Weak solutions to the equations of quantum fluids

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Euler-Korteweg-Poisson system

Mass conservation - equation of continuity

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum equations - Newton's second law

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho)$$

$$= \boxed{\varrho \nabla_x \left(K(\varrho)\Delta_x \varrho + \frac{1}{2} K'(\varrho)|\nabla_x \varrho|^2 \right)} - \varrho \mathbf{u} + \varrho \nabla_x V$$

Poisson equation

$$\Delta_x V = \varrho - \overline{\varrho}$$

Alternative formulation

Korteweg tensor

$$\varrho \nabla_x \left(K(\varrho)\Delta_x \varrho + \frac{1}{2} K'(\varrho) |\nabla_x \varrho|^2 \right)$$

$$K(\varrho) = \overline{K} \text{ -capillarity, } \ K(\varrho) = \frac{\hbar}{4\varrho} \text{ -quantum fluids}$$

Korteweg tensor in divergence form

$$\varrho \nabla_x \left(K(\varrho)\Delta_x \varrho + \frac{1}{2} K'(\varrho) |\nabla_x \varrho|^2 \right) = \mathrm{div}_x \mathcal{K}(\varrho, \nabla_x \varrho)$$

$$\mathcal{K}(\varrho, \nabla_x \varrho) = \left[\chi(\varrho)\Delta_x \varrho + \frac{1}{2} \chi'(\varrho) |\nabla_x \varrho|^2 \right] \mathbb{I} - 4\chi(\varrho) \nabla_x \sqrt{\varrho} \otimes \nabla_x \sqrt{\varrho}$$

Motivation: Quantum fluids

Unknown variables

$$\varrho, \ \mathbf{J} = \varrho \mathbf{u}$$

System of equations

$$\partial_t \varrho + \mathrm{div}_x \mathbf{J} = 0$$

$$\partial_t \mathbf{J} + \mathrm{div}_x \left(\frac{\mathbf{J} \times \mathbf{J}}{\varrho} \right) + \nabla_x p(\varrho) + \mathbf{J} = \frac{\hbar}{2} \varrho \nabla_x \left(\frac{\Delta_x \sqrt{\varrho}}{\sqrt{\varrho}} \right) + \varrho \nabla_x V$$

$$\Delta_x V = \varrho - \overline{\varrho}$$

Alternative description

Ansatz

$$\varrho = |\Psi|^2, \ \mathbf{J} = \hbar \Im[\overline{\psi} \nabla_x \psi]$$

Schrödinger equation

$$\imath \hbar \partial_t \psi = -\frac{\hbar^2}{2} \Delta_x \psi - V\psi + a|\psi|^{\gamma-1}\psi - \boxed{\imath \hbar \log\left(\psi / \overline{\psi}\right)}$$

Poisson equation

$$\Delta_x V = |\psi|^2, \ \overline{\varrho} = 0$$

Pressure

$$p(\varrho) = \frac{\gamma - 1}{\gamma + 1} \varrho^{(\gamma+1)/2}$$

Weak solutions?

Density

Density ϱ must be sufficiently regular

Vacuum zones

Density ϱ may vanish on some non-trivial subset of Ω

Singularities ?

Shock waves for the momentum field $\mathbf{J}$?

Boundary and initial conditions

Geometry

$$t \in (0,T),\ x \in \mathbb{T}^3 = \left([0,1]\Big|_{\{0;1\}} \right)^3 - \text{periodic b.c.}$$

Initial conditions

$$\varrho(0,\cdot) = \varrho_0 = r_0^2,\ r_0 \in C^2,\ \mathrm{meas}\left\{ x \in \mathbb{T}^3 \ \Big|\ r_0(x) = 0 \right\} = 0$$

$$\mathbf{J}(0,\cdot) = \mathbf{J}_0 = \varrho_0 \mathbf{U}_0,\ \mathbf{U}_0 \in C^3$$

