5], we say that $[\varrho, \mathbf{u}]$ is a *renormalized solution* of the continuity equation (1.1) if the integral identity

$$\int_\Omega b(\varrho)\varphi \ \mathrm{d}x \Big|_{t=\tau_1}^{t=\tau_2} = \int_0^T \int_\Omega \Big[b(\varrho)\partial_t \varphi + b(\varrho)\mathbf{u} \cdot \nabla_x \varphi + \Big(b(\varrho) - b'(\varrho)\varrho \Big) \mathrm{div}_x \mathbf{u} \varphi \Big] \ \mathrm{d}x \tag{2.15}$$

holds for any $0 \leq \tau_1 < \tau_2 \leq T$, any test function $\varphi \in C_c^\infty([0,T] \times \overline{\Omega})$, and any continuously differentiable function b with the derivative b' vanishing outside a compact set.

Lemma 2.1. *Let* $[\varrho, \mathbf{u}]$,

$$\varrho \in L^\infty(0,T; L^1(\Omega)),$$

$$\mathbf{u} \in L^2(0,T; W^{1,2}(\Omega; R^3)), \ \mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0, \ \mathrm{div}_x \mathbf{u} \in L^1(0,T; L^\infty(\Omega)), \tag{2.16}$$

be a renormalized solution of the continuity equation (1.1) such that

$$b(\varrho)(0,\cdot) = b(\varrho_0), \ 0 < \operatorname*{ess\,inf}_\Omega \varrho_0 \leq \varrho_0 \leq \operatorname*{ess\,sup}_\Omega \varrho_0 < \infty.$$

Then

$$\operatorname*{ess\,inf}_\Omega \varrho_0 \exp\left(- \int_0^t \|\mathrm{div}_x \mathbf{u}\|_{L^\infty(\Omega)} \ \mathrm{d}s \right) \leq \varrho(t,\cdot) \leq \quad \operatorname*{ess\,sup}_\Omega \varrho_0 \exp\left(\int_0^t \|\mathrm{div}_x \mathbf{u}\|_{L^\infty(\Omega)} \ \mathrm{d}s \right) \tag{2.17}$$

for any $t \in [0,T]$.

Proof:

Step 1

Although the proof seems obvious at the level of uniform bounds, the fact that the equation of continuity is satisfied only in the renormalized form (2.15) requires certain effort. To begin, we take a spatially homogenous φ in (2.15) to deduce that

$$\int_\Omega b(\varrho)(\tau,\cdot) \ \mathrm{d}x \leq \int_\Omega b(\varrho_0) \ \mathrm{d}x + \int_0^\tau \int_\Omega \left| b(\varrho) - b'(\varrho)\varrho \right| |\mathrm{div}_x \mathbf{u}| \ \mathrm{d}x \ \mathrm{d}t \tag{2.18}$$

for a.a. $\tau \in (0,T)$. Moreover, as the velocity satisfies (2.16), the validity of (2.18) can be extended to any b with uniformly bounded derivative.

Step 2

Now, we take

$$b_m(\varrho) = \left\{ \begin{array}{l} \varrho^\alpha \text{ for } 0 \leq \varrho < m, \\ \\ m^\alpha + \alpha m^{\alpha-1}(\varrho - m) \text{ for } \varrho \geq m. \end{array} \right.$$

It is easy to check that

$$\left| b_m(\varrho) - b_m'(\varrho)\varrho \right| \leq c\alpha \ b_m(\varrho)$$

for all ϱ, $\alpha, m > 1$, where the constant c is independent of m and α.

Thus, going back to (2.18), we may use the Gronwall's inequality to deduce

$$\int_\Omega b_m(\varrho)(\tau,\cdot) \ \mathrm{d}x \leq \exp\left(c\alpha \int_0^\tau \|\mathrm{div}_x \mathbf{u}\|_{L^\infty(\Omega)} \right) \int_\Omega b_m(\varrho_0) \ \mathrm{d}x;$$

whence, letting $m \to \infty$,

$$\|\varrho(\tau,\cdot)\|_{L^\alpha(\Omega)} \leq \exp\left(c\int_0^\tau \|{\rm div}_x \mathbf{u}\|_{L^\infty(\Omega)}\right)\|\varrho_0\|_{L^\infty(\Omega)}.$$

Since the previous estimate holds for *any* $\alpha > 0$, we deduce that ϱ is uniformly bounded in $(0,T)\times\Omega$. Thus, in particular, $\varrho \in C([0,T];L^1(\Omega))$ and the norm is bounded uniformly for all t.

To complete the proof, we make use of the regularizing procedure of DiPerna and Lions [5].

Step 3

Using the regularization method of DiPerna and Lions [5], we deduce that

$$\partial_t B(\varrho + M(t)) + {\rm div}_x \left[B\Big(\varrho + M(t)\Big){\rm div}_x \mathbf{u}\right] \tag{2.19}$$

$$= \Big[B'(\varrho + M(t))(\varrho + M(t)) - B(\varrho + M(t))\Big]{\rm div}_x\mathbf{u} - B'(\varrho)\Big(\partial_t M(t) + M(t){\rm div}_x \mathbf{u}\Big)$$

for any sublinear B.

Step 4

Taking

$$M(t) = \operatorname*{ess\,sup}_\Omega \varrho_0 \exp\left(L\int_0^t \|{\rm div}_x\mathbf{u}\|_{L^\infty(\Omega)}\ {\rm d}s\right)$$

we observe that

$$\Big(\partial_t M(t) + M(t){\rm div}_x\mathbf{u}\Big) \geq 0;$$

whence the choice $B(z) = z^+$ and integration (2.19) over Ω gives rise to the right inequality in (2.17). The left inequality can be deduced in a similar fashion.

□

It follows from the hypothesis (2.10) and the regularity of the weak solutions considered in Theorem 2.1 that $[\varrho, \mathbf{u}]$ is a renormalized solution of the equation of continuity (1.1). Consequently, under the hypotheses of Theorem 2.1, we have that the density ϱ complies with the bounds (2.17). We note that no extra regularity of $\partial\Omega$ was needed in the proof.

