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Very weak solutions to the rotating Stokes and Oseen problems in weighted spaces

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Abstract

We consider the linearized problems arising from the motion of fluid around the rotating rigid body. We are interested in the very weak solution of these problems.

Acknowlegment: This work is devoted to our friend Miroslav Krbec which cannot finished with us our paper since he has passed away in June 16, 2012. We will remember him.

1 Introduction

We shall study the basic systems of equations describing the flow around a rotating body – the time-periodic Stokes system in an exterior domain or in $\mathbb{R}^3$, and the time-periodic Oseen system in $\mathbb{R}^3$. The aim of this paper is to prove the existence of very weak solutions in weighted L^q-spaces.

The definition of very weak solutions is strongly based on duality arguments, constructing at first strong or sufficiently regular solutions to the linearized related model. A very weak solution has no differentiability. A very weak solution is not necessarily a weak solution. Another approach was introduced by Farwig et all [11] where they assume that their solution belongs to Serrin's class of uniqueness. In that sense the very weak solution coincides with a weak solution. In the very general setting, to define a priori boundary values is impossible, but it is possible to generalize functionals on the boundary which give normal and tangential traces.

This work is motivated by the study in weighted L^q-spaces of very weak solutions to the time-periodic incompressible nonlinear Navier-Stokes system.

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The first step concerns its linearized and steady versions, so in the present paper we will investigate both strong and very weak solutions of the following modified Stokes and Oseen problems. We will turn later to a consideration of the nonlinear case, our results will be discussed in a prepared second paper [23]. After transformations in the coordinates attached to the rotating frame, in a fixed exterior domain $D \subset \mathbb{R}^3$ or/and in the whole space $\mathbb{R}^3$, the cited linear systems are
either **the modified Stokes systems**,

$$\begin{aligned} -\nu\Delta\mathbf{u} - ((\omega \wedge \mathbf{x})\cdot\nabla)\mathbf{u} + \omega\wedge\mathbf{u} = \mathbf{f} - \nabla p \quad & \text{in } D \text{ or } \mathbb{R}^3,\\ \text{div }\mathbf{u} = 0 \text{ or } g \quad & \text{in } D \text{ or } \mathbb{R}^3,\\ \mathbf{u} = \mathbf{0} \text{ or } \mathbf{h} \quad & \text{on } \partial D \text{ if } D \neq \mathbb{R}^3,\\ \mathbf{u} \to \mathbf{0} \quad & \text{as } |\mathbf{x}| \to \infty, \end{aligned} \tag{1}$$

or **the modified Oseen systems**,

$$\begin{aligned} -\nu\Delta\mathbf{u} - ((\boldsymbol{\omega} \wedge \mathbf{x})\cdot\nabla)\mathbf{u} + \omega\wedge\mathbf{u} + \mathbf{k}\boldsymbol{\partial_3}\mathbf{u} = \mathbf{f} - \nabla p \quad & \text{in } D \text{ or } \mathbb{R}^3,\\ \text{div }\mathbf{u} = 0 \text{ or } g \quad & \text{in } D \text{ or } \mathbb{R}^3,\\ \mathbf{u} = \mathbf{0} \text{ or } \mathbf{h} \quad & \text{on } \partial D \text{ if } D \neq \mathbb{R}^3,\\ \mathbf{u} \to \mathbf{0} \quad & \text{as } |\mathbf{x}| \to \infty. \end{aligned} \tag{2}$$

See Appendix 1 for a short review on the concept of very weak solution and on the considered initial nonlinear problem.

The set $\mathbb{R}^3 \setminus D$ corresponding to the rotating body is assumed to move in the direction of its axis of rotation, and the angular velocity is assumed to be $\boldsymbol{\omega} = |\boldsymbol{\omega}|\mathbf{e}_3$; ν is the coefficient of viscosity.

Solving the previous systems on an exterior domain D, it is well-known that the inhomogeneous divergence conditions $(1)_2$, $(2)_2$ with $g \neq 0$ appear when we make use of the localization procedure. Indeed chosen $\rho > \rho_0 > 0$ such that $\mathbb{R}^3 \setminus D \subseteq B(0,\rho_0)$, by use of the appropriate cut-off function χ, $\chi|_{B(0,\rho_0)} = 1$, one decompose the systems (1) (or (2)) in an inhomogeneous modified Stokes (or Oseen) system in the whole space coupled with an usual inhomogeneous Stokes system in $D \cap B(0,\rho)$.

Remark 1. *The behavior of the solutions at infinity (conditions $(1)_4, (2)_4$) can be precisely specified by the choice of the weights and of the functional spaces. In particular the weights must control the rate of convergence of the solutions at infinity.*

As a consequence of the model and of the remark, if we denote by $\mathbf{v}_\infty$ the corresponding value at infinity, inhomogeneous boundary conditions in the form $\mathbf{u}_\infty = \boldsymbol{\omega} \wedge \mathbf{x} - \mathbf{v}_\infty$ can also appear (see (43) in Appendix 1).

The paper is organized as follows. In Section 2 we collect mathematical preliminaries. Section 3 is devoted to the modified Stokes problem in a 3D-exterior domain : we recall the know results on the strong solution; we introduce the definition of a very weak solution and we prove the existence of a very weak solution in the whole space and in the exterior domain. Last part of Section 3 deals with nonhomogeneous data. Section 4 is devoted to the problem of very weak solution to the modified Oseen problem in the whole space.

Remark 2. *Let us mention that for the rotating Stokes problem in a 2D-exterior domain, the situation is very different, we cannot treat it in the same functional frame as in a 3D-exterior domain.*

2 Mathematical Preliminaries

2.1 Basic notations and weighted Function Spaces

Let A_q, $1 \leq q < \infty$, the classes of Muckenhoupt weights, be given by all functions $w \in L^1_{loc}(\mathbb{R}^n)$, which are almost everywhere positive and for which the numbers $A_q(w)$ have the properties

$$A_1(w) := \sup_{Q \ni \mathbf{x_0}} \left(|Q|^{-1} w(Q)\right) \leq C w(\mathbf{x}_0) \text{ for a.a. } \mathbf{x}_0 \in \mathbb{R}^n, \tag{3}$$

$$A_q(w) := \sup_{Q} \left(|Q|^{-1} w(Q)\right) \left(|Q|^{-1} w'(Q)\right)^{q-1} \text{ is finite,} \tag{4}$$

where $w' := w^{-\frac{1}{q-1}}$ and $w(Q) = \int_Q w(x)dx$; the supremum is taken over all cubes Q in $\mathbb{R}^n$. We have excluded the case where w vanishes almost everywhere. It is easily seen from (4) that $|Q|^{-1}(w(Q))^{1/q}(w'(Q))^{1/q'} \leq C$, as a consequence $w' \in A_{q'}$, $\frac{1}{q} + \frac{1}{q'} = 1$. By [31], for $1 < q < \infty$ and $w \in A_q$ there exists s such that $1 \leq s < q$ and $w \in A_s$.

Let Ω be an open set in $\mathbb{R}^3$. For $q \in (1, \infty)$, $w \in A_q$, we define the weighted Lebesgue spaces $L^q_w(\Omega) := \left\{ f \in L^1_{loc}(\overline{\Omega}) \ , \ \int_\Omega |f|^q w \, dx \ < \infty \right\}$, with the norm $\|f\|_{q,w} := \left(\int_\Omega |f|^q w \, dx \right)^{\frac{1}{q}}$, and for $k \in \mathbb{N}, k \geq 1$, the weighted Sobolev spaces $H^{k,q}_w(\Omega) := \{ f \in L^1_{loc}(\overline{\Omega}) \ , \ \nabla^j f \in L^q_w(\Omega), \ j \leq k \}$, equipped with the norm $|||u|||_{k,q,w} := \sum_{j=0}^k \|\nabla^j u\|_{q,w}$. We also use the homogeneous Sobolev spaces $\widehat{H}^{k,q}_w(\Omega) := \{ f \in L^1_{loc}(\overline{\Omega}) \ , \ \nabla^k f \in L^q_w(\Omega) \}$, with the norm $\|u\|_{\widehat{H}^{k,q}_w} := \|\nabla^k u\|_{q,w}$, and finally, to take advantage of the denseness properties, the spaces $\widehat{H}^{k,q}_{w,0}(\Omega) := \overline{C^\infty_0(\Omega)}^{\|\cdot\|_{\widehat{H}^{k,q}_w}}$, $H^{k,q}_{w,0}(\Omega) := \overline{C^\infty_0(\Omega)}^{\|\cdot\|_{H^{k,q}_w}}$. In all cases, we will add the subscript σ to denote the divergence free counterpart. We also can consider the respective dual spaces.

If Ω is a bounded domain, let $1 < s \leq q < \infty$ and $w \in A_s$. It follows from Hölder's inequality that the weighted Lebesgue spaces are embedded into unweighted ones as follows : $L^q_w(\Omega) \hookrightarrow L^r(\Omega)$ for every $r < q/s$. On the other hand, the Sobolev embedding $H^{k,r}_w(\Omega) \hookrightarrow L^q_w(\Omega)$ holds under Lipschitz regularity for Ω and under the condition $\frac{1}{r} \leq \frac{1}{q} + \frac{k}{3s}$ with $k < 3$. That property is shown by Fröhlich in [13].