Reformulation, Step 1

Extending the density

$$\partial_t \varrho + \mathrm{div}_x \tilde{\mathbf{J}} = 0, \ \varrho(0,\cdot) = \varrho_0$$

Flux ansatz

$$\tilde{\mathbf{J}} = \varrho(\mathbf{U}_0 - Z), \ Z = Z(t)$$

$$\partial_t \int_{\mathbb{T}^3} \mathbf{H}[\tilde{\mathbf{J}}] \ \mathrm{d}x + \int_{\mathbb{T}^3} \mathbf{H}[\tilde{\mathbf{J}}] \ \mathrm{d}x = 0$$

$\mathbf{H}$ – standard Helmholtz projection

$$\mathrm{meas} \left\{ x \in \mathbb{T}^3 \ \middle| \ \varrho(t,x) = 0 \right\} = 0 \text{ for any } t \in [0,T]$$

Reformulation, Step 2

Flux ansatz

$$\mathbf{J} = \tilde{\mathbf{J}} + \mathbf{w},\ \mathrm{div}_x \mathbf{w} = 0,\ \mathbf{w}(0, \cdot) = 0$$

$$\mathbf{w} \in \boxed{C_{\rm weak}([0,T], L^2(\Omega; R^3))} \cup L^\infty((0,T) \times \Omega; R^3)$$

Equations

$$\partial_t \left(\mathbf{w} + \tilde{\mathbf{J}} \right) + \mathrm{div}_x \left(\frac{(\mathbf{w} + \tilde{\mathbf{J}}) \otimes (\mathbf{w} + \tilde{\mathbf{J}})}{\varrho} \right) + \nabla_x p(\varrho) + \left(\mathbf{w} + \tilde{\mathbf{J}} \right) =$$

$$\nabla_x \Big(\chi(\varrho) \Delta_x \varrho \Big) + \frac{1}{2} \nabla_x \Big(\chi'(\varrho) |\nabla_x \varrho|^2 \Big) - 4 \mathrm{div}_x \Big(\chi(\varrho) \nabla_x \sqrt{\varrho} \otimes \nabla_x \sqrt{\varrho} \Big)$$
$$+ \varrho \nabla_x V$$

Reformulation, Step 3

Final flux ansatz

$$\tilde{\mathbf{J}} = \mathbf{H}[\tilde{\mathbf{J}}] + \nabla_x M, \ \mathbf{v} = e^t \left(\mathbf{w} + \mathbf{H}[\tilde{\mathbf{J}}] \right),$$

Equations

$$\mathrm{div}_x \mathbf{v} = 0, \ \mathbf{v}(0,\cdot) = \mathbf{H}[\mathbf{J}_0]$$

$$\partial_t \mathbf{v} + \mathrm{div}_x \left(\frac{(\mathbf{v}+\mathbf{h}) \otimes (\mathbf{v}+\mathbf{h})}{r} + \mathbb{H} \right) + \nabla_x \Pi = 0$$

Coefficients

$$r = e^t \varrho, \ \mathbf{h} = e^t \nabla_x M$$

Driving terms

Convective term

$$\mathbb{H}(t,x) = 4e^t \left(\chi(\varrho) \nabla_x \sqrt{\varrho} \otimes \nabla_x \sqrt{\varrho} - \frac{1}{3} \chi(\varrho) |\nabla_x \sqrt{\varrho}|^2 \mathbb{I} \right)$$

$$4e^t \left(\frac{1}{3} |\nabla_x V|^2 \mathbb{I} - \nabla_x V \otimes \nabla_x V \right), \ \mathbb{H} \in R^{3\times 3}_{0,\mathrm{sym}}$$

Pressure term

$$\Pi(t,x) = e^t \Big(p(\varrho) + \partial_t M + M - \chi(\varrho) \Delta_x \varrho \Big)$$

$$-e^t \left(\frac{1}{2} \chi'(\varrho) |\nabla_x \varrho|^2 - \frac{4}{3} \chi(\varrho) |\nabla_x \sqrt{\varrho}|^2 + \overline{\varrho} V + \frac{1}{3} |\nabla_x V|^2 \right) + \boxed{\Lambda}$$

