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ON THE LINEAR PROBLEM ARISING FROM MOTION OF A FLUID AROUND A MOVING RIGID BODY

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Abstract. We study a linear system of equations arising from a fluid motion around a moving rigid body. The aim of the present paper is the proof of the existence of a strong solution in a weighted Lebesgue space. In particular, we prove the existence of a global pressure gradient in L^2.

Keywords: incompressible fluid, rotating rigid body, strong solution

MSC 2010: 35Q35

1. Mathematical formulation

In the present paper we study the initial-boundary value problem of the motion of a viscous fluid around a moving rigid body. First we will give the mathematical formulation of the problem.

Let $\mathcal{B}$ denote an open, connected and bounded C^2 domain, representing a rigid body in a fluid motion in $\mathcal{D} := \mathbb{R}^3 \setminus \overline{\mathcal{B}}$. Clearly, $\mathcal{D}$ defines an exterior C^2 domain in $\mathbb{R}^3$ with boundary $\Sigma = \partial\mathcal{D} = \partial\mathcal{B}$.

The motion of the fluid and the body will be governed by the following system of equations.

Equations of fluid

$$
\left.\begin{aligned}
&\operatorname{div} \boldsymbol{w} = 0\\
&\frac{\partial \boldsymbol{w}}{\partial t} + (\boldsymbol{w} - \boldsymbol{U}) \cdot \nabla \boldsymbol{w} + \boldsymbol{\omega} \times \boldsymbol{w} =\\
&\qquad = \operatorname{div} \mathbb{T}(\boldsymbol{w}, \pi) + \boldsymbol{Q}^{\top} \cdot \mathcal{F}(x,t)
\end{aligned}\right\} \quad \text{in} \quad \mathcal{D} \times (0,T), \tag{1}
$$

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where $\boldsymbol{U} = \boldsymbol{\xi} + \boldsymbol{\omega} \times y$. Here by $\boldsymbol{w}$ we denote the velocity of the fluid and by $\boldsymbol{U}$ the velocity of the body, where $\boldsymbol{\xi}$ stands for its translation and $\boldsymbol{\omega}$ for its rotation. Furthermore,

$$\mathbb{T}(\boldsymbol{w}, p) = 2\nu \boldsymbol{D}\boldsymbol{w} - \boldsymbol{I}p\,^{1)} \qquad \textit{the Cauchy stress tensor},$$

where the constant $\nu > 0$ denotes the viscosity. In addition, the term $\boldsymbol{Q}^\top \mathcal{F}$ represents a given external force, while the tensor $Q^\top$ is related to ω in the following way

$$(2) \qquad \frac{d\boldsymbol{Q}^\top}{dt} = \Omega(\omega)\boldsymbol{Q}^\top, \quad \boldsymbol{Q}^\top(0) = \boldsymbol{I}, \quad \Omega(\omega) = \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix}.$$

The above system will be completed by the following boundary and initial conditions

$$(3) \qquad \boldsymbol{w} = \boldsymbol{w}_* + \boldsymbol{U}, \quad \text{on} \quad \partial\mathcal{D} \times (0, T),$$

$$(4) \qquad \lim_{|y| \to \infty} \boldsymbol{w}(y, t) = 0,$$

$$(5) \qquad \boldsymbol{w}(0) = \boldsymbol{w}_0.$$

Equations of body

$$(6) \qquad m\dot{\boldsymbol{\xi}} + m\boldsymbol{\omega} \times \boldsymbol{\xi} = \boldsymbol{Q} \cdot \boldsymbol{F} - \int_{\partial\mathcal{D}} \mathbb{T}(\boldsymbol{w}, \pi) \cdot \boldsymbol{n} - \boldsymbol{w}(\boldsymbol{w} - \boldsymbol{U}) \cdot \boldsymbol{n} dS,$$

$$(7) \qquad \boldsymbol{J}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \boldsymbol{J}\boldsymbol{\omega} = \boldsymbol{Q}^\top \cdot \boldsymbol{M}_C - \int_{\partial\mathcal{D}} y \times (\mathbb{T}(\boldsymbol{w}, \pi) \cdot \boldsymbol{n} - \boldsymbol{w}(\boldsymbol{w} - \boldsymbol{U}) \cdot \boldsymbol{n}) dS,$$

where $\boldsymbol{F}$ is the external force acting on the body and $\boldsymbol{M}_C$ is the external torque, while $\boldsymbol{n}$ stands for the outward unit normal on $\partial\mathcal{D}$. Finally, $\boldsymbol{J}$ is the inertial tensor with respect to the center of mass.

For the sake of simplicity we assume $\mathcal{F} = 0, \boldsymbol{F} = 0, \boldsymbol{M}_C = 0$ and $\boldsymbol{w}_* = 0$. The above system of equations in a fixed exterior domain are obtained by applying the so-called global transformation to the equations of the moving body in a fluid motion in the whole space, which clearly coincides with the classical Navier-Stokes equation in an time dependent exterior domain combined with appropriate boundary condition and asymptotic condition as $|x| \to +\infty$. In particular, the conservation of energy is invariant under this transformation. Thus, using the usual energy method global existence of weak solutions and local existence of strong solutions to the above system can be proved similar as in case of the Navier-Stokes equations. There are several results in this direction. The existence of a global weak solution of the Leray-Hopf type has been proved by Borchers see [1] (see also [18]). The asymptotic behaviour in time of such solution was investigated by Chen and Miyakawa in [2]. The first result of existence for more regular data is due to Hishida [11]. The generalization of Hishida's results in L^p spaces was done by Hieber, Heck and Geissert in [10]. They proved the existence of a unique local mild solution to the Navier-Stokes problem. The existence of a global strong solution under a smallness assumption on the data with respect to the L^2-norm has been studied by Galdi and Silvestre [6, 7] and by

1) Here $\boldsymbol{I}$ denotes the identity matrix (δ_{ij}).

Takahashi and Tucsnak [20] for a rigid body being a disk in the two-dimensional situation. Local in time existence and uniqueness of the strong solution has been proved by Cumsille and Tucsnak [3]. The global time existence and uniqueness was investigated in work of Cumsille and Takahashi [4]. However in three dimensional case the uniqueness is valid only under a smallness assumption of data.

Alternatively, the problem has been studied in [3, 19, 20, 5] by using the local transformation introduced by Inue and Wakimoto in [13], and in domains depending on time in [14, 15, 16, 17].

The aim of this paper will be the study of the corresponding linear system by neglecting the nonlinear term $(\boldsymbol{w}\cdot\nabla)\boldsymbol{w}$ in the momentum equation of (1) and moving the term $-\boldsymbol{U}\cdot\nabla\boldsymbol{w}+\omega\times\boldsymbol{w}$ to the right hand side. Our main result will be the existence of global strong solution to this linear problem in a suitable weighted Sobolev space together with estimates of the pressure and the pressure gradient as well. This result will be used for the study of global strong solutions to the full nonlinear problem which will be the subject of a forthcoming paper. In Section 2 we introduce the notion of a weak solution belonging to an appropriate weighted Sobolev space and state our main result (cf. Thm. 2.1). The proof of the main theorem will be divided into two parts. The first part concerns the existence of a strong solution to the linear problem coupled with a motion of body with a right hand side $\boldsymbol{f}$ in $\boldsymbol{L}^2$. Second part deals with a weighted approach of the heat equation with right hand side f in weighted L^2 space.

2. The linearized problem

In this section we study the following linear problem which describes the movement of a rigid body inside of a fluid, neglecting the nonlinear term $(\boldsymbol{w}\cdot\nabla)\boldsymbol{w}$ and moving the term $\boldsymbol{U}\cdot\nabla\boldsymbol{w}+\boldsymbol{\omega}\times\boldsymbol{w}$ to the right hand side. The equation of the fluid is given by the following Stokes system

$$\left.\begin{aligned} &\operatorname{div}\boldsymbol{u}=0\\ &\frac{\partial\boldsymbol{u}}{\partial t}-\Delta\boldsymbol{u}=\boldsymbol{f}-\nabla p\end{aligned}\right\}\ \text{in }\ \mathcal{D}\times(0,T), \tag{2.1}$$

with the boundary and initial conditions

$$\boldsymbol{u}=\boldsymbol{U}\quad\text{on}\quad\partial\mathcal{D}\times(0,T), \tag{2.2}$$

$$\lim_{|x|\to\infty}\boldsymbol{u}(x,t)=0, \tag{2.3}$$

$$\boldsymbol{u}(0)=\boldsymbol{u}_0, \tag{2.4}$$

where $\boldsymbol{U}=\boldsymbol{\xi}+\boldsymbol{\omega}\times x$. The equation of the movement of the body is given by

$$m\dot{\boldsymbol{\xi}}=\boldsymbol{\gamma}_1-\int_{\partial\mathcal{D}}\mathbb{T}(\boldsymbol{u},p)\cdot\boldsymbol{n}dS, \tag{2.5}$$

$$\boldsymbol{J}\dot{\boldsymbol{\omega}}=\boldsymbol{\gamma}_2-\int_{\partial\mathcal{D}}x\times(\mathbb{T}(\boldsymbol{u},p)\cdot\boldsymbol{n})dS. \tag{2.6}$$

Here $\boldsymbol{f}(x,t),\boldsymbol{u}_0(x),\boldsymbol{\gamma}_1(t)$ and $\boldsymbol{\gamma}_2(t)$ are given data, while $\boldsymbol{u},p,\boldsymbol{\xi}$ and $\boldsymbol{\omega}$ [2)] denote the unknown quantities.

[2)] Recall that $\boldsymbol{\xi}$ stands for the translation and $\boldsymbol{\omega}$ stands for the rotation of the body.

Our aim is to study the above system for a right hand side $\boldsymbol{f} = \mathbf{rot}(\mathbf{rot}\,\boldsymbol{a} \times \boldsymbol{b})$, where $\boldsymbol{a}$ denotes a smooth vector field such that $|a(x)|$ behaves like $|x|^2$ $(x \in \mathbb{R}^3)$.

