Mathematics of fluids in motion: Analysis and/or numerics

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Navier-Stokes-Fourier system

Mass conservation

$$\partial_t \varrho + \mathrm{div}_x (\varrho \mathbf{u}) = 0$$

Momentum balance

$$\partial_t (\varrho \mathbf{u}) + \mathrm{div}_x (\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u})$$

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \eta \mathrm{div}_x \mathbf{u} \mathbb{I}$$

Thermal energy equation

$$c_v \left[\partial_t (\varrho \vartheta) + \mathrm{div}_x (\varrho \vartheta \mathbf{u}) \right] - \mathrm{div}_x (\kappa(\vartheta) \nabla_x \vartheta)$$

$$= \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \varrho p_\vartheta(\varrho, \vartheta) \mathrm{div}_x \mathbf{u}$$

Numerical scheme

Implicit time discretization

$$v_h(t,\cdot) = v_h^k \text{ for } t \in [k\Delta h, (k+1)\Delta h),\ k = 0,1,\dots,N$$

$$D_t v_h^k = \frac{v_h^k - v_h^{k-1}}{\Delta t}$$

Triangulation

$$\text{triangulation } K \in K_h,\ h \approx \Delta t,\ \text{edges } \Gamma \in \Gamma_h$$

Finite element spaces

$$Q_h(\Omega) = \left\{ v \ \middle|\ v|_K = \mathrm{const}\ K \in K_h \right\}$$

$$V_h(\Omega) = \left\{ \mathbf{v} \in L^2_{\mathrm{div}}(\Omega; R^3) \ \middle|\ \mathbf{v}\cdot\mathbf{n}|_{\partial\Omega} = 0,\ \mathbf{v}|_K = a_k\mathbf{x} + \mathbf{b}_k,\ K \in K_h \right\}$$

Numerical scheme [Karlsen-Karper], I

Equation of continuity

$$\int_\Omega D_t \varrho_h^k \varphi_h \ \mathrm{d}x \equiv \int_\Omega \frac{\varrho_h^k - \varrho_h^{k-1}}{\Delta t} \varphi_h \ \mathrm{d}x = \sum_{\Gamma \in \Gamma_h} \int_\Gamma \mathrm{Up}[\varrho_h^k, u_h^k] \ [[\varphi_h]] \ \mathrm{dS}_x$$

for all $\varphi_h \in Q_h(\Omega)$

Momentum equation

$$\int_\Omega D_t \left(\varrho_h^k \widehat{\mathbf{u}}_h^k \right) \cdot \varphi_h \ \mathrm{d}x - \sum_{\Gamma \in \Gamma_h} \int_\Gamma \mathrm{Up}[\varrho_h^k \widehat{\mathbf{u}}_h^k, \mathbf{u}_h^k] \ \cdot [[\widehat{\varphi}_h]] \mathrm{dS}_x$$

$$- \int_\Omega p(\varrho_h^k, \vartheta_h^k) \mathrm{div}_x \varphi_h \ \mathrm{d}x$$

$$= - \int_\Omega \left(\mu \mathbf{curl}^*[\mathbf{u}_h^k] \cdot \mathbf{curl}^*[\varphi_h] + (\lambda + \mu) \mathrm{div}_x \mathbf{u}_h^k \mathrm{div}_x \varphi_h \right) \ \mathrm{d}x$$

for all $\varphi_h \in V_h(\Omega)$

Numerical scheme [Karlsen-Karper], II

Energy equation

$$c_v \int_\Omega D_t \left(\varrho_h^k \vartheta_h^k \right) \varphi_h \ \mathrm{d}x - c_v \sum_{\Gamma \in \Gamma_h} \int_\Gamma \mathrm{Up}[\varrho_h^k \vartheta_h^k, \mathbf{u}_h^k] \ [[\varphi_h]] \mathrm{dS}_x$$

