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LINEARIZED NAVIER-STOKES EQUATIONS IN $\mathbb{R}^3$: AN APPROACH IN WEIGHTED SOBOLEV SPACES

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Abstract. In this work, we study the linearized Navier-Stokes equations in $\mathbb{R}^3$, the Oseen equations. We are interested in the existence and the uniqueness of generalized and strong solutions in L^p-theory which makes analysis more difficult. Our approach rests on the use of weighted Sobolev spaces.

1. **Introduction.** We consider the Oseen equations in $\mathbb{R}^3$ obtained formally by linearising of the Navier-Stokes equations:

$$-\Delta \boldsymbol{u} + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{u}) + \nabla \pi = \boldsymbol{f} \quad \text{and} \quad \operatorname{div} \boldsymbol{u} = h \quad \text{in} \quad \mathbb{R}^3, \tag{1}$$

where, $\boldsymbol{v}$ is a given velocity field belonging to $\boldsymbol{L}^3(\mathbb{R}^3)$ with divergence free. When (1) is posed in the bounded domain Ω, *i.e* with a boundary condition, the existence and uniqueness of weak solutions of problem (1) are studied in the classical Sobolev spaces $\boldsymbol{W}^{m,p}(\Omega)$, see [4], [9] for instance. It is well known that it is not possible to extend this result to the case of unbounded domains, for example the exterior domain or the whole space $\mathbb{R}^3$ since the spaces $\boldsymbol{W}^{m,p}(\Omega)$ are not appropriate. Therefore, a specific functional framework is necessary which also has to take into account the behaviour of the functions at infinity. Our approach is based on the weighted Sobolev spaces $\boldsymbol{W}^{m,p}_{\alpha}(\mathbb{R}^3)$ introduced by Hanouzet [10] and Cantor [5] (see section 2 for the details). Another approach we can refer to Galdi [8] or Farwig [6, 7]. In the last years, different methods have been devoloped to study the problem (1). One idea is to suppose in addition that the norm of $\boldsymbol{v}$ in $\boldsymbol{L}^3(\mathbb{R}^3)$ is controled by a positive constant:

$$||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)} < k, \tag{2}$$

for more details see [2]. Observe that this condition of smallness is very strong. The basic idea of our work consists on improving the work done by Amrouche and Consiglieri [2] by dropping the condition (2).

2. **Basic concepts on weighted Sobolev spaces.** Throughout this paper, all functions and distributions are defined on the 3-dimentional real Euclidean space $\mathbb{R}^3$. Let $x = (x_1, x_2, x_3)$ be a typical point in $\mathbb{R}^3$ and let $r = |x| = (x_1^2 + x_2^2 + x_3^2)^{1/2}$ denote its distance to the origin. In order to control the behaviour at infinity of our functions and distributions we use for basic weight the quantity $\rho(x) = (1 + r^2)^{1/2}$ which is equivalent to r at infinity, and to one on any bounded subset of $\mathbb{R}^3$. We define $\mathcal{D}(\mathbb{R}^3)$ to be the linear space of infinite differentiable functions with compact support on $\mathbb{R}^3$. Now, let $\mathcal{D}'(\mathbb{R}^3)$ denote the dual space of $\mathcal{D}(\mathbb{R}^3)$, ofen called the space of distributions on $\mathbb{R}^3$. We denote by $< .,. >$ the duality pairing between $\mathcal{D}'(\mathbb{R}^3)$ and $\mathcal{D}(\mathbb{R}^3)$. For

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each $p \in \mathbb{R}$ and $1 < p < \infty$, the conjugate exponent p' is given by the relation $\frac{1}{p} + \frac{1}{p'} = 1$. Then, for any nonnegative integers m and real numbers $p > 1$ and α, setting

$$k = k(m,p,\alpha) = \begin{cases} -1, & \text{if } \frac{3}{p} + \alpha \notin \{1,...,m\}, \\ m - \frac{3}{p} - \alpha, & \text{if } \frac{3}{p} + \alpha \in \{1,...,m\}, \end{cases}$$

we define the following space:

$$\begin{aligned} W^{m,p}_{\alpha}(\mathbb{R}^3) = \{u \in \mathcal{D}'(\mathbb{R}^3);\ & \\ & \forall\, \lambda \in \mathbb{N}^3 : 0 \le |\lambda| \le k,\ \rho^{\alpha - m + |\lambda|}(\ln(1+\rho))^{-1} D^{\lambda} u \in L^p(\mathbb{R}^3); \\ & \forall\, \lambda \in \mathbb{N}^3 : k+1 \le |\lambda| \le m, \rho^{\alpha - m + |\lambda|} D^{\lambda} u \in L^p(\mathbb{R}^3)\}. \end{aligned}$$

It is a reflexive Banach space equipped with its natural norm:

$$\begin{aligned} ||u||_{W^{m,p}_{\alpha}(\mathbb{R}^3)} = \Bigg(& \sum_{0 \le |\lambda| \le k} ||\rho^{\alpha - m + |\lambda|}(\ln(1+\rho))^{-1} D^{\lambda} u||^p_{L^p(\mathbb{R}^3)} \\ & + \sum_{k+1 \le |\lambda| \le m} ||\rho^{\alpha - m + |\lambda|} D^{\lambda} u||^p_{L^p(\mathbb{R}^3)} \Bigg)^{1/p}. \end{aligned}$$

For $m = 0$, we set

$$W^{0,p}_{\alpha}(\mathbb{R}^3) = \{u \in \mathcal{D}'(\mathbb{R}^3);\ \ \rho^{\alpha}\, u \in L^p(\mathbb{R}^3)\}.$$

We note that the logarithmic weight only appears if $p = 3$ or $p = \frac{3}{2}$ and all the local properties of $W^{m,p}_{\alpha}(\mathbb{R}^3)$ coincide with those of the classical Sobolev space $W^{m,p}(\mathbb{R}^3)$. We set $\overset{\circ}{W}{}^{m,p}_{\alpha}(\mathbb{R}^3)$ as the adherence of $\mathcal{D}(\mathbb{R}^3)$ for the norm $\| \,.\, \|_{W^{m,p}_{\alpha}(\mathbb{R}^3)}$. Then, the dual space of $\overset{\circ}{W}{}^{m,p}_{\alpha}(\mathbb{R}^3)$, denoting by $W^{-m,p'}_{-\alpha}(\mathbb{R}^3)$, is a space of distributions. On the other hand, these spaces obey the following imbedding

$$W^{m,p}_{\alpha}(\mathbb{R}^3) \hookrightarrow W^{m-1,p}_{\alpha-1}(\mathbb{R}^3)$$

if and only if $m > 0$ and $3/p + \alpha \neq 1$ or $m \le 0$ and $3/p + \alpha \neq 3$.
In addition, we have for $\alpha = 0$ or $\alpha = 1$

$$W^{1,p}_{\alpha}(\mathbb{R}^3) \hookrightarrow W^{0,p*}_{\alpha}(\mathbb{R}^3) \quad \text{where} \quad p* = \frac{3p}{3-p} \quad \text{and} \quad 1 < p < 3. \tag{3}$$

Consequently, by duality, we have

$$W^{0,q}_{-\alpha}(\mathbb{R}^3) \hookrightarrow W^{-1,p'}_{-\alpha}(\mathbb{R}^3) \quad \text{where} \quad q = \frac{3p'}{3+p'} \quad \text{and} \quad p' > 3/2.$$

Moreover, the Hardy inequality holds,

$$\forall u \in W^{1,p}_{\alpha}(\mathbb{R}^3), \qquad \begin{cases} ||u||_{W^{1,p}_{\alpha}(\mathbb{R}^3)} \le C||\nabla\, u||_{\boldsymbol{W}^{0,p}_{\alpha}(\mathbb{R}^3)}, & \text{if} \quad 3/p + \alpha > 1, \\ ||u||_{W^{1,p}_{\alpha}(\mathbb{R}^3)/\mathcal{P}_0} \le C||\nabla\, u||_{\boldsymbol{W}^{0,p}_{\alpha}(\mathbb{R}^3)}, & \text{otherwise,} \end{cases}$$

where $\mathcal{P}_0$ stands for the space of constant functions in $W^{1,p}_{\alpha}(\mathbb{R}^3)$ when $3/p + \alpha \le 1$ with C satisfying $C = C(p,\alpha) > 0$.

3. **Generalized solutions in $\boldsymbol{W}^{1,p}_0(\mathbb{R}^3)$.** We are interested in the existence and the uniqueness of generalized solutions $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}_0(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$, with $1 < p < \infty$, to the problem (1). We will consider the following data:

$$\boldsymbol{f} \in \boldsymbol{W}^{-1,p}_0(\mathbb{R}^3), \quad \boldsymbol{v} \in \boldsymbol{L}^3_{\sigma}(\mathbb{R}^3) \quad \text{and} \quad h \in L^p(\mathbb{R}^3).$$

On the one hand if $\boldsymbol{u} \in \boldsymbol{W}^{1,p}_0(\mathbb{R}^3)$, then we have $\boldsymbol{u} \in \boldsymbol{L}^{3/2}_{\text{loc}}(\mathbb{R}^3)$ and thus $\boldsymbol{v} \otimes \boldsymbol{u}$ belongs to $\boldsymbol{L}^1_{\text{loc}}(\mathbb{R}^3)$. It means that $\text{div}(\boldsymbol{v} \otimes \boldsymbol{u})$ is well defined as a distribution in $\mathbb{R}^3$. On the other hand, if $p \ge 3/2$, we deduce that the term $\boldsymbol{v} \cdot \nabla\, \boldsymbol{u}$ is well defined and we can write $\text{div}(\boldsymbol{v} \otimes \boldsymbol{u}) = \boldsymbol{v} \cdot \nabla\, \boldsymbol{u}$. Moreover, if $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}_0(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$ with $p < 3$ is a solution to (1), we have for any $\boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}(\mathbb{R}^3)$:

$$\int_{\mathbb{R}^3} ((\nabla \boldsymbol{u} + \boldsymbol{v} \otimes \boldsymbol{u}) : \nabla \boldsymbol{\varphi} - \pi\, \text{div} \boldsymbol{\varphi}) = \langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle_{W^{-1,p}_0(\mathbb{R}^3) \times \overset{\circ}{W}{}^{1,p'}_0(\mathbb{R}^3)}. \tag{4}$$

Observe that in this case, $\boldsymbol{u} \in \boldsymbol{L}^{p*}(\mathbb{R}^3)$ with $\frac{1}{p*} = \frac{1}{p} - \frac{1}{3}$, so $\boldsymbol{v} \otimes \boldsymbol{u} \in \boldsymbol{L}^p(\mathbb{R}^3)$. Because $\boldsymbol{\mathcal{D}}(\mathbb{R}^3)$ is dense in $\boldsymbol{W}_0^{1,p'}(\mathbb{R}^3)$, this last relation holds for any $\boldsymbol{\varphi} \in \boldsymbol{W}_0^{1,p'}(\mathbb{R}^3)$. As this last space contains the constant vectors when $p' \geq 3$, the force $\boldsymbol{f}$ must satisfies the following compatibility condition:

$$\langle f_i, 1\rangle_{W_0^{-1,p}(\mathbb{R}^3)\times W_0^{1,p'}(\mathbb{R}^3)} = 0 \quad \text{for any } i = 1,2,3 \quad \text{if} \quad p \leq 3/2. \tag{5}$$

If $p \geq 3$, (1) is equivalent to the following variational problem:

$$\int_{\mathbb{R}^3} \left(\nabla \boldsymbol{u} : \nabla \boldsymbol{\varphi} - \pi \operatorname{div}\boldsymbol{\varphi} + \boldsymbol{v} \cdot \nabla \boldsymbol{u} \cdot \boldsymbol{\varphi}\right) = \langle \boldsymbol{f}, \boldsymbol{\varphi}\rangle_{W_0^{-1,p}(\mathbb{R}^3)\times W_0^{1,p'}(\mathbb{R}^3)}. \tag{6}$$

Remark 1.