Systematically, vector functions and spaces of vector functions will be denoted by boldface letters.

2.2 Some required properties

• **Bogowski's properties:** The following lemmas and corollaries are a weighted analogue to Bogowski's theorem in various situations (bounded domain, whole space, exterior domain).

Lemma 1. *Let $\Omega \subseteq \mathbb{R}^n$ $(n \geq 2)$ be a bounded Lipschitz domain, and let $k \in \mathbb{N}$, $1 < q < \infty$ and $w \in A_q$. Then for every $g \in H^{k,q}_{w,0}(\Omega)$ with $\int_\Omega g(x)dx = 0$, there exists some $\mathbf{u} \in \mathbf{H}^{k+1,q}_{w,0}(\Omega)$ such that*

$$div\ \mathbf{u} = g \quad and \quad \|\mathbf{u}\|_{k+1,q,w} \leq C\|g\|_{k,q,w}, \tag{5}$$

where $C = C(\Omega, k, q, w) > 0$.

Proof. This result has been proved in [28]. □

Lemma 2. *For every $g \in \widehat{H}^{-1,q}_w(\mathbb{R}^3)$ there exists a function $\mathbf{u} \in \mathbf{L}^q_w(\mathbb{R}^3)$ that solves*

$$div\ \mathbf{u} = g \quad and \quad \|\mathbf{u}\|_{q,w} \leq C\|g\|_{\widehat{H}^{-1,q}_w(\mathbb{R}^3)}$$

with C independent of g.

Proof. See [24]. □

Corollary 1. *Let D be an exterior $C^{1,1}$-domain in $\mathbb{R}^3$. For every $g \in \widehat{H}^{-1,q}_w(D)$ there exists a function $\mathbf{u} \in \mathbf{L}^q_w(D)$ that solves*

$$div\ \mathbf{u} = g \quad and \quad \|\mathbf{u}\|_{q,w} \leq C\|g\|_{\widehat{H}^{-1,q}_w(D)}$$

with C independent of g.

Proof. See [24]. □

Observation:

- Increasing the regularity of g, e.g. in $H^{1,q}_{w,0}(D)$, we receive $\mathbf{u}$ in $\mathbf{H}^{2,q}_{w,0}(D)$ satisfying the corresponding inequality.
- In Corollary 1. we consider $u = 0$ on $\partial\Omega$. In the case that $u = h$ on $\partial\Omega$ then
$$\|\mathbf{u}\|_{q,w} \leq C(\|g\|_{\widehat{H}^{-1,q}_w(D)} + \|h\|_{T^{0,q}_w(\partial D)}).$$

For definition of $T^{0,q}_w(\partial D)$ see Section 2.3.

• **Convenient weight functions:** We define a subclass of Muckenhoupt weight functions called q-weights functions.

In the cylindrical coordinates system (ρ, θ, x_3) attached to the axis of revolution e_3, it is well known that $(\mathbf{e}_3 \wedge \mathbf{x}) \cdot \nabla \mathbf{u}$ is expressed by $\partial_\theta \mathbf{u}$. Then the rotating term will force us to deal with a more restricted class of weight functions on $\mathbb{R}^3$.

Namely, we will consider q-weight functions $w(\mathbf{x})$ defined in $\mathbb{R}^3$ that satisfy the following condition depending on q

$$\begin{array}{lll} \text{if } 2 \le q < \infty, & w^r \in A_{rq/2} & \text{for some } r \in [1, \infty), \\ \text{if } 1 < q < 2, & w^r \in A_{rq/2} & \text{for some } r \in \left(\frac{2}{q}, \frac{2}{2-q}\right], \end{array} \tag{6}$$

and which are independent of the angular variable θ (after the corresponding transformation into the cylindrical coordinates).
Examples for q-weight functions on $\mathbb{R}^3$ that fulfill (6) were shown in [10] :

(i) radially symmetric weights in the form $w_\alpha(\mathbf{x}) = |\mathbf{x}|^\alpha$ or $= (1 + |\mathbf{x}|)^\alpha$, where $-3 < \alpha < \frac{3q}{2}$ if $2 \le q < \infty$ and $-\frac{3q}{2} < \alpha < 3(q-1)$ if $1 < q < 2$.

(ii) anisotropic, axially symmetric weights in the form $(1 + |\mathbf{x}|)^\alpha (1 + \rho(\mathbf{x}))^\beta$, with $\rho(\mathbf{x}) = \sqrt{x_1^2 + x_2^2}$ provided that

$$\begin{array}{llll} -2 < \beta < q & \text{and} & -3 < \alpha + \beta < \frac{3q}{2} & \text{if } 2 \le q < \infty \\ -q < \beta < 2(q-1) & \text{and} & -\frac{3q}{2} < \alpha + \beta < 3(q-1) & \text{if } 1 < q < 2. \end{array}$$

Lemma 3. *Let $(T_j)_{j \in \mathcal{J}}$ be a family of linear operators such that $T_j : L^q_w(\mathbb{R}^3) \to L^q_w(\mathbb{R}^3)$ is continuous for every $q \ge 2$ and every $w \in A_{\frac{q}{2}}$ and such that $\|T_j\|_{\mathcal{L}(L^q_w)}$ is uniformly bounded in j. Moreover, we assume for the adjoint operators T_j^* that, for every $j \in \mathcal{J}$,*

$$T_j^* v = T_j v \quad \textit{for every} \quad v \in L^q_w(\mathbb{R}^3) \cap L^{q'}_{w'}(\mathbb{R}^3). \tag{7}$$

Then $T_j : L^p_w(\mathbb{R}^3) \to L^p_w(\mathbb{R}^3)$ is continuous for $p \in (1, 2)$ and for every q-weight w fulfilling (6), and $\|T_j\|_{\mathcal{L}(L^p_w)}$ is uniformly bounded in j.

Proof. As in the proof of [10][Prop. 3.1] one uses dualization and (7) to show that $T_j : L^p_w(\mathbb{R}^3) \to L^p_w(\mathbb{R}^3)$ is continuous for $p \in (1, 2)$, if the weight function w fulfills $w^{\frac{2}{2-q}} \in A_{\frac{q}{2-q}}$. The constant of continuity of T_j is equal to $\|T_j\|_{\mathcal{L}(L^{q'}_{w'})}$ which is uniformly bounded in j.
Now one uses the same interpolation procedure as in the proof of [10] [Theorem 1.2.(i)] to conclude the continuity of T_j for an arbitrary q-weight function w fulfilling (6). Since the complex interpolation does not increase the continuity constants, we get the statement of the Lemma. □

Corollary 2. *Any q-weight function chosen according to (6) belongs to A_q.*

Proof. Let ϕ be nonnegative, radial and radially decreasing, with $\int_{\mathbb{R}^3} \phi = 1$. We set $\phi_\epsilon(\mathbf{x}) := \frac{1}{\epsilon^3} \phi(\frac{\mathbf{x}}{\epsilon})$ and define the family of operators T_ϵ by $T_\epsilon f := f * \phi_\epsilon$, with small parameters $\epsilon > 0$.

By [31, Prop. V.2.1] the operators $T_\epsilon : L^r_w(\mathbb{R}^3) \to L^r_w(\mathbb{R}^3)$ are continuous and the family $(T_\epsilon)_{\epsilon>0}$ is uniformly bounded in $\mathcal{L}(L^r_w(\mathbb{R}^3))$ if and only if $w \in A_r$; in particular, this is true for $r \geq 2$ and $w \in A_{\frac{r}{2}} \subset A_r$.
Thus by Lemma 3 we obtain that $T_\epsilon : L^q_w(\mathbb{R}^n) \to L^q_w(\mathbb{R}^n)$ is continuous for all $q \in (1, \infty)$ and q-weight w according to (6). Thus again by [31, Prop. V.2.1] we obtain that $w \in A_q$. □

2.3 Appropriate extension and trace operators

Let $D \subset \mathbb{R}^n$ be a Lipschitz unbounded domain. In the L^q_w setting, the existence of a continuous extension operator has been shown by Chua [5] and is stated in the following theorem

Theorem 1. *Let $w_i \in A_{q_i}$, $1 \leq q_i < \infty$ for $i = 1, ...I_{max}$. Then there exists an extension operator on D*

$$\Upsilon : \bigcap_{i=1}^{I_{max}} L^{q_i}_{w_i}(D) \rightarrowtail \bigcap_{i=1}^{I_{max}} L^{q_i}_{w_i}(\mathbb{R}^n),$$

such that $\|\nabla^{k_i} \Upsilon f\|_{q_i,w_i;R^n} \leq C_i \|\nabla^{k_i} f\|_{q_i,w_i;D}$ for all i and for every f

Proof. See [5]. The result holds for an unbounded (ϵ, ∞)-domain, see the work of Jones [21]. Lipschitz domains are a particular example of (ϵ, ∞)-domains. □

We shall also need trace operators. We shall work with them in a formal way rather than considering particular cases. Some important progress has been done during last thirty years : Let us mention the trace theorem due to Nikol'skii for weights of type a power of distance to the boundary, followed by more recent paper [1] with weights equal to a power of distance to a point at the boundary and also with some weights from A_q. Hereafter we will recall the result of Chua.