Remark 2.1 In order to treat the nonlinear system we may move the term $(\boldsymbol{w}-\boldsymbol{U})\cdot\nabla\boldsymbol{w} + \boldsymbol{\omega}\times\boldsymbol{w}$ of equation (1) to the right hand side. Neglecting the convective term $\boldsymbol{w}\cdot\nabla\boldsymbol{w}$ we end up with a linearized system with $\boldsymbol{f} = \boldsymbol{U}\cdot\nabla\boldsymbol{w} - \boldsymbol{\omega}\times\boldsymbol{w}$. Calculating

$$\boldsymbol{\omega}\times\boldsymbol{w} = w_i\omega\times e_i = w_i\frac{\partial}{\partial x_i}(\boldsymbol{\omega}\times x) = \boldsymbol{w}\cdot\nabla\boldsymbol{U}$$

$$\boldsymbol{U} = \boldsymbol{\xi} + \boldsymbol{\omega}\times x = \mathbf{rot}\,\boldsymbol{\psi}, \quad \text{where} \quad \boldsymbol{\psi} = \frac{1}{2}(\boldsymbol{\xi}\times x - \boldsymbol{\omega}|x|^2), \quad x \in \mathbb{R}^3,$$

we see that

$$\boldsymbol{U}\cdot\nabla\boldsymbol{w} - \boldsymbol{\omega}\times\boldsymbol{w} = \boldsymbol{U}\cdot\nabla\boldsymbol{w} - \boldsymbol{w}\cdot\nabla\boldsymbol{U} = \mathbf{rot}(\mathbf{rot}\,\boldsymbol{\psi}\times\boldsymbol{w}), \tag{2.7}$$

which has the desired form.

Clearly, for such forces $\boldsymbol{f}$ we cannot expect the existence of weak solutions in the usual Sobolev spaces rather then in appropriated weighted Sobolev spaces. For the notion of such weak solutions we will introduce the following weight function

$$\eta(x) = (1+|x|^2)^{-1/2}, \quad x \in \mathbb{R}^3.$$

Then we define the spaces

$$\boldsymbol{L}^2_\eta(\mathcal{D}) = \{\boldsymbol{v} \in \boldsymbol{L}^2_{\mathrm{loc}}(\mathcal{D}) \mid \eta\boldsymbol{v} \in \boldsymbol{L}^2(\mathcal{D})\},$$
$$\boldsymbol{W}^{1,2}_\eta(\mathcal{D}) = \{\boldsymbol{v} \in \boldsymbol{W}^{1,2}_{\mathrm{loc}}(\mathcal{D}) \mid \eta\boldsymbol{v} \in \boldsymbol{W}^{1,2}(\mathcal{D})\}.$$

In addition, by $\mathcal{C}(\mathcal{D})$ we denote the space of all solenoidal smooth vector fields $\boldsymbol{\varphi} \in \boldsymbol{C}^\infty_{0,\sigma}(\mathbb{R}^3)$ for which there exist constant vectors Φ_1 and Φ_2, such that

$$\boldsymbol{\varphi} = \boldsymbol{\Phi}_1 + \boldsymbol{\Phi}_2\times x \quad \text{in a neighbourhood of } \partial\mathcal{D}.$$

Then we define $\mathcal{V}(\mathcal{D})$ and $\mathcal{V}_\eta(\mathcal{D})$ as closure of $\mathcal{C}(\mathcal{D})$ with respect to the norm in $\boldsymbol{W}^{1,2}(\mathcal{D})$ and $\boldsymbol{W}^{1,2}_\eta(\mathcal{D})$ respectively.

Definition 2.1 (**Weak solution**) Let $\boldsymbol{u}_0 \in \mathcal{V}(\mathcal{D})$ with $\boldsymbol{u}_0 = \boldsymbol{\xi}_0 + \boldsymbol{\omega}_0\times x$ on $\partial\mathcal{D}$. We assume $\boldsymbol{f} = \mathbf{rot}\,\boldsymbol{g}$, where $\boldsymbol{g} \in L^2(0,T;\boldsymbol{W}^{1,2}_\eta(\mathcal{D}))$. A triple $(\boldsymbol{u},\boldsymbol{\xi},\boldsymbol{\omega})$ is called a *weak solution* to (2.1)-(2.6) if

(i) $\boldsymbol{u} \in L^2(0,T;\mathcal{V}_\eta(\mathcal{D})) \cap C_w([0,T);\boldsymbol{L}^2_\eta(\mathcal{D}))$,
(ii) $\boldsymbol{\xi},\boldsymbol{\omega} \in \boldsymbol{C}([0,T])$,
(iii) for every $\boldsymbol{\varphi} \in C^\infty(0,T;\mathcal{C}(\mathcal{D}))$ there holds the identity

$$\int_0^t\int_{\mathcal{D}} -\boldsymbol{u}\cdot\frac{\partial\varphi}{\partial t} + \boldsymbol{D}\boldsymbol{u} : \boldsymbol{D}\boldsymbol{\varphi}\,dx\,ds$$
$$+\int_{\mathcal{D}}\boldsymbol{u}(t)\cdot\boldsymbol{\varphi}(t)\,dx + \boldsymbol{\xi}(t)\cdot\boldsymbol{\Phi}_1(t) + \boldsymbol{J}\boldsymbol{\omega}(t)\cdot\boldsymbol{\Phi}_2(t)$$
$$= \int_0^t m\boldsymbol{\xi}\cdot\dot{\boldsymbol{\Phi}}_1 - \boldsymbol{\gamma}_1\cdot\boldsymbol{\Phi}_1 + \boldsymbol{J}\boldsymbol{\omega}\cdot\dot{\boldsymbol{\Phi}}_2 - \boldsymbol{\gamma}_2\cdot\boldsymbol{\Phi}_2 ds$$
$$+\int_{\mathcal{D}}\boldsymbol{u}_0\cdot\boldsymbol{\varphi}(0)dx + \boldsymbol{\xi}_0\cdot\boldsymbol{\Phi}_1(0) + \boldsymbol{J}\boldsymbol{\omega}_0\cdot\boldsymbol{\Phi}_2(0)dx + \int_0^t\int_{\mathcal{D}}\boldsymbol{f}\cdot\boldsymbol{\varphi}\,dx\,ds \tag{2.8}$$

for all $0 < t < T$.

Our main result is the following

Theorem 2.1. *Let $\boldsymbol{u}_0 \in \mathcal{V}(\mathcal{D})$ with $\boldsymbol{u}_0 = \boldsymbol{\xi}_0 + \boldsymbol{\omega}_0 \times x$ on $\partial\mathcal{D}$. Let $\boldsymbol{f} = \mathbf{rot}\,\boldsymbol{g}$, such that $\boldsymbol{g} \in L^2(0,T; \boldsymbol{W}^{1,2}_\eta(\mathcal{D}))$. Then there exists a weak solution $(\boldsymbol{u}, \boldsymbol{\xi}, \boldsymbol{\omega})$ to* (2.1)-(2.6) *according to Def. 2.1, such that*

$$\frac{\partial \boldsymbol{u}}{\partial t}, \frac{\partial^2 \boldsymbol{u}}{\partial x_i \partial x_j} \in L^2(0,T; \boldsymbol{L}^2_\eta(\mathcal{D})), \quad (i,j = 1,2,3) \tag{2.9}$$

and there exists a pressure $p \in L^2(0,T; L^2_{\mathrm{loc}}(\overline{\mathcal{D}}))$ with

$$\nabla p \in L^2(0,T; \boldsymbol{L}^2(\mathcal{D})). \tag{2.10}$$

Furthermore, there holds

$$\begin{aligned} &\Big\| \frac{\partial \boldsymbol{u}}{\partial t} \Big\|_{L^2(0,T;\boldsymbol{L}^2_\eta)} + \|\boldsymbol{u}\|_{L^2(0,T;\boldsymbol{W}^{2,2}_\eta)} + \|\nabla \boldsymbol{u}\|_{L^\infty(0,T;L^2_\eta)} + \\ &\quad + \|\boldsymbol{\xi}\|_{\boldsymbol{W}^{1,2}(0,T)} + \|\boldsymbol{\omega}\|_{\boldsymbol{W}^{1,2}(0,T)} + \|\nabla p\|_{L^2(0,T;\boldsymbol{L}^2)} \le cK_0, \end{aligned} \tag{2.11}$$

where $K_0 := \|\boldsymbol{u}_0\|_{\boldsymbol{W}^{1,2}} + \|\boldsymbol{f}\|_{L^2(0,T;\boldsymbol{L}^2_\eta)} + |\boldsymbol{\omega}_0| + |\boldsymbol{\xi}_0| + \|\gamma_1\|_{\boldsymbol{L}^2(0,T)} + \|\gamma_2\|_{\mathbf{L}^2(0,T)}$ and $c = \mathrm{const}$ depending on $\mathcal{D}$ only.

Remark 2.2 Since $\boldsymbol{f} \notin \boldsymbol{L}^2$ we are not allowed to test equation $(2.1)_2$ with the solution $\boldsymbol{u}$. Therefore an estimate based on the usual energy method is not possible. To overcome this difficulties we divide the problem into a Stokes like problem in the whole space with non decaying right hand side and a linear problem (2.1)–(2.6) with a right hand side belonging to $L^2(0,T; \boldsymbol{L}^2(\mathcal{D}))$.

3. Estimates for auxiliary problems

Our first result is related to the a-priori estimate of weak solutions of the Stokes like system in weighted Sobolev space with solenoidal right hand side. As we will see below such system coincides with the system of heat equations. Therefore it will be sufficient to consider the case of the heat equation.

Lemma 3.1. *Let $f \in L^2(0,T; L^2_\eta(\mathbb{R}^3))$. Then there exists a weak solution*

$$z \in L^2(0,T; W^{1,2}_\eta(\mathbb{R}^3)) \cap L^\infty(0,T; L^2_\eta(\mathbb{R}^3))$$

to the heat equation

$$\frac{\partial z}{\partial t} - \Delta z = f \quad \text{in} \quad \mathbb{R}^3 \times (0,T), \tag{3.1}$$

$$z(0) = 0. \tag{3.2}$$

In addition, there holds $\frac{\partial z}{\partial t}, \frac{\partial^2 z}{\partial x_i \partial x_j} \in L^2(0,T; L^2_\eta(\mathbb{R}^3))$, $(i,j = 1,2,3)$ together with the estimate

$$\Big\| \frac{\partial z}{\partial t} \Big\|_{L^2(0,T;L^2_\eta)} + \|z\|_{L^2(0,T;W^{2,2}_\eta)} \le c\|f\|_{L^2(0,T;L^2_\eta)} {}^{3)}. \tag{3.3}$$

Proof. We divide the proof into two steps. First, we consider the case $f \in L^2(0,T;L^2(\mathbb{R}^3))$ and prove the a-priori estimate (3.3). Secondly, for general f we get an approximate weak solution z_m for the truncated right hand side f_m and pass to the limit $m \to \infty$ by using a-priori estimate (3.3).