$$+ \sum_{\Gamma \in \Gamma_h} \int_\Gamma \frac{1}{h} [[K(\vartheta_h^k)]] \ [[\varphi_h]] \mathrm{dS}_x$$

$$= \int_\Omega \left(\mu |\mathbf{curl}^* [\mathbf{u}_h^k]|^2 + (\lambda + \mu) |\mathrm{div}_x \mathbf{u}_h^k|^2 \right) \varphi_h \ \mathrm{d}x$$

$$- \int_\Omega \vartheta_h^k \partial_\vartheta p(\varrho_h^k, \vartheta_h^k) \mathrm{div}_x \mathbf{u}_h^k \varphi_h \ \mathrm{d}x$$

for all $\varphi_h \in Q_h(\Omega)$, $K'(\vartheta) = \kappa(\vartheta)$

Frequently asked questions...

Analysis

- Are there *a priori* bounds for solutions?
- Does the problem admit a solution and in which sense?
- Is the solution unique and regular?

Numerics

- Is the numerical method stable?
- Is the numerical method convergent?
- Is the convergence unconditional; what are the error estimates?

Navier-Stokes-Fourier system (weak form)

Mass conservation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum balance

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u})$$

Thermal energy balance

$$c_v \partial_t(\varrho \vartheta) + c_v \mathrm{div}_x(\varrho \vartheta \mathbf{u}) - \Delta K(\vartheta)$$

$$\boxed{\geq} \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \varrho p_\vartheta(\varrho, \vartheta) \mathrm{div}_x \mathbf{u}$$

Total energy balance

$$\frac{\mathrm{d}}{\mathrm{d}t} \int \left[\frac{1}{2} \varrho |\mathbf{u}|^2 + c_v \varrho \vartheta + P(\varrho) \right] \mathrm{d}x \boxed{\leq} 0$$

Renormalization

Renormalized equation of continuity

$$\partial_t b(\varrho) + \mathrm{div}_x (b(\varrho)\mathbf{u}) + \Big(b'(\varrho)\varrho - b(\varrho) \Big) \mathrm{div}_x \mathbf{u} = 0$$

Renormalized thermal energy balance

$$c_v \partial_t (\varrho H(\vartheta)) + c_v \mathrm{div}_x (\varrho H(\vartheta)\mathbf{u}) - \mathrm{div}_x \Big(H'(\vartheta)\kappa(\vartheta)\nabla_x \vartheta \Big) \geq$$

$$H'(\vartheta)\mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - H''(\vartheta)\kappa(\vartheta)|\nabla_x \vartheta|^2 - aH'(\vartheta)\vartheta\varrho \mathrm{div}_x \mathbf{u}$$

Entropy balance

$$\partial_t (\varrho s(\varrho, \vartheta)) + \mathrm{div}_x (\varrho s(\varrho, \vartheta)\mathbf{u}) - \mathrm{div}_x \left(\frac{\kappa(\vartheta)}{\vartheta} \nabla_x \vartheta \right)$$

$$\geq \frac{1}{\vartheta} \left(\mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} + \frac{\kappa(\vartheta)}{\vartheta} |\nabla_x \vartheta|^2 \right)$$

Compactness - convergence

Density oscillations

$$\partial_t \overline{\varrho \log(\varrho)} + \mathrm{div}_x \left(\overline{\varrho \log(\varrho)} \right) \mathbf{u}) + \overline{\varrho \mathrm{div}_x \mathbf{u}} = 0$$

$$\partial_t (\varrho \log(\varrho)) + \mathrm{div}_x \left(\varrho \log(\varrho) \right) \mathbf{u}) + \varrho \mathrm{div}_x \mathbf{u} = 0$$

Effective viscous flux

$$0 \leq \overline{p(\varrho)\varrho} - \overline{p(\varrho)}\ \varrho = \overline{\varrho \mathrm{div}_x \mathbf{u}} - \varrho \mathrm{div}_x \mathbf{u}$$