For $k \geq 1$ one has $\mathbf{H}^{k,q}_w(D) \subset \mathbf{H}^{k,1}_{loc}(\overline{D})$, hence the restriction $\mathbf{u} \mapsto \mathbf{u}|_{\partial D}$ is well defined. Thus we may define $\mathbf{T}^{k,q}_w(\partial D) := (\mathbf{H}^{k,q}_w(D))|_{\partial D}$ equipped with the norm of the quotient space

$$\|\mathbf{h}\|_{\mathbf{T}^{k,q}_w(\partial D)} := \inf\{\mathbf{u} \in \mathbf{H}^{k,q}_w(D)\,,\ \mathbf{u}|_{\partial D} = \mathbf{h}\}.$$

Then $\mathbf{T}^{k,q}_w(\partial D)$ is also a reflexive Banach space, in which $C^\infty(\bar{D})|_{\partial D}$ is dense. Passing to the dual, we set $\mathbf{T}^{-1,q'}_w(\partial D) := (\mathbf{T}^{1,q}_{w'}(\partial D))'$. In accordance with the usual notations we can also denote these trace spaces $\mathbf{T}^{k,q}_w(\partial D)$ by $\mathbf{H}^{k-1/q,q}_w(\partial D)$. Then we have

Theorem 2. *Let $k \geq 1$ in $\mathbb{N}$, and $w \in A_q$, $1 < q < \infty$. Then, in the case of the half-space, there exists a continuous linear operator*

$$\mathcal{T} : \prod_{j=0}^{k-1} H^{k-j/q,q}_w(\mathbb{R}^n_+)|_{\mathbb{R}^{n-1}} \to H^{k,q}_w(\mathbb{R}^n_+) \tag{8}$$

such that $\frac{\partial^j}{\partial x_n^j}\mathcal{T}(g)|_{x_n=0} = g_j$*, where* $g = (g_0, ..., g_{k-1})$*. And in the case of a* $C^{k-1,1}$*-domain* $D \subset \mathbb{R}^n$*, there exists a continuous linear operator*

$$\mathcal{L} : \prod_{j=0}^{k-1} H_w^{k-j/q,q}(D)|_{\partial D} \to H_w^{k,q}(D) \tag{9}$$

such that $\frac{\partial^j}{\partial N^j}\mathcal{L}(g) = g_j$*, where* $g = (g_0, ..., g_{k-1})$*.*

Proof. see [5]. □

Corollary 3. *Let* $k \geq 1$ *in* $\mathbb{N}$*, and* $w \in A_q$*,* $1 < q < \infty$*. Let* $C^{k-1,1}$*-domain* $D \subset \mathbb{R}^n$*, there exists a continuous linear operator*

$$\mathcal{L} : \prod_{j=0}^{k-1} \widehat{H}_w^{k-j/q,q}(D)|_{\partial D} \to \widehat{H}_w^{k,q}(D) \tag{10}$$

such that $\frac{\partial^j}{\partial N^j}\mathcal{L}(g) = g_j$*, where* $g = (g_0, ..., g_{k-1})$*.*

3 The modified Stokes problem

3.1 Strong solutions

Let us briefly recall in this subsection the known results from [10] and [27] about the strong solvability of the modified Stokes system in an exterior domain $D \subset \mathbb{R}^3$ of class $C^{1,1}$:

$$\begin{aligned} -\nu\Delta\mathbf{u} - |\omega|(\mathbf{e}_3 \wedge \mathbf{x})\cdot\nabla\mathbf{u} + |\omega|\mathbf{e}_3 \wedge \mathbf{u} &= \mathbf{f} - \nabla p && \text{in } D, \\ \text{div } \mathbf{u} &= 0 && \text{in } D, \\ \mathbf{u} &= \mathbf{0} && \text{on } \partial D. \end{aligned} \tag{11}$$

One solves this system in homogeneous weighted Sobolev spaces, it means that the asymptotic behaviour of strong solutions is given by some linear polynomial; the weight function growing as $|x| \to \infty$ controls the rate of convergence.

Let us denote by $\mathcal{R}$ the operator $|\omega|((\mathbf{e}_3 \wedge \mathbf{x})\cdot\nabla - \mathbf{e}_3\wedge)$ and by $S_{\omega,\nu}$ the operator $-\nu\Delta - \mathcal{R}$. Equation $(11)_1$ can be simply rewritten in the form $S_{\omega,\nu}\mathbf{u} + \nabla p = \mathbf{f}$.

Moreover we consider the null space $\mathbf{K} = \mathbf{K}_{q,w}$ of the same system which is given by

$$\mathbf{K} = \Big\{\mathbf{v} \in \widehat{\mathbf{H}}^{2,q}_{w,\sigma}(D),\ S_{\omega,\nu}\mathbf{v} + \nabla\pi = \mathbf{0}\ ,\mathbf{v}|_{\partial D} = \mathbf{0},\ \text{for some } \pi \in \widehat{H}^{1,q}_w(D)\Big\} \tag{12}$$

Note that the definition of $\mathbf{K}$ do not require any explicit information about the asymptotic behaviour of $\mathbf{v}$, it is hidden in the framework of function spaces, (see Proposition 1 below). Similarly we denote by $\mathbf{K}' = \mathbf{K}_{q',w'}$ the null space associated to the dual system, $S'_{\omega,\nu}\mathbf{v} - \nabla\pi = \mathbf{0},\ \text{div } \mathbf{v} = 0,\ \mathbf{v}|_{\partial\Omega} = \mathbf{0}$.

Both null spaces $\mathbf{K}$ and $\mathbf{K}'$ with respect to the velocity fields, have the finite dimension 3, independently on q and w. Let us also define:

$$\mathcal{K} = \mathcal{K}_{q,w} = \{(\mathbf{\Phi}, \Psi\ ),\ \mathbf{\Phi} \in K_{q,w} \text{ and } \Psi \text{ is the associate pressure to } \mathbf{\Phi}\}\,.$$

The following theorem and propositions summarize the known results :

Theorem 3. *Let w be a q-weight function according to (6), and $\mathbf{K} = \mathbf{K}_{q,w}$ the null space defined in* (12)*.*
1. For every $\mathbf{f} \in \mathbf{L}^q_w(D)$, and for $\omega = |\omega|\mathbf{e_3}$, there exists a strong solution $\{\mathbf{u}, p\} \in \widehat{\mathbf{H}}^{2,q}_{w,\sigma}(D) \times \widehat{H}^{1,q}_w(D)$ to Problem (11)*, that satisfies the estimate*

$$\|\nabla^2\mathbf{u}\|_{q,w} + \|\mathcal{R}\mathbf{u}\|_{q,w} + \|\nabla p\|_{q,w} \le c\|\mathbf{f}\|_{q,w} \tag{13}$$

with some $c > 0$.
2. There is not more than one strong solution of Problem (11) *in the factor spaces modulo K and $\mathbb{R}$*

$$\widehat{\mathbf{H}}^{2,q}_{w,\sigma}(D)/K\ \times\ \widehat{H}^{1,q}_w(D)/\mathbb{R}$$

Remark 3. *The sign of the angular velocity ω plays no role in the structure of Problem* (11)*, except that for the dual operator $S'_{\omega,\nu} = S_{-\omega,\nu}$, therefore from Theorem 3 we also receive $\mathbf{v} \in \widehat{\mathbf{H}}^{2,q}_{w,\sigma}(D)/K$ and $\pi \in \widehat{H}^{1,q}_w(D)/\mathbb{R}$ as the unique strong solution of the equation $S_{-\omega,\nu}\mathbf{v} - \nabla\pi = \mathbf{f}$ where $\mathbf{f} \in \mathbf{L}^q_w(D)$, with $\mathbf{v} = \mathbf{0}$ on ∂D.*

Proposition 1. *Suppose that $\{\mathbf{u}, p\} \in \widehat{\mathbf{H}}^{2,q}_w(D) \times \widehat{H}^{1,q}_w(D)$ is a strong solution to Problem (11) with respect to $\mathbf{f} = \mathbf{0}$. Then there exists a polynomial of the form*

$$\mathcal{P}_1(\mathbf{u}) = \alpha\boldsymbol{\omega} + \beta\boldsymbol{\omega} \wedge \mathbf{x} + \gamma(x_1, x_2, -2x_3)^T,$$

with $\alpha, \beta, \gamma \in \mathbb{R}$, such that $\mathbf{u} - \mathcal{P}_1(\mathbf{u}) \xrightarrow{|\mathbf{x}|\to\infty} \mathbf{0}$. On the other hand, fixing α, β, and γ, solution $\mathbf{u}$ is uniquely determined in $\widehat{\mathbf{H}}^{2,q}_{w,\sigma}(D)$ with this behavior, and the associated pressure is $\nabla p = -S_{\omega,\nu}\mathbf{u}$.