1° Let $f \in L^2(0,T;L^2(\mathbb{R}^3))$. Clearly, there exists a weak solution

$$z \in L^2(0,T;W^{1,2}(\mathbb{R}^3)) \cap C([0,T];L^2(\mathbb{R}^3)),$$

such that

$$\frac{\partial z}{\partial t}, \frac{\partial^2 z}{\partial x_i \partial x_j} \in L^2(0,T;L^2(\mathbb{R}^3)), \quad (i,j = 1,2,3).$$

Setting $h(x,t) = z(x,t)\eta(x)$ using the product rule the equation (3.1) turns into

$$\frac{\partial h}{\partial t} - \Delta h = \eta f - 2\nabla z \cdot \nabla \eta - z\Delta\eta \quad \text{in} \quad \mathbb{R}^3 \times (0,T). \tag{3.4}$$

By an elementary calculus, the equation (3.4) can be rewritten as

$$\frac{\partial h}{\partial t} - \Delta h = \eta f + 2x \cdot \eta^2 \nabla h + (\eta^4 + 2\eta^2)h \quad \text{in} \quad \mathbb{R}^3 \times (0,T). \tag{3.5}$$

Next, we multiply both sides of (3.5) by h, integrate the obtained equation over $\mathbb{R}^3 \times (0,t)$ $(t \in (0,T))$ and apply integration by parts. This yields

$$\frac{1}{2}\|h(t)\|^2_{L^2} + \int_0^t \int_{\mathbb{R}^3} |\nabla h|^2 dxds = \int_0^t \int_{\mathbb{R}^3} \eta f h dxds + \int_0^t \int_{\mathbb{R}^3} (\eta^2 - \eta^4)|h|^2 dxds$$

for a.e. $t \in (0,T)$. Using Young's inequality and Gronwall's lemma we obtain the following a-priori estimate

$$\|h\|_{L^\infty(0,T;L^2)} + \|\nabla h\|_{L^2(0,T;L^2)} \le c\|\eta f\|_{L^2(0,T;L^2)}. \tag{3.6}$$

Recalling the definition of h from (3.6) we immediately obtain

$$\|z\|_{L^\infty(0,T;L^2_\eta)} + \|\nabla z\|_{L^2(0,T;\boldsymbol{L}^2_\eta)} \le c\|\eta f\|_{L^2(0,T;L^2)}. \tag{3.7}$$

On the other hand, multiplying equation (3.5) by $\frac{\partial h}{\partial t}$ and with Δh respectively, applying integration by parts, observing (3.6) and (3.7) we get

$$\left\|\frac{\partial z}{\partial t}\right\|_{L^2(0,T;L^2_\eta)} + \|z\|_{L^2(0,T;W^{2,2}_\eta)} \le c\|\eta f\|_{L^2(0,T;L^2)}. \tag{3.8}$$

2° Now, let $f \in L^2(0,T;L^2_\eta(\mathbb{R}^3))$. We define

$$f_\varepsilon(x) = (1 + \varepsilon|x|)^{-1} f, \quad x \in \mathbb{R}^3, \quad \varepsilon > 0.$$

Clearly, $f_\varepsilon \in L^2(0,T;L^2(\mathbb{R}^3))$ and $\|f_\varepsilon\|_{L^2(0,T;L^2_\eta)} \le \|f\|_{L^2(0,T;L^2_\eta)}$ for all $\varepsilon > 0$. As it has been shown in 1° for each $\varepsilon > 0$ there exists a weak solution $z_\varepsilon \in$

[3)] Here by $W^{2,2}_\eta(\mathbb{R}^3)$ we denote the space of all $v \in W^{2,2}_{\mathrm{loc}}(\mathbb{R}^3)$ such that $\eta D^\alpha v \in L^2(\mathbb{R}^3)$ for every multi-index $\alpha \le 2$.

$L^2(0,T;W^{1,2}(\mathbb{R}^3))\cap C([0,T];L^2(\mathbb{R}^3))$ to (3.1), (3.2) replacing f by f_ε therein. In addition, we have $\frac{\partial z_\varepsilon}{\partial t}\in L^2(0,T;L^2(\mathbb{R}^3))$ and $\frac{\partial^2 z_\varepsilon}{\partial x_i\partial x_j}\in L^2(0,T;L^2(\mathbb{R}^3))$ $(i,j=1,2,3)$. From (3.7) and (3.8) it follows that

$$\Big\|\frac{\partial z_\varepsilon}{\partial t}\Big\|_{L^2(0,T;L^2_\eta)}+\|z_\varepsilon\|_{L^2(0,T;W^{2,2}_\eta)}\le c\|\eta f_\varepsilon\|_{L^2(0,T;L^2)}\le c\|f\|_{L^2(0,T;L^2_\eta)}. \tag{3.9}$$

By means of reflexivity of $L^2(0,T;W^{2,2}_\eta)$ there exists a sequence (ε_k) with $\varepsilon_k\to 0^+$ as $k\to\infty$ and $z\in L^2(0,T;W^{2,2}_\eta)$ with $\frac{\partial z}{\partial t}\in L^2(0,T;L^2_\eta)$ such that

$$z_{\varepsilon_k}\to z\quad \textit{weakly in}\quad L^2(0,T;W^{2,2}_\eta)\quad \textit{as}\quad k\to\infty.$$

In the equation for z_{ε_k} taking the passage to the limit $\varepsilon_k\to 0^+$ on both sides we see that z solves (3.1), (3.2) in weak sense. Finally, by virtue of (3.9) using the lower semi continuity of the norm we get (3.3). □

Next, let us consider the problem (2.1)–(2.6) with $\boldsymbol{f}\in L^2(0,T;\boldsymbol{L}^2(\mathcal{D}))$. In this case we have the following existence result.

Lemma 3.2. *Let $\boldsymbol{f}\in L^2(0,T;\boldsymbol{L}^2(\mathcal{D}))$ and let $\boldsymbol{u}_0\in\mathcal{V}(\mathcal{D})$ with $\boldsymbol{u}_0=\boldsymbol{\xi}_0+\boldsymbol{\omega}_0\times x$ on $\partial\mathcal{D}$, where $\boldsymbol{\xi}_0,\boldsymbol{\omega}_0\in\mathbb{R}$ are given. In addition, let $\boldsymbol{\gamma}_1,\boldsymbol{\gamma}_2\in\boldsymbol{L}^2(0,T)$. Then there exists a weak solution $(\boldsymbol{u},\boldsymbol{\xi},\boldsymbol{\omega})$ to (2.1)–(2.6), such that*

$$\begin{aligned}&\|\nabla\boldsymbol{u}\|_{\boldsymbol{L}^2}+\|\boldsymbol{u}\|_{L^\infty(0,T;\boldsymbol{L}^2)}+\|\boldsymbol{\xi}\|_{\boldsymbol{L}^\infty(0,T)}+\|\boldsymbol{\omega}\|_{\boldsymbol{L}^\infty(0,T)}\\&\qquad\le c\|\boldsymbol{u}_0\|_{\boldsymbol{L}^2}+\|\boldsymbol{f}\|_{\boldsymbol{L}^2}+|\boldsymbol{\xi}_0|+|\boldsymbol{\omega}_0|+|\boldsymbol{\gamma}_1|_{\boldsymbol{L}^2(0,T)}+|\boldsymbol{\gamma}_2|_{\boldsymbol{L}^2(0,T)}.\end{aligned}\tag{3.10}$$

In addition, we have

$$\frac{\partial\boldsymbol{u}}{\partial t},\frac{\partial^2\boldsymbol{u}}{\partial x_i\partial x_j},\nabla p\in L^2(0,T;\boldsymbol{L}^2(\mathcal{D}))\quad (i,j=1,2,3),\quad \dot{\boldsymbol{\xi}},\dot{\boldsymbol{\omega}}\in\boldsymbol{L}^2(0,T)$$

and there holds

$$\begin{aligned}&\Big\|\frac{\partial\boldsymbol{u}}{\partial t}\Big\|_{\boldsymbol{L}^2}+\|\nabla^2\boldsymbol{u}\|_{\boldsymbol{L}^2}+\|\nabla p\|_{\boldsymbol{L}^2}+\|\dot{\boldsymbol{\xi}}\|_{\boldsymbol{L}^2(0,T)}+\|\dot{\boldsymbol{\omega}}\|_{\boldsymbol{L}^2(0,T)}\\&\qquad\le c\|\boldsymbol{u}_0\|_{\boldsymbol{W}^{1,2}}+\|\boldsymbol{f}\|_{\boldsymbol{L}^2}+|\boldsymbol{\xi}_0|+|\boldsymbol{\omega}_0|+\|\dot{\boldsymbol{\gamma}}_1\|_{\boldsymbol{L}^2(0,T)}+\|\dot{\boldsymbol{\gamma}}_2\|_{\boldsymbol{L}^2(0,T)}.\end{aligned}\tag{3.11}$$

Proof. 1° The existence and uniqueness of a weak solution $(\boldsymbol{u},\boldsymbol{\xi},\boldsymbol{\omega})$ can be shown easily by applying the linear theory of evolutionary equations in Hilbert spaces (e. g. see in [8]).