To simplify the study of problem (1), we can suppose at first that $h = 0$. Indeed, if h in $L^p(\mathbb{R}^3)$, there exists $\chi \in W_0^{2,p}(\mathbb{R}^3)$ such that $\Delta \chi = h$ (see [1]) and satisfying

$$||\nabla \chi||_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)} \leqslant C||h||_{L^p(\mathbb{R}^3)}. \tag{7}$$

Set $\boldsymbol{w}_h = \nabla \chi \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^3)$ and $\boldsymbol{z} = \boldsymbol{u} - \boldsymbol{w}_h$. Then problem (1) becomes:

$$-\Delta \boldsymbol{z} + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{z}) + \nabla \pi = \boldsymbol{f} + \Delta \boldsymbol{w}_h - \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{w}_h) \quad \text{and} \quad \operatorname{div} \boldsymbol{z} = 0 \quad \text{in} \quad \mathbb{R}^3.$$

If $1 < p < 3$, we have $\boldsymbol{w}_h \in \boldsymbol{L}^{p*}(\mathbb{R}^3)$ and $\boldsymbol{v} \otimes \boldsymbol{w}_h$ belongs to $\boldsymbol{L}^p(\mathbb{R}^3)$. Consequently $\operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{w}_h)$ belongs to $\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)$. However when $p \geq 3$, $\operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{w}_h) = \boldsymbol{v} \cdot \nabla \boldsymbol{w}_h$ belongs to $\boldsymbol{L}^r(\mathbb{R}^3)$, with $\frac{1}{r} = \frac{1}{3} + \frac{1}{p}$ and $\boldsymbol{L}^r(\mathbb{R}^3) \hookrightarrow \boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)$. This means that $\mathbf{F} := \boldsymbol{f} + \Delta \boldsymbol{w}_h - \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{w}_h)$ belongs to $\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)$. In addition, we have for any $i = 1,2,3$ and $p \leqslant \frac{3}{2}$ the equivalence

$$\langle f_i, 1\rangle_{W_0^{-1,p}(\mathbb{R}^3)\times W_0^{1,p'}(\mathbb{R}^3)} = 0 \Longleftrightarrow \langle F_i, 1\rangle_{W_0^{-1,p}(\mathbb{R}^3)\times W_0^{1,p'}(\mathbb{R}^3)} = 0. \tag{8}$$

This means that to solve (1), it is sufficient to solve the following problem:

$$-\Delta \boldsymbol{u} + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{u}) + \nabla \pi = \boldsymbol{f} \quad \text{and} \quad \operatorname{div} \boldsymbol{u} = 0 \quad \text{in} \quad \mathbb{R}^3. \tag{9}$$

In the following theorem, we establishe the existence of generalized solutions to problem (1) in the case $1 < p \leq 2$. The uniqueness of the solutions will be studied later.

Theorem 3.1. *Let $1 < p \leqslant 2$. Assume that $\boldsymbol{f} \in \boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)$ satisfies the compatibility condition (5) and let $\boldsymbol{v} \in \boldsymbol{L}^3_\sigma(\mathbb{R}^3)$. Then the Oseen problem (9) has a solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$ such that*

$$||\boldsymbol{u}||_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)} + ||\pi||_{L^p(\mathbb{R}^3)} \leqslant C(1 + ||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)})||\boldsymbol{f}||_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)}. \tag{10}$$

Proof. First, the case $p = 2$ is an immediate consequence of the following property

$$\forall \boldsymbol{w} \in \boldsymbol{W}_0^{1,2}(\mathbb{R}^3), \quad \int_{\mathbb{R}^3} (\boldsymbol{v} \cdot \nabla)\, \boldsymbol{w} \cdot \boldsymbol{w} = 0$$

and Lax-Milgram's lemma. Then we can suppose that $1 < p < 2$.
The main idea of the proof is to observe that $\boldsymbol{v} \in \boldsymbol{L}^3_\sigma(\mathbb{R}^3)$ can be approximated by a smooth function $\boldsymbol{\psi} \in \boldsymbol{\mathcal{D}}_\sigma(\mathbb{R}^3)$. Given ε, there is $\boldsymbol{\psi}_\epsilon \in \boldsymbol{\mathcal{D}}_\sigma(\mathbb{R}^3)$ such that

$$||\boldsymbol{v} - \boldsymbol{\psi}_\varepsilon||_{\boldsymbol{L}^3(\mathbb{R}^3)} < \varepsilon, \tag{11}$$

where $\varepsilon > 0$ is a constant which will be fixed as below. By (5) and [3], we have $\boldsymbol{f} = \operatorname{div} \mathbf{F}$ with $\mathbf{F} \in \boldsymbol{L}^p(\mathbb{R}^3)$. Let $\rho \in \mathcal{D}(\mathbb{R}^3)$, be a smooth $\mathcal{C}^\infty$ function with compact support in $B(0,1)$, such that $\rho \geq 0$, $\int_{\mathbb{R}^3} \rho(x)\, dx = 1$. For $t \in (0,1)$, let ρ_t denote the function $x \longmapsto (\frac{1}{t^3})\rho(\frac{x}{t})$. Let $\varphi \in \mathcal{D}(\mathbb{R}^3)$ such that $0 \leqslant \varphi(x) \leqslant 1$ for any $x \in \mathbb{R}^3$, and

$$\varphi(x) = \begin{cases} 1 & \text{if} \quad 0 \leqslant |\, x \,| \leqslant 1, \\ 0 & \text{if} \quad |\, x \,| \geqslant 2. \end{cases}$$

We begin with applying the cut off functions φ_k defined on $\mathbb{R}^3$ for any $k \in \mathbb{N}^*$, as $\varphi_k(x) = \varphi(\frac{x}{k})$. Set $\mathbf{F}_k = \varphi_k \mathbf{F}$. Thus we obtain

$$\mathbf{G}_{t,k} = \rho_t * \mathbf{F}_k \in \boldsymbol{\mathcal{D}}(\mathbb{R}^3) \quad \text{and} \quad \lim_{t\to 0} \lim_{k\to\infty} \mathbf{G}_{t,k} = \mathbf{F} \quad \text{in} \quad \boldsymbol{L}^p(\mathbb{R}^3). \tag{12}$$

Now, observe that using Young inequality, we have

$$||\rho_t * \mathbf{F}_k||_{\boldsymbol{L}^2(\mathbb{R}^3)} \leqslant ||\rho_t||_{\boldsymbol{L}^q(\mathbb{R}^3)} ||\mathbf{F}_k||_{\boldsymbol{L}^p(\mathbb{R}^3)}, \tag{13}$$

with $q = \dfrac{2p}{3p-2}$. Observe that $q > 1$ is equivalent to $p < 2$. After an easy calculation, we obtain that

$$||\rho_t * \mathbf{F}_k||_{\boldsymbol{L}^2(\mathbb{R}^3)} \leqslant \frac{4}{3}\pi t^{\frac{-3}{q'}} ||\mathbf{F}_k||_{\boldsymbol{L}^p(\mathbb{R}^3)}. \tag{14}$$

We choose $t = k^{-\alpha}$ with $\alpha > 0$ which will be precised later. We set $\boldsymbol{f}_k = \operatorname{div} \mathbf{G}_{t,k}$ for any $k \in \mathbb{N}^*$. Then we have

$$\boldsymbol{f}_k \to \boldsymbol{f} \quad \text{in} \quad \boldsymbol{W}_0^{-1,p}(\mathbb{R}^3).$$

It is clear that $\boldsymbol{f}_k$ satisfies the condition (5).

Step 1. We suppose that $\boldsymbol{v} \in \boldsymbol{\mathcal{D}}_\sigma(\mathbb{R}^3)$. Thanks to Lemma 4.1 see [2], there exists a unique solution

$$\boldsymbol{u}_k \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \cap \boldsymbol{W}_0^{1,2}(\mathbb{R}^3), \quad \pi_k \in L^2(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$$

satisfying

$$-\Delta\, \boldsymbol{u}_k + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{u}_k) + \nabla\, \pi_k = \boldsymbol{f}_k, \quad \operatorname{div} \boldsymbol{u}_k = 0 \quad \text{in} \quad \mathbb{R}^3. \tag{15}$$

Set $B_\varepsilon = \operatorname{supp} \boldsymbol{\psi}_\varepsilon$, then from the Stokes theory (see [1] Theorem 3.3), we obtain

$$||\boldsymbol{u}_k||_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)} + ||\pi_k||_{L^p(\mathbb{R}^3)} \leqslant C_1 \left(||\boldsymbol{f}_k||_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)} + ||\boldsymbol{v} \otimes \boldsymbol{u}_k||_{\boldsymbol{L}^p(\mathbb{R}^3)} \right), \tag{16}$$

where C_1 doesn't depend on $k, \boldsymbol{f}_k$ and $\boldsymbol{v}$. Using Hölder inequality, we have

$$\begin{aligned} ||\boldsymbol{v} \otimes \boldsymbol{u}_k||_{\boldsymbol{L}^p(\mathbb{R}^3)} &\leqslant ||(\boldsymbol{v} - \boldsymbol{\psi}_\epsilon) \otimes \boldsymbol{u}_k||_{\boldsymbol{L}^p(\mathbb{R}^3)} + ||\boldsymbol{\psi}_\varepsilon \otimes \boldsymbol{u}_k||_{\boldsymbol{L}^p(\mathbb{R}^3)} \\ &\leqslant ||\boldsymbol{v} - \boldsymbol{\psi}_\varepsilon||_{\boldsymbol{L}^3(\mathbb{R}^3)} ||\boldsymbol{u}_k||_{\boldsymbol{L}^{p*}(\mathbb{R}^3)} + ||\boldsymbol{\psi}_\varepsilon||_{\boldsymbol{L}^3(B_\varepsilon)} ||\boldsymbol{u}_k||_{\boldsymbol{L}^{p*}(B_\varepsilon)}. \end{aligned}$$

Using the Sobolev inequality, we obtain

$$||\boldsymbol{u}_k||_{\boldsymbol{L}^{p*}(\mathbb{R}^3)} \leqslant C_2 ||\boldsymbol{u}_k||_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)}. \tag{17}$$

By the assumption (11), and from (16), (17) and (17) it follows that

$$(1 - C_1 C_2 \varepsilon) ||\boldsymbol{u}_k||_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)} + ||\pi_k||_{L^p(\mathbb{R}^3)} \leqslant C_1 (||\boldsymbol{f}_k||_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)} + ||\boldsymbol{\psi}_\varepsilon||_{\boldsymbol{L}^3(B_\varepsilon)} ||\boldsymbol{u}_k||_{\boldsymbol{L}^{p*}(B_\varepsilon)}). \tag{18}$$

Taking $0 < \varepsilon < 1/2C_1C_2$, we obtain

$$||\boldsymbol{u}_k||_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)} + ||\pi_k||_{L^p(\mathbb{R}^3)} \leqslant 2C_1 (||\boldsymbol{f}_k||_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)} + ||\boldsymbol{\psi}_\varepsilon||_{\boldsymbol{L}^3(B_\varepsilon)} ||\boldsymbol{u}_k||_{\boldsymbol{L}^{p*}(B_\varepsilon)}). \tag{19}$$

From (19), we prove that there exists $C > 0$ not depending of k and $\boldsymbol{v}$ such that for any $k \in \mathbb{N}^*$ we have

$$||\boldsymbol{u}_k||_{\boldsymbol{L}^{p*}(B_\varepsilon)} \leqslant C ||\boldsymbol{f}_k||_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)}. \tag{20}$$