Remark 4. *Another version of the result given by Proposition 1 is the following: Let $\mathbf{f} \in \mathbf{L}^{q_1}_{w_1}(D) \cap \mathbf{L}^{q_2}_{w_2}(D)$ where w_i is a q_i-weight function according to (6), $i = 1, 2$. Let $(\mathbf{u}_i, p_i)$ be strong solutions to Problem (11) that satisfy the respective estimate $(13)_{q_i,w_i}$. Then $\mathbf{u}_1$ coincides with $\mathbf{u}_2$ up to a polynomial field $\alpha\boldsymbol{\omega} + \beta\boldsymbol{\omega} \wedge \mathbf{x} + \gamma(x_1, x_2, -2x_3)^T$ for some $\alpha, \beta, \gamma \in \mathbb{R}$.*

Proposition 2. *Suppose $\{\mathbf{u}, p\} \in \widehat{\mathbf{H}}^{2,q}_{w,\sigma}(D) \times \widehat{H}^{1,q}_w(D)$ is a strong solution to Problem* (11) *with $\mathbf{f} \in \mathbf{L}^q_w(D)$. Then the following estimate holds*

$$\inf_{\{\mathbf{v},\pi\}\in\mathcal{K}} (\|\mathbf{u} + \mathbf{v}\|_{\widehat{\mathbf{H}}^{2,q}_w} + \|\mathcal{R}(\mathbf{u} + \mathbf{v})\|_{q,w} + \|\nabla p + \nabla\pi\|_{q,w}) \le c\|\mathbf{f}\|_{q,w}.$$

Moreover, at least one representant of $\{\mathbf{u}, p\} + \mathcal{K}$ fulfills the estimate (13), let us denote it by $\{\mathbf{u} + \mathbf{u}_K,\ p + p_K\}$.

Remark 5. *Propositions 1 and 2 have the following interesting corollary. For* $\mathbf{f} \in \mathbf{L}^q_w$ *there exists a strong solution of the problem* (11)*, given by Theorem 3, which fulfills its decay at infinity, assuming that there exists* $r > 1$ *such that* $\|\boldsymbol{\phi}\|_r \leq \|\nabla^2 \boldsymbol{\phi}\|_{q,w}$ *for every* $\boldsymbol{\phi} \in \mathbf{C}^\infty_0(D)$*. This holds in particular in the case* $w = 1$ *and* $q \in (1, \frac{3}{2})$*.*

Proof. Let us justify the Remark: Approximate $\mathbf{f}$ by a sequence $(\mathbf{f}_n)$ from $\mathbf{C}^\infty_0(D)$. Then one uses the weighted Poincaré inequality (see Appendix 2) in $D \cap B_n(0)$, $\operatorname{supp} \mathbf{f}_n \subset B_n$, to show that $\mathbf{f}_n \in \widehat{\mathbf{H}}^{-1,2}(D)$. Thus by [20] there exist corresponding weak solutions $(\mathbf{u}_n)$ to (11). We know that these solutions also satisfy $\nabla^2 \mathbf{u}_n \in \mathbf{L}^q_w(D)$. In particular $\mathbf{u}_n \in \mathbf{L}^r(D)$. Thus by the estimate from Theorem 3 we find that there exists some $\mathbf{u} \in \widehat{\mathbf{H}}^{2,q}_w(D)$ such that $\mathbf{u}_n \xrightarrow{n\to\infty} \mathbf{u}$ in $\widehat{\mathbf{H}}^{2,q}_w(D)/\mathbf{K}$. Since $\mathbf{K}$ is finite dimensional, there exists $\mathbf{v}_n$ with $\|\nabla^2 \mathbf{v}_n\|_{q,w} = \|\mathbf{u}_n\|_{\widehat{H}^{2,q}_w/\mathbf{K}}$. Thus the sequence of solutions $\mathbf{v}_n$ converges in the Banach space $(\widehat{\mathbf{H}}^{2,q}_w(D) \cap \mathbf{L}^r(D)) + \mathbf{K}$, and we obtain that there exists $\mathbf{u}_\mathbf{K} \in \mathbf{K}$ such that $\mathbf{u} - \mathbf{u}_\mathbf{K} \in \mathbf{L}^r(D)$. Then $\mathbf{u} - \mathbf{u}_\mathbf{K}$ is the (strong) solution which decays at infinity. □

Remark 6. *In case of the nonhomogeneous modified Stokes problem with non zero divergence, div* $\mathbf{u} = g \neq 0$*, it is not difficult to prove that the respective problem is solvable with the estimate*

$$\|\nabla^2 \mathbf{u}\|_{q,w} + \|\nabla p\|_{q,w} + \|\mathcal{R}\mathbf{u}\|_{q,w} \leq c\,(\|\mathbf{f}, \nabla g, (\omega \wedge \mathbf{x}) \cdot \mathbf{g}\|_{\mathbf{q},\mathbf{w}}) \tag{14}$$

under sufficiently smoothness of g*, see [27].*

3.2 Very weak solutions

In the lowest regularity context the data are given by functionals which now are in general no distributions. The corresponding functional spaces must be precisely defined.

Let $D \subseteq \mathbb{R}^3$ be always an exterior domain of type $C^{1,1}$ or the whole space, and w always a q-weight function according to (6). At first, we note that the q-property (6) is "closed" with respect to the dual weight function $w' = w^{-\frac{1}{q-1}}$ if q is replaced by q' (This follows from a straight forward calculation in the case of radially symmetric weights). Let us introduce the following spaces of functions and functionals,

$$\begin{aligned}
\widehat{\mathbf{H}}^{2,q'}_{w',K}(D) &:= \widehat{\mathbf{H}}^{2,q'}_{w'}(D)/\mathbf{K}_{q',w'} \\
\mathcal{H}^{-1,q}_{w,0}(D) &:= \left(\widehat{H}^{1,q'}_{w'}(D)\right)' \\
\widehat{\mathbf{H}}^{-2,q}_{w,K}(D) &:= (\widehat{\mathbf{H}}^{2,q'}_{w',K}(D))' = \widehat{\mathbf{H}}^{-2,q}_{w}(D) \cap \mathbf{K}^\perp_{q',w'},
\end{aligned}$$

and we will add if necessarily the subscript σ in the case of divergence free vector fields.

Then we consider the nonhomogeneous modified Stokes systems in $D \subset \mathbb{R}^3$ or in the whole space $\mathbb{R}^3$,

$$\begin{aligned} S_{\omega,\nu}\mathbf{u} + \nabla p &= \mathbf{f} && \text{in } D \subseteq \mathbb{R}^3, \\ \text{div } \mathbf{u} &= g && \text{in } D, \\ \mathbf{u} &= \mathbf{0} && \text{on } \partial D, \quad \text{if } D \neq \mathbb{R}^3. \end{aligned} \tag{15}$$

Definition 1. *Let $\mathbf{f} \in \widehat{\mathbf{H}}_w^{-2,q}(D)$ and $g \in \mathcal{H}_{w,0}^{-1,q}(D)$ be given. Then a function $\mathbf{u} \in \mathbf{L}_w^q(D)$ is called a very weak solution to the system (15), if the equation*

$$\left\langle \mathbf{u}, S_{\omega,\nu}^{'}\boldsymbol{\phi} - \nabla\psi \right\rangle_w = \langle \mathbf{f}, \boldsymbol{\phi}\rangle_w + \langle g, \psi\rangle_w \tag{16}$$

holds for all test functions $\boldsymbol{\phi} \in \widehat{\mathbf{H}}_{w',\sigma}^{2,q'}(D)$ with $\mathcal{R}\boldsymbol{\phi} \in \mathbf{L}_{w'}^{q'}(D)$ and $\psi \in \widehat{H}_{w'}^{1,q'}(D)$. The pairing $\langle \cdot, \cdot \rangle_w$ denotes convenient dualities (between $\mathbf{L}_w^q$ and $\mathbf{L}_{w'}^{q'}$, between $\widehat{\mathbf{H}}_{w,\sigma}^{-2,q}$ and $\widehat{\mathbf{H}}_{w',\sigma}^{2,q'}$, between $\mathcal{H}_{w,0}^{-1,q}$ and $\widehat{H}_{w'}^{1,q'}$).

Setting $\psi = 1$, it follows that a first necessary condition for the existence of a very weak solution $\mathbf{u}$ is $\langle g, 1\rangle = 0$. Adding in (16), $S_{\omega,\nu}^{'}\boldsymbol{\Phi} - \nabla\Psi = \mathbf{0}$, with $\{\boldsymbol{\Phi}, \Psi\} \in \mathcal{K}'$, it follows that two other necessary conditions (the compatibility conditions) for the existence of a very weak solution are $\langle \mathbf{f}, \boldsymbol{\Phi}\rangle_w = 0$ for all $\boldsymbol{\Phi} \in \mathbf{K}'$ and $\langle g, \Psi\rangle_w = 0$ for all $\Psi \in \Pi$, where $\Pi = \left\{\Psi : \nabla\Psi = S'_{\omega,\nu}\boldsymbol{\Phi},\ \boldsymbol{\Phi} \in \mathbf{K}'\right\}$.