$\Lambda -$ a suitable constant

Convex integration

Field equations, constitutive relations

$$\partial_t \mathbf{u} + \mathrm{div}_x \mathbb{V} = 0, \ \ \mathbb{V} = \mathbb{F}(\mathbf{u})$$

Reformulation, subsolutions

$$\mathbb{V} = \mathbb{F}(\mathbf{u}) \Leftrightarrow G(\mathbf{u}, \mathbb{V}) = E(\mathbf{u}), E(\mathbf{u}) \leq G(\mathbf{u}, \mathbb{V}) < \overline{e}(\mathbf{u})$$

E convex, $\overline{e}$ "concave"

Oscillatory lemma, oscillatory increments

$$\partial_t \mathbf{u}_\varepsilon + \mathrm{div}_x \mathbb{V}_\varepsilon = 0, \ \ \mathbf{u}_\varepsilon \boxed{\rightharpoonup} 0$$

$$E(\mathbf{u} + \mathbf{u}_\varepsilon) \leq G\left(\mathbf{u} + \mathbf{u}_\varepsilon, \mathbb{V} + \mathbb{V}_\varepsilon\right) < \overline{e}(\mathbf{u} + \mathbf{u}_\varepsilon)$$

$$\liminf \int E(\mathbf{u}_\varepsilon) \boxed{\geq} \int (\overline{e}(\mathbf{u}) - E(\mathbf{u}))^\alpha$$

Applications to incompressible flows

Incompressible Euler system - DeLellis, Székelyhidi [2008]

$$\mathbf{h} = 0, \ \mathbb{H} = 0, \ r = 1, \ \Pi = e(t, x)$$

Compressible Euler with solenoidal data - Chiodaroli [2013]

$$r = r(x), \ \mathbf{h} = 0, \ \mathbb{H} = \mathbb{H}(x), \ \Pi = e(t, x)$$

Present situation

r, $\mathbf{h}$, $\mathbb{H}$, Π continuous functions of both t and x on the $\boxed{open}$ set

$$\left\{ (t, x) \ \middle| \ \varrho(t, x) > 0 \right\}$$

Basic ideas of analysis

Localization

Localizing the result of DeLellis and Széhelyhidi to "small" cubes by means of scaling arguments

Linearization

Replacing all continuous functions by their means on any of the "small" cubes

Covering the non-vacuum set

Applying Whitney's decomposition lemma to the non-vacuum set $\{\varrho > 0\}$

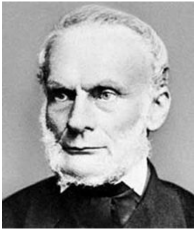
Existence results

Good news

The problem admits global-in-time (finite energy) weak solutions of any (large) initial data

Bad news

There are infinitely many solutions for given initial data



Rudolph Clausius
[1822-1888]

Die Energie der Welt ist constant;
Die Entropie der Welt strebt einem Maximum zu

Energy

$$E(\varrho, \nabla_x \varrho, \mathbf{J}) = \frac{1}{2} \frac{|\mathbf{J}|^2}{\varrho} + P(\varrho) + 2\chi(\varrho)|\nabla_x \sqrt{\varrho}|^2 + \frac{1}{2}|\nabla_x V|^2$$

What's wrong?

Energy production

"Most" solutions constructed by convex integration produce energy!

Admissible solutions

Admissible solutions should conserve or at least dissipate the total energy. Admissible solutions do comply with the weak strong uniqueness principle. Weak and strong solutions emanating from the same initial data coincide as long as the latter exists.

Infinitely many admissible solutions

For any regular ϱ_0 there exists a (non-smooth) $\mathbf{u}_0$ such that the problem has infinitely many admissible solutions.