Indeed, assuming, per absurdum, the invalidity of (20). Then for any $m \in \mathbb{N}^*$ there exists $\ell_m \in \mathbb{N}$, $\boldsymbol{f}_{\ell_m} \in \boldsymbol{W}_0^{-1,p}(\mathbb{R}^3) \cap \boldsymbol{W}_0^{-1,2}(\mathbb{R}^3)$ and $\boldsymbol{v}_m \in \boldsymbol{\mathcal{D}}_\sigma(\mathbb{R}^3)$ such that, if $(\boldsymbol{u}_{\ell_m}, \pi_{\ell_m}) \in (\boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \cap \boldsymbol{W}_0^{1,2}(\mathbb{R}^3)) \times (L^p(\mathbb{R}^3) \cap L^2(\mathbb{R}^3))$ denotes the corresponding solution to the following problem:

$$-\Delta\, \boldsymbol{u}_{\ell_m} + \operatorname{div}(\boldsymbol{v}_m \otimes \boldsymbol{u}_{\ell_m}) + \nabla\, \pi_{\ell_m} = \boldsymbol{f}_{\ell_m}, \quad \operatorname{div} \boldsymbol{u}_{\ell_m} = 0 \quad \text{in} \quad \mathbb{R}^3, \tag{21}$$

the inequality

$$||\boldsymbol{u}_{\ell_m}||_{\boldsymbol{L}^{p*}(B_\epsilon)} > m ||\boldsymbol{f}_{\ell_m}||_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)}, \tag{22}$$

would hold. Note that $\boldsymbol{f}_{\ell_m} = \operatorname{div}(\rho_t * \mathbf{F}_{\ell_m})$ with $\mathbf{F}_{\ell_m} = \varphi_{\ell_m} \mathbf{F}$. Set $\boldsymbol{w}_m = \dfrac{\boldsymbol{u}_{\ell_m}}{||\boldsymbol{u}_{\ell_m}||_{\boldsymbol{L}^{p*}(B_\varepsilon)}}$, $\theta_m = \dfrac{\pi_{\ell_m}}{||\boldsymbol{u}_{\ell_m}||_{\boldsymbol{L}^{p*}(B_\varepsilon)}}$ and $\mathbf{R}_m = \dfrac{\boldsymbol{f}_{\ell_m}}{||\boldsymbol{u}_{\ell_m}||_{\boldsymbol{L}^{p*}(B_\varepsilon)}}$. Then for any $m \in \mathbb{N}^*$ we have

$$-\Delta\, \boldsymbol{w}_m + \operatorname{div}(\boldsymbol{v}_m \otimes \boldsymbol{w}_m) + \nabla\, \theta_m = \mathbf{R}_m \quad \text{and} \quad \operatorname{div} \boldsymbol{w}_m = 0 \quad \text{in} \quad \mathbb{R}^3. \tag{23}$$

Now, using (23) and the fact that $\operatorname{div}(\boldsymbol{v}_m \otimes \boldsymbol{w}_m) = \boldsymbol{v}_m \cdot \nabla\, \boldsymbol{w}_m$, we obtain for any $m \in \mathbb{N}^*$ and $t > 0$

$$\int_{\mathbb{R}^3} |\nabla\, \boldsymbol{w}_m|^2\, dx = -\frac{1}{||\boldsymbol{u}_{\ell_m}||_{\boldsymbol{L}^{p*}(B_\varepsilon)}} \int_{\mathbb{R}^3} \rho_t * \mathbf{F}_{\ell_m} : \nabla\, \boldsymbol{w}_m\, dx.$$

Using (22) and Cauchy Schwartz inequality, we have

$$||\nabla\, \boldsymbol{w}_m||_{\boldsymbol{L}^2(\mathbb{R}^3)} < \frac{1}{m ||\boldsymbol{f}_{\ell_m}||_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)}} ||\rho_t * \mathbf{F}_{\ell_m}||_{\boldsymbol{L}^2(\mathbb{R}^3)}. \tag{24}$$

Using (14) and choosing $t = \dfrac{1}{m^\alpha}$ with $0 < \alpha < \frac{q'}{3}$, we deduce that

$$||\nabla \boldsymbol{w}_m||_{\boldsymbol{L}^2(\mathbb{R}^3)} \leqslant \frac{4\pi}{3m^{1-\frac{3\alpha}{q'}}||\boldsymbol{f}_{\ell_m}||_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)}}||\mathbf{F}_{\ell_m}||_{\boldsymbol{L}^p(\mathbb{R}^3)}. \tag{25}$$

Because the semi-norm $||\nabla \cdot||_{\boldsymbol{L}^2(\mathbb{R}^3)}$ is equivalent to the full norm $||\cdot||_{\boldsymbol{W}_0^{1,2}(\mathbb{R}^3)}$ and the right hand side of the last inequality tends to zero when m goes to ∞, we deduce that

$$\boldsymbol{w}_m \to \mathbf{0} \text{ in } \quad \boldsymbol{W}_0^{1,2}(\mathbb{R}^3). \tag{26}$$

Then, $\boldsymbol{w}_m \to \mathbf{0}$ in $\boldsymbol{L}^6(\mathbb{R}^3)$ and in particular in $\boldsymbol{L}^{p*}(B_\varepsilon)$. On the other hand, we have $||\boldsymbol{w}_m||_{\boldsymbol{L}^{p*}(B_\varepsilon)} = 1$, leading to a contradiction. Inequality (20) is therefore established. From (19), (20) and (11) we obtain for any $k \in \mathbb{N}^*$

$$||\boldsymbol{u}_k||_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)} + ||\pi_k||_{L^p(\mathbb{R}^3)} \leqslant 2C_1(1 + C||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)})||\boldsymbol{f}_k||_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)}. \tag{27}$$

Thus we can extract subsequences of $\boldsymbol{u}_k$ and π_k, still denoted by $\boldsymbol{u}_k$ and π_k, such that

$$\boldsymbol{u}_k \rightharpoonup \boldsymbol{u} \quad \text{in} \quad \boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \quad \text{and} \quad \pi_k \rightharpoonup \pi \quad \text{in} \quad L^p(\mathbb{R}^3),$$

where $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$ verify (9) and the following estimate

$$||\boldsymbol{u}||_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)} + ||\pi||_{L^p(\mathbb{R}^3)} \leqslant 2C_1(1 + C||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)})||\boldsymbol{f}||_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)}. \tag{28}$$

Step 2. We suppose that $\boldsymbol{v}$ belongs only to $\boldsymbol{L}^3_\sigma(\mathbb{R}^3)$. Let $\boldsymbol{v}_\lambda \in \boldsymbol{\mathcal{D}}_\sigma(\mathbb{R}^3)$ such that

$$\boldsymbol{v}_\lambda \longrightarrow \boldsymbol{v} \qquad \text{in } \boldsymbol{L}^3(\mathbb{R}^3). \tag{29}$$

Using the first step, there exists $(\boldsymbol{u}_\lambda, \pi_\lambda) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$ satisfying

$$-\Delta \boldsymbol{u}_\lambda + \operatorname{div}(\boldsymbol{v}_\lambda \otimes \boldsymbol{u}_\lambda) + \nabla \pi_\lambda = \boldsymbol{f} \quad \text{and} \quad \operatorname{div} \boldsymbol{u}_\lambda = 0 \quad \text{in} \quad \mathbb{R}^3, \tag{30}$$

and satisfying the estimate

$$||\boldsymbol{u}_\lambda||_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)} + ||\pi_\lambda||_{L^p(\mathbb{R}^3)} \leqslant 2C_1(1 + C||\boldsymbol{v}_\lambda||_{\boldsymbol{L}^3(\mathbb{R}^3)})||\boldsymbol{f}||_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)}. \tag{31}$$

We can finally extract a subsequence converging to $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$ which is a solution of the Oseen problem (9) and verifying the estimate (10) when $1 < p < 2$. For $p = 2$, estimate (10) was proved in Theorem 3.4 of [2]. ☐

Remark 2.

1) If h belongs to $L^p(\mathbb{R}^3)$ with $1 < p \leq 2$ *i.e.* we are in the case of problem (1) and the estimate (10) becomes:

$$||\boldsymbol{u}||_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)} + ||\pi||_{L^p(\mathbb{R}^3)} \leqslant C(1 + ||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)})\big(||\boldsymbol{f}||_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)} + (1 + ||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)})||h||_{L^p(\mathbb{R}^3)}\big). \tag{32}$$

The proof of (32) when $1 < p < 2$ is a simple consequence of Remark 1 (7). Note that the proof of (32) when $p = 2$ is done in Theorem 3.3 of [2].

2) For $p = 2$ and $h = 0$, the velocity $\boldsymbol{u}$ of the Oseen problem (9) satisfies the estimate

$$||\boldsymbol{u}||_{\boldsymbol{W}_0^{1,2}(\mathbb{R}^3)} \leqslant C||\boldsymbol{f}||_{\boldsymbol{W}_0^{-1,2}(\mathbb{R}^3)},$$

and the energy equality

$$\int_{\mathbb{R}^3} |\nabla \boldsymbol{u}|^2 \, d\mathbf{x} = \langle \boldsymbol{f}, \boldsymbol{u} \rangle_{\boldsymbol{W}_0^{-1,2}(\mathbb{R}^3) \times \boldsymbol{W}_0^{1,2}(\mathbb{R}^3)}.$$

In addition, the pressure π of the Oseen problem (9) satisfies the following estimate:

$$||\pi||_{L^2(\mathbb{R}^3)} \leqslant C(1 + ||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)})||\boldsymbol{f}||_{\boldsymbol{W}_0^{-1,2}(\mathbb{R}^3)}.$$

See Theorem 3.3 of [2] for more details.

We will prove now some regularity results, when the external forces belong to the intersection of negative weighted Sobolev spaces. The first result is given by the following theorem.

Theorem 3.2. *Let $1 < p < 2$. Let $\boldsymbol{f}$ belong to $\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3) \cap \boldsymbol{W}_0^{-1,2}(\mathbb{R}^3)$ satisfying the compatibility condition (5) and let $\boldsymbol{v} \in \boldsymbol{L}^3_\sigma(\mathbb{R}^3)$. Then the Oseen problem (9) has a unique solution $(\boldsymbol{u}, \pi) \in (\boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \cap \boldsymbol{W}_0^{1,2}(\mathbb{R}^3)) \times (L^p(\mathbb{R}^3) \cap L^2(\mathbb{R}^3))$ such that*

$$\begin{aligned} ||\boldsymbol{u}||_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)} &+ ||\boldsymbol{u}||_{\boldsymbol{W}_0^{1,2}(\mathbb{R}^3)} + ||\pi||_{L^p(\mathbb{R}^3)} + ||\pi||_{L^2(\mathbb{R}^3)} \\ &\leqslant C\big(1 + ||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)}\big)\big(||\boldsymbol{f}||_{\boldsymbol{W}_0^{-1,2}(\mathbb{R}^3)} + ||\boldsymbol{f}||_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)}\big). \end{aligned} \tag{33}$$

Proof. As in Theorem 3.1, we can suppose that $\boldsymbol{v} \in \boldsymbol{\mathcal{D}}_\sigma(\mathbb{R}^3)$. Let $\boldsymbol{f}$ belongs to $\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3) \cap \boldsymbol{W}_0^{-1,2}(\mathbb{R}^3)$ satisfying the compatibility condition (5). Then $\boldsymbol{f}$ can be written as $\boldsymbol{f} = \operatorname{div}\boldsymbol{F}$ with $\boldsymbol{F} \in \boldsymbol{L}^p(\mathbb{R}^3) \cap \boldsymbol{L}^2(\mathbb{R}^3)$. Take the same sequence $\boldsymbol{f}_k$, as in Theorem 3.1, which converges to $\boldsymbol{f}$ in $\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3) \cap \boldsymbol{W}_0^{-1,2}(\mathbb{R}^3)$. Proceeding as in the first step of Theorem 3.1, we deduce that there exists a unique solution

$$\boldsymbol{u}_k \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \cap \boldsymbol{W}_0^{1,2}(\mathbb{R}^3), \quad \pi_k \in L^p(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$$

satisfying

$$-\Delta\, \boldsymbol{u}_k + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{u}_k) + \nabla\, \pi_k = \boldsymbol{f}_k, \quad \operatorname{div} \boldsymbol{u}_k = 0 \quad \text{in} \quad \mathbb{R}^3 \tag{34}$$

and with the following estimate

$$||\boldsymbol{u}_k||_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)} + ||\pi_k||_{L^p(\mathbb{R}^3)} \leqslant C(1 + ||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)})||\boldsymbol{f}_k||_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)}, \tag{35}$$

where C doesn't depend on k. On the other hand, multiplying by $\boldsymbol{u}_k$, we have also the following estimate

$$||\boldsymbol{u}_k||_{\boldsymbol{W}_0^{1,2}(\mathbb{R}^3)} + ||\pi_k||_{L^2(\mathbb{R}^3)} \leqslant C||\boldsymbol{f}_k||_{\boldsymbol{W}_0^{-1,2}(\mathbb{R}^3)}. \tag{36}$$