2.3.2 Renormalization of the thermal energy inequality

Multiplying equation (1.3) on $\chi'(\vartheta)$, $\chi' \geq 0$ and passing, formally, to the weak formulation, we obtain a renormalized modification of the thermal energy inequality (2.6):

$$c_v \int_\Omega \varrho\chi(\vartheta)\varphi\ {\rm d}x\Big|_{t=\tau_1}^{t=\tau_2} \tag{2.20}$$

$$\geq \int_{\tau_1}^{\tau_2}\int_\Omega \left[c_v\left(\varrho\chi(\vartheta)\partial_t\varphi + \varrho\chi(\vartheta)\mathbf{u}\cdot\nabla_x\varphi\right) + \chi'(\vartheta)\kappa(\vartheta)\nabla_x\vartheta\nabla_x\varphi\right]\ {\rm d}x\ {\rm d}t$$

$$+\int_{\tau_1}^{\tau_2}\int_\Omega \left[\chi'(\vartheta)\mathbb{S}(\nabla_x\mathbf{u}):\nabla_x\mathbf{u}\varphi-\kappa(\vartheta)\chi''(\vartheta)|\nabla_x\vartheta|^2\varphi-\varrho\vartheta\chi'(\vartheta)\mathrm{div}_x\mathbf{u}\varphi\right]\ \mathrm{d}x\ \mathrm{d}t$$

for any $\varphi\in C_c^\infty([0,T]\times\overline{\Omega})$, $\varphi\geq 0$.

Lemma 2.2. *Suppose that*

$$\mathbf{u}\in L^2(0,T;W^{1,2}(\Omega;R^3))\cap L^\infty((0,T)\times\Omega;R^3),\ \ \mathbf{u}\cdot\mathbf{n}|_{\partial\Omega}=0,\ \ \mathrm{div}_x\mathbf{u}\in L^1(0,T;L^\infty(\Omega)), \tag{2.21}$$

and

$$\varrho,\ \vartheta\in L^\infty((0,T)\times\Omega),\ \ \mathrm{ess\,inf}\,\varrho>0,\ \ \vartheta\in L^2(0,T;W^{1,2}(\Omega)). \tag{2.22}$$

satisfy the weak form of the continuity equation (2.4), together with the thermal energy inequality (2.6).

Then the renormalized thermal energy inequality (2.20) holds for any continuously differentiable function χ.

Proof: To begin, we note that, in accordance with the hypothesis (2.22), any composition $b(\vartheta)$ with a continuously differentiable b belongs to the class

$$b(\vartheta)\in L^\infty((0,T)\times\Omega)\cap L^2(0,T;W^{1,2}(\Omega)).$$

Next, we introduce the space $W_0^{-1,p}(\Omega)$ as the *dual* to $W^{1,p'}(\Omega)$, $\frac{1}{p}+\frac{1}{p'}=1$.

Step 1

It follows from the equation of continuity (2.4) and the hypotheses (2.21), (2.22) that

$$\partial_t\varrho\in L^\infty(0,T;W_0^{-1,p}(\Omega))\ \text{for any}\ 1<p<\infty, \tag{2.23}$$

in particular, the density ϱ itself can be interpreted as a Lipschitz mapping of t ranging in the Banach space $W_0^{-1,p}(\Omega)$,

$$\|\varrho(t,\cdot)-\varrho(s,\cdot)\|_{W_0^{-1,p}(\Omega)}=\sup_{\Phi\in W^{1,p'}(\Omega)}\int_\Omega\Big(\varrho(t,\cdot)-\varrho(s,\cdot)\Big)\Phi\ \mathrm{d}x\leq L|t-s|,\ \ t,s\in[0,T]. \tag{2.24}$$

Step 2

In accordance with the variational inequality (2.6), we may extend

$$\varrho\vartheta(\tau,\cdot)=\mathrm{ess}\lim_{t\to 0+}\varrho\vartheta(t,\cdot)\ \text{for all}\ \tau\leq 0,\ \ \varrho\vartheta(\tau,\cdot)=\mathrm{ess}\lim_{t\to T-}\varrho\vartheta(t,\cdot)\ \text{for all}\ \tau\geq 0, \tag{2.25}$$

setting, finally,

$$\vartheta(\tau,\cdot)=\left\{\begin{array}{l}\varrho\vartheta/\varrho_0\ \text{for}\ \tau\leq 0,\\ \\ \varrho\vartheta/\varrho(T,\cdot)\ \text{for}\ \tau\geq T.\end{array}\right.$$

Here, relation (2.25) is understood in the weak sense. We note that, thanks to the regularity properties (2.21), (2.22), the mapping $t \mapsto \varrho(t,\cdot)$ is (strongly) continuous in $L^p(\Omega)$ for any finite p.

Accordingly, we define a regularization $[v]_\delta$ by

$$[v]_\delta(t,\cdot) = h_\delta * v = \int_{-\infty}^{\infty} h_\delta(t-s)v(s,\cdot)\ \mathrm{d}s,$$

where $\{h_\delta = h_\delta(t)\}_{\delta>0}$ is a standard family of regularizing kernels in the time variable supported in a $\delta-$neighborhood of zero.

Step 3

We take the quantity $[\varphi\chi'([\vartheta]_\delta)]_\delta$, $\varphi \in C_c^\infty((0,T)\times\overline{\Omega})$, $\varphi,\ \chi' \geq 0$, as a test function in (2.6) to obtain

$$-\int_0^T \int_\Omega \left[c_v \left(\varrho\vartheta\partial_t \left[\varphi\chi'([\vartheta]_\delta)\right]_\delta + \varrho\vartheta\mathbf{u}\cdot\nabla_x \left[\varphi\chi'([\vartheta]_\delta)\right]_\delta\right) + \kappa(\vartheta)\nabla_x\vartheta\cdot\nabla_x\left[\varphi\chi'([\vartheta]_\delta)\right]_\delta\right]\ \mathrm{d}x\ \mathrm{d}t \tag{2.26}$$

$$\geq \int_0^T \int_\Omega \left[\int \mathbb{S}(\nabla_x\mathbf{u}) : \nabla_x\mathbf{u}\left[\varphi\chi'([\vartheta]_\delta)\right]_\delta - \varrho\vartheta\mathrm{div}_x\mathbf{u}\left[\varphi\chi'([\vartheta]_\delta)\right]_\delta\right]\ \mathrm{d}x\ \mathrm{d}t$$

as soon as δ is small enough. Note that this step can be fully justified by means of a density argument.