Biting limit of the temperature

$$\lim K_\alpha(\vartheta_\varepsilon) = K_\alpha(\vartheta),\ \ K_\alpha \nearrow K$$

Conditional regularity

Hypotheses

$$\text{regular initial data } \varrho_0 > 0,\ \vartheta_0 > 0,\ \mathbf{u}_0$$

$$\mathrm{div}_x \mathbf{u} \in L^1(0,T; L^\infty(\Omega)),\ \mathbf{u} \in L^\infty((0,T) \times \Omega; R^3)$$

$$\vartheta < \overline{\vartheta}$$

Conclusion

The weak solution is regular (smooth); whence unique

Corollary

Bounded numerical solutions converge unconditionally to the unique strong solution

Relative entropy (modulated energy)

Relative entropy functional

$$\mathcal{E}\left(\varrho, \vartheta, \mathbf{u} \;\Big|\; r, \Theta, \mathbf{U}\right)$$

$$= \int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u} - \mathbf{U}|^2 + H_\Theta(\varrho, \vartheta) - \frac{\partial H_\Theta(r, \Theta)}{\partial \varrho}(\varrho - r) - H_\Theta(r, \Theta) \right) \, \mathrm{d}x$$

Ballistic free energy

$$H_\Theta(\varrho, \vartheta) = \varrho\Big(e(\varrho, \vartheta) - \Theta s(\varrho, \vartheta)\Big)$$

Coercivity of the ballistic free energy

$$\varrho \mapsto H_\Theta(\varrho, \Theta) \text{ strictly convex}$$

$$\vartheta \mapsto H_\Theta(\varrho, \vartheta) \text{ decreasing for } \vartheta < \Theta \text{ and increasing for } \vartheta > \Theta$$

Dissipative solutions

Relative entropy inequality

$$\left[\mathcal{E}\left(\varrho,\vartheta,\mathbf{u}\middle| r,\Theta,\mathbf{U}\right)\right]_{t=0}^{\tau}$$

$$+\int_0^\tau \int_\Omega \frac{\Theta}{\vartheta}\left(\mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{\mathbf{q}(\vartheta,\nabla_x\vartheta)\cdot\nabla_x\vartheta}{\vartheta}\right)\ \mathrm{d}x\ \mathrm{d}t$$

$$\leq \int_0^\tau \mathcal{R}(\varrho,\vartheta,\mathbf{u},r,\Theta,\mathbf{U})\ \mathrm{d}t$$

for any $r > 0$, $\Theta > 0$, $\mathbf{U}$ satisfying relevant boundary conditions

Remainder

$$\boxed{\mathcal{R}(\varrho, \vartheta, \mathbf{u}, r, \Theta, \mathbf{U})}$$

$$= \int_\Omega \left(\varrho \Big(\partial_t \mathbf{U} + \mathbf{u} \cdot \nabla_x \mathbf{U} \Big) \cdot (\mathbf{U} - \mathbf{u}) + \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{U} \right) \ \mathrm{d}x$$

$$+ \int_\Omega \left[\Big(p(r, \Theta) - p(\varrho, \vartheta) \Big) \mathrm{div} \mathbf{U} + \frac{\varrho}{r} (\mathbf{U} - \mathbf{u}) \cdot \nabla_x p(r, \Theta) \right] \ \mathrm{d}x$$

$$- \int_\Omega \left(\varrho \Big(s(\varrho, \vartheta) - s(r, \Theta) \Big) \partial_t \Theta + \varrho \Big(s(\varrho, \vartheta) - s(r, \Theta) \Big) \mathbf{u} \cdot \nabla_x \Theta \right.$$

$$\left. + \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta)}{\vartheta} \cdot \nabla_x \Theta \right) \ \mathrm{d}x$$

$$+ \int_\Omega \frac{r - \varrho}{r} \Big(\partial_t p(r, \Theta) + \mathbf{U} \cdot \nabla_x p(r, \Theta) \Big) \ \mathrm{d}x$$