Finally, $(\boldsymbol{u}_k, \pi_k)$ is bounded in $(\boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \cap \boldsymbol{W}_0^{1,2}(\mathbb{R}^3)) \times (L^p(\mathbb{R}^3) \cap L^2(\mathbb{R}^3))$ and we can extract a subsequence denoted again by $(\boldsymbol{u}_k, \pi_k)$ and satisfying

$$\boldsymbol{u}_k \rightharpoonup \boldsymbol{u} \quad \text{in} \quad \boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \cap \boldsymbol{W}_0^{1,2}(\mathbb{R}^3) \quad \text{and} \quad \pi_k \rightharpoonup \pi \quad \text{in} \quad L^p(\mathbb{R}^3) \cap L^2(\mathbb{R}^3). \tag{37}$$

We then verify that $(\boldsymbol{u}, \pi)$ is solution of (9) and we have the estimate (33). To finish we observe that the uniqueness is immediate because $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,2}(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$. □

In Theorem 3.1, we have studied the existence of weak solution of the Oseen problem when $1 < p \leq 2$. Now the question that will be discussed: If the solution given by Theorem 3.1 is unique? If it is unique, is it for all $1 < p \leq 2$? The first answer is given in the following proposition:

Proposition 1. *Let $6/5 < p < 2$. Let $\boldsymbol{f} \in \boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)$ satisfy the compatibility condition (5) and $\boldsymbol{v} \in \boldsymbol{L}^3_\sigma(\mathbb{R}^3)$. Then the solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$ given by Theorem 3.1 is unique.*

Proof. Suppose that there exist two solutions $(\boldsymbol{u}_1, \pi_1)$ and $(\boldsymbol{u}_2, \pi_2)$ belonging to $\boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$ and verifying Problem (9). Set $\boldsymbol{u} = \boldsymbol{u}_1 - \boldsymbol{u}_2$ and $\pi = \pi_1 - \pi_2$ then we have

$$-\Delta\, \boldsymbol{u} + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{u}) + \nabla\, \pi = \boldsymbol{0} \quad \text{and} \quad \operatorname{div} \boldsymbol{u} = 0 \quad \text{in} \quad \mathbb{R}^3. \tag{38}$$

Our aim is to prove that $(\boldsymbol{u}, \pi) = (\boldsymbol{0}, 0)$. Observe that for any $\varepsilon > 0$, $\boldsymbol{v}$ can be decomposed as: $\boldsymbol{v} = \boldsymbol{v}_1 + \boldsymbol{v}_2$ with

$$\boldsymbol{v}_1 \in \boldsymbol{L}^3_\sigma(\mathbb{R}^3), \quad ||\boldsymbol{v}_1||_{\boldsymbol{L}^3(\mathbb{R}^3)} < \varepsilon \qquad \text{and} \quad \boldsymbol{v}_2 \in \boldsymbol{\mathcal{D}}_\sigma(\mathbb{R}^3). \tag{39}$$

The parameter ε will be fixed at the end of the proof.

Note that $\boldsymbol{v}_2 \in \boldsymbol{L}^1(\mathbb{R}^3) \cap \boldsymbol{L}^\infty(\mathbb{R}^3)$. Now, since $\boldsymbol{u} \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \hookrightarrow \boldsymbol{L}^{p*}(\mathbb{R}^3)$ we prove that $\boldsymbol{v}_2 \otimes \boldsymbol{u}$ belongs to $\boldsymbol{L}^{p*}(\mathbb{R}^3) \cap \boldsymbol{L}^1(\mathbb{R}^3)$. As $6/5 < p < 2$, then $2 < p* < 6$ and thus $\operatorname{div}(\boldsymbol{v}_2 \otimes \boldsymbol{u}) = \boldsymbol{v}_2 \cdot \nabla\, \boldsymbol{u}$ belongs to $\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3) \cap \boldsymbol{W}_0^{-1,2}(\mathbb{R}^3)$ and satisfies the compatibilty condition (5). Then it follows from Theorem 3.2 that there exists a unique $\boldsymbol{z} \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \cap \boldsymbol{W}_0^{1,2}(\mathbb{R}^3)$ and $\theta \in \boldsymbol{L}^p(\mathbb{R}^3) \cap \boldsymbol{L}^2(\mathbb{R}^3)$ such that

$$-\Delta\, \boldsymbol{z} + \operatorname{div}(\boldsymbol{v}_1 \otimes \boldsymbol{z}) + \nabla\, \theta = -\boldsymbol{v}_2 \cdot \nabla\, \boldsymbol{u} \quad \text{and} \quad \operatorname{div} \boldsymbol{z} = 0 \quad \text{in} \quad \mathbb{R}^3. \tag{40}$$

Because of (38) and (40), the functions $\boldsymbol{w} = \boldsymbol{z} - \boldsymbol{u}$ and $q = \theta - \pi$ satisfy:

$$-\Delta\, \boldsymbol{w} + \operatorname{div}(\boldsymbol{v}_1 \otimes \boldsymbol{w}) + \nabla\, q = \boldsymbol{0} \quad \text{and} \quad \operatorname{div} \boldsymbol{w} = 0 \quad \text{in} \quad \mathbb{R}^3. \tag{41}$$

From the Stokes theory see ([1]) and Sobolev imbeddings, we obtain

$$\begin{aligned} ||\boldsymbol{w}||_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)} \leq C||\boldsymbol{v}_1 \otimes \boldsymbol{w}||_{\boldsymbol{L}^p(\mathbb{R}^3)} &\leq C||\boldsymbol{v}_1||_{\boldsymbol{L}^3(\mathbb{R}^3)}||\boldsymbol{w}||_{\boldsymbol{L}^{p*}(\mathbb{R}^3)} \\ &\leq CC^*||\boldsymbol{v}_1||_{\boldsymbol{L}^3(\mathbb{R}^3)}||\boldsymbol{w}||_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)} \\ &\leq CC^*\varepsilon||\boldsymbol{w}||_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)}. \end{aligned}$$

Taking $0 < \varepsilon < 1/(CC^*)$, we conclude that $\boldsymbol{w} = \boldsymbol{0}$ and so $q = 0$. Thus $(\boldsymbol{u}, \pi)$ belongs to $\boldsymbol{W}_0^{1,2}(\mathbb{R}^3) \times \boldsymbol{L}^2(\mathbb{R}^3)$ and we can write that $\operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{u}) = \boldsymbol{v} \cdot \nabla\, \boldsymbol{u}$. Using (38), we deduce that

$$\langle -\Delta\, \boldsymbol{u} + \boldsymbol{v} \cdot \nabla\, \boldsymbol{u} + \nabla\, \pi, \boldsymbol{u} \rangle_{\boldsymbol{W}_0^{-1,2}(\mathbb{R}^3) \times \boldsymbol{W}_0^{1,2}(\mathbb{R}^3)} = \boldsymbol{0},$$

and so

$$||\nabla \boldsymbol{u}||_{\boldsymbol{L}^2(\mathbb{R}^3)} + \int_{\mathbb{R}^3} \boldsymbol{v}\cdot\nabla\,\boldsymbol{u}\cdot\boldsymbol{u}\,dx = 0.$$

Since $\int_{\mathbb{R}^3} \boldsymbol{v}\cdot\nabla\,\boldsymbol{u}\cdot\boldsymbol{u}\,dx = \frac{1}{2}\int_{\mathbb{R}^3} \boldsymbol{v}\cdot\nabla\,|\boldsymbol{u}^2|\,dx = 0$, we prove that $||\nabla\,\boldsymbol{u}||_{\boldsymbol{L}^2(\mathbb{R}^3)} = 0$ and thus $\boldsymbol{u} = \boldsymbol{0}$ and so $\pi = 0$. Finally, we have proved that $(\boldsymbol{u},\pi) = (\boldsymbol{0}, 0)$ for any $6/5 < p < 2$. □

The second regularity result is stated in the following theorem.

Theorem 3.3. *Let $1 < p < r < 2$. Suppose that $\boldsymbol{f}$ belongs to $\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3) \cap \boldsymbol{W}_0^{-1,r}(\mathbb{R}^3)$ satisfying the compatibility condition (5) with respect to p and r and let $\boldsymbol{v} \in \boldsymbol{L}^3_\sigma(\mathbb{R}^3)$. Then the Oseen problem (9) has a solution $(\boldsymbol{u},\pi) \in (\boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \cap \boldsymbol{W}_0^{1,r}(\mathbb{R}^3)) \times (L^p(\mathbb{R}^3) \cap L^r(\mathbb{R}^3))$ such that*

$$\begin{aligned} ||\boldsymbol{u}||_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)} + ||\boldsymbol{u}||_{\boldsymbol{W}_0^{1,r}(\mathbb{R}^3)} + ||\pi||_{L^p(\mathbb{R}^3)} + ||\pi||_{L^r(\mathbb{R}^3)} \\ \leqslant C(1 + ||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)})(||\boldsymbol{f}||_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)} + ||\boldsymbol{f}||_{\boldsymbol{W}_0^{-1,r}(\mathbb{R}^3)}). \end{aligned} \tag{42}$$

Proof. Let $\boldsymbol{f}$ belongs to $\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3) \cap \boldsymbol{W}_0^{-1,r}(\mathbb{R}^3)$ and satisfying the compatibility condition (5) with respect to p and with r. Then $\boldsymbol{f}$ can be written as $\boldsymbol{f} = \operatorname{div}\boldsymbol{F}$ with $\boldsymbol{F} \in \boldsymbol{L}^p(\mathbb{R}^3) \cap \boldsymbol{L}^r(\mathbb{R}^3)$. Take the same sequence $\boldsymbol{f}_k$, as in the proof of Theorem 3.1, which now converges to $\boldsymbol{f}$ in $\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3) \cap \boldsymbol{W}_0^{-1,r}(\mathbb{R}^3)$. Proceeding as in the first step of Theorem 3.1, we can suppose that $\boldsymbol{v} \in \boldsymbol{\mathcal{D}}_\sigma(\mathbb{R}^3)$ and then there exists a unique solution

$$\boldsymbol{u}_k \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \cap \boldsymbol{W}_0^{1,2}(\mathbb{R}^3), \quad \pi_k \in L^p(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$$

such that

$$-\,\Delta\,\boldsymbol{u}_k + \operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{u}_k) + \nabla\,\pi_k = \boldsymbol{f}_k, \quad \operatorname{div}\boldsymbol{u}_k = 0 \quad \text{in} \quad \mathbb{R}^3 \tag{43}$$

and satisfying the estimate

$$||\boldsymbol{u}_k||_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)} + ||\pi_k||_{L^p(\mathbb{R}^3)} \leqslant C_p(1 + ||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)})||\boldsymbol{f}_k||_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)}, \tag{44}$$

where C_p doesn't depend on k. On the other hand, using an interpolation argument, we have also $\boldsymbol{u}_k \in \boldsymbol{W}_0^{1,r}(\mathbb{R}^3)$, because $p < r < 2$. Now proceeding as in Theorem 3.1, we prove that

$$||\boldsymbol{u}_k||_{\boldsymbol{W}_0^{1,r}(\mathbb{R}^3)} + ||\pi_k||_{L^r(\mathbb{R}^3)} \leqslant C_r(1 + ||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)})||\boldsymbol{f}_k||_{\boldsymbol{W}_0^{-1,r}(\mathbb{R}^3)}, \tag{45}$$

where C_r doesn't depend on k.

