CERGE-EI, Charles University and Czech Academy of Sciences
Term: Summer Preparatory Semester 2011
Course Title: Mathematics
Instructor: Sherzod Tashpulatov
Teaching Assistant: Svyatoslav Vovchak

Final Exam
Student Name: ______________________
Date: August 15, 2011

1 (20%) *Limits, Derivation, and Integration*

a) Using the implicit function theorem, find $\frac{dy}{dx}$ at $x_0 = 1$, $y_0 = 0$ for $x^2 e^y + ye^x - 5 = 0$.

b) $\int\limits_0^1 \ln(1+\sqrt{x})\, dx$

c) $\frac{d}{dr}\left(\int\limits_0^r (t^{51} \cdot e^{5t} + \frac{1+r}{\sqrt{t}})\, dt\right)$

2 (12%) Extremize $f(x,y) = xy^2 + x^3y - xy + 1$. Classify the stationary points (i.e., solutions to first-order conditions) based on the analysis of eigenvalues from the derived Hessian matrix. Provide a summary of x^*, y^*, and f^*.

3 (8%) Consider the system of ordinary differential equations

$$\begin{cases} \dot{x} &= x + z \\ \dot{y} &= y + z \\ \dot{z} &= x + y + 2z \end{cases}$$

Solve the system. Discuss the stability of the solution as $t \to \infty$.

4 (25%) *Phase Diagrams*

a) Find the steady states of $\dot{x} = x^3 - 13x + 12$. Using the phase diagram, characterize the stability of the steady states.

b) Find the steady states of the following dynamic system

$$\begin{cases} \dot{x} = (x-2)(x+y-4) \\ \dot{y} = (y-3)(x-y+2) \end{cases}$$

Assume that $x \geq 0$ and $y \geq 0$, which means that you are asked to analyze only the first quadrant. Draw the phase diagram, which includes steady state points, loci, and directional arrows. Characterize the stability of the steady states.

5 (20%) Extremize $2x + y + z^2$ subject to the constraints $x^2 + y^2 + z^2 = 5$ and $y \geq 1$ using the Lagrangian. In particular, find the FOC solution. Apply the constraint qualification test. Provide a summary of the optimal solution.

6 (15%) Maximize $\int\limits_0^1 (-2x^2 - 4xu - 5u^2)\, dt$ subject to $\dot{x} = 2x + 3u$ and $x(0) = 0$, $x(1) = 1$.