Thus, letting $\delta \to 0$ in (2.26), we deduce

$$-\lim_{\delta\to 0}\int_0^T \int_\Omega c_v\varrho\vartheta\partial_t\left[\varphi\chi'([\vartheta]_\delta)\right]_\delta\ \mathrm{d}x\ \mathrm{d}t - \int_0^T\int_\Omega \left[c_v\varrho\vartheta\mathbf{u}\cdot\nabla_x\left(\varphi\chi'(\vartheta)\right) + \kappa(\vartheta)\chi'(\vartheta)\nabla_x\vartheta\cdot\nabla_x\varphi\right]\ \mathrm{d}x\ \mathrm{d}t \tag{2.27}$$

$$\geq \int_0^T\int_\Omega\left[\int \mathbb{S}(\nabla_x\mathbf{u}) : \nabla_x\mathbf{u}\chi'(\vartheta)\varphi - \kappa(\vartheta)\chi''(\vartheta)|\nabla_x\vartheta|^2\varphi - \varrho\vartheta\mathrm{div}_x\mathbf{u}\chi'(\vartheta)\varphi\right]\ \mathrm{d}x\ \mathrm{d}t.$$

Step 4

It remains to identify the limit

$$\lim_{\delta\to 0}\int_0^T\int_\Omega \varrho\vartheta\partial_t\left[\varphi\chi'([\vartheta]_\delta)\right]_\delta\ \mathrm{d}x\ \mathrm{d}t = \lim_{\delta\to 0}\int_0^T\int_\Omega [\varrho\vartheta]_\delta\,\partial_t\left(\varphi\chi'([\vartheta]_\delta)\right)\ \mathrm{d}x\ \mathrm{d}t$$

$$= -\lim_{\delta\to 0}\int_0^T\int_\Omega \partial_t\left[\varrho\vartheta\right]_\delta\chi'([\vartheta]_\delta)\varphi\ \mathrm{d}x\ \mathrm{d}t.$$

To this end, we estimate the commutator

$$\partial_t\left[\varrho\vartheta\right]_\delta - \partial_t\left(\varrho[\vartheta]_\delta\right).$$

Assume, for a moment, that we can show

$$\partial_t\left[\varrho\vartheta\right]_\delta - \partial_t\left(\varrho[\vartheta]_\delta\right) \to 0 \text{ in } L^2(0,T;W_0^{-1,2}(\Omega)). \tag{2.28}$$

Taking (2.28) for granted and with (2.23) in mind, we deduce

$$\lim_{\delta\to 0}\int_0^T\int_\Omega \partial_t\left[\varrho\vartheta\right]_\delta \chi'([\vartheta]_\delta)\varphi \ \mathrm{d}x\ \mathrm{d}t = \lim_{\delta\to 0}\int_0^T\int_\Omega \partial_t\left(\varrho[\vartheta]_\delta\right)\chi'([\vartheta]_\delta)\varphi \ \mathrm{d}x\ \mathrm{d}t$$

$$= \int_0^T\int_\Omega \varrho\mathbf{u}\cdot\nabla_x\left(\vartheta\chi'(\vartheta)\varphi\right)\ \mathrm{d}x\ \mathrm{d}t - \int_0^T\int_\Omega \varrho\mathbf{u}\cdot\nabla_x\left(\chi(\vartheta)\varphi\right)\ \mathrm{d}x\ \mathrm{d}t$$

$$= \int_0^T\int_\Omega \varrho\mathbf{u}\cdot\nabla_x\left(\vartheta\chi'(\vartheta)\varphi\right)\ \mathrm{d}x + \int_0^T\int_\Omega \varrho\chi(\vartheta)\partial_t\varphi\ \mathrm{d}x\ \mathrm{d}t| + \lim_{\delta\to 0}\int_0^T\int_0^T\int_\Omega \varrho\partial_t\left(\chi([\vartheta]_\delta)\varphi\right)\ \mathrm{d}x\ \mathrm{d}t$$

$$= \int_0^T\int_\Omega \varrho\mathbf{u}\cdot\nabla_x\left(\vartheta\chi'(\vartheta)\varphi\right)\ \mathrm{d}x + \int_0^T\int_\Omega \varrho\chi(\vartheta)\partial_t\varphi\ \mathrm{d}x\ \mathrm{d}t - \int_0^T\int_0^T\int_\Omega \varrho\mathbf{u}\cdot\nabla_x\left(\chi(\vartheta)\varphi\right)\ \mathrm{d}x\ \mathrm{d}t,$$

which, combined with (2.27), yields the desired conclusion (2.20), at least for a.a. $0<\tau_1<\tau_2<T$. Finally, under present circumstances, the relations (2.9) imply *strong* continuity of the temperature at the time $t=0$; whence $\tau_1=0$ may be included by standard arguments.

Step 5

To conclude the proof, we have to show (2.28). To this end, we employ the arguments similar to those for the so-called Friedrichs lemma. Note that similar strategy was employed by Frehse et al. [12] in a slightly different context. Seeing that (2.28) obviously holds for a dense set of (smooth) functions ϑ in $L^2(0,T;W^{1,2}(\Omega))$, it is enough to establish the estimate

$$\left\|\partial_t\left[\varrho\vartheta\right]_\delta - \partial_t\left(\varrho[\vartheta]_\delta\right)\right\|_{L^2(0,T;W_0^{-1,2}(\Omega))} \leq c(\varrho,\mathbf{u})\left\|\vartheta\right\|_{L^2(0,T;W^{1,2}(\Omega))}. \tag{2.29}$$

We write

$$\int_0^T\int_\Omega\Big(\partial_t\left[\varrho\vartheta\right]_\delta - \partial_t\left(\varrho[\vartheta]_\delta\right)\Big)\Phi(x)\ \mathrm{d}x\ \mathrm{d}t = \int_0^T\int_\Omega\Phi(x)\int_{-\infty}^{\infty}\Big(\varrho(s,x)-\varrho(t,x)\Big)\vartheta(s,x)h'_\delta(t-s)\ \mathrm{d}s\ \mathrm{d}x\ \mathrm{d}t$$

$$-\int_0^T\int_\Omega\varrho\mathbf{u}\cdot\nabla_x\left([\vartheta]_\delta\Phi\right)\ \mathrm{d}x\ \mathrm{d}t.$$