Finally, $(\boldsymbol{u}_k,\pi_k)$ is bounded in $(\boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \cap \boldsymbol{W}_0^{1,r}(\mathbb{R}^3)) \times (L^p(\mathbb{R}^3) \cap L^r(\mathbb{R}^3))$ and we can extract a subsequence denoted again by $(\boldsymbol{u}_k,\pi_k)$ and satisfying

$$\boldsymbol{u}_k \rightharpoonup \boldsymbol{u} \quad \text{in} \quad \boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \cap \boldsymbol{W}_0^{1,r}(\mathbb{R}^3) \quad \text{and} \quad \pi_k \rightharpoonup \pi \quad \text{in} \quad L^p(\mathbb{R}^3) \cap L^r(\mathbb{R}^3). \tag{46}$$

We can verify that $(\boldsymbol{u},\pi)$ is a solution of (9) and it impies that the estimate (52) is valid. □

Now, we study the uniqueness of generalized solution when $1 < p \leqslant 6/5$:

Proposition 2. *Let $1 < p \leqslant 6/5$. Let $\boldsymbol{f} \in \boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)$ satisfy the compatibility condition (5) and $\boldsymbol{v} \in \boldsymbol{L}^3_\sigma(\mathbb{R}^3)$. Then the solution $(\boldsymbol{u},\pi) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$ given by Theorem 3.1 is unique.*

Proof. We proceed as in Proposition 1. Let $(\boldsymbol{u},\pi)$ belongs to $\boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$ and satisfying (38). We know that $\boldsymbol{v}_2 \otimes \boldsymbol{u}$ belongs to $\boldsymbol{L}^{p*}(\mathbb{R}^3) \cap \boldsymbol{L}^p(\mathbb{R}^3)$, with $3/2 < p* \leq 2$ and thus $\operatorname{div}(\boldsymbol{v}_2 \otimes \boldsymbol{u})$ belongs to $\boldsymbol{W}_0^{-1,p*}(\mathbb{R}^3) \cap \boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)$. Moreover $\operatorname{div}(\boldsymbol{v}_2 \otimes \boldsymbol{u})$ satisfies the compatibility condition (5). Using Theorem 3.3, we deduce that there exists $(\boldsymbol{\xi},\varphi) \in (\boldsymbol{W}_0^{1,p*}(\mathbb{R}^3) \cap \boldsymbol{W}_0^{1,p}(\mathbb{R}^3)) \times (L^{p*}(\mathbb{R}^3) \cap L^p(\mathbb{R}^3))$ such that

$$-\,\Delta\,\boldsymbol{\xi} + \operatorname{div}(\boldsymbol{v}_1 \otimes \boldsymbol{\xi}) + \nabla\,\varphi = -\operatorname{div}(\boldsymbol{v}_2 \otimes \boldsymbol{u}) \quad \text{and} \quad \operatorname{div}\boldsymbol{\xi} = 0 \quad \text{in} \quad \mathbb{R}^3. \tag{47}$$

Set $\boldsymbol{\lambda} = \boldsymbol{\xi} - \boldsymbol{u}$ and $\psi = \varphi - \pi$, we have

$$-\Delta\,\boldsymbol{\lambda} + \operatorname{div}(\boldsymbol{v}_1 \otimes \boldsymbol{\lambda}) + \nabla\,\psi = \boldsymbol{0} \quad \text{and} \quad \operatorname{div}\boldsymbol{\lambda} = 0 \quad \text{in} \quad \mathbb{R}^3.$$

As in Proposition 1, we prove that $(\boldsymbol{\lambda},\psi) = (\boldsymbol{0}, 0)$. Then we deduce that $(\boldsymbol{u},\pi)$ belongs to $\boldsymbol{W}_0^{1,p*}(\mathbb{R}^3) \times L^{p*}(\mathbb{R}^3)$. Using again Proposition 1, we prove that $(\boldsymbol{u},\pi) = (\boldsymbol{0}, 0)$. □

We can now summarize our existence, uniqueness and regularity results as below.

Theorem 3.4. *Assume that $\boldsymbol{v} \in \boldsymbol{L}^3_\sigma(\mathbb{R}^3)$.*

i) Let $1 < p \leq 2$, $h \in L^p(\mathbb{R}^3)$ and $\boldsymbol{f} \in \boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)$ satisfy the compatibility condition (5). Then the Oseen problem (1) *has a unique solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$ such that*

$$||\boldsymbol{u}||_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)} + ||\pi||_{L^p(\mathbb{R}^3)} \leqslant C(1+||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)}) \left(||\boldsymbol{f}||_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)} + (1+||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)})||h||_{L^p(\mathbb{R}^3)}\right). \quad (48)$$

ii) Let $1 < p < r \leqslant 2$. Suppose that $\boldsymbol{f}$ belongs to $\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3) \cap \boldsymbol{W}_0^{-1,r}(\mathbb{R}^3)$ and satisfying the compatibility condition (5) with respect to p and r. Then the Oseen problem (9) *has a unique solution $(\boldsymbol{u}, \pi) \in (\boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \cap \boldsymbol{W}_0^{1,r}(\mathbb{R}^3)) \times (L^p(\mathbb{R}^3) \cap L^r(\mathbb{R}^3))$ such that*

$$\begin{aligned} ||\boldsymbol{u}||_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)} + ||\boldsymbol{u}||_{\boldsymbol{W}_0^{1,r}(\mathbb{R}^3)} + ||\pi||_{L^p(\mathbb{R}^3)} + ||\pi||_{L^r(\mathbb{R}^3)} \\ \leqslant C(1+||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)})(||\boldsymbol{f}||_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)} + ||\boldsymbol{f}||_{\boldsymbol{W}_0^{-1,r}(\mathbb{R}^3)}). \end{aligned} \quad (49)$$

Finally the following existence result can be stated via a dual argument.

Theorem 3.5. *For $p > 2$, let $\boldsymbol{f} \in \boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)$, $h \in L^p(\mathbb{R}^3)$ and $\boldsymbol{v} \in \boldsymbol{L}^3_\sigma(\mathbb{R}^3)$. Then, the Oseen problem* (1) *has a unique solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$ if $p < 3$ and if $p \geq 3$, $\boldsymbol{u}$ is unique up to an additive constant vector. In addition, we have*

$$||\boldsymbol{u}||_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)/\mathcal{P}_{[1-3/p]}} + ||\pi||_{L^p(\mathbb{R}^3)} \leqslant C(1+||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)})^2 \left(||\boldsymbol{f}||_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)} + ||h||_{L^p(\mathbb{R}^3)}\right). \quad (50)$$

Proof. On one hand, Green's formula yields, for all $\boldsymbol{w} \in \boldsymbol{W}_0^{1,p'}(\mathbb{R}^3)$ and $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$

$$\begin{aligned} &\langle -\Delta\, \boldsymbol{u} + \boldsymbol{v}\cdot\nabla\, \boldsymbol{u} + \nabla\, \pi, \boldsymbol{w}\rangle_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)\times \boldsymbol{W}_0^{1,p'}(\mathbb{R}^3)} \\ &\quad = \langle \boldsymbol{u}, -\Delta\, \boldsymbol{w} - \operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{w})\rangle_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)\times \boldsymbol{W}_0^{-1,p'}(\mathbb{R}^3)} - \langle \pi, \operatorname{div} \boldsymbol{w}\rangle_{L^p(\mathbb{R}^3)\times L^{p'}(\mathbb{R}^3)}. \end{aligned}$$

Taking into account that if $p > 2$, we have $\boldsymbol{w} \in \boldsymbol{W}_0^{1,p'}(\mathbb{R}^3) \hookrightarrow \boldsymbol{L}^{3p/(2p-3)}(\mathbb{R}^3)$ and since $\boldsymbol{v} \in \boldsymbol{L}^3_\sigma(\mathbb{R}^3)$ we can conclude that $\boldsymbol{v}\otimes\boldsymbol{w} \in \boldsymbol{L}^{p'}(\mathbb{R}^3)$ and consequently $\operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{w}) \in \boldsymbol{W}_0^{-1,p'}(\mathbb{R}^3)$. On the other hand, for all $\eta \in L^{p'}(\mathbb{R}^3)$,

$$\langle \boldsymbol{u}, \nabla\, \eta\rangle_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)\times \boldsymbol{W}_0^{-1,p'}(\mathbb{R}^3)} = -\langle \operatorname{div} \boldsymbol{u}, \eta\rangle_{L^p(\mathbb{R}^3)\times L^{p'}(\mathbb{R}^3)}.$$

Then problem (1) has the following equivalent variational formulation:
Find $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$ such that for all $(\boldsymbol{w}, \eta) \in \boldsymbol{W}_0^{1,p'}(\mathbb{R}^3) \times L^{p'}(\mathbb{R}^3)$,

$$\begin{aligned} &\langle \boldsymbol{u}, -\Delta\, \boldsymbol{w} - \operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{w}) + \nabla\, \eta\rangle_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)\times \boldsymbol{W}_0^{-1,p'}(\mathbb{R}^3)} - \langle \pi, \operatorname{div} \boldsymbol{w}\rangle_{L^p(\mathbb{R}^3)\times L^{p'}(\mathbb{R}^3)} \\ &\quad = \langle \boldsymbol{f}, \boldsymbol{w}\rangle_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)\times \boldsymbol{W}_0^{1,p'}(\mathbb{R}^3)} - \langle h, \eta\rangle_{L^p(\mathbb{R}^3)\times L^{p'}(\mathbb{R}^3)}. \end{aligned} \quad (51)$$

According to Theorem 3.4, for each $(\boldsymbol{f}', h') \in \boldsymbol{W}_0^{-1,p'}(\mathbb{R}^3) \times L^{p'}(\mathbb{R}^3)$ satisfying

$$\langle f_i', 1\rangle_{W_0^{-1,p'}(\mathbb{R}^3)\times W_0^{1,p}(\mathbb{R}^3)} = 0 \quad \text{if} \quad p' \leqslant \frac{3}{2},$$

there exists a unique solution $(\boldsymbol{w}, \eta) \in \boldsymbol{W}_0^{1,p'}(\mathbb{R}^3) \times L^{p'}(\mathbb{R}^3)$ such that

$$-\Delta\, \boldsymbol{w} - \operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{w}) + \nabla\, \eta = \boldsymbol{f}', \quad \operatorname{div} \boldsymbol{w} = h' \quad \text{in} \quad \mathbb{R}^3,$$

with the estimate

$$||\boldsymbol{w}||_{\boldsymbol{W}_0^{1,p'}(\mathbb{R}^3)} + ||\eta||_{L^{p'}(\mathbb{R}^3)} \leq C(1+||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)})\big(||\boldsymbol{f}'||_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)} + (1+||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)})||h'||_{L^p(\mathbb{R}^3)}\big).$$