Let us introduce the space $\mathcal{H}_{w,0,\mathcal{K}}^{-1,q}(D) = \mathcal{H}_{w,0}^{-1,q}(D) \cap \Pi^{\perp} \cap \mathbb{R}^{\perp}$, where $D \subseteq \mathbb{R}^3$.

Now we are in a position to prove the existence and uniqueness theorem of very weak solutions.

Theorem 4. *Let $1 < q < \infty$ and w be a radially symmetric q-weight function according to (6). Then, for every $\mathbf{f} \in \widehat{\mathbf{H}}_{w,K}^{-2,q}(D)$, and $g \in \mathcal{H}_{w,0,\mathcal{K}}^{-1,q}(D)$, there exists an unique very weak solution $\mathbf{u} \in \mathbf{L}_w^q(D)$ in the sense of Definition 1. Moreover, there exists a pressure functional $p \in \mathcal{H}_{w,0}^{-1,q}(D)$ such that $(\mathbf{u}, p)$ solves*

$$\left\langle \mathbf{u}, S_{\omega,\nu}^{'}\boldsymbol{\phi} \right\rangle_w - \langle p, \mathit{div}\ \boldsymbol{\phi}\rangle_w = \langle \mathbf{f}, \boldsymbol{\phi}\rangle_w \ \text{ for all } \boldsymbol{\phi} \in \widehat{\mathbf{H}}_{w',\sigma}^{2,q'}(D), \mathcal{R}\boldsymbol{\phi} \in \mathbf{L}_{w'}^{q'}(D).$$

$\mathbf{u}$ and p fulfill the inequality

$$\|\mathbf{u}\|_{q,w} + \|p\|_{\mathcal{H}_{w,0}^{-1,q}} \leq c\left(\|\mathbf{f}\|_{\widehat{\mathbf{H}}_w^{-2,q}} + \|g\|_{\mathcal{H}_{w,0}^{-1,q}}\right). \tag{17}$$

Proof. We know that w' is a radially symmetric q'-weight function. For every $\mathbf{v} \in \mathbf{L}_{w'}^{q'}(D)$, in accordance with Theorem 3, there exists a couple $\{\phi, \psi\}$ in $\widehat{\mathbf{H}}_{w',\sigma}^{2,q'}(D) \times \widehat{H}_{w'}^{1,q'}(D)/\mathbb{R}$, strong solution to

$$S_{\omega,\nu}^{'}\phi - \nabla\psi = \mathbf{v},\ \text{div } \boldsymbol{\phi} = 0,\ \boldsymbol{\phi}|_{\partial D} = \mathbf{0}. \tag{18}$$

Thanks to Proposition 2, let $\{\boldsymbol{\phi}_v, \psi_v\}$ be a representant of $\{\boldsymbol{\phi}, \psi\} + \mathcal{K}'$ which satisfies the estimate

$$\|\nabla^2\boldsymbol{\phi}_v\|_{q',w'} + \|\mathcal{R}\boldsymbol{\phi}_v\|_{q',w'} + \|\nabla\psi_v\|_{q',w'} \leq c\|\mathbf{v}\|_{q',w'},$$

we now define $\mathbf{u} := \mathbf{u}(\mathbf{f}, g)$ setting

$$\langle \mathbf{u}, \mathbf{v} \rangle_w := \langle \mathbf{f}, \boldsymbol{\phi}_v \rangle_w + \langle g, \psi_v \rangle_w \,,$$

where $\mathbf{f} \in \widehat{\mathbf{H}}_w^{-2,q}(D)$ and $g \in \mathcal{H}_{w,0}^{-1,q}(D)$ are given. Then one has

$$\begin{aligned} |\langle \mathbf{u}, \mathbf{v} \rangle_w| &\leq |\langle \mathbf{f}, \boldsymbol{\phi}_v \rangle_w + \langle g, \psi_v \rangle_w| \\ &\leq \|\mathbf{f}\|_{\widehat{\mathbf{H}}_w^{-2,q}} \|\nabla^2 \boldsymbol{\phi}_v\|_{q',w'} + \|g\|_{\mathcal{H}_{w,0}^{-1,q}} \|\nabla \psi_v\|_{q',w'} \\ &\leq c \left(\|\mathbf{f}\|_{\widehat{\mathbf{H}}_w^{-2,q}} + \|g\|_{\mathcal{H}_{w,0}^{-1,q}} \right) \|\mathbf{v}\|_{q',w'}. \end{aligned}$$

Thus $\mathbf{u} = \mathbf{u}(\mathbf{f}, g)$ belongs to $(\mathbf{L}_{w'}^{q'}(D))' = \mathbf{L}_w^q(D)$ and fulfills the stated estimate.

It remains to show that this function $\mathbf{u}(\mathbf{f}, g)$ solves very weakly of our problem. To see this, let $\boldsymbol{\phi}, \psi$ be the test functions with $\boldsymbol{\phi} \in \widehat{\mathbf{H}}_{w',\sigma}^{2,q'}(D)$, $\mathcal{R}\boldsymbol{\phi} \in \mathbf{L}_{w'}^{q'}(D)$ and $\psi \in \widehat{H}_{w'}^{1,q'}(D)$ independently. We can choose $\mathbf{v} := S'_{\omega,\nu}\phi - \nabla\psi \in \mathbf{L}_{\mathbf{w}'}^{\mathbf{q}'}(\boldsymbol{\Omega})$, then

$$\left\langle \mathbf{u}(\mathbf{f}, g), S'_{\omega,\nu}\boldsymbol{\phi} - \nabla\psi \right\rangle_w = \langle \mathbf{u}(\mathbf{f}, g), \mathbf{v} \rangle_w = \langle \mathbf{f}, \boldsymbol{\phi} \rangle_w + \langle g, \psi \rangle_w \,.$$

Therefore $\mathbf{u}(\mathbf{f}, g)$ is a very weak solution to our problem.

Note that the relation $\left\langle \mathbf{u}(\mathbf{f}, g), S'_{\omega,\nu}(\boldsymbol{\phi} + \boldsymbol{\Phi}) - \nabla(\psi + \Psi) \right\rangle_w = \langle \mathbf{f}, \boldsymbol{\phi} \rangle_w + \langle g, \psi \rangle_w$ holds for all $\{\boldsymbol{\Phi}, \Psi\} \in \mathcal{K}'$. The compatibility conditions being satisfied by $\mathbf{f}$ and g, we get $\langle \mathbf{f}, \boldsymbol{\phi} \rangle_w + \langle g, \psi \rangle_w = \langle \mathbf{f}, \boldsymbol{\phi} + \boldsymbol{\Phi} \rangle_w + \langle g, \psi + \Psi \rangle_w$, it means that the previous relation accords to the given definition of the very weak solution.

To prove the uniqueness of $\mathbf{u} = \mathbf{u}(\mathbf{f}, g)$, let $\mathbf{U} = \mathbf{U}(\mathbf{f}, g) \in \mathbf{L}_w^q(D)$ be another very weak solution to the system (15). We have

$$\left\langle \mathbf{U}, S'_{\omega,\nu}\boldsymbol{\phi} - \nabla\psi \right\rangle_w = \langle \mathbf{f}, \boldsymbol{\phi} \rangle_w + \langle g, \psi \rangle_w \tag{19}$$

for all test functions $\phi \in \widehat{\mathbf{H}}_{w',\sigma}^{2,q'}(D)$ with $\mathcal{R}\phi \in \mathbf{L}_{w'}^{q'}(D)$ and $\psi \in \widehat{H}_{w'}^{1,q'}(D)$. Thus

$$\left\langle \mathbf{U} - \mathbf{u},\, S'_{\omega,\nu}\boldsymbol{\phi} - \nabla\psi \right\rangle_w \;=\; 0.$$

Here the quantity $S'_{\omega,\nu}\boldsymbol{\phi} - \nabla\psi$ is an arbitrary element from $\mathbf{L}_{w'}^{q'}(D)$, it follows that $\mathbf{U} - \mathbf{u} = \mathbf{0}$ in $\mathbf{L}_w^q(D)$. □

Remark 7. $D \subseteq \mathbb{R}^3$. *The proofs are similar in the case of an exterior domain and in the case of the whole space. The main idea is only the duality with the respective strong solution. Two important features of our results are the required compatibility conditions for the data and on the other hand the role of the nullspaces* $\mathcal{K}$ *and* $\mathcal{K}'$.