Using (2.23) and taking the supremum of the last integral over Φ belonging to a unit ball in $W^{1,2}(\Omega)$ we observe that this term complies with (2.29). Finally, writing

$$\int_0^T\int_\Omega\Phi(x)\int_{-\infty}^{\infty}\Big(\varrho(s,x)-\varrho(t,x)\Big)\vartheta(s,x)h'_\delta(t-s)\ \mathrm{d}s\ \mathrm{d}x\ \mathrm{d}t$$

$$=\int_0^T\int_\Omega\Phi(x)\int_{-\infty}^{\infty}\frac{\varrho(s,x)-\varrho(t,x)}{s-t}\vartheta(s,x)(s-t)h'_\delta(t-s)\ \mathrm{d}s\ \mathrm{d}x\ \mathrm{d}t$$

we use (2.24) to conclude the proof of (2.29).

□

We remark that, similarly to Lemma 2.1, no regularity properties of $\partial\Omega$ were needed in the proof.

3 Entropy and relative energy inequality

We start by deriving the standard entropy inequality from (2.20). To this end, we first show that, under the hypotheses of Theorem 2.1, the temperature remains bounded below away from zero.

Lemma 3.1. *Let $[\varrho, \vartheta, \mathbf{u}]$ be a weak solution of the Navier-Stokes-Fourier system in $(0,T) \times \Omega$ belonging to the class (1.16). Suppose that*

$$\operatorname{ess}\inf_{\Omega} \vartheta_0 > 0.$$

Then there exists a constant $\underline{\vartheta} > 0$, depending only on the quantities $\overline{\varrho}$, $\overline{\vartheta}$, L from (1.16) and on T, such that

$$\vartheta(t, \cdot) \geq \underline{\vartheta} > 0 \text{ for a.a. } t \in [0,T]. \tag{3.1}$$

Proof:

As shown in Lemma 2.2, the solution $[\varrho, \vartheta, \mathbf{u}]$ satisfies the renormalized thermal energy balance (2.20). Taking

$$\chi_\delta(\vartheta) = -\frac{1}{(\vartheta + \delta)^\alpha}, \ \alpha > 1, \ \delta > 0$$

in (2.20), we deduce that

$$\int_\Omega \frac{1}{(\vartheta + \delta)^\alpha}(\tau, \cdot) \ \mathrm{d}x \leq \int_\Omega \frac{1}{(\vartheta_0 + \delta)^\alpha} \ \mathrm{d}x + \alpha \overline{\varrho}\overline{\vartheta} \int_0^\tau \int_\Omega |\mathrm{div}_x \mathbf{u}| \frac{1}{(\vartheta + \delta)^\alpha} \ \mathrm{d}x \ \mathrm{d}t$$

for a.a. $\tau \in [0,T]$.

Consequently, applying Gronwall's lemma and letting $\delta \to 0$, we obtain

$$\sup_{\alpha > 1} \left\{ \operatorname{ess} \sup_{\tau \in (0,T)} \left\| \frac{1}{\vartheta}(\tau, \cdot) \right\|_{L^\alpha(\Omega)} \right\} < \infty,$$

which yields the desired lower bound for ϑ.

□

In view of the results obtained in Lemmas 2.1 - 3.1 we may infer that, under the hypotheses of Theorem 2.1,

$$0 < \underline{\varrho} \leq \varrho(t,x) \leq \overline{\varrho}, \ 0 < \underline{\vartheta} \leq \vartheta(t,x) \leq \overline{\vartheta} \text{ for a.a. } (t,x) \in (0,T) \times \Omega. \tag{3.2}$$

3.1 Entropy inequality

The temperature being bounded below, we can take $\chi(\vartheta) = \log(\vartheta)$ in (2.20) to obtain the *entropy inequality*

$$\int_\Omega \varrho s(\varrho, \vartheta) \varphi \ \mathrm{d}x \Big|_{t=\tau_1}^{t=\tau_2} \tag{3.3}$$

$$\geq \int_{\tau_1}^{\tau_2} \int_\Omega \left[(\varrho s(\varrho,\vartheta)\partial_t \varphi + \varrho s(\varrho,\vartheta)\mathbf{u}\cdot\nabla_x\varphi) + \frac{\kappa(\vartheta)}{\vartheta}\nabla_x\vartheta\cdot\nabla_x\varphi \right] \ \mathrm{d}x \ \mathrm{d}t$$

$$+ \int_{\tau_1}^{\tau_2} \int_\Omega \frac{1}{\vartheta} \left[\mathbb{S}(\nabla_x\mathbf{u}) : \nabla_x\mathbf{u}\varphi + \frac{\kappa(\vartheta)}{\vartheta}|\nabla_x\vartheta|^2\varphi \right] \ \mathrm{d}x \ \mathrm{d}t$$

for any $\varphi \in C_c^\infty([0,T]\times\overline{\Omega})$, $\varphi \geq 0$, where

$$s(\varrho,\vartheta) = c_v \log(\vartheta) - \log(\varrho).$$

3.2 Relative energy (entropy)

Having collected all the necessary tools we are able to use the technique of relative entropies adapted to the Navier-Stokes-Fourier system as in [10]. We introduce the *relative energy functional*

$$\mathcal{E}\left(\varrho,\vartheta,\mathbf{u} \,\middle|\, r,\Theta,\mathbf{U}\right) = \int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u}-\mathbf{U}|^2 + H_\Theta(\varrho,\vartheta) - \frac{\partial H_\Theta(r,\Theta)}{\partial\vartheta} - H_\Theta(r,\Theta) \right] \ \mathrm{d}x,$$

where

$$H_\Theta(\varrho,\vartheta) = \varrho e(\varrho,\vartheta) - \varrho\Theta s(\varrho,\vartheta) = c_v\varrho\vartheta + \varrho P(\varrho) - \varrho\Theta\log\left(\frac{\vartheta^{c_v}}{\varrho}\right).$$