Observe that the mapping

$$T:\ (\boldsymbol{f}', h') \mapsto \langle \boldsymbol{f}, \boldsymbol{w}\rangle_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)\times \boldsymbol{W}_0^{1,p'}(\mathbb{R}^3)} - \langle h, \eta\rangle_{L^p(\mathbb{R}^3)\times L^{p'}(\mathbb{R}^3)},$$

is linear and continuous with

$$\begin{aligned} |T(\boldsymbol{f}', h')| &\leq ||\boldsymbol{f}||_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)}||\boldsymbol{w}||_{\boldsymbol{W}_0^{1,p'}(\mathbb{R}^3)} + ||h||_{L^p(\mathbb{R}^3)}||\eta||_{L^{p'}(\mathbb{R}^3)} \\ &\leq C(1+||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)})^2 \left(||\boldsymbol{f}||_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)} + ||h||_{L^p(\mathbb{R}^3)}\right) \left(||\boldsymbol{f}'||_{\boldsymbol{W}_0^{-1,p'}(\mathbb{R}^3)} + ||h'||_{L^{p'}(\mathbb{R}^3)}\right). \end{aligned}$$

Note that $\boldsymbol{f}'$ belongs to $\boldsymbol{W}_0^{-1,p'}(\mathbb{R}^3)$ and $\boldsymbol{f}' \perp \mathbb{R}^3$ if $p \geq 3$. Thus there exists of unique $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$ if $2 < p < 3$, and a unique $(\boldsymbol{u}, \pi) \in \left(\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)/\mathcal{P}_{[1-3/p]}) \times L^p(\mathbb{R}^3)\right)$ if $p \geq 3$, such that

$$T(\boldsymbol{f}', h') = \langle \boldsymbol{u}, \boldsymbol{f}'\rangle_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)\times \boldsymbol{W}_0^{-1,p'}(\mathbb{R}^3)} - \langle \pi, h'\rangle_{L^p(\mathbb{R}^3)\times L^{p'}(\mathbb{R}^3)},$$

with

$$||\boldsymbol{u}||_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)/\mathcal{P}_{[1-3/p]}} + ||\pi||_{L^p(\mathbb{R}^3)} \leq C(1+||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)})^2 \left(||\boldsymbol{f}||_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)} + ||h||_{L^p(\mathbb{R}^3)} \right).$$

By definition of T, it follows that

$$\langle \boldsymbol{f}, \boldsymbol{w} \rangle_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)\times \boldsymbol{W}_0^{1,p'}(\mathbb{R}^3)} - \langle h, \eta \rangle_{L^p(\mathbb{R}^3)\times L^{p'}(\mathbb{R}^3)} =$$

$$\langle \boldsymbol{u}, \boldsymbol{f}' \rangle_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)\times \boldsymbol{W}_0^{-1,p'}(\mathbb{R}^3)} - \langle \pi, h' \rangle_{L^p(\mathbb{R}^3)\times L^{p'}(\mathbb{R}^3)},$$

which is the variational formulation (51). □

Remark 3.

Suppose in the assumption of Theorem 3.5 that $h = 0$ and proceeding as in the proof of Theorem 3.5. Then problem (1) has the following equivalent variational formulation: Find $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$ such that for all $\boldsymbol{w} \in \boldsymbol{V}_{p'}(\mathbb{R}^3)$ and $\eta \in L^{p'}(\mathbb{R}^3)$,

$$\langle \boldsymbol{u}, -\Delta\, \boldsymbol{w} - \operatorname{div}(\boldsymbol{v}\otimes \boldsymbol{w}) + \nabla\, \eta \rangle_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)\times \boldsymbol{W}_0^{-1,p'}(\mathbb{R}^3)} = \langle \boldsymbol{f}, \boldsymbol{w} \rangle_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)\times \boldsymbol{W}_0^{1,p'}(\mathbb{R}^3)}$$

According to Theorem 3.4, for each $\boldsymbol{f}' \in \boldsymbol{W}_0^{-1,p'}(\mathbb{R}^3)$ satisfying

$$\langle f_i', 1 \rangle_{W_0^{-1,p'}(\mathbb{R}^3)\times W_0^{1,p}(\mathbb{R}^3)} = 0 \quad \text{if} \quad p' \leqslant \frac{3}{2},$$

there exists a unique solution $(\boldsymbol{w}, \eta) \in \boldsymbol{W}_0^{1,p'}(\mathbb{R}^3) \times L^{p'}(\mathbb{R}^3)$ satisfies the problem (1) with the estimate

$$||\boldsymbol{w}||_{\boldsymbol{W}_0^{1,p'}(\mathbb{R}^3)} + ||\eta||_{L^{p'}(\mathbb{R}^3)} \leq C(1+||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)})||\boldsymbol{f}'||_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)}.$$

Observe that the mapping

$$T:\ \boldsymbol{f}' \mapsto \langle \boldsymbol{f}, \boldsymbol{w} \rangle_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)\times \boldsymbol{W}_0^{1,p'}(\mathbb{R}^3)}$$

is linear and continuous with

$$|T(\boldsymbol{f}')| \leq ||\boldsymbol{f}||_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)} ||\boldsymbol{w}||_{\boldsymbol{W}_0^{1,p'}(\mathbb{R}^3)} \leq C(1+||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)})||\boldsymbol{f}||_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)} ||\boldsymbol{f}'||_{\boldsymbol{W}_0^{-1,p'}(\mathbb{R}^3)}.$$

Thus there exists a unique velocity $\boldsymbol{u}$ in $\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)/\mathcal{P}_{[1-3/p]}$ of problem (1) satisfies the estimate:

$$||\boldsymbol{u}||_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)/\mathcal{P}_{[1-3/p]}} \leqslant C(1+||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)})||\boldsymbol{f}||_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)}.$$

In addition, we have $-\Delta\, \boldsymbol{u} + \operatorname{div}(\boldsymbol{v}\otimes \boldsymbol{u}) - \boldsymbol{f}$ belongs to $\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)$ and satisfies

$$\langle -\Delta\, \boldsymbol{u} + \operatorname{div}(\boldsymbol{v}\otimes \boldsymbol{u}) - \boldsymbol{f}, \boldsymbol{w} \rangle_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)\times \boldsymbol{W}_0^{1,p'}(\mathbb{R}^3)} = 0$$

for all $\boldsymbol{w}$ in $\boldsymbol{V}_{p'}(\mathbb{R}^3)$. Thus we use Theorem 1 of [1] to deduce the existence of a unique pressure π in $L^p(\mathbb{R}^3)$ of problem (1).

Now, we prove an other regularity result when $2 < r < p < \infty$:

Lemma 3.6. *Supposing that* $\boldsymbol{f} \in \boldsymbol{W}_0^{-1,p}(\mathbb{R}^3) \cap \boldsymbol{W}_0^{-1,r}(\mathbb{R}^3)$, $h \in L^p(\mathbb{R}^3) \cap L^r(\mathbb{R}^3)$ *and* $\boldsymbol{v} \in \boldsymbol{L}^3_\sigma(\mathbb{R}^3)$, *with* $2 < r < p < \infty$ *the Oseen problem* (1) *has a unique solution* $(\boldsymbol{u}, \pi) \in (\boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \cap \boldsymbol{W}_0^{1,r}(\mathbb{R}^3)) \times (L^p(\mathbb{R}^3) \cap L^r(\mathbb{R}^3))$ *satisfying the following estimate*

$$\begin{aligned} ||\boldsymbol{u}||_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)} + ||\boldsymbol{u}||_{\boldsymbol{W}_0^{1,r}(\mathbb{R}^3)} + ||\pi||_{L^p(\mathbb{R}^3)} + ||\pi||_{L^r(\mathbb{R}^3)} \leqslant C(1+||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)})^2 \times \\ \left(||\boldsymbol{f}||_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)} + ||\boldsymbol{f}||_{\boldsymbol{W}_0^{-1,r}(\mathbb{R}^3)} + ||h||_{L^p(\mathbb{R}^3)} + ||h||_{L^r(\mathbb{R}^3)} \right). \end{aligned} \tag{52}$$

Proof. We suppose that $\boldsymbol{v} \in \boldsymbol{\mathcal{D}}_\sigma(\mathbb{R}^3)$. Let $\boldsymbol{f} \in \boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)$ and $h \in L^p(\mathbb{R}^3)$, from Theorem 3.5 there exists a unique solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^3)/\mathcal{P}_{[1-3/p]} \times L^p(\mathbb{R}^3)$ to the Oseen problem (1) such that

$$||\nabla\, \boldsymbol{u}||_{\boldsymbol{L}^p(\mathbb{R}^3)} + ||\pi||_{L^p(\mathbb{R}^3)} \leqslant C(1+||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)})^2(||\boldsymbol{f}||_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)} + ||h||_{L^p(\mathbb{R}^3)}). \tag{53}$$

Note that $\mathcal{P}_{[1-3/p]}$ is equal to zero if $p < 3$. Since $\boldsymbol{v} \in \boldsymbol{\mathcal{D}}_\sigma(\mathbb{R}^3)$, we prove that $\boldsymbol{v}\cdot\nabla\, \boldsymbol{u}$ belongs to $\boldsymbol{L}^1(\mathbb{R}^3) \cap \boldsymbol{L}^p(\mathbb{R}^3)$ and using the fact that $r < p$, we prove that $\boldsymbol{v}\cdot\nabla\, \boldsymbol{u}$ belongs to $\boldsymbol{L}^r(\mathbb{R}^3)$ and has a compact support. Then $\boldsymbol{v}\cdot\nabla\boldsymbol{u}\in \boldsymbol{W}_0^{-1,r}(\mathbb{R}^3)$ and according to Theorem 3.3 of [1], there exists a unique solution $(\boldsymbol{u}',\pi')\in \boldsymbol{W}_0^{1,r}(\mathbb{R}^3)/\mathcal{P}_{[1-3/r]}\times L^r(\mathbb{R}^3)$ such that

$$-\Delta\boldsymbol{u}'+\nabla\pi'=\boldsymbol{f}-\boldsymbol{v}\cdot\nabla\boldsymbol{u}\quad\text{and}\quad \operatorname{div}\boldsymbol{u}'=h\quad\text{in}\quad\mathbb{R}^3, \tag{54}$$

taking into account that $\boldsymbol{f}$ belongs also to $\boldsymbol{W}_0^{-1,r}(\mathbb{R}^3)$ and h belongs to $L^r(\mathbb{R}^3)$.
Set $\boldsymbol{z}=\boldsymbol{u}-\boldsymbol{u}'$ and $\theta=\pi-\pi'$, we obtain

$$-\Delta\boldsymbol{z}+\nabla\theta=\boldsymbol{0}\quad\text{and}\quad\operatorname{div}\boldsymbol{z}=0\quad\text{in}\quad\mathbb{R}^3. \tag{55}$$

The uniqueness argument implies first that the harmonic function θ belonging to $L^p(\mathbb{R}^3)+L^r(\mathbb{R}^3)$ is necessarily equal to zero and with similar argument, we obtain also $\nabla\boldsymbol{u}=\nabla\boldsymbol{u}'\in\boldsymbol{L}^p(\mathbb{R}^3)\cap L^r(\mathbb{R}^3)$. Note that $\boldsymbol{u}'=\boldsymbol{u}$ if $2<r<p<3$ and $\boldsymbol{u}=\boldsymbol{u}'+\boldsymbol{k}\in\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)$ with $\boldsymbol{k}\in\mathbb{R}^3$, if $2<r<3<p$. Then problem (54) becomes

$$-\Delta\boldsymbol{u}'+\nabla\pi'=\boldsymbol{f}-\boldsymbol{v}\cdot\nabla\boldsymbol{u}'\quad\text{and}\quad\operatorname{div}\boldsymbol{u}'=0\quad\text{in}\quad\mathbb{R}^3. \tag{56}$$

According to Theorem 3.5, we have

$$||\nabla\boldsymbol{u}'||_{\boldsymbol{L}^r(\mathbb{R}^3)}+||\pi'||_{L^r(\mathbb{R}^3)}\leqslant C(1+||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)})^2(||\boldsymbol{f}||_{\boldsymbol{W}_0^{-1,r}(\mathbb{R}^3)}+||h||_{L^r(\mathbb{R}^3)}). \tag{57}$$

Replacing $\nabla\boldsymbol{u}'$ with $\nabla\boldsymbol{u}$ and π' with π in (57) and using (53), we deduce (52). □

Remark 4.