2° Assume $\boldsymbol{u}_0=0$. Let $(\boldsymbol{u},\boldsymbol{\xi},\boldsymbol{\omega})$ be a weak solution to (2.1)–(2.6). Firstly, we assume

$$\frac{\partial\boldsymbol{u}}{\partial t}\in L^2(0,T;\boldsymbol{L}^2(\mathcal{D})),\quad \dot{\boldsymbol{\xi}},\dot{\boldsymbol{\omega}}\in\boldsymbol{L}^2(0,T).\tag{3.12}$$

Next, let $\zeta\in C_0^\infty(\mathbb{R}^3)$ such that $\zeta\equiv 1$ in a neighborhood of $\mathcal{B}$. Set

$$\Psi(x,t):=\frac{1}{2}\,\mathbf{rot}((\boldsymbol{\xi}(t)\times x-\boldsymbol{\omega}(t)|x|^2)\zeta(x)),\quad (x,t)\in\mathbb{R}^3\times(0,T).$$

Since

$$\frac{1}{2}\,\mathbf{rot}(\boldsymbol{\xi}\times x-\boldsymbol{\omega}|x|^2)=\boldsymbol{\xi}+\boldsymbol{\omega}\times x\quad \forall\, x\in\mathbb{R}^3$$

it follows that $\boldsymbol{u} - \Psi = 0$ on $\partial\mathcal{D}$. Thus, for almost all $t \in (0,T)$ the function $\boldsymbol{v} := \boldsymbol{u}(\cdot,t) - \Psi(\cdot,t)$ is a solution to the Stokes system

$$\begin{cases} \operatorname{div} \boldsymbol{v} = 0 \quad \text{in} \quad \mathcal{D}, \\ -\Delta \boldsymbol{v} = \boldsymbol{f}(t) + \Delta\Psi(t) - \dfrac{\partial \boldsymbol{u}}{\partial t}(t) - \nabla p(t) \quad \text{in} \quad \mathcal{D}, \\ \boldsymbol{v}|_{\partial\mathcal{D}} = 0, \qquad \lim\limits_{|x|\to\infty} \boldsymbol{v}(x) = 0. \end{cases}$$

By the well-known theory of the Stokes equation one gets $\nabla p(t) \in \boldsymbol{L}^2(\mathcal{D})$ together with the estimate

$$\|\nabla p(t)\|_{\boldsymbol{L}^2} \le c\Big(\|\boldsymbol{f}(t)\|_{\boldsymbol{L}^2} + \Big\|\frac{\partial \boldsymbol{u}}{\partial t}(t)\Big\|_{\boldsymbol{L}^2} + |\boldsymbol{\xi}(t)| + |\boldsymbol{\omega}(t)|\Big), \tag{3.13}$$

where $c =$ const independent of $t \in (0,T)$. Hence, from (3.13), the equation $(2.1)_2$ and the assumption $\frac{\partial \boldsymbol{u}}{\partial t} \in L^2(0,T;\boldsymbol{L}^2(\mathcal{D}))$ it follows that $\nabla p, \Delta \boldsymbol{u} \in L^2(0,T;\boldsymbol{L}^2(\mathcal{D}))$. Moreover there holds

$$\begin{aligned} &\|\nabla p\|_{L^2(0,T;\boldsymbol{L}^2)} + \|\nabla^2 \boldsymbol{u}\|_{L^2(0,T;\boldsymbol{L}^2)} \le \\ &\quad \le c\Big(\|\boldsymbol{f}\|_{L^2(0,T;\boldsymbol{L}^2)} + \Big\|\frac{\partial \boldsymbol{u}}{\partial t}\Big\|_{L^2(0,T;\boldsymbol{L}^2)} + \|\boldsymbol{\xi}\|_{\boldsymbol{L}^\infty} + \|\boldsymbol{\omega}\|_{\boldsymbol{L}^\infty}\Big). \end{aligned} \tag{3.14}$$

Then, multiplying both sides of $(2.1)_2$ with $\frac{\partial \boldsymbol{u}}{\partial t}$, integrating the result over $\mathcal{D} \times (0,t)$ $(t \in (0,T))$ and applying integration by parts we are led to

$$\begin{aligned} &\int_0^t \int_{\mathcal{D}} \Big|\frac{\partial \boldsymbol{u}}{\partial t}\Big|^2 dxds + \frac{1}{2}\int_{\mathcal{D}} |\nabla \boldsymbol{u}(t)|^2 dx = \\ &\quad = \int_0^t \int_{\partial\mathcal{D}} (\mathbb{T}(\boldsymbol{u},p)\cdot\boldsymbol{n})\cdot(\dot{\boldsymbol{\xi}} + \dot{\boldsymbol{\omega}} \times x)dSds + \int_0^t \int_{\mathcal{D}} \boldsymbol{f}\cdot\frac{\partial \boldsymbol{u}}{\partial t} dxds. \end{aligned} \tag{3.15}$$ [4)]

Next, from (2.5) we induce

$$\int_{\partial\mathcal{D}} (\mathbb{T}(\boldsymbol{u},p)\cdot\boldsymbol{n})\cdot\dot{\boldsymbol{\xi}}(s)dS = \boldsymbol{\gamma}_1(s)\cdot\dot{\boldsymbol{\xi}}(s) - m|\dot{\boldsymbol{\xi}}(s)|^2 \quad \text{for a. e. } s \in (0,T).$$

Moreover, from (2.6) we obtain

$$\begin{aligned} \int_{\partial\mathcal{D}} (\mathbb{T}(\boldsymbol{u},p)\cdot\boldsymbol{n})\cdot\dot{\boldsymbol{\omega}}(s)\times x)dS &= \int_{\partial\mathcal{D}} x \times (\mathbb{T}(\boldsymbol{u},p)\cdot\boldsymbol{n})dS\cdot\dot{\boldsymbol{\omega}}(s) \\ &= (\boldsymbol{\gamma}_2(s) - \boldsymbol{J}\dot{\boldsymbol{\omega}}(s))\cdot\dot{\boldsymbol{\omega}}(s) \end{aligned}$$

for a. e. $s \in (0,T)$. Inserting these identities into (3.15) and applying integration by parts we see that

$$\begin{aligned} &\int_0^t \int_{\mathcal{D}} \Big|\frac{\partial \boldsymbol{u}}{\partial t}\Big|^2 dxds + \frac{1}{2}\int_{\mathcal{D}} |\nabla \boldsymbol{u}(t)|^2 dx + \int_0^t m|\dot{\boldsymbol{\xi}}|^2 + |\boldsymbol{R}\dot{\boldsymbol{\omega}}|^2 ds \\ &\quad = \int_0^t \dot{\boldsymbol{\xi}}\cdot\boldsymbol{\gamma}_1 + \dot{\boldsymbol{\omega}}\cdot\boldsymbol{\gamma}_2 ds + \int_0^t \int_{\mathcal{D}} \boldsymbol{f}\cdot\frac{\partial \boldsymbol{u}}{\partial t} dxds \end{aligned}$$

[4)] Note that by virtue of (3.13) the trace of $\mathbb{T}(\boldsymbol{u},p)$ upon $\partial\mathcal{D}$ is well defined.

for a. e. $t \in (0,T)$. Here $\boldsymbol{R}$ denotes the square root of $\boldsymbol{J}$, i. e $\boldsymbol{R}^2 = \boldsymbol{J} = \boldsymbol{J}^\top$. By the aid of Cauchy-Schwarz's inequality and Young's inequality one finds

$$
\begin{aligned}
&\Big\|\frac{\partial \boldsymbol{u}}{\partial t}\Big\|_{L^2(0,T;\boldsymbol{L}^2)} + \|\dot{\boldsymbol{\xi}}\|_{\boldsymbol{L}^2(0,T)} + \|\dot{\boldsymbol{\omega}}\|_{\boldsymbol{L}^2(0,T)} + \|\nabla \boldsymbol{u}\|_{L^\infty(0,T;\boldsymbol{L}^2)} \le \\
&\qquad \le 2\Big(\|\boldsymbol{f}\|_{L^2(0,T;\boldsymbol{L}^2)} + \|\boldsymbol{\gamma}_1\|_{L^2(0,T)} + \|\boldsymbol{\gamma}_2\|_{L^2(0,T)}\Big).
\end{aligned} \tag{3.16}
$$

Finally, combining (3.14) and (3.16) we obtain the a-priori estimate

$$
\begin{aligned}
&\|\nabla p\|_{L^2(0,T;\boldsymbol{L}^2)} + \|\nabla^2 \boldsymbol{u}\|_{L^2(0,T;\boldsymbol{L}^2)} \\
&\qquad \le c\Big(\|\boldsymbol{f}\|_{L^2(0,T;\boldsymbol{L}^2)} + \|\boldsymbol{\gamma}_1\|_{L^2(0,T)} + \|\boldsymbol{\gamma}_2\|_{L^2(0,T)}\Big).
\end{aligned} \tag{3.17}
$$

Secondly, let us consider the general case, without assuming (3.12). To begin with we introduce the Steklov mean as follows. Let $f \in L^1(0,T;L^1(\mathcal{D}))$. Define,

$$
f_\lambda(x,t) = -\frac{1}{\lambda} \int\limits_{\max\{t+\lambda,0\}}^{t} f(x,s)ds, \quad (x,s) \in \mathcal{D} \times (0,T), \quad \lambda < 0.
$$

Applying the Steklov mean to both sides of the equations $(2.1)_2$, (2.5) and (2.6) recalling that $\boldsymbol{u}(0) = 0$ we see that $(\boldsymbol{u}_\lambda, \boldsymbol{\xi}_\lambda, \boldsymbol{\omega}_\lambda)$ is a weak solution to (2.1)–(2.6) with $\boldsymbol{f}_\lambda, \boldsymbol{\gamma}_{1,\lambda}, \boldsymbol{\gamma}_{2,\lambda}, \boldsymbol{\omega}_\lambda(0), \boldsymbol{\xi}_\lambda(0)$ instead of $\boldsymbol{f}, \boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \boldsymbol{\omega}_0, \boldsymbol{\xi}_0$. In addition, this weak solution satisfies (3.12). Thus, from (3.16) and (3.17) it follows that

$$
\begin{aligned}
&\Big\|\frac{\partial \boldsymbol{u}_\lambda}{\partial t}\Big\|_{L^2(0,T;\boldsymbol{L}^2)} + \|\nabla^2 \boldsymbol{u}_\lambda\|_{L^2(0,T;\boldsymbol{L}^2)} + \|\nabla \boldsymbol{u}_\lambda\|_{L^\infty(0,T;\boldsymbol{L}^2)} + \\
&\qquad + \|\nabla p_\lambda\|_{L^2(0,T;\boldsymbol{L}^2)} + \|\dot{\boldsymbol{\xi}}_\lambda\|_{\boldsymbol{L}^2(0,T)} + \|\dot{\boldsymbol{\omega}}_\lambda\|_{\boldsymbol{L}^2(0,T)} \\
&\qquad \le 2\Big(\|\boldsymbol{f}\|_{L^2(0,T;\boldsymbol{L}^2)} + \|\boldsymbol{\gamma}_1\|_{L^2(0,T)} + \|\boldsymbol{\gamma}_2\|_{L^2(0,T)}\Big).
\end{aligned} \tag{3.18}
$$

Here $c = \text{const} > 0$ depending on the geometry $\mathcal{D}$ only. Thus, by means of reflexivity of $L^2(0,T;\boldsymbol{L}^2(\mathcal{D}))$ and $\boldsymbol{L}^2(0,T)$ from (3.18) the assertion of the lemma follows.