Now, exactly as in [10] we may deduce the *relative energy inequality* in the form:

$$\left[\mathcal{E}\left(\varrho,\vartheta,\mathbf{u}\middle| r,\Theta,\mathbf{U}\right)\right]_{t=0}^{t=\tau} + \int_0^\tau \int_\Omega \frac{\Theta}{\vartheta}\left(\mathbb{S}(\nabla_x\mathbf{u}) : \nabla_x\mathbf{u} + \frac{\kappa(\vartheta)|\nabla_x\vartheta|^2}{\vartheta}\right) \ \mathrm{d}x \ \mathrm{d}t \tag{3.4}$$

$$\leq \int_0^\tau \int_\Omega \Big(\varrho(\mathbf{U}-\mathbf{u})\cdot\partial_t\mathbf{U} + \varrho(\mathbf{U}-\mathbf{u})\otimes\mathbf{u} : \nabla_x\mathbf{U} - p(\varrho,\vartheta)\mathrm{div}_x\mathbf{U} + \mathbb{S}(\nabla_x\mathbf{u}) : \nabla_x\mathbf{U} \Big) \ \mathrm{d}x \ \mathrm{d}t$$

$$- \int_0^\tau \int_\Omega \left(\varrho\Big(s(\varrho,\vartheta) - s(r,\Theta)\Big)\partial_t\Theta + \varrho\Big(s(\varrho,\vartheta) - s(r,\Theta)\Big)\mathbf{u}\cdot\nabla_x\Theta - \frac{\kappa(\vartheta)\nabla_x\vartheta}{\vartheta}\cdot\nabla_x\Theta \right) \ \mathrm{d}x \ \mathrm{d}t$$

$$+ \int_0^\tau \int_\Omega \left(\left(1 - \frac{\varrho}{r}\right)\partial_t p(r,\Theta) - \frac{\varrho}{r}\mathbf{u}\cdot\nabla_x p(r,\Theta) \right) \ \mathrm{d}x \ \mathrm{d}t$$

for any trio of sufficiently regular test functions $[r,\Theta,\mathbf{U}]$, where $\mathbf{U}$ satisfies the same boundary conditions as $\mathbf{u}$. Note that, in contrast with [10], the weak solution $[\varrho,\vartheta,\mathbf{u}]$ satisfying the hypotheses of Theorem 2.1 is already quite regular, in particular bounded, so that the class of test functions can be extended considerably. As a matter of fact, we only need $[r,\Theta,\mathbf{U}]$ to be bounded with first derivatives square integrable.

3.3 Weak-strong uniqueness

Following further [10], we suppose that $r = \tilde{\varrho}$, $\vartheta = \tilde{\vartheta}$, $\mathbf{U} = \tilde{\mathbf{u}}$ is a strong solution of the Navier-Stokes-Fourier system emanating from the same initial data. After some manipulation, making use of the bounds already obtained, we arrive at

$$\left[\mathcal{E} \left(\varrho, \vartheta, \mathbf{u} \Big| \tilde{\varrho}, \tilde{\vartheta}, \tilde{\vartheta} \right) \right]_{t=0}^{t=\tau} \tag{3.5}$$

$$+ \int_0^\tau \int_\Omega \left(\frac{\tilde{\vartheta}}{\vartheta} - 1 \right) \left(\mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} + \frac{\kappa(\vartheta)|\nabla_x \vartheta|^2}{\vartheta} \right) \ \mathrm{d}x \ \mathrm{d}t$$

$$+ \int_0^\tau \int_\Omega \Big(\mathbb{S}(\nabla_x \mathbf{u}) - \mathbb{S}(\nabla_x \tilde{\mathbf{u}}) \Big) : \Big(\nabla_x \mathbf{u} - \nabla_x \tilde{\mathbf{u}} \Big) \ \mathrm{d}x \ \mathrm{d}t + \int_0^\tau \int_\Omega \frac{\kappa(\vartheta) \nabla_x \vartheta}{\vartheta} \cdot \Big(\nabla_x \vartheta - \nabla_x \tilde{\vartheta} \Big) \ \mathrm{d}x \ \mathrm{d}t$$

$$\leq c \int_0^\tau \left[\left\| \partial_t \tilde{\vartheta} \right\|_{L^\infty(\Omega)} + \|\partial_t \tilde{\mathbf{u}}\|_{L^\infty(\Omega;R^3)} + \left\| \nabla_x \tilde{\vartheta} \right\|_{L^\infty(\Omega;R^3)} + \|\nabla_x \tilde{\varrho}\|_{L^\infty(\Omega;R^3)} + \|\nabla_x \tilde{\mathbf{u}}\|_{L^\infty(\Omega;R^{3\times 3})} \right] \times$$

$$\times \left[\|\varrho - \tilde{\varrho}\|^2_{L^2(\Omega)} + \left\| \vartheta - \tilde{\vartheta} \right\|^2_{L^2(\Omega)} + \|\mathbf{u} - \tilde{\mathbf{u}}\|^2_{L^2(\Omega;R^3)} \right] \ \mathrm{d}t$$

$$+ \int_0^\tau \int_\Omega \tilde{\varrho} \Big(s(\tilde{\varrho}, \tilde{\vartheta}) - s(\varrho, \vartheta) \Big) \Big(\partial_t \tilde{\vartheta} + \tilde{\mathbf{u}} \cdot \nabla_x \tilde{\vartheta} \Big) \ \mathrm{d}x \ \mathrm{d}t$$

$$+ \int_0^\tau \int_\Omega \Big(p(\varrho, \vartheta) - p(\tilde{\varrho}, \tilde{\vartheta}) \Big) \mathrm{div}_x \tilde{\mathbf{u}} \ \mathrm{d}x \ \mathrm{d}t + \int_0^\tau \int_\Omega \left(1 - \frac{\varrho}{\tilde{\varrho}} \right) \Big(\partial_t p(\tilde{\varrho}, \tilde{\vartheta}) + \tilde{\mathbf{u}} \cdot \nabla_x p(\tilde{\varrho}, \tilde{\vartheta}) \Big) \ \mathrm{d}x \ \mathrm{d}t$$

where the constant depends only on the $L^\infty-$norms of both the strong and the weak solution considered.