3.3 The case of nonhomogeneous Dirichlet conditions

We are interested in the case of nonhomogeneous Dirichlet boundary conditions $\mathbf{u}|_{\partial D} = \omega \times \mathbf{x} := \mathbf{h}$ where $D \subset \mathbb{R}^3$ is always an exterior domain of type $C^{1,1}$. Let w be always a q-weight function according to (6), we consider the modified Stokes system with $\nu = 1$

$$\begin{aligned} S_{\omega,1}\mathbf{u} + \nabla p &= \mathbf{f} && \text{in } D \\ \text{div } \mathbf{u} &= g && \text{in } D \\ \mathbf{u}|_{\partial D} &= \omega \times \mathbf{x} := \mathbf{h}. \end{aligned} \tag{20}$$

It is easily seen that if $\mathbf{h} \in \mathbf{T}^{0,q}_w(\partial D)$ then the functionals "supported by the boundary"

$$\tilde{\mathbf{f}_h} : \ \boldsymbol{\phi} \to \ \langle \mathbf{h}, (\mathbf{N}\cdot\nabla)\boldsymbol{\phi}|_{\partial D}\rangle_{w,\partial D}$$

$$\text{and} \qquad \tilde{g}_h : \ \psi \to \ \langle \mathbf{h}, \psi|_{\partial D}\ \mathbf{N}\rangle_{w,\partial D}$$

belong to $\big(\widehat{\mathbf{H}}^{2,q'}_{w'}(D)\big)'$ and $\big(\widehat{H}^{1,q'}_{w'}(D)\big)'$ respectively. In order to solve very weakly the system (20), we must introduce appropriate functionals $\mathbf{F} = \mathbf{F}(\mathbf{f},\mathbf{h})$ and $G = G(g,\mathbf{h})$ with firstly the properties $\mathbf{F}(\mathbf{f},\mathbf{0}) = \mathbf{f}$ and $G(g,\mathbf{0}) = g$, and secondly such that, for each $\phi \in \widehat{\mathbf{H}}^{2,q'}_{w',\sigma}(D)$, $\mathcal{R}\boldsymbol{\phi} \in \mathbf{L}^{q'}_{w'}(D)$ and $\psi \in \widehat{H}^{1,q'}_{w'}(D)$, the relations

$$\left\langle \mathbf{u}, S'_{\omega,1}\boldsymbol{\phi}\right\rangle_w = \langle \mathbf{f},\boldsymbol{\phi}\rangle_w \ \text{ and } \ -\langle \mathbf{u},\nabla\psi\rangle_w = \langle g,\psi\rangle_w \tag{21}$$

not only give us a very weak solution, but also generalize Definition 1 for a nonhomogeneous boundary condition $\mathbf{u}|_{\partial D} = \mathbf{h} \neq \mathbf{0}$.

Applying formally Green's formula, we have

$$\left\langle \mathbf{u}, S'_{\omega,1}\boldsymbol{\phi}\right\rangle_w = \langle S_{\omega,1}\mathbf{u},\boldsymbol{\phi}\rangle_w - \langle \mathbf{u}|_{\partial D}, (\mathbf{N}\cdot\nabla)\boldsymbol{\phi}\rangle_{w,\partial D}. \tag{22}$$

Using the boundary condition " $\mathbf{u}|_{\partial D} = \mathbf{h}$ ", we derive the definition of $\mathbf{F}$

$$\left\langle \mathbf{u}, S'_{\omega,1}\boldsymbol{\phi}\right\rangle_w = \langle \mathbf{f},\boldsymbol{\phi}\rangle_w \ := \left\langle \mathbf{F} - \tilde{\mathbf{f}}_h, \boldsymbol{\phi}\right\rangle_w. \tag{23}$$

Note that we took $\nu = 1$ in order to have $\tilde{\mathbf{f}_h}$ independent on the viscosity.

Applying now formally Stokes-Ostrogradski's formula, we similarly derive the definition of G

$$\begin{aligned} -\langle \mathbf{u},\nabla\psi\rangle_w &= \langle \text{div }\mathbf{u},\psi\rangle_w - \langle \mathbf{u}|_{\partial D}, \psi|_{\partial D}\,\mathbf{N}\rangle_{w,\partial D} \\ &= \langle g,\psi\rangle_w \ := \langle G - \tilde{g}_h,\psi\rangle_w. \end{aligned} \tag{24}$$

The functionals $\mathbf{F}$ and G are a priori so irregular as the given data $\mathbf{f}$ and g, it is difficult to distinguish between their parts which are more regular and consist in forces and their other parts which characterize a divergence or boundary values. The following assertion is important

Proposition 3. *The decompositions* $\mathbf{f} = \mathbf{F} - \tilde{\mathbf{f}}_h$ *and* $g = G - \tilde{g}_h$ *are uniquely defined by the data* $\mathbf{f} \in \widehat{\mathbf{H}}_w^{-2,q}(D)$, $g \in \mathcal{H}_{w,0}^{-1,q}(D)$ *and* $\mathbf{h} \in \mathbf{T}_w^{0,q}(\partial D)$.

Proof. Let us assume that exist two decompositions

$$\langle \mathbf{f}, \boldsymbol{\phi} \rangle_w = \langle \mathbf{F}_i, \boldsymbol{\phi} \rangle_w - \langle \mathbf{h}_i, (\mathbf{N} \cdot \nabla)\boldsymbol{\phi} \rangle_{w,\partial D} \quad \text{for} \quad i = 1, 2.$$

So we have

$$\langle \mathbf{F}_1 - \mathbf{F}_2, \boldsymbol{\phi} \rangle_w - \langle \mathbf{h}_1 - \mathbf{h}_2, (\mathbf{N} \cdot \nabla)\boldsymbol{\phi}|_{\partial D} \rangle_{w,\partial D} = 0 \quad \text{for } \boldsymbol{\phi} \in \widehat{\mathbf{H}}_{w'}^{2,q'}(D).$$

The previous functional $\langle \mathbf{F}_1 - \mathbf{F}_2, \cdot \,\rangle_w$ vanishes on $\mathbf{C}_0^\infty(D)$: it follows that $\mathbf{F}_1 - \mathbf{F}_2 = \mathbf{0}$ as a distribution on the domain D. Consequently, for every $\boldsymbol{\phi}$, $\langle \mathbf{h}_1 - \mathbf{h}_2, (\mathbf{N} \cdot \nabla)\boldsymbol{\phi}|_{\partial D} \rangle_{w,\partial D} = 0$. The mapping defining on ∂D the normal trace of $\nabla\boldsymbol{\phi}$, from $\widehat{\mathbf{H}}_{w'}^{2,q'}(D)$ to $\mathbf{T}_{w'}^{1,q'}(\partial D)$ is surjective, hence $\mathbf{h}_1 - \mathbf{h}_2 = \mathbf{0}$.

Similarly, one can verify the uniquely defined decomposition for g. The functionals $\mathbf{F}$ and G can be in smaller spaces, like respectively $\widehat{\mathbf{H}}_w^{-1,q}(D)$ or maybe $\mathbf{L}_w^q(D)$, and $L_w^q(D)$ or maybe $H_w^{1,q}(D)$. □

Theorem 5. *Let the data* $\mathbf{f}$, g *and* $\mathbf{h}$ *be regular enough to justify that, in the decompositions given by Proposition 3, smooth functions* $\mathbf{F} \in \mathbf{L}_w^q(D)$ *and* $G \in \widehat{H}_w^{1,q}(D)$ *exist. Let* $\mathbf{u}$ *be given as a very weak solution to problem (20) with* $\mathbf{h} \in \mathbf{T}_w^{2,q}(\partial D)$, *where* $\mathbf{f}$ *and* g *satisfy the compatibility conditions. Then, this very weak solution* $\mathbf{u}$ *to the nonhomogeneous modified Stokes system which data are* $\mathbf{f}$, g *and* $\mathbf{h}$ *coincides with the strong solution of the nonhomogeneous modified Stokes system which data are* $\mathbf{F}$, G *and* $\mathbf{h}$. *In particular* $\mathbf{u} \in \widehat{\mathbf{H}}_w^{2,q}(D)$ *and*

$$\|\nabla^2 \mathbf{u}\|_{q,w} + \|\mathcal{R}\mathbf{u}\|_{q,w} \le c(\|\mathbf{F}\|_{q,w} + \|G\|_{\widehat{H}_w^{1,q}} + \|\mathbf{h}\|_{T_w^{2,q}}). \tag{25}$$

Proof. There exists an extension of $\mathbf{h} \in \mathbf{T}_w^{2,q}(\partial D)$, say $\mathbf{v}_{1h}$, which satisfies $\mathbf{v}_{1h} \in \widehat{\mathbf{H}}_w^{2,q}(D)$, $\mathbf{v}_{1h}|_{\partial D} = \mathbf{h}$ and $\|\mathbf{v}_{1h}\|_{\widehat{H}_w^{2,q}} \le c\|\mathbf{h}\|_{T_w^{2,q}}$. Then we can solve $\text{div } \mathbf{b}_1 = G - \text{div } \mathbf{v}_{1h}$, by Corollary 1 and Observation (ii) there exists $\mathbf{b}_1 \in \widehat{\mathbf{H}}_{w,0}^{2,q}(D)$ such that $\|\mathbf{b}_1\|_{\widehat{H}_w^{2,q}} \le c(\|G\|_{\widehat{H}_w^{1,q}} + \|\mathbf{h}\|_{T_w^{2,q}})$.