3° Let $\boldsymbol{u}_0 \in \mathcal{V}(\mathcal{D}) \cap \boldsymbol{W}^{2,2}(\mathcal{D})$. Let $(\boldsymbol{u}, \boldsymbol{\xi}, \boldsymbol{\omega})$ be a weak solution to (2.1)–(2.6). Clearly, $\boldsymbol{u} - \boldsymbol{u}_0$ solves the system (2.1)–(2.6) too, with vanishing initial data and right hand side in $L^2(0,T;\boldsymbol{L}^2(\mathcal{D}))$. Hence, applying the result of step 2°, wee see that

$$
\frac{\partial \boldsymbol{u}}{\partial t}, \Delta \boldsymbol{u}, \nabla p \in L^2(0,T;\boldsymbol{L}^2(\mathcal{D})), \quad \dot{\boldsymbol{\xi}}, \dot{\boldsymbol{\omega}} \in \boldsymbol{L}^2(0,T).
$$

Now, we are in a position to multiply the equation $(2.1)_2$ by $\frac{\partial \boldsymbol{u}}{\partial t}$ and integrate both sides over $\mathcal{D} \times (0,t)$ $(t \in (0,T))$. Then arguing similar as in step 2° by using integration by parts we achieve the identity

$$
\begin{aligned}
&\int\limits_0^t \int\limits_{\mathcal{D}} \Big|\frac{\partial \boldsymbol{u}}{\partial t}\Big|^2 dxds + \frac{1}{2}\int\limits_{\mathcal{D}} |\nabla \boldsymbol{u}(t)|^2 dx + \int\limits_0^t |\dot{\boldsymbol{\xi}}|^2 + |\boldsymbol{R}\dot{\boldsymbol{\omega}}|^2 ds \\
&\qquad = \frac{1}{2}\int\limits_{\mathcal{D}} |\nabla \boldsymbol{u}_0|^2 dx + \int\limits_0^t \int\limits_{\mathcal{D}} \boldsymbol{f} \cdot \frac{\partial \boldsymbol{u}}{\partial t} dxds + \int\limits_0^t \dot{\boldsymbol{\xi}} \cdot \boldsymbol{\gamma}_1 + \dot{\boldsymbol{\omega}} \cdot \boldsymbol{\gamma}_2 ds.
\end{aligned}
$$

for a. e. $t \in (0,T)$. As above from this identity the assertion easily follows.

4° Finally, in case $\boldsymbol{u}_0 \in \mathcal{V}(\mathcal{D})$ the proof will be completed by a standard density argument using the a-priori estimate obtained in 3°. $\square$

4. Proof of Main Theorem

We divide the proof into two steps. First, we prove the assertion for the case when $\boldsymbol{f}$ belongs to $L^2(0,T;\boldsymbol{L}^2(\mathcal{D}))$. Then, we will complete the proof by applying a standard approximation argument, passing to the limit on the basis of the a-priori estimate obtained by the first step.

1° Let $\boldsymbol{f} = \mathbf{rot}\,\boldsymbol{g}$, with $\boldsymbol{g} \in L^2(0,T;\boldsymbol{W}^{1,2}(\mathcal{D}))$, $\boldsymbol{u}_0 \in \mathcal{V}(\mathcal{D})$ with $\boldsymbol{u}_0 = \boldsymbol{\xi} + \boldsymbol{\omega} \times x$ a.e. on $\partial\mathcal{D}$ and $\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2 \in \boldsymbol{L}^2(0,T)$. According to Lemma 3.2 there exists a weak solution $(\boldsymbol{u}, \boldsymbol{\xi}, \boldsymbol{\omega})$ to (2.1)–(2.6) which is strong in the sense that $\frac{\partial \boldsymbol{u}}{\partial t}, \Delta \boldsymbol{u}, \nabla p \in L^2(0,T;\boldsymbol{L}^2(\mathcal{D}))$ and $\dot{\boldsymbol{\xi}}, \dot{\boldsymbol{\omega}} \in \boldsymbol{L}^2(0,T)$.

Fix $0 < R_0 < \infty$, such that $\mathcal{B} \subset B_{R_0/2}(0)$. Let $\zeta \in C_0^\infty(\mathbb{R}^3)$ denote a cut off function with $\operatorname{supp}(\zeta) \subset B_{R_0}$, such that $\zeta \equiv 1$ in a neighborhood of $\mathcal{B}$. We write $\boldsymbol{f}$ as the sum $\boldsymbol{f}_1 + \boldsymbol{f}_2$, where $\boldsymbol{f}_1 = \mathbf{rot}(\zeta \boldsymbol{g})$, $\quad \boldsymbol{f}_2 = \mathbf{rot}((1-\zeta)\boldsymbol{g})$ $\quad$ a.e. in $\quad \mathcal{D} \times (0,T)$. Once more applying Lemma 3.2 there exists a strong solution $(\boldsymbol{u}_1, \boldsymbol{\xi}_1, \boldsymbol{\omega}_1)$ to the system (2.1)–(2.6) with right hand side $\boldsymbol{f}_1$ in place of $\boldsymbol{f}$. In particular, we have the a-priori estimate

$$\begin{aligned} &\Big\|\frac{\partial \boldsymbol{u}_1}{\partial t}\Big\|_{L^2(0,T;\boldsymbol{L}^2)} + \|\nabla^2 \boldsymbol{u}_1\|_{L^2(0,T;\boldsymbol{L}^2)} + \|\nabla p_1\|_{L^2(0,T;\boldsymbol{L}^2)} + \|\dot{\boldsymbol{\xi}}_1\|_{\boldsymbol{L}^2} + \|\dot{\boldsymbol{\omega}}_1\|_{\boldsymbol{L}^2} \\ &\le c\Big(\|\nabla \boldsymbol{u}_0\|_{\boldsymbol{L}^2} + \|\boldsymbol{f}\|_{L^2(0,T;\boldsymbol{L}^2(\mathcal{D}_{R_0}))} + \|\boldsymbol{\gamma}_1\|_{\boldsymbol{L}^2} + \|\boldsymbol{\gamma}_2\|_{\boldsymbol{L}^2}\Big). \end{aligned} \tag{4.1}$$

Additionally, by means of (3.10) we get

$$\begin{aligned} &\|\nabla \boldsymbol{u}_1\|_{L^2(0,T;\boldsymbol{L}^2)} + \|\boldsymbol{u}_1\|_{L^\infty(0,T;\boldsymbol{L}^2)} + \|\boldsymbol{\xi}_1\|_{\boldsymbol{L}^\infty} + \|\boldsymbol{\omega}_1\|_{\boldsymbol{L}^\infty} \\ &\le c\Big(\|\boldsymbol{u}_0\|_{\boldsymbol{L}^2} + \|\boldsymbol{f}\|_{L^2(0,T;\boldsymbol{L}^2(\mathcal{D}_{R_0}))} + |\boldsymbol{\xi}_0| + |\boldsymbol{\omega}_0| + \|\boldsymbol{\gamma}_1\|_{\boldsymbol{L}^2} + \|\boldsymbol{\gamma}_2\|_{\boldsymbol{L}^2}\Big). \end{aligned} \tag{4.2}$$

Next, let $\boldsymbol{z} \in L^2(0,T;\boldsymbol{W}^{2,2}(\mathbb{R}^3)) \cap W^{1,2}(0,T;\boldsymbol{L}^2(\mathbb{R}^3))$, such that z_j is the strong solution to the heat equation

$$\begin{aligned} \frac{\partial z_j}{\partial t} - \Delta z_j &= f_{2,j} \quad \text{in} \quad \mathbb{R}^3 \times (0,T), \\ z_j(0) &= 0 \quad \text{in} \quad \mathbb{R}^3 \end{aligned}$$

($j = 1,2,3$) (cf. Lemma 3.1). Owing to $\operatorname{div} \boldsymbol{f}_2 = 0$ one sees that $\operatorname{div} \boldsymbol{z}$ is a weak solution to the heat equation with zero data. By a standard uniqueness argument it follows that $\operatorname{div} \boldsymbol{z} = 0$ a.e. in $\mathbb{R}^3 \times (0,T)$. Furthermore, applying Lemma 3.1 one gets the estimate

$$\Big\|\frac{\partial \boldsymbol{z}}{\partial t}\Big\|_{L^2(0,T;\boldsymbol{L}^2_\eta)} + \|\boldsymbol{z}\|_{L^2(0,T;\boldsymbol{W}^{2,2}_\eta)} \le c\|\boldsymbol{f}\|_{L^2(0,T;\boldsymbol{L}^2_\eta)}. \tag{4.3}$$