Next, we observe that the above inequality keeps the same form if we replace

$$s(\varrho, \vartheta) - s(\tilde{\varrho}, \tilde{\vartheta}) \approx \frac{\partial s(\tilde{\varrho}, \tilde{\vartheta})}{\partial \varrho} (\varrho - \tilde{\varrho}) + \frac{\partial s(\tilde{\varrho}, \tilde{\vartheta})}{\partial \vartheta} (\vartheta - \tilde{\vartheta}),$$

$$p(\varrho, \vartheta) - p(\tilde{\varrho}, \tilde{\vartheta}) \approx \frac{\partial p(\tilde{\varrho}, \tilde{\vartheta})}{\partial \varrho} (\varrho - \tilde{\varrho}) + \frac{\partial p(\tilde{\varrho}, \tilde{\vartheta})}{\partial \vartheta} (\vartheta - \tilde{\vartheta}).$$

Using the fact that $[\tilde{\varrho}, \tilde{\vartheta}, \tilde{\mathbf{u}}]$ solves the equations we can reduce (3.5) to the following inequality:

$$\left[\mathcal{E} \left(\varrho, \vartheta, \mathbf{u} \Big| \tilde{\varrho}, \tilde{\vartheta}, \tilde{\mathbf{u}} \right) \right]_{t=0}^{t=\tau} \tag{3.6}$$

$$+ \int_0^\tau \int_\Omega \left(\frac{\tilde{\vartheta}}{\vartheta} - 1 \right) \left(\mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} + \frac{\kappa(\vartheta)|\nabla_x \vartheta|^2}{\vartheta} \right) \ \mathrm{d}x \ \mathrm{d}t$$

$$+ \int_0^\tau \int_\Omega \left(\frac{\vartheta}{\tilde{\vartheta}} - 1 \right) \left(\mathbb{S}(\nabla_x \tilde{\mathbf{u}}) : \nabla_x \tilde{\mathbf{u}} + \frac{\kappa(\tilde{\vartheta})|\nabla_x \tilde{\vartheta}|^2}{\tilde{\vartheta}} \right) \ \mathrm{d}x \ \mathrm{d}t$$

$$+\int_0^\tau \int_\Omega \Big(\mathbb{S}(\nabla_x \mathbf{u}) - \mathbb{S}(\nabla_x \tilde{\mathbf{u}}) \Big) : \Big(\nabla_x \mathbf{u} - \nabla_x \tilde{\mathbf{u}} \Big) \ {\rm d}x \ {\rm d}t$$

$$+\int_0^\tau \int_\Omega \left(\frac{\kappa(\vartheta)\nabla_x \vartheta}{\vartheta} - \frac{\kappa(\tilde\vartheta)\nabla_x \tilde\vartheta}{\tilde\vartheta} \right) \cdot \Big(\nabla_x \vartheta - \nabla_x \tilde\vartheta \Big) \ {\rm d}x \ {\rm d}t$$

$$\leq c \int_0^\tau \left[\left\| \partial_t \tilde\vartheta \right\|_{L^\infty(\Omega)} + \| \partial_t \tilde{\mathbf{u}} \|_{L^\infty(\Omega;R^3)} + \left\| \nabla_x \tilde\vartheta \right\|_{L^\infty(\Omega;R^3)} + \| \nabla_x \tilde\varrho \|_{L^\infty(\Omega;R^3)} + \| \nabla_x \tilde{\mathbf{u}} \|_{L^\infty(\Omega;R^{3\times 3})} \right] \times$$

$$\times \left[\| \varrho - \tilde\varrho \|^2_{L^2(\Omega)} + \left\| \vartheta - \tilde\vartheta \right\|^2_{L^2(\Omega)} + \| \mathbf{u} - \tilde{\mathbf{u}} \|^2_{L^2(\Omega;R^3)} \right] \ {\rm d}t.$$

Finally, eliminating the terms with transport coefficients on the left hand side of (3.6) in the same way as in [11, Section 6], we may infer that

$$\left[\mathcal{E} \left(\varrho, \vartheta, \mathbf{u} \Big| \tilde\varrho, \tilde\vartheta, \tilde{\mathbf{u}} \right) \right]_{t=0}^{t=\tau} \tag{3.7}$$

$$\leq c \int_0^\tau \left[\left\| \partial_t \tilde\vartheta \right\|_{L^\infty(\Omega)} + \| \partial_t \tilde{\mathbf{u}} \|_{L^\infty(\Omega;R^3)} + \left\| \nabla_x \tilde\vartheta \right\|_{L^\infty(\Omega;R^3)} + \| \nabla_x \tilde\varrho \|_{L^\infty(\Omega;R^3)} + \| \nabla_x \tilde{\mathbf{u}} \|_{L^\infty(\Omega;R^{3\times 3})} \right] \times$$

$$\times \mathcal{E} \left(\varrho, \vartheta, \mathbf{u} \Big| \tilde\varrho, \tilde\vartheta, \tilde{\mathbf{u}} \right) \ {\rm d}t.$$

Applying Gronwall's lemma we obtain the desired conclusion $\varrho = \tilde\varrho$, $\vartheta = \tilde\vartheta$, $\mathbf{u} = \tilde{\mathbf{u}}$.

Remark 3.1. *Note that (3.7) requires very mild assumptions concerning the* strong *solution* $[\tilde\varrho, \tilde\vartheta, \tilde{\mathbf{u}}]$*, namely*

$$\tilde\varrho, \ \tilde\vartheta \in L^\infty((0,T)\times\Omega), \nabla_x \tilde\varrho, \nabla_x \tilde\vartheta \in L^1(0,T;L^\infty(\Omega;R^3)) \cap L^2((0,T)\times\Omega;R^3), \ \partial_t \tilde\vartheta \in L^1(0,T;L^\infty(\Omega)) \tag{3.8}$$

$$\tilde{\mathbf{u}} \in L^\infty((0,T)\times\Omega;R^3), \ \partial_t \tilde{\mathbf{u}} \in L^1(0,T;L^\infty(\Omega;R^3)), \ \nabla_x \tilde{\mathbf{u}} \in L^1(0,T;L^\infty(\Omega;R^{3\times 3})) \cap L^2((0,T)\times\Omega;R^{3\times 3}). \tag{3.9}$$

4 Conditional regularity of smooth solutions

Our ultimate goal is to show that the weak solution $[\varrho, \vartheta, \mathbf{u}]$ enjoys the regularity claimed in (2.12 - 2.14). In view of the weak-strong uniqueness result established in the previous section, it is enough to work with the (local-in-time) strong solution of the same problem, the existence of which was established by Valli [21], and Valli and Zajaczkowski [22]. We follow step by step the arguments of [11], preforming the necessary modifications as the case may be. In what follows, $[\varrho, \vartheta, \mathbf{u}]$ will denote the local-tin-time strong solution emanating from the regular initial data (2.11).