By Theorem 3, there exists a strong solution $(\mathbf{u}_1, p_1) \in \widehat{\mathbf{H}}_w^{2,q}(D) \times \widehat{H}_w^{1,q}(D)$ to the following homogeneous modified Stokes system

$$\begin{aligned} S_{\omega,1}\mathbf{u}_1 + \nabla p_1 &= -S_{\omega,1}(\mathbf{b}_1 + \mathbf{v}_{1h}) && \text{in } D \\ \text{div } \mathbf{u}_1 &= 0 && \text{in } D \\ \mathbf{u}_1|_{\partial D} &= \mathbf{0}, \end{aligned} \tag{26}$$

which satisfies the estimate (13).
It is now clear that

$$\begin{aligned} S_{\omega,1}(\mathbf{u}_1 + \mathbf{b}_1 + \mathbf{v}_{1h}) + \nabla p_1 &= \mathbf{0} && \text{in } D \\ \text{div } (\mathbf{u}_1 + \mathbf{b}_1 + \mathbf{v}_{1h}) &= G && \text{in } D \\ (\mathbf{u}_1 + \mathbf{b}_1 + \mathbf{v}_{1h})|_{\partial D} &= \mathbf{h}, \end{aligned} \tag{27}$$

therefore $\mathbf{v}_1 := \mathbf{u}_1 + \mathbf{b}_1 + \mathbf{v}_{1h} \in \widehat{\mathbf{H}}^{2,q}_w(D)$ solves strongly the previous modified Stokes system (27) and satisfies the inequality

$$\|\nabla^2 \mathbf{v}_1\|_{q,w} + \|\mathcal{R}\mathbf{v}_1\|_{q,w} \leq c\Big(\|G\|_{\widehat{H}^{1,q}_w} + \|\mathbf{h}\|_{T^{2,q}_w}\Big).$$

Finally, we have another strong solution $(\mathbf{v}_2, p_2)$ of the modified Stokes system

$$\begin{aligned} S_{\omega,1}\mathbf{v}_2 + \nabla p_2 &= \mathbf{F} && \text{in } D \\ \operatorname{div} \mathbf{v}_2 &= 0 && \text{in } D \\ \mathbf{v}_2|_{\partial D} &= \mathbf{0}, && \end{aligned} \tag{28}$$

with $\|\nabla^2 \mathbf{v}_2\|_{q,w} + \|\mathcal{R}\mathbf{v}_2\|_{q,w} \leq c\|\mathbf{F}\|_{q,w}$.

Thus $\mathbf{u} := \mathbf{v}_1 + \mathbf{v}_2$ and $p := p_1 + p_2$ solve strongly the system

$$\begin{aligned} S_{\omega,1}\mathbf{u} + \nabla p &= \mathbf{F} && \text{in } D \\ \operatorname{div} \mathbf{u} &= G && \text{in } D \\ \mathbf{u}|_{\partial D} &= \mathbf{h}, && \end{aligned} \tag{29}$$

with the estimate

$$\|\mathbf{u}\|_{\widehat{H}^{2,q}_w} + \|\mathcal{R}\mathbf{u}\|_{\widehat{H}^{1,q}_w} \leq c(\|\mathbf{F}\|_{q,w} + \|G\|_{\widehat{H}^{1,q}_w} + \|\mathbf{h}\|_{T^{2,q}_w}). \tag{30}$$

By means of Stokes-Ostrogradski's and Green's formulas, we can easily verify that the strong solutions $\mathbf{v}_1$ and $\mathbf{v}_2$ are also very weak solutions of related modified Stokes models: indeed the following relations hold $-\langle \mathbf{v}_1, \nabla\psi\rangle_w = \langle G - \tilde{g}_h, \psi\rangle_w = \langle g, \psi\rangle_w$, $0 = \Big\langle \mathbf{v}_1, S'_{\omega,1}\,\boldsymbol{\phi}\Big\rangle_w + \Big\langle \tilde{\mathbf{f}}_h, \boldsymbol{\phi}\Big\rangle_w$, and $-\langle \mathbf{v}_2, \nabla\psi\rangle_w = 0$, $\langle \mathbf{F}, \boldsymbol{\phi}\rangle_w = \Big\langle \mathbf{v}_2, S'_{\omega,1}\,\phi\Big\rangle_w$. Adding the respective formulas, we conclude that

$$\Big\langle \mathbf{u}, S'_{\omega,1}\,\boldsymbol{\phi}\Big\rangle_w = \langle \mathbf{f}, \boldsymbol{\phi}\rangle_w \ \text{ and } \ -\langle \mathbf{u}, \nabla\psi\rangle_w = \langle g, \psi\rangle_w\,.$$

These equations hold for $\boldsymbol{\phi} \in \widehat{\mathbf{H}}^{2,q'}_{w',\sigma}$ with $\mathcal{R}\boldsymbol{\phi} \in \mathbf{L}^{q'}_{w'}$ and $\psi \in \widehat{H}^{1,q'}_{w'}$. Then $\mathbf{u}$ is also a very weak solution to the initial system (20). □

Remark 8. *Under the made assumptions,* $\mathbf{F} \in \mathbf{L}^q_w(D)$ *and* $G \in \widehat{H}^{1,q}_w(D)$*, we have proved a result on the regularity of the obtained very weak solution (which is unique as soon as* $\mathbf{f} \in \widehat{\mathbf{H}}^{-2,q}_{w,K}(D)$ *and* $g \in \mathcal{H}^{-1,q}_{w,0,\mathcal{K}}(D)$*, see Theorem 4).*

In agreement with Proposition 3, assuming $\mathbf{f} \in \widehat{\mathbf{H}}^{-2,q}_{w,K}(D)$, $g \in \mathcal{H}^{-1,q}_{w,0,\mathcal{K}}(D)$ and $\mathbf{h} \in \mathbf{T}^{1,q}_w(\partial D)$, it is clear that $\mathbf{v}_1$ and $\mathbf{v}_2$ as very weak solutions of the related modified Stokes systems (27) and (28) from the proof of Theorem 5 satisfy the following estimates

$$\|\mathbf{v}_1\|_{q,w} + \|p_1\|_{\mathcal{H}^{-1,q}_{w,0}} \leq c(\|G\|_{\mathcal{H}^{-1,q}_{w,0}} + \|\mathbf{h}\|_{T^{0,q}_w}) \leq c(\|g\|_{\mathcal{H}^{-1,q}_{w,0}} + \|\mathbf{h}\|_{T^{0,q}_w})$$

$$\|\mathbf{v}_2\|_{q,w} + \|p_2\|_{\mathcal{H}^{-1,q}_{w,0}} \leq c\|\mathbf{F}\|_{\widehat{H}^{-2,q}_w}) \leq c(\|\mathbf{f}\|_{\widehat{H}^{-2,q}_w}).$$

So we have proved the theorem

Theorem 6. *Let $1 < q < \infty$ and w a q-weight function that fulfills (6). Assume that*

$$\mathbf{f} \in \widehat{\mathbf{H}}^{-2,q}_{w,K}(D), \quad g \in \mathcal{H}^{-1,q}_{w,0,\mathcal{K}}(D), \ \textit{and} \ \ \mathbf{h} \in \mathbf{T}^{0,q}_{w}(\partial D).$$

Then there exists an unique very weak solution $\mathbf{u} \in \mathbf{L}^q_w(D)$ to problem (20). The associated pressure is obtained in $\mathcal{H}^{-1,q}_{w,0}(D)$ by

$$\langle p, div\, \boldsymbol{\phi} \rangle_w = \left\langle \mathbf{u}, S'_{\omega,\nu}\ \boldsymbol{\phi} \right\rangle_w - \langle \mathbf{f}, \boldsymbol{\phi} \rangle_w, \quad \forall \boldsymbol{\phi} \in \widehat{\mathbf{H}}^{2,q'}_{w'}(D).$$

These functions $\mathbf{u}$ and p satisfy the estimate

$$\|\mathbf{u}\|_{q,w} + \|p\|_{\mathcal{H}^{-1,q}_{w,0}} \le c \left(\|\mathbf{f}\|_{\widehat{H}^{-2,q}_w} + \|g\|_{\mathcal{H}^{-1,q}_{w,0}} + \|\mathbf{h}\|_{T^{0,q}_w} \right).$$

4 The modified Oseen problem

We will investigate the modified Oseen problem in the whole space $\mathbb{R}^3$. First we defined new convenient weight functions, then we recall the known results from [9] about the strong solvability of the problem

$$\begin{aligned} S_{\omega,\nu}\mathbf{u} + k\partial_3\mathbf{u} + \nabla p = \mathbf{f} \qquad & \text{in } \mathbb{R}^3, \\ \text{div } \mathbf{u} = 0 \text{ or } g \qquad & \text{in } \mathbb{R}^3. \end{aligned} \tag{31}$$

Again the asymptotic behavior of $\mathbf{u}$ at infinity will be intrinsic in the chosen weighted L^q-spaces. On the other hand, important here is the asymmetry or anisotropy of the fields $\mathbf{u}$.