Setting $\boldsymbol{u}_2 = \boldsymbol{u} - \boldsymbol{u}_1 - \boldsymbol{z}$ and $p_2 = p - p_1$ we see that $\boldsymbol{u}_2 \in L^2(0,T;\boldsymbol{W}^{1,2}(\mathcal{D})) \cap L^\infty(0,T;\boldsymbol{L}^2(\mathcal{D}))$ solves the system

$$\left.\begin{aligned} \operatorname{div} \boldsymbol{u}_2 &= 0 \\ \frac{\partial \boldsymbol{u}_2}{\partial t} - \Delta \boldsymbol{u}_2 &= -\nabla p_2 \end{aligned}\right\} \text{ in } \mathcal{D} \times (0,T), \tag{4.4}$$

fulfilling the following boundary and initial condition

$$\boldsymbol{u}_2 = \boldsymbol{\xi} - \boldsymbol{\xi}_1 + (\boldsymbol{\omega} - \boldsymbol{\omega}_1) \times x - \boldsymbol{z} \quad \text{on} \quad \partial\mathcal{D} \times (0,T), \tag{4.5}$$

$$\lim_{|x| \to \infty} \boldsymbol{u}_2(x,t) = 0, \tag{4.6}$$

$$\boldsymbol{u}_2(0) = 0. \tag{4.7}$$

We multiply the equation $(4.4)_2$ by $\boldsymbol{u}_2$, integrate both sides over $\mathcal{D}\times(0,t)$ $(t\in(0,T))$ and apply integration by parts. This leads to

$$\begin{aligned}&\frac{1}{2}\|\boldsymbol{u}_2(t)\|^2_{2,\mathcal{D}}-\int_0^t\int_{\mathcal{D}}\operatorname{div}\mathbb{T}(\boldsymbol{u}_2,p_2)\cdot\boldsymbol{u}_2\zeta dxds\\ &\qquad -\int_0^t\int_{\mathcal{D}}\operatorname{div}\mathbb{T}(\boldsymbol{u}_2,p_2-(p_2)_{\mathcal{D}_{R_0}})\cdot\boldsymbol{u}_2(1-\zeta)dxds=0\end{aligned}\tag{4.8}$$

for all $t\in(0,T)$. Recalling the definition of $\mathbb{T}$ we calculate

$$-\mathbb{T}(\boldsymbol{u}_2,p_2)=-\mathbb{T}(\boldsymbol{u},p)+\mathbb{T}(\boldsymbol{u}_1,p_1)+\boldsymbol{D}\boldsymbol{z}.\tag{4.9}$$

Noticing $\boldsymbol{u}_2=\boldsymbol{u}-\boldsymbol{u}_1-\boldsymbol{z}$ and taking into account $\operatorname{div}\mathbb{T}(\boldsymbol{u},p)=\frac{\partial\boldsymbol{u}}{\partial t}-\boldsymbol{f}$ (cf. $(2.1)_2$ for $\boldsymbol{u}$) the first integral on the left of (4.8) satisfies the following identity

$$\begin{aligned}&-\int_0^t\int_{\mathcal{D}}\operatorname{div}\mathbb{T}(\boldsymbol{u}_2,p_2)\cdot\boldsymbol{u}_2\zeta dxds\\ &\quad=-\int_0^t\int_{\mathcal{D}}\operatorname{div}\mathbb{T}(\boldsymbol{u},p)\cdot\boldsymbol{u}\zeta dxds+\int_0^t\int_{\mathcal{D}}\Big(\frac{\partial\boldsymbol{u}}{\partial t}-\boldsymbol{f}\Big)\cdot(\boldsymbol{u}_1+\boldsymbol{z})\zeta dxds\\ &\qquad\quad+\int_0^t\int_{\mathcal{D}}\Big(\operatorname{div}\mathbb{T}(\boldsymbol{u}_1,p_1)+\Delta\boldsymbol{z}\Big)\cdot\boldsymbol{u}_2\zeta dxds.\end{aligned}\tag{4.10}$$

Using integration by parts, observing equations (2.5) and (2.6) assigned to $\boldsymbol{\xi}$ and $\boldsymbol{\omega}$, we infer

$$\begin{aligned}&-\int_0^t\int_{\mathcal{D}}\operatorname{div}\mathbb{T}(\boldsymbol{u},p)\cdot\boldsymbol{u}\zeta dxds\\ &\quad=\int_0^t\int_{\mathcal{D}}|\nabla\boldsymbol{u}|^2\zeta dxds+\frac{1}{2}(|\boldsymbol{\xi}(t)|^2+|\boldsymbol{R}\boldsymbol{\omega}(t)|^2-|\boldsymbol{\xi}_0|^2-|\boldsymbol{R}\boldsymbol{\omega}_0|^2)\\ &\qquad+\int_0^t\int_{\mathcal{D}}\mathbb{T}(\boldsymbol{u},p-p_{\mathcal{D}_{R_0}}):\boldsymbol{u}\otimes\nabla\zeta dxds^{\,5)}-\int_0^t\boldsymbol{\gamma}_1\cdot\boldsymbol{\xi}+\boldsymbol{\gamma}_2\cdot\boldsymbol{\omega}ds.\end{aligned}\tag{4.11}$$

On the other hand, integration by parts gives

$$\begin{aligned}&-\int_0^t\int_{\mathcal{D}}\operatorname{div}\mathbb{T}(\boldsymbol{u}_2,p_2)\cdot\boldsymbol{u}_2(1-\zeta)dxds=\int_0^t\int_{\mathcal{D}}|\nabla\boldsymbol{u}_2|^2(1-\zeta)dxds\\ &\qquad-\int_0^t\int_{\mathcal{D}}\mathbb{T}(\boldsymbol{u}_2,p_2-(p_2)_{\mathcal{D}_{R_0}}):\boldsymbol{u}_2\otimes\nabla\zeta dxds.\end{aligned}\tag{4.12}$$

---

[5] Note that for every $\boldsymbol{v}\in\mathcal{V}(\mathcal{D})$ there holds $\int_{\mathcal{D}}\boldsymbol{v}\cdot\nabla\zeta dx=\int_{\partial\mathcal{D}}\boldsymbol{v}\cdot\boldsymbol{n}dS=0$.

Combining identities (4.10), (4.11), (4.12) and inserting the result into the left hand side of (4.8) we get

$$
\begin{aligned}
&\frac{1}{2}\Big(\|\boldsymbol{u}_2(t)\|^2_{\boldsymbol{L}^2} + |\boldsymbol{\xi}(t)|^2 + |\boldsymbol{R}\boldsymbol{\omega}(t)|^2\Big) + \int\limits_0^t \int\limits_{\mathcal{D}} |\nabla \boldsymbol{u}|^2\zeta + |\nabla \boldsymbol{u}_2|^2(1-\zeta)dxds \\
&= \frac{1}{2}\Big(\|\boldsymbol{u}_0\|^2_{\boldsymbol{L}^2} + |\boldsymbol{\xi}_0|^2 + |\boldsymbol{R}\boldsymbol{\omega}_0|^2\Big) + \int\limits_0^t \boldsymbol{\gamma}_1 \cdot \boldsymbol{\xi} + \boldsymbol{\gamma}_2 \cdot \boldsymbol{\omega} ds \\
&+ \int\limits_0^t \int\limits_{\mathcal{D}} \Big(\mathbb{T}(\boldsymbol{u}_2, p_2 - (p_2)_{\mathcal{D}_{R_0}}) \cdot \boldsymbol{u}_2 - \mathbb{T}(\boldsymbol{u}, p - p_{\mathcal{D}_{R_0}}) \cdot \boldsymbol{u}\Big) \cdot \nabla\zeta dxds \\
&- \int\limits_0^t \int\limits_{\mathcal{D}} \Big(\frac{\partial \boldsymbol{u}}{\partial t} - \boldsymbol{f}\Big) \cdot (\boldsymbol{u}_1 + \boldsymbol{z})\zeta dxds - \int\limits_0^t \int\limits_{\mathcal{D}} \Big(\operatorname{div}\mathbb{T}(\boldsymbol{u}_1, p_1) + \Delta \boldsymbol{z}\Big) \cdot \boldsymbol{u}_2 \zeta dxds \\
&= \frac{1}{2}\Big(\|\boldsymbol{u}_0\|^2_{\boldsymbol{L}^2} + |\boldsymbol{\xi}_0|^2 + |\boldsymbol{R}\boldsymbol{\omega}_0|^2\Big) + I_1 + I_2 + I_3 + I_4
\end{aligned}
\tag{4.13}
$$

for almost all $t \in (0,T)$.

(i) Using Cauchy-Schwarz's inequality and Young's inequality taking into account (4.1), (4.2) and (4.3) we easily get

$$
I_1 + I_4 \le cK_0^2 + \frac{1}{8}\Big(\|\boldsymbol{u}_2\|^2_{L^\infty(0,T;\boldsymbol{L}^2)} + \|\boldsymbol{\omega}\|^2_{\boldsymbol{L}^\infty(0,T)} + \|\boldsymbol{\xi}\|^2_{\boldsymbol{L}^\infty(0,T)}\Big)^{6)}.
$$

(ii) In order to estimate I_2 we first notice the following identity

$$
\begin{aligned}
&\mathbb{T}(\boldsymbol{u}_2, p_2 - (p_2)_{\mathcal{D}_{R_0}}) \cdot \boldsymbol{u}_2 - \mathbb{T}(\boldsymbol{u}, p - p_{\mathcal{D}_{R_0}}) \cdot \boldsymbol{u} \\
&\quad = -(\boldsymbol{D}\boldsymbol{u}_1 + \boldsymbol{D}\boldsymbol{z}) \cdot \boldsymbol{u}_2 - \boldsymbol{D}\boldsymbol{u}_2 \cdot (\boldsymbol{u}_1 + \boldsymbol{z}) - (\boldsymbol{D}\boldsymbol{u}_1 + \boldsymbol{D}\boldsymbol{z}) \cdot (\boldsymbol{u}_1 + \boldsymbol{z}) \\
&\qquad + \boldsymbol{I}(p_1 - (p_1)_{\mathcal{D}_{R_0}}) \cdot \boldsymbol{u}_2 - \boldsymbol{I}(p_1 - (p_1)_{\mathcal{D}_{R_0}}) \cdot (\boldsymbol{u}_1 + \boldsymbol{z}) \\
&\qquad - \boldsymbol{I}(p_2 - (p_2)_{\mathcal{D}_{R_0}}) \cdot (\boldsymbol{u}_1 + \boldsymbol{z}).
\end{aligned}
$$

Again using Cauchy-Schwarz's, Young's inequality and the Poincaré inequality for the term involving the pressure $p_1 - (p_1)_{\mathcal{D}_{R_0}}$ together with (4.2) and (4.3) we get

$$
I_2 \le cK_0^2 + c\|p_2 - (p_2)_{\mathcal{D}_{R_0}}\|_{L^2(0,T;L^2(\mathcal{D}_{R_0}))}K_0 + \frac{1}{8}\|\boldsymbol{u}_2\|^2_{L^\infty(0,T;\boldsymbol{L}^2)}.
$$

(iii) For the estimation of I_3 we apply integration by parts. This gives

$$
\begin{aligned}
&-\int\limits_0^t \int\limits_{\mathcal{D}} \frac{\partial \boldsymbol{u}_2}{\partial t} \cdot (\boldsymbol{u}_1 + \boldsymbol{z})\zeta dxds \\
&\quad = \int\limits_0^t \int\limits_{\mathcal{D}} \boldsymbol{u}_2 \cdot \frac{\partial}{\partial t}(\boldsymbol{u}_1 + \boldsymbol{z})\zeta dxds - \int\limits_{\mathcal{D}} \boldsymbol{u}_2(t) \cdot (\boldsymbol{u}_1(t) + \boldsymbol{z}(t))\zeta dx.
\end{aligned}
$$

---

6) For the definition of K_0 see p. 5.