Reasoning as in Lemma 3.6, we prove that if $\boldsymbol{f}\in\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)\cap\boldsymbol{W}_0^{-1,r}(\mathbb{R}^3)$ satisfies the compatibility condition (5) if $r\le 3/2$ and $h\in L^p(\mathbb{R}^3)\cap L^r(\mathbb{R}^3)$ with $1<r\le 2<p$ and $\boldsymbol{v}\in\boldsymbol{L}^3_\sigma(\mathbb{R}^3)$, then there exists a unique solution $(\boldsymbol{u},\pi)\in(\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)\cap\boldsymbol{W}_0^{1,r}(\mathbb{R}^3))\times(L^p(\mathbb{R}^3)\cap L^r(\mathbb{R}^3))$ to the Oseen problem (1).

4. **Strong solutions in $\boldsymbol{W}_0^{2,p}(\mathbb{R}^3)$ and in $\boldsymbol{W}_1^{2,p}(\mathbb{R}^3)$.** We begin by proving the existence of a unique strong solution in $\boldsymbol{W}_0^{2,p}(\Omega)\times\boldsymbol{W}_0^{1,p}(\Omega)$ for $1<p<3$:

Theorem 4.1. *For $1<p<3$, let $\boldsymbol{f}\in\boldsymbol{L}^p(\mathbb{R}^3)$, $h\in W_0^{1,p}(\mathbb{R}^3)$ and $\boldsymbol{v}\in\boldsymbol{L}^3_\sigma(\mathbb{R}^3)$. Then problem (1) has a unique solution $(\boldsymbol{u},\pi)\in\boldsymbol{W}_0^{2,p}(\mathbb{R}^3)/\mathcal{P}_{[2-3/p]}\times W_0^{1,p}(\mathbb{R}^3)$ such that*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_0^{2,p}(\mathbb{R}^3)/\mathcal{P}_{[2-3/p]}}+\|\pi\|_{W_0^{1,p}(\mathbb{R}^3)}\le C(1+||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)})^3\left(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\mathbb{R}^3)}+\|h\|_{W_0^{1,p}(\mathbb{R}^3)}\right). \tag{58}$$

Proof. The proof is similar to that of Theorem 5.1 of [2]. Note that in Theorem 4.1 we don't need to suppose that $\boldsymbol{v}$ satisfies (2). Observe first that if $1<p<3$ we have

$$L^p(\mathbb{R}^3)\hookrightarrow W_0^{-1,3p/(3-p)}(\mathbb{R}^3),$$

because $W_0^{1,t'}(\mathbb{R}^3)\hookrightarrow L^{p'}(\mathbb{R}^3)$ with $t=\dfrac{3p}{3-p}$ and $\dfrac{1}{p'}=\dfrac{1}{t'}-\dfrac{1}{3}$.
Since $h\in L^{3p/(3-p)}(\mathbb{R}^3)$ and $\boldsymbol{f}\in\boldsymbol{W}_0^{-1,3p/(3-p)}(\mathbb{R}^3)$, Theorem 3.4 and Theorem 3.5 guarantees the existence of a solution $(\boldsymbol{u},\pi)\in\boldsymbol{W}_0^{1,3p/(3-p)}(\mathbb{R}^3)\times L^{3p/(3-p)}(\mathbb{R}^3)$ to the Oseen problem (1). Moreover, we have

$$\begin{aligned}&\|\boldsymbol{u}\|_{\boldsymbol{W}_0^{1,3p/(3-p)}(\mathbb{R}^3)/\mathcal{P}_{[2-3/p]}}+\|\pi\|_{L^{3p/(3-p)}(\mathbb{R}^3)}\le\\ &C\left(1+\|\boldsymbol{v}\|_{\boldsymbol{L}^3(\mathbb{R}^3)}\right)^2\left(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\mathbb{R}^3)}+\|h\|_{W_0^{1,p}(\mathbb{R}^3)}\right).\end{aligned} \tag{59}$$

Note that the compatibility condition (5) is not required because we have $3p/(3-p)>3/2$. Using the fact that $\operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{u})=\boldsymbol{v}\cdot\nabla\boldsymbol{u}$ belongs to $\boldsymbol{L}^p(\mathbb{R}^3)$, we can apply the Stokes regularity theory, see Theorem 3.8 of [1], to deduce the existence of $(\boldsymbol{z},\eta)\in\boldsymbol{W}_0^{2,p}(\mathbb{R}^3)\times W_0^{1,p}(\mathbb{R}^3)$ unique up to an element of $\mathcal{P}_{[2-3/p]}\times\{0\}$ verifying:

$$-\Delta\boldsymbol{z}+\nabla\eta=\boldsymbol{f}-\boldsymbol{v}\cdot\nabla\boldsymbol{u}\quad\text{and}\quad\operatorname{div}\boldsymbol{z}=h\quad\text{in }\mathbb{R}^3.$$

Moreover, we have

$$\begin{aligned}&\inf_{\boldsymbol{\lambda}\in\mathcal{P}_{[2-3/p]}}\|\boldsymbol{z}+\boldsymbol{\lambda}\|_{\boldsymbol{W}_0^{2,p}(\mathbb{R}^3)}+\|\eta\|_{W_0^{1,p}(\mathbb{R}^3)}\\ &\le C\left(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\mathbb{R}^3)}+\|\boldsymbol{v}\|_{\boldsymbol{L}^3(\mathbb{R}^3)}\|\nabla\boldsymbol{u}\|_{\boldsymbol{L}^{3p/(3-p)}(\mathbb{R}^3)}+\|h\|_{W_0^{1,p}(\mathbb{R}^3)}\right),\\ &\le C\left(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\mathbb{R}^3)}+C_1\|\boldsymbol{v}\|_{\boldsymbol{L}^3(\mathbb{R}^3)}\left(1+\|\boldsymbol{v}\|_{\boldsymbol{L}^3(\mathbb{R}^3)}\right)^2\left(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\mathbb{R}^3)}+\|h\|_{W_0^{1,p}(\mathbb{R}^3)}\right)+\|h\|_{W_0^{1,p}(\mathbb{R}^3)}\right),\end{aligned} \tag{60}$$

with C denoting a constant only dependent on p. Set $\boldsymbol{w} = \boldsymbol{z} - \boldsymbol{u}$ and $\theta = \eta - \pi$, then we have

$$-\Delta \boldsymbol{w} + \nabla \theta = 0 \quad \text{and} \quad \operatorname{div} \boldsymbol{w} = 0 \quad \text{in} \quad \mathbb{R}^3.$$

Since $\nabla \boldsymbol{z} \in \boldsymbol{L}^{3p/(3-p)}(\mathbb{R}^3)$, there exists a constant $\mathbf{k} \in \mathbb{R}^3$, depending on $\boldsymbol{z}$ ($\mathbf{k} = \mathbf{0}$ if $p \geq 3/2$), such that $\boldsymbol{z} + \mathbf{k} \in \boldsymbol{W}_0^{1,3p/(3-p)}(\mathbb{R}^3)$ and thus $\boldsymbol{w} + \mathbf{k} \in \boldsymbol{W}_0^{1,3p/(3-p)}(\mathbb{R}^3)$. As $\Delta\theta = 0$ in $\mathbb{R}^3$ and $\theta \in L^{3p/(3-p)}(\mathbb{R}^3)$, then $\theta = 0$ and so $\boldsymbol{w}$ is a harmonic function belonging to $\boldsymbol{W}_0^{2,p}(\mathbb{R}^3) + \boldsymbol{W}_0^{1,3p/(3-p)}(\mathbb{R}^3)$. Then if $p < 3/2$, we would have $3p/(3-p) < 3$ and thus $\boldsymbol{u} = \boldsymbol{z} \in \boldsymbol{W}_0^{2,p}(\mathbb{R}^3)$. If $p \geq 3/2$, there exists a polynomial $\boldsymbol{\lambda} \in \mathcal{P}_{[2-3/p]} \subset \boldsymbol{W}_0^{2,p}(\mathbb{R}^3)$ such that $\boldsymbol{u} = \boldsymbol{z} + \boldsymbol{\lambda}$. Consequently, $\boldsymbol{u} \in \boldsymbol{W}_0^{2,p}(\mathbb{R}^3)$ and $\pi \in W_0^{1,p}(\mathbb{R}^3)$ and we obtain (58). □

Remark 5.

1) Under the assumptions of Theorem 4.1 and supposing that $1 < p \leq 2$, the solution $(\boldsymbol{u}, \pi)$ satisfies the estimate:

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_0^{2,p}(\mathbb{R}^3)/\mathcal{P}_{[2-3/p]}} + \|\pi\|_{W_0^{1,p}(\mathbb{R}^3)} \leq$$
$$C(1 + ||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)})^2 \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\mathbb{R}^3)} + (1 + ||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)}) \|h\|_{W_0^{1,p}(\mathbb{R}^3)} \right).$$

2) If we suppose in the assumption of Theorem 4.1 that $h = 0$, we prove that the solution $(\boldsymbol{u}, \pi)$ satisfies the estimate:

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_0^{2,p}(\mathbb{R}^3)/\mathcal{P}_{[2-3/p]}} + \|\pi\|_{W_0^{1,p}(\mathbb{R}^3)} \leq C(1 + ||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)})^2 \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\mathbb{R}^3)}. \tag{61}$$

Estimate (61) is an easy consequence of estimate (33) and Remark (3).

3) For $p \geq 3$, the hypothesis of $\boldsymbol{f} \in \boldsymbol{L}^p(\mathbb{R}^3)$, $h \in W_0^{1,p}(\mathbb{R}^3)$ and $\boldsymbol{v} \in \boldsymbol{L}^3_\sigma(\mathbb{R}^3)$ is not sufficient to ensure the existence of strong solutions for problem (1) in $\boldsymbol{W}_0^{2,p}(\mathbb{R}^3) \times W_0^{1,p}(\mathbb{R}^3)$. Indeed, suppose that under this assumptions it would be possible to find $\boldsymbol{u} \in \boldsymbol{W}_0^{2,p}(\mathbb{R}^3)$ and $\pi \in W_0^{1,p}(\mathbb{R}^3)$ such that

$$\boldsymbol{v} \cdot \nabla \boldsymbol{u} = \Delta \boldsymbol{u} - \nabla \pi + \boldsymbol{f} \in \boldsymbol{L}^p(\mathbb{R}^3).$$

This is a contradiction, since $\boldsymbol{v} \in \boldsymbol{L}^3(\mathbb{R}^3)$ and $\nabla \boldsymbol{u} \notin \boldsymbol{L}^{3p/(3-p)}(\mathbb{R}^3)$. Thus, it is necessary to suppose in addition that $\boldsymbol{f} \in \boldsymbol{L}^q(\mathbb{R}^3)$, $h \in W_0^{1,q}(\mathbb{R}^3)$ and $\boldsymbol{v} \in \boldsymbol{L}^{3pq/q(3+p)-3p}(\mathbb{R}^3)$ for some $3p/(3+p) \leq q < 3$. Under this assumptions, we deduce that the solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{2,q}(\mathbb{R}^3) \times W_0^{1,q}(\mathbb{R}^3)$ given by Theorem 4.1 belongs also to $\boldsymbol{W}_0^{2,p}(\mathbb{R}^3) \times W_0^{1,p}(\mathbb{R}^3)$ and it satisfies

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_0^{2,p}(\mathbb{R}^3)} + \|\pi\|_{W_0^{1,p}(\mathbb{R}^3)} \leq C(1 + ||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)})^3 (\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\mathbb{R}^3)} + \|h\|_{W_0^{1,p}(\mathbb{R}^3)}).$$

Finally, we take $\boldsymbol{f}$ in weighted $\boldsymbol{L}^p(\mathbb{R}^3)$, more precisely $\boldsymbol{f} \in \boldsymbol{W}_1^{0,p}(\mathbb{R}^3)$, and the data h in the corresponding weighted Sobolev space $W_1^{1,p}(\mathbb{R}^3)$.