4.1 Definition of "hybrid Muckenhaupt class"

We have to take into account the anisotropic nature of Problem (31) in terms of weights: q-weight functions chosen according to analogous conditions with (6) are required to ensure the independence on the angular variable θ, but non only, their restrictions to the third variable must belong to the *one-sided Muckenhoupt class* corresponding to *one-sided maximal operators* on the real line.

In order to describe such convenient weight functions, we first modify the referred set of cubes with edges parallel to the coordinate axes x_1, x_2, x_3 in the definition of A_q. So, denoting by $\mathcal{J}$ the set of all bounded intervals, (rectangles) in $\mathbb{R}^3$ we consider the following subclass of A_q (always denoted by A_q in this section) given by all functions $w \in L^1_{loc}(\mathbb{R}^3)$, almost everywhere positive, for which exists a constant $C > 0$ such that the number

$$A_q(w) = \sup_{R \in \mathcal{J}} (|R|^{-1} w(R))(|R|^{-1} w'(R))^{q-1} \le C, \ 1 < q < \infty.$$

We need to introduce some notations: We take $w_r(\theta, x_3) := w(r\cos\theta,\ r\sin\theta,\ x_3)$ for $r = |(x_1, x_2)|$. We assume w_r to be θ-independent for a.a. $r \in [0, \infty)$. Let

$M_h^+ w_r$ and $M_h^- w_r$ be defined by $M_h^+ w_r(x_3) = h^{-1}\int_{x_3}^{x_3+h} w_r(t)\,\mathrm{d}t$, $M_h^- w_r(x_3) = h^{-1}\int_{x_3-h}^{x_3} w_r(t)\,\mathrm{d}t$, and let

$$A_q^-(w_r) := \sup_{x_3\in\mathbb{R}}\ \sup_{h>0} \left(M_h^+ w_r(x_3)\right)\left(M_h^- w_r'(x_3)\right)^{q-1}.$$

The following definition will be used

Definition 2. *(i) An a.e. positive weight function w from $L^q_{\mathrm{loc}}(\mathbb{R}^3)$, $1 < q < \infty$ belongs to A_q^- if the following three conditions are satisfied:*

$$w \in A_q,\ \ w_r \ \text{ is independent on } \theta, \ \text{ and}$$
$$\operatorname*{ess\,sup}_{r\geq 0}\ A_q^-(w_r) < \infty.$$

(ii) Finally $\widetilde{A}_q^-$ will be the set of all weight functions satisfying the following q-conditions

$$\begin{aligned} 2 \leq q < \infty: &\quad w^\tau \in A^-_{\tau q/2} \quad \text{for some} \quad \tau \in [1,\infty), \\ 1 \leq q < 2: &\quad w^\tau \in A^-_{\tau q/2} \quad \text{for some} \quad \tau \in \left(\tfrac{2}{q}, \tfrac{2}{2-q}\right]. \end{aligned} \tag{32}$$

Any $w \in \widetilde{A}_q^-$ is called anisotropic q-weight function according to (32). $\widetilde{A}_q^+$ will be the corresponding set changing all signs $-$ by signs $+$ and vice versa.

4.2 Strong solution

We recall the known results from [9] about the strong solvability of the modified Oseen system in the whole space $\mathbb{R}^3$:

Theorem 7. *Let $w \in \widetilde{A}_q^-$ be given. We assume $g = 0$ and $\mathbf{f} \in \mathbf{L}^q_w(\mathbb{R}^3)$.*
(i) There exists a strong solution $\mathbf{u}, p \in \widehat{\mathbf{H}}^{2,q}_w(\mathbb{R}^3) \times \widehat{H}^{1,q}_w(\mathbb{R}^3)$ to Problem (31) that satisfies the estimates

$$\|\nabla^2\mathbf{u}\|_{q,w} + \|\nabla p\|_{q,w} \leq c\|\mathbf{f}\|_{q,w} \tag{33}$$

$$\|k\partial_3\mathbf{u}\|_{q,w} + \|\mathcal{R}\mathbf{u}\|_{q,w} \leq c\left(1 + \frac{k^5}{\nu^{5/2}|\omega|^{5/2}}\right)\|\mathbf{f}\|_{q,w} \tag{34}$$

with both constants $c = c(q,w) > 0$ independent of ν, k and ω.

(ii) Let $\mathbf{f} \in \mathbf{L}^{q_1}_{w_1}(\mathbb{R}^3) \cap \mathbf{L}^{q_2}_{w_2}(\mathbb{R}^3)$ where $w_1 \in \widetilde{A}^-_{q_1}$ and $w_2 \in \widetilde{A}^-_{q_2}$, and let $\mathbf{u}_1, \mathbf{u}_2$ together with associated pressure functions p_1, p_2 be strong solutions to Problem (31), satisfying (33) for (q_1, w_1) and (q_2, w_2) respectively.

Then there are $\alpha, \beta \in \mathbb{R}$ such that $\mathbf{u}_1$ coincides with $\mathbf{u}_2$ up to an affine linear field $\alpha\mathbf{e}_3 + \beta\omega \wedge \mathbf{x}$, and p_1 coincides with p_2 up to a constant.

Let $\mathbf{K}_{\mathcal{O}}$ be the nullspace of the system (31), with respect to the velocity field, i.e. the set of all $\mathbf{v} \in \widehat{\mathbf{H}}^{2,q}_{w,\sigma}(D)$, $S_{\omega,\nu,k}\,\mathbf{v} + \nabla\pi = \mathbf{0}$, $\mathbf{v}|_{\partial D} = \mathbf{0}$, for some $\pi \in \widehat{H}^{1,q}_{w}(D)$; it has the finite dimension 2 independently on q and w. $\mathcal{K}_{\mathcal{O}}$ will denote the set of $\{\mathbf{\Phi}, \Psi\}, \mathbf{\Phi} \in \mathbf{K}_{\mathcal{O}}$, Ψ being the associate pressure to $\mathbf{\Phi}$. We have also the property analogous to Proposition 2: Let $\{\mathbf{u}, p\}$ be some strong solution to system (31), ($\{\mathbf{u}, p\}$ not necessarily satisfying the estimates (33), (34)). Then at least one representant of $\{\mathbf{u}, p\} + \mathcal{K}_{\mathcal{O}}$, say $\{\mathbf{u}_*, p_*\}$, fulfills the estimate

$$\|\nabla^2\mathbf{u}_*\|_{q,w} + \|\mathcal{R}\mathbf{u}_*\|_{q,w} + \|\partial_3\mathbf{u}_*\|_{q,w} + \|\nabla p_*\|_{q,w} \leq C\|\mathbf{f}\|_{q,w}\ .$$

Remark 9. *Concerning the nonhomogeneous case with $g \neq 0$, let us mention that similarly as in [9] we can get the following estimate*

$$\|\nabla^2\mathbf{u}, \nabla p\|_{q,w} + \|\mathcal{R}\mathbf{u}\|_{q,w} \leq c\|\mathbf{f}, \nabla g, (\omega \wedge \mathbf{x})g, kg\,\mathbf{e}_3\|_{q,w}. \tag{35}$$

Remark 10. *Let us remark that Theorem 7 was stated in [9] for more general type of weights which are refereed to cubes and that we need restriction to weights referred to rectangles only for the estimate (34). Since our aim hereafter is to introduce the notion of very weak solution for Problem (31), we have to consider this subclass $\widetilde{A}^-_q$ of weights.*

4.3 Very weak solution

Denoting

$$S_{\omega,\nu,k} = S_{\omega,\nu} + k\partial_3 \tag{36}$$

and $S^{'}_{\omega,\nu,k} = S_{-\omega,\nu,-k}$, and denoting the $\langle\cdot,\cdot\rangle_w$ denotes convenient dualities as in Section 3, we take the following definition

Definition 3. *Let $\mathbf{f} \in \widehat{\mathbf{H}}^{-2,q}_{w}(\mathbb{R}^3)$ and $g \in \mathcal{H}^{-1,q}_{w,0}(\mathbb{R}^3)$ be given. Then a function $\mathbf{u} \in \mathbf{L}^q_w(\mathbb{R}^3)$ is called a very weak solution to Problem (31), if the equation*

$$\left\langle \mathbf{u}, S^{'}_{\omega,\nu,k}\boldsymbol{\phi} - \nabla\psi \right\rangle_w = \langle \mathbf{f}, \boldsymbol{\phi}\rangle_w + \langle g, \psi\rangle_w \tag{37}$$

hold for all test functions $\phi \in \widehat{\mathbf{H}}^{2,q'}_{w',\sigma}(\mathbb{R}^3)$ with $\mathcal{R}\boldsymbol{\phi} \in \mathbf{L}^{q'}_{w'}(\mathbb{R}^3)$, $\partial_3\boldsymbol{\phi} \in \mathbf{L}^{q'}_{w'}(\mathbb{R}^3)$ and $\psi \in \widehat{H}^{1,q'}_{w'}(\mathbb{R}^3)$.

The spaces $\widehat{\mathbf{H}}^{-2,q}_{w,K_{\mathcal{O}}}(\mathbb{R}^3)$ and $\mathcal{H}^{-1,q}_{w,0,\mathcal{K}_{\mathcal{O}}}(\mathbb{R}