By the aid of Cauchy-Schwarz's inequality and Young's inequality again observing (4.1), (4.1) and (4.3) we get

$$I_3 \le cK_0^2 + \frac{1}{8}\|\boldsymbol{u}_2\|^2_{L^\infty(0,T;\boldsymbol{L}^2)}.$$

Now, inserting the estimates of I_1–I_4 into (4.13) we obtain the estimate

$$\begin{aligned}&\|\boldsymbol{u}_2\|^2_{L^\infty(0,T;\boldsymbol{L}^2)} + \|\nabla \boldsymbol{u}_2\|^2_{L^2(0,T;\boldsymbol{L}^2)} + \|\boldsymbol{\xi}\|^2_{\boldsymbol{L}^2(0,T)} + \|\boldsymbol{\omega}\|^2_{\boldsymbol{L}^2(0,T)}\\ &\quad\le cK_0^2 + c\|p_2 - (p_2)_{\mathcal{D}_{R_0}}\|_{L^2(0,T;L^2(\mathcal{D}_{R_0}))}K_0.\end{aligned} \tag{4.14}$$

Next, we are going to estimate the L^2 norm of $\frac{\partial \boldsymbol{u}_2}{\partial t}$. To begin with, we multiply both sides of the equation $(4.4)_2$ with $\frac{\partial \boldsymbol{u}_2}{\partial t}$ and integrate the obtained result over $\mathcal{D}\times(0,t)\,(t\in(0,T))$. Then, using integration by parts we are led to

$$\begin{aligned}&\Big\|\frac{\partial \boldsymbol{u}_2}{\partial t}\Big\|^2_{L^2(0,t;\boldsymbol{L}^2)} - \int_0^t\int_{\mathcal{D}} \operatorname{div}\mathbb{T}(\boldsymbol{u}_2,p_2)\cdot\frac{\partial \boldsymbol{u}_2}{\partial t}\zeta dxds + \frac{1}{2}\int_{\mathcal{D}}|\nabla \boldsymbol{u}_2(t)|^2(1-\zeta)dx\\ &= \int_0^t\int_{\mathcal{D}} \mathbb{T}(\boldsymbol{u}_2, p_2 - (p_2)_{\mathcal{D}_{R_0}})\cdot\frac{\partial \boldsymbol{u}_2}{\partial t}\otimes\nabla\zeta dxds + \frac{1}{2}\int_{\mathcal{D}}|\nabla \boldsymbol{u}_0|^2(1-\zeta)dx\end{aligned} \tag{4.15}$$

for almost all $t\in(0,T)$. By an elementary calculus we see that

$$\begin{aligned}&-\int_0^t\int_{\mathcal{D}} \operatorname{div}\mathbb{T}(\boldsymbol{u}_2,p_2)\cdot\frac{\partial \boldsymbol{u}_2}{\partial t}\zeta dxds = -\int_0^t\int_{\mathcal{D}} \operatorname{div}\mathbb{T}(\boldsymbol{u},p)\cdot\frac{\partial \boldsymbol{u}}{\partial t}\zeta dxds\\ &+\int_0^t\int_{\mathcal{D}}\Big(\frac{\partial \boldsymbol{u}}{\partial t} - \boldsymbol{f}\Big)\cdot\frac{\partial(\boldsymbol{u}_1+\boldsymbol{z})}{\partial t}\zeta dxds + \int_0^t\int_{\mathcal{D}}\Big(\operatorname{div}\mathbb{T}(\boldsymbol{u}_1,p_1)+\Delta\boldsymbol{z}\Big)\cdot\frac{\partial \boldsymbol{u}_2}{\partial t}\zeta dxds.\end{aligned}$$

Observing (2.5) and (2.6) arguing as in the proof of Lemma 3.2 we infer

$$\begin{aligned}&-\int_0^t\int_{\mathcal{D}} \operatorname{div}\mathbb{T}(\boldsymbol{u},p)\cdot\frac{\partial \boldsymbol{u}}{\partial t}\zeta dxds = \int_0^t |\dot{\boldsymbol{\xi}}|^2 - \dot{\boldsymbol{\xi}}\cdot\boldsymbol{\gamma}_1 + |\boldsymbol{R}\dot{\boldsymbol{\omega}}|^2 - \dot{\boldsymbol{\omega}}\cdot\boldsymbol{\gamma}_2 ds\\ &+\frac{1}{2}\int_{\mathcal{D}}|\nabla \boldsymbol{u}(t)|^2dx - \frac{1}{2}\int_{\mathcal{D}}|\nabla \boldsymbol{u}_0|^2dx + \int_0^t\int_{\mathcal{D}}\mathbb{T}(\boldsymbol{u}, p - p_{\mathcal{D}_{R_0}}) : \frac{\partial \boldsymbol{u}}{\partial t}\otimes\nabla\zeta dxds.^{7)}\end{aligned}$$

Combining the above identities together with (4.15) we obtain

$$
\begin{aligned}
&\Big\|\frac{\partial \boldsymbol{u}_2}{\partial t}\Big\|^2_{L^2(0,t;\boldsymbol{L}^2)} + \frac{1}{2}\int\limits_{\mathcal{D}} |\nabla \boldsymbol{u}(t)|^2\zeta + |\nabla \boldsymbol{u}_2(t)|^2(1-\zeta)dx + \int\limits_0^t |\dot{\boldsymbol{\xi}}|^2 + |\boldsymbol{R}\dot{\boldsymbol{\omega}}|^2 ds \\
&\quad = \int\limits_0^t\int\limits_{\mathcal{D}} \Big(\mathbb{T}(\boldsymbol{u}_2, p_2 - (p_2)_{\mathcal{D}_{R_0}})\cdot\frac{\partial \boldsymbol{u}_2}{\partial t} - \mathbb{T}(\boldsymbol{u}, p - p_{\mathcal{D}_{R_0}})\cdot\frac{\partial \boldsymbol{u}}{\partial t}\Big)\cdot\nabla\zeta dxds \\
&\qquad\quad + \int\limits_0^t\int\limits_{\mathcal{D}} \Big(\frac{\partial \boldsymbol{u}}{\partial t} - \boldsymbol{f}\Big)\cdot\frac{\partial(\boldsymbol{u}_1+\boldsymbol{z})}{\partial t}\zeta + \Big(\operatorname{div}\mathbb{T}(\boldsymbol{u}_1,p_1) + \Delta\boldsymbol{z}\Big)\cdot\frac{\partial \boldsymbol{u}_2}{\partial t}\zeta dxds \\
&\qquad\quad + \frac{1}{2}\|\nabla \boldsymbol{u}_0\|^2_{\boldsymbol{L}^2} + \int\limits_0^t \dot{\boldsymbol{\xi}}\cdot\gamma_1 + \dot{\boldsymbol{\omega}}\cdot\gamma_2 ds.
\end{aligned} \tag{4.16}
$$

Clearly, recalling $\boldsymbol{u} = \boldsymbol{u}_1 + \boldsymbol{u}_2 + \boldsymbol{z}$ one calculates

$$
\begin{aligned}
&\mathbb{T}(\boldsymbol{u}_2, p_2 - (p_2)_{\mathcal{D}_{R_0}})\cdot\frac{\partial \boldsymbol{u}_2}{\partial t} - \mathbb{T}(\boldsymbol{u}, p - p_{\mathcal{D}_{R_0}})\cdot\frac{\partial \boldsymbol{u}}{\partial t} \\
&\quad = -(\boldsymbol{D}\boldsymbol{u}_1 + \boldsymbol{D}\boldsymbol{z})\cdot\frac{\partial \boldsymbol{u}_2}{\partial t} - \boldsymbol{D}\boldsymbol{u}_2\cdot\frac{\partial(\boldsymbol{u}_1+\boldsymbol{z})}{\partial t} - (\boldsymbol{D}\boldsymbol{u}_1 + \boldsymbol{D}\boldsymbol{z})\cdot\frac{\partial(\boldsymbol{u}_1+\boldsymbol{z})}{\partial t} \\
&\qquad + \boldsymbol{I}(p_1 - (p_1)_{\mathcal{D}_{R_0}})\cdot\frac{\partial \boldsymbol{u}_2}{\partial t} - \boldsymbol{I}(p_1 - (p_1)_{\mathcal{D}_{R_0}})\cdot\frac{\partial(\boldsymbol{u}_1+\boldsymbol{z})}{\partial t} \\
&\qquad\quad - \boldsymbol{I}(p_2 - (p_2)_{\mathcal{D}_{R_0}})\cdot\frac{\partial(\boldsymbol{u}_1+\boldsymbol{z})}{\partial t}.
\end{aligned}
$$