Theorem 4.2. *Suppose that $1 < p < 3$ and $p \neq 3/2$. Let $h \in W_1^{1,p}(\mathbb{R}^3)$ and $\boldsymbol{f} \in \boldsymbol{W}_1^{0,p}(\mathbb{R}^3)$ such that*

$$\int_{\mathbb{R}^3} \boldsymbol{f}(\boldsymbol{x})\, d\boldsymbol{x} = 0 \quad \text{if} \quad p < 3/2, \tag{62}$$

and let $\boldsymbol{v} \in \boldsymbol{L}^3_\sigma(\mathbb{R}^3)$. Then the Oseen problem (1) has a unique solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_1^{2,p}(\mathbb{R}^3) \times W_1^{1,p}(\mathbb{R}^3)$ satisfying the following estimate:

$$||\boldsymbol{u}||_{\boldsymbol{W}_1^{2,p}(\mathbb{R}^3)} + ||\pi||_{W_1^{1,p}(\mathbb{R}^3)} \leqslant C(1 + ||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)})^6 (||\boldsymbol{f}||_{\boldsymbol{W}_1^{0,p}(\mathbb{R}^3)} + \|h\|_{W_1^{1,p}(\mathbb{R}^3)}). \tag{63}$$

Proof. First, note that we have $\boldsymbol{W}_1^{0,p}(\mathbb{R}^3) \hookrightarrow \boldsymbol{L}^1(\mathbb{R}^3)$ if $p < 3/2$ and thus $\int_{\mathbb{R}^3} \boldsymbol{f}(\mathbf{x})\, d\mathbf{x}$ is well defined. On the other hand, observe that $h \in W_1^{1,p}(\mathbb{R}^3) \hookrightarrow L^p(\mathbb{R}^3)$ and for $p \neq 3/2$, we have $\boldsymbol{f} \in \boldsymbol{W}_1^{0,p}(\mathbb{R}^3) \hookrightarrow \boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)$. Then thanks to Theorem 3.4 and Theorem 3.5, there exists a unique solution

$$\boldsymbol{u} \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^3), \quad \pi \in L^p(\mathbb{R}^3)$$

satisfying

$$-\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} - \boldsymbol{v} \cdot \nabla \boldsymbol{u} \quad \text{and} \quad \operatorname{div} \boldsymbol{u} = h \quad \text{in} \quad \mathbb{R}^3,$$

and we have

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)} + \|\pi\|_{L^p(\mathbb{R}^3)} \leq C \left(1 + ||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)}\right)^2 \left(\|\boldsymbol{f}\|_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)} + \|h\|_{L^p(\mathbb{R}^3)} \right). \tag{64}$$

We suppose that $\boldsymbol{v} \in \boldsymbol{\mathcal{D}}_\sigma(\mathbb{R}^3)$. Observe that $\boldsymbol{v}\cdot\nabla\,\boldsymbol{u}$ belongs to $\boldsymbol{W}^{0,p}_1(\mathbb{R}^3)$ and reasoning as in Theorem 4.1 we deduce that $(\boldsymbol{u},\pi)\in \boldsymbol{W}^{2,p}_1(\mathbb{R}^3)\times W^{1,p}_1(\mathbb{R}^3)$. After an easy calculation, we obtain that the pair $(\rho\,\boldsymbol{u},\rho\,\pi)\in \boldsymbol{W}^{2,p}_0(\mathbb{R}^3)\times W^{1,p}_0(\mathbb{R}^3)$ satisfies the following equations in $\mathbb{R}^3$:

$$-\Delta(\rho\,\boldsymbol{u})+\boldsymbol{v}\cdot\nabla(\rho\,\boldsymbol{u})+\nabla(\rho\,\pi):=\boldsymbol{\chi}_\rho \quad\text{and}\quad \operatorname{div}(\rho\,\boldsymbol{u}):=\xi_\rho \quad\text{in}\quad \mathbb{R}^3,$$

with

$$\boldsymbol{\chi}_\rho=\rho\,\boldsymbol{f}-2\nabla\,\rho\cdot\nabla\,\boldsymbol{u}-(\Delta\,\rho)\boldsymbol{u}+(\nabla\,\rho)\pi+(\boldsymbol{v}\cdot\nabla\,\rho)\boldsymbol{u}\quad\text{and}\quad \xi_\rho=\rho\,h+\nabla\,\rho\cdot\boldsymbol{u}. \tag{65}$$

Let us mention that ρ is the weight defined by $\rho(\mathbf{x})=(1+|\mathbf{x}|^2)^{1/2}$. It is clear that $(\boldsymbol{\chi}_\rho,\xi_\rho)$ belongs to $\boldsymbol{L}^p(\mathbb{R}^3)\times W^{1,p}_0(\mathbb{R}^3)$, and using Theorem 4.1 we obtain

$$\begin{aligned}\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}_1(\mathbb{R}^3)}+\|\pi\|_{W^{1,p}_1(\mathbb{R}^3)}&\le C\|\rho\,\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}_0(\mathbb{R}^3)}+\|\rho\,\pi\|_{W^{1,p}_0(\mathbb{R}^3)}\\&\le C(1+||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)})^3\left(\|\boldsymbol{\chi}_\rho\|_{\boldsymbol{L}^p(\mathbb{R}^3)}+\|\xi_\rho\|_{W^{1,p}_0(\mathbb{R}^3)}\right),\end{aligned}\tag{66}$$

Using (64), and that $\boldsymbol{W}^{1,p}_0(\mathbb{R}^3)\hookrightarrow \boldsymbol{L}^{p*}(\mathbb{R}^3)$ we deduce that

$$\begin{aligned}&\|\boldsymbol{\chi}_\rho\|_{\boldsymbol{L}^p(\mathbb{R}^3)}+\|\xi_\rho\|_{W^{1,p}_0(\mathbb{R}^3)}\\&\quad\le C\left(||\boldsymbol{f}||_{\boldsymbol{W}^{0,p}_1(\mathbb{R}^3)}+\|h\|_{W^{1,p}_1(\mathbb{R}^3)}+\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}_0(\mathbb{R}^3)}+\|\pi\|_{L^p(\mathbb{R}^3)}+||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)}\|\boldsymbol{u}\|_{\boldsymbol{L}^{p*}(\mathbb{R}^3)}\right)\\&\quad\le C\left(||\boldsymbol{f}||_{\boldsymbol{W}^{0,p}_1(\mathbb{R}^3)}+\|h\|_{W^{1,p}_1(\mathbb{R}^3)}+\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}_0(\mathbb{R}^3)}+\|\pi\|_{L^p(\mathbb{R}^3)}+||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)}\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}_0(\mathbb{R}^3)}\right)\\&\quad\le C\left(||\boldsymbol{f}||_{\boldsymbol{W}^{0,p}_1(\mathbb{R}^3)}+\|h\|_{W^{1,p}_1(\mathbb{R}^3)}+(1+||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)})(\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}_0(\mathbb{R}^3)}+\|\pi\|_{L^p(\mathbb{R}^3)})\right)\\&\quad\le C\left(||\boldsymbol{f}||_{\boldsymbol{W}^{0,p}_1(\mathbb{R}^3)}+\|h\|_{W^{1,p}_1(\mathbb{R}^3)}+(1+||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)})^3(\|\boldsymbol{f}\|_{\boldsymbol{W}^{-1,p}_0(\mathbb{R}^3)}+\|h\|_{L^p(\mathbb{R}^3)})\right).\end{aligned}\tag{67}$$

From (66) and (67) and using the fact that $W^{1,p}_1(\mathbb{R}^3)\hookrightarrow L^p(\mathbb{R}^3)$ and $\boldsymbol{W}^{0,p}_1(\mathbb{R}^3)\hookrightarrow \boldsymbol{W}^{-1,p}_0(\mathbb{R}^3)$ for $p\neq 3/2$, we deduce that

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}_1(\mathbb{R}^3)}+\|\pi\|_{W^{1,p}_1(\mathbb{R}^3)}\le$$

$$C(||\boldsymbol{f}||_{\boldsymbol{W}^{0,p}_1(\mathbb{R}^3)}+\|h\|_{W^{1,p}_1(\mathbb{R}^3)})(1+||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)})^3\left(1+(1+||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)})^3\right).$$

Then $(\boldsymbol{u},\pi)\in\boldsymbol{W}^{2,p}_1(\mathbb{R}^3)\times W^{1,p}_1(\mathbb{R}^3)$ satisfies the estimate (63).

To finish, observe that the uniqueness of the solution $(\boldsymbol{u},\pi)\in\boldsymbol{W}^{2,p}_1(\mathbb{R}^3)\times W^{1,p}_1(\mathbb{R}^3)$ is immediate because $\boldsymbol{W}^{2,p}_1(\mathbb{R}^3)\times W^{1,p}_1(\mathbb{R}^3)\subset \boldsymbol{W}^{1,p}_0(\mathbb{R}^3)\times L^p(\mathbb{R}^3)$ and that $(\boldsymbol{u},\pi)$ is unique in $\boldsymbol{W}^{1,p}_0(\mathbb{R}^3)\times L^p(\mathbb{R}^3)$. □

Remark 6.

1) For $p=3/2$, the existence result of Theorem 4.2 holds if we suppose in addition that $\boldsymbol{f}\in\boldsymbol{W}^{0,3/2}_1(\mathbb{R}^3)\cap\boldsymbol{W}^{-1,3/2}_0(\mathbb{R}^3)$.
2) Under the assumptions of Theorem 4.1 and supposing that $1<p\le 2$, the solution $(\boldsymbol{u},\pi)$ satisfies the estimate:

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}_1(\mathbb{R}^3)}+\|\pi\|_{W^{1,p}_1(\mathbb{R}^3)}\le C(1+||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)})^5\left(\|\boldsymbol{f}\|_{\boldsymbol{W}^{0,p}_1(\mathbb{R}^3)}+(1+||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)})\|h\|_{W^{1,p}_1(\mathbb{R}^3)}\right).$$

3) If we suppose in the assumption of Theorem 4.2 that $h=0$, we prove that the solution $(\boldsymbol{u},\pi)$ satisfies the estimate:

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}_1(\mathbb{R}^3)}+\|\pi\|_{W^{1,p}_1(\mathbb{R}^3)}\le C(1+||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)})^4\|\boldsymbol{f}\|_{\boldsymbol{W}^{0,p}_1(\mathbb{R}^3)}. \tag{68}$$

4) For $p\ge 3$, the hypothesis of $\boldsymbol{f}\in\boldsymbol{W}^{0,p}_1(\mathbb{R}^3)$, $h\in W^{1,p}_1(\mathbb{R}^3)$ and $\boldsymbol{v}\in\boldsymbol{L}^3_\sigma(\mathbb{R}^3)$ is not sufficient to study the existence of strong solutions for problem (1) in $\boldsymbol{W}^{2,p}_1(\mathbb{R}^3)\times W^{1,p}_1(\mathbb{R}^3)$. Indeed, suppose that under this assumptions it would be possible to find $\boldsymbol{u}\in\boldsymbol{W}^{2,p}_1(\mathbb{R}^3)$ and $\pi\in W^{1,p}_1(\mathbb{R}^3)$ such that

$$\boldsymbol{v}\cdot\nabla\,\boldsymbol{u}=\Delta\,\boldsymbol{u}-\nabla\,\pi+\boldsymbol{f}\in\boldsymbol{W}^{0,p}_1(\mathbb{R}^3).$$

This is a contradiction, since $\boldsymbol{v}\in\boldsymbol{L}^3(\mathbb{R}^3)$ and $\nabla\,\boldsymbol{u}\notin\boldsymbol{W}^{0,p*}_1(\mathbb{R}^3)$.

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