4.1 Energy bounds for the velocity

To begin, we rewrite the momentum equation (1.2) in the form

$$\varrho \partial_t \mathbf{u} + \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}) = -\varrho \mathbf{u} \cdot \nabla_x \mathbf{u} - \varrho \nabla_x \vartheta - \varrho^{\gamma-1} \nabla_x \varrho = h_1 - \varrho^{\gamma-1} \nabla_x \varrho, \tag{4.1}$$

where, in accordance with the previous estimates

$$h_1 \in L^2(0,T; L^2(\Omega; R^3)).$$

By taking inner product with $\partial_t \mathbf{u}$ on both side of (4.1) and integrating on Ω, we find for any $t \in (0,T)$,

$$\int_\Omega |\partial_t \mathbf{u}(t,\cdot)|^2 \ \mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |\nabla_x \mathbf{u}(t,\cdot)|^2 \ \mathrm{d}x \leq c \int_\Omega |\nabla_x \varrho(t,\cdot)|^2 \ \mathrm{d}x.$$

Thus, combining standard elliptic estimates, we get

$$\sup_{t\in(0,T)} \|\mathbf{u}(t,\cdot)\|^2_{W^{1,2}(\Omega;R^3)} + \int_0^T \left[\|\mathbf{u}\|^2_{W^{2,2}(\Omega;R^3)} + \|\partial_t \mathbf{u}\|^2_{L^2(\Omega;R^3)} \right] \ \mathrm{d}t \leq c \int_0^T \|\nabla_x \varrho\|^2_{L^2(\Omega;R^3)} \ \mathrm{d}t. \tag{4.2}$$

On the other hand, we may differentiate the equation of continuity to deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |\nabla_x \varrho|^2 \ \mathrm{d}x \leq c \int_\Omega \left(|\nabla_x \varrho|^2 + |\nabla_x \varrho| |\nabla_x \mathrm{div}_x \mathbf{u}| \right) \ \mathrm{d}x \ \mathrm{d}t. \tag{4.3}$$

Combining (4.2), (4.3) we infer that

$$\sup_{t\in(0,T)} \left[\|\mathbf{u}(t,\cdot)\|_{W^{1,2}(\Omega;R^3)} + \|\nabla_x \varrho(t,\cdot)\|_{L^2(\Omega;R^3)} \right] + \int_0^T \left[\|\partial_t \mathbf{u}(t,\cdot)\|^2_{L^2(\Omega;R^3)} + \|\mathbf{u}(t,\cdot)\|^2_{W^{2,2}(\Omega;R^3)} \right] \ \mathrm{d}t \leq c. \tag{4.4}$$

Moreover, going back to the equation of continuity, we get

$$\sup_{t\in(0,T)} \|\partial_t \varrho(t,\cdot)\|_{L^2(\Omega)} \leq c. \tag{4.5}$$

Remark 4.1. *The bound above requires certain regularity of the domain, namely the validity of the elliptic estimates*

$$\|\mathbf{u}\|_{W^{2,2}(\Omega)} \leq \left(\|\mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u})\|_{L^2(\Omega;R^{3\times 3})} + \|\mathbf{u}\|_{L^2(\Omega;R^3)} \right) \tag{4.6}$$

4.2 Energy bounds for the temperature

We start by estimating the term $\mathbb{S} : \nabla_x \mathbf{u}$ by means of the Gagliardo-Nirenberg inequality, namely,

$$\|\mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u}\|_{L^2(\Omega)} \leq c_1 \|\nabla_x \mathbf{u}\|^2_{L^4(\Omega;R^{3\times 3})} \leq c_2 \|\mathbf{u}\|_{L^\infty(\Omega;R^3)} \|\Delta \mathbf{u}\|_{L^2(\Omega;R^3)}. \tag{4.7}$$

Thus, going back to (1.3) and using the standard parabolic estimates (cf. the proof in [7, Chapter 7, Lemma 7.4], we conclude

$$\sup_{t\in(0,T)} \|\vartheta(t,\cdot)\|_{W^{1,2}(\Omega)} + \int_0^T \left(\|\partial_t \vartheta(t,\cdot)\|^2_{L^2(\Omega)} + \|\vartheta(t,\cdot)\|^2_{W^{2,2}(\Omega)} \right) \ \mathrm{d}t \leq c. \tag{4.8}$$

Remark 4.2. *Here, similarly to the preceding step, the elliptic regularity*

$$\|\vartheta\|_{W^{2,2}(\Omega)} \leq c \left(\|\Delta \vartheta\|_{L^2(\Omega)} + \|\vartheta\|_{L^2(\Omega)} \right) \tag{4.9}$$

is needed.

4.3 Estimates of the time derivatives of ϱ and $\mathbf{u}$

Taking the time derivative of the momentum equation and denoting $\mathbf{v} = \partial_t \mathbf{u}$, we get

$$\varrho \left(\partial_t \mathbf{v} + \mathbf{u} \cdot \nabla_x \mathbf{v} \right) - \mathrm{div}_x \Big(\mathbb{S}(\nabla_x \mathbf{v}) \Big) = -\partial_t \varrho \mathbf{v} + \varrho \mathbf{v} \cdot \nabla_x \mathbf{u} - \partial_t \varrho \mathbf{u} \cdot \nabla_x \mathbf{u} - \nabla_x \partial_t p(\varrho, \vartheta);$$

whence

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \varrho |\mathbf{v}|^2 \ \mathrm{d}x + \int_\Omega \mathbb{S}(\nabla_x \mathbf{v}) : \nabla_x \mathbf{v} \ \mathrm{d}x = \int_\Omega \Big[\varrho \mathrm{div}_x \mathbf{u} |\mathbf{v}|^2 + \varrho \mathbf{u} \cdot \nabla_x \Big(\mathbf{u} \cdot \nabla_x \mathbf{u} \cdot \mathbf{v} \Big) + \partial_t p(\varrho, \vartheta) \mathrm{div}_x \mathbf{v} \Big] \ \mathrm{d}x.$$