By the aid of this identity using Young's inequality taking into account estimates (4.1), (4.2) and (4.3) we obtain

$$
\begin{aligned}
&\Big\|\frac{\partial \boldsymbol{u}_2}{\partial t}\Big\|^2_{L^2(0,T;\boldsymbol{L}^2)} + \|\nabla \boldsymbol{u}_2\|^2_{L^\infty(0,T;\boldsymbol{L}^2)} + \|\dot{\boldsymbol{\xi}}\|^2_{\boldsymbol{L}^2(0,T)} + \|\dot{\boldsymbol{\omega}}\|^2_{\boldsymbol{L}^2(0,T)} \\
&\quad \leq cK_0^2 + c\|p_2 - (p_2)_{\mathcal{D}_{R_0}}\|_{L^2(0,T;L^2(\mathcal{D}_{R_0}))}K_0.
\end{aligned} \tag{4.17}
$$

For the estimation of the pressure on the right hand side of (4.17) we make use of $(4.4)_2$. That is

$$
-\nabla p_2 = \frac{\partial \boldsymbol{u}_2}{\partial t} - \Delta \boldsymbol{u}_2 \quad \text{a.e. in} \quad \mathcal{D}_{R_0}\times(0,T).
$$

Consulting, [9] (Th. III. 3.1, Th. III. 5.2) we obtain the estimate

$$
\|p_2 - (p_2)_{L^2(0,T;L^2(\mathcal{D}_{R_0}))}\|_{L^2(0,T;L^2(\mathcal{D}_{R_0}))} \leq c\Big(\Big\|\frac{\partial \boldsymbol{u}_2}{\partial t}\Big\|_{L^2(0,T;\boldsymbol{L}^2)} + \|\nabla \boldsymbol{u}_2\|_{L^2(0,T;\boldsymbol{L}^2)}\Big).
$$

Inserting this estimate into the right hand side of (4.17) using Young's inequality taking into account (4.14) we get

$$
\Big\|\frac{\partial \boldsymbol{u}_2}{\partial t}\Big\|_{L^2(0,T;\boldsymbol{L}^2)} + \|\boldsymbol{u}_2\|_{L^\infty(0,T;\boldsymbol{W}^{1,2})} + \|\boldsymbol{\xi}\|_{\boldsymbol{W}^{1,2}} + \|\boldsymbol{\omega}\|_{\boldsymbol{W}^{1,2}} \leq cK_0. \tag{4.18}
$$

---

7) Remark, $p_{\mathcal{D}_{R_0}}\int\limits_{\mathcal{D}}\frac{\partial \boldsymbol{u}}{\partial t}\cdot\nabla\zeta dx = 0$ (cf. also footnote 4).

Finally, it remains to estimate ∇p_2 and $\nabla^2 \boldsymbol{u}_2$. Let $t \in (0,T)$ be fixed such that $\frac{\partial \boldsymbol{u}_2}{\partial t}(t), \frac{\partial^2 \boldsymbol{z}}{\partial x_i \partial x_j}(t) \in \boldsymbol{L}^2$ $(i,j=1,2,3)$. We define

$$\boldsymbol{\Psi}_2(x) = \frac{1}{2}\,\mathbf{rot}\Big(\zeta(\boldsymbol{\xi}_2(t)\times x - \boldsymbol{\omega}_2(t)|x|^2)\Big), \quad x\in\mathbb{R}^3, t\geq 0,$$

where $\boldsymbol{\xi}_2 = \boldsymbol{\xi} - \boldsymbol{\xi}_1$ and $\boldsymbol{\omega}_2 = \boldsymbol{\omega} - \boldsymbol{\omega}_1$. Then $\boldsymbol{v} = \boldsymbol{u}_2(\cdot,t) - \boldsymbol{\Psi}_2(\cdot,t) - \zeta\boldsymbol{z}(\cdot,t)$ solves the steady problem

$$\begin{aligned}
\operatorname{div}\boldsymbol{v} &= -\nabla\zeta\cdot\boldsymbol{z}(t) \quad \text{in} \quad \mathcal{D},\\
-\Delta\boldsymbol{v} &= \frac{\partial \boldsymbol{u}_2}{\partial t}(t) + \Delta\boldsymbol{\Psi}_2(t) + \Delta(\zeta\boldsymbol{z}(t)) - \nabla p_2(t) \quad \text{in} \quad \mathcal{D},\\
\boldsymbol{v} &= 0 \quad \text{on} \quad \partial\mathcal{D}.
\end{aligned}$$

Consulting [9] we see that $\frac{\partial^2 \boldsymbol{v}}{\partial x_i\partial x_j}, \nabla p_2 \in \boldsymbol{L}^2(\mathcal{D})$ together with the estimate

$$\begin{aligned}
&\|\nabla^2\boldsymbol{v}\|^2_{\boldsymbol{L}^2} + \|\nabla p_2(t)\|^2_{\boldsymbol{L}^2}\\
&\leq \; c\Big(\Big\|\frac{\partial\boldsymbol{u}_2}{\partial t}(t)\Big\|^2_{\boldsymbol{L}^2} + \|\Delta(\zeta\boldsymbol{z}(t))\|^2_{\boldsymbol{L}^2(\mathcal{D}_{R_0})} + \|\Delta\boldsymbol{\Psi}_2(t)\|^2_{\boldsymbol{L}^2}\Big).
\end{aligned} \tag{4.19}$$

Integrating, both sides of (4.19) over $(0,T)$ using (4.18), (4.3) and (4.1) we see that

$$\|\nabla^2\boldsymbol{u}_2\|_{L^2(0,T;\boldsymbol{L}^2)} + \|\nabla p_2\|_{L^2(0,T;\boldsymbol{L}^2)} \leq cK_0. \tag{4.20}$$

2° *Proof of the theorem for general* $\boldsymbol{f} = \mathbf{rot}(\boldsymbol{g})$ *for* $\boldsymbol{g}\in L^2(0,T;\boldsymbol{W}^{1,2}_\eta(\mathcal{D}))$. For $\varepsilon>0$ we define

$$\boldsymbol{g}_\varepsilon(x,t) = (1+\varepsilon|x|)^{-1}\boldsymbol{g}(x,t), \quad (x,t)\in\mathcal{D}\times(0,T), \quad \boldsymbol{f}_\varepsilon = \mathbf{rot}(\boldsymbol{g}_\varepsilon).$$

Clearly, $\boldsymbol{g}_\varepsilon \in L^2(0,T;\boldsymbol{W}^{1,2}(\mathcal{D}))$ and $\boldsymbol{f}_\varepsilon\in L^2(0,T;\boldsymbol{L}^2(\mathcal{D}))$ for all $\varepsilon>0$. In addition, we immediately see that for a. e. $t\in(0,T)$

$$\|\boldsymbol{g}_\varepsilon(t)\|_{\boldsymbol{W}^{1,2}_\eta(\mathcal{D})} \leq 2\|\boldsymbol{g}(t)\|_{\boldsymbol{W}^{1,2}_\eta(\mathcal{D})} \quad \forall\,\varepsilon>0.$$

By Lebesgue's theorem of dominated convergence it follows that

$$\boldsymbol{f}_\varepsilon \to \boldsymbol{f} \quad \text{in} \quad L^2(0,T;\boldsymbol{L}^2_\eta(\mathcal{D})) \quad \text{as} \quad \varepsilon\to 0.$$

As it has been proved above in 1° there exists a strong solution $(\boldsymbol{u}_\varepsilon, \boldsymbol{\xi}_\varepsilon, \boldsymbol{\omega}_\varepsilon)$ to the system (2.1)–(2.6) with $\boldsymbol{f}_\varepsilon$ in place of $\boldsymbol{f}$. Furthermore, there exists a pressure $p_\varepsilon \in L^2(0,T;L^2_{\text{loc}}(\overline{\mathcal{D}}))$ with $\nabla p_\varepsilon \in L^2(0,T;\boldsymbol{L}^2(\mathcal{D}))$. Owing to (4.1), (4.2), (4.3), (4.18) and (4.20) we get the a-priori bound

$$\begin{aligned}
&\Big\|\frac{\partial\boldsymbol{u}_\varepsilon}{\partial t}\Big\|_{L^2(0,T;\boldsymbol{L}^2_\eta)} + \|\nabla^2\boldsymbol{u}_\varepsilon\|_{L^2(0,T;\boldsymbol{L}^2_\eta)} + \|\boldsymbol{u}_\varepsilon\|_{L^\infty(0,T;\boldsymbol{W}^{1,2}_\eta)} +\\
&\quad + \|\boldsymbol{\xi}_\varepsilon\|_{\boldsymbol{W}^{1,2}} + \|\boldsymbol{\omega}_\varepsilon\|_{\boldsymbol{W}^{1,2}} + \|\nabla p_\varepsilon\|_{L^2(0,T;\boldsymbol{L}^2)} \leq cK_0,
\end{aligned} \tag{4.21}$$

where c denotes a constant depending on $\mathcal{D}$ only. Hence, by means of reflexivity we may choose a sequence of positive numbers $\varepsilon_j\to 0$ as $j\to\infty$, such that

$$\begin{aligned}
\boldsymbol{u}_{\varepsilon_j} &\to \boldsymbol{u} \quad \textit{weakly in} \quad L^2(0,T;\boldsymbol{W}^{2,2}_\eta(\mathcal{D})),\\
\frac{\partial\boldsymbol{u}_{\varepsilon_j}}{\partial t} &\to \frac{\partial\boldsymbol{u}}{\partial t} \quad \textit{weakly in} \quad L^2(0,T;\boldsymbol{L}^2_\eta(\mathcal{D})),\\
\boldsymbol{\xi}_{\varepsilon_j}, \boldsymbol{\omega}_{\varepsilon_j} &\to \boldsymbol{\xi}, \boldsymbol{\omega} \quad \textit{weakly in} \quad \boldsymbol{W}^{1,2}(0,T),\\
\nabla p_{\varepsilon_j} &\to \nabla p \quad \textit{weakly in} \quad L^2(0,T;\boldsymbol{L}^2(\mathcal{D})) \quad \textit{as} \quad j\to\infty.
\end{aligned}$$

As one easily checks the triple $(\boldsymbol{u}, \boldsymbol{\xi}, \boldsymbol{\omega})$ is a strong solution to the system (2.1)–(2.6) with pressure p satisfying (2.9), (2.10). ■

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