In accordance with the uniform bounds (4.5-4.8), the integral on the right-hand side can be "absorbed" by means of Gronwall's type argument, and we may conclude that

$$\sup_{t \in (0,T)} \|\partial_t \mathbf{u}(t, \cdot)\|_{L^2(\Omega;R^3)} + \int_0^T \|\nabla_x (\partial_t \mathbf{u})(t, \cdot)\|^2_{L^2(\Omega;R^3)} \leq c. \tag{4.10}$$

We note that the most difficult term proportional to $\varrho \mathbf{u} |\nabla_x \mathbf{u}|^2$ can be handled with help of the Gagliardo-Nirenberg inequality

$$\left\| \varrho \mathbf{u} |\nabla_x \mathbf{u}|^2 \right\|^2_{L^2(\Omega;R^3)} \leq c \|\varrho\|_{L^\infty(\Omega)} \|\mathbf{u}\|^3_{L^\infty(\Omega;R^3)} \|\Delta \mathbf{u}\|^2_{L^2(\Omega;R^3)}.$$

Now, going back to (4.1), we get

$$\mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}) \in L^\infty(0,T; L^2(\Omega; R^3));$$

whence, by standard elliptic regularity estimates (4.6),

$$\sup_{t \in (0,T)} \|\mathbf{u}(t, \cdot)\|_{W^{2,2}(\Omega)} \leq c. \tag{4.11}$$

Since $W^{1,2} \circlearrowleft L^6$ we deduce from (4.10) and the other estimates already shown that

$$\int_0^\tau \|\mathbf{u}(t, \cdot)\|^2_{W^{2,6}(\Omega;R^3)} \ \mathrm{d}t \leq c \left(1 + \int_0^\tau \|\nabla_x \varrho(t, \cdot)\|^2_{L^6(\Omega;R^3)} \ \mathrm{d}t \right). \tag{4.12}$$

On the other hand, the equation of continuity (1.1) yields

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\nabla_x \varrho(t, \cdot)\|^6_{L^6(\Omega;R^3)} \leq c \left(\|\nabla_x \varrho(t, \cdot)\|^6_{L^6(\Omega;R^3)} + \|\mathbf{u}(t, \cdot)\|_{W^{2,6}(\Omega;R^3)} \|\nabla_x \varrho(t, \cdot)\|^5_{L^6(\Omega;R^3)} \right),$$

which, combined with (4.12), gives rise to

$$\sup_{t\in(0,T)}\left[\|\partial_t\varrho(t,\cdot)\|_{L^6(\Omega)}+\|\nabla_x\varrho(t,\cdot)\|_{L^6(\Omega;R^3)}\right]+\int_0^\tau\|\mathbf{u}(t,\cdot)\|^2_{W^{2,6}(\Omega;R^3)}\le c \tag{4.13}$$

Remark 4.3. *The bound (4.13) requires the elliptic estimates*

$$\|\mathbf{u}\|_{W^{2,q}(\Omega;R^3)}\le c\left(\|\mathrm{div}_x\mathbb{S}(\nabla_x\mathbf{u})\|_{L^q(\Omega;R^3)}+\|\mathbf{u}\|_{L^q(\Omega;R^3)}\right) \tag{4.14}$$

for $1<q\le\infty$.

4.4 Hölder continuity of the temperature

Since the principal part of the thermal energy balance (1.3) is non-linear, the higher order estimates of ϑ require a refined technique based on the L^p maximal regularity. To this end, we show that ϑ is Hölder continuous. We remark that ϱ enjoys Hölder regularity as a consequence of (4.13).

We rewrite the thermal energy balance in the form

$$c_v\partial_t\vartheta+\left(c_v\mathbf{u}-\frac{\kappa(\vartheta)}{\varrho^2}\nabla_x\varrho\right)\cdot\nabla_x\vartheta-\mathrm{div}_x\left(\frac{\kappa(\vartheta)}{\varrho}\nabla_x\vartheta\right)=\frac{1}{\varrho}\mathbb{S}(\nabla_x\mathbf{u}):\nabla_x\mathbf{u}-\vartheta\mathrm{div}_x\mathbf{u} \tag{4.15}$$

that can be viewed as a linear parabolic equation with non-constant coefficients.

Now, by virtue of (4.11), (4.13) we have

$$\frac{1}{\varrho}\mathbb{S}(\nabla_x\mathbf{u}):\nabla_x\mathbf{u}-\vartheta\mathrm{div}_x\mathbf{u}\in L^\infty(0,T;L^3(\Omega)),\ \varrho\in L^\infty(0,T;W^{1,6}(\Omega)); \tag{4.16}$$

whence the nowadays standard parabolic theory (see e.g. Ladyzhenskaya et al. [17]) yields

$$\vartheta \text{ Hölder continuous in } [0,T]\times\overline{\Omega}. \tag{4.17}$$

4.5 L^p-L^q parabolic estimates

Seeing that both ϱ and ϑ are Hölder continuous, we are ready to apply the machinery of the L^p-L^q estimates (see Amann [1], Denk, Hieber, and Prüss [4], Krylov [16]) to the parabolic problems (4.1) and (4.15).

First, in accordance with (4.16), (4.17), we obtain

$$\vartheta\in L^p(0,T;W^{2,3}(\Omega)),\ \partial_t\vartheta\in L^p(0,T;L^3(\Omega)),\ \text{for any } 1<p<\infty$$

in particular,

$$\nabla_x\vartheta\in L^p(0,T;L^q(\Omega;R^3)) \text{ for all } 1<p,q<\infty. \tag{4.18}$$

Now, using (4.18) and applying the same treatment to (4.1), we deduce

$$\mathbf{u}\in L^p(0,T;W^{2,6}(\Omega;R^3)),\ \partial_t\mathbf{u}\in L^p(0,T;L^6(\Omega)) \text{ for all } 1<p<\infty. \tag{4.19}$$

Finally, since (4.19) implies

$$\nabla_x \mathbf{u} \in L^\infty((0,T) \times \Omega), \tag{4.20}$$

we are in the situation treated in [11]. Thus the proof of Theorem 2.1 is completed by the same arguments as in [11, Section 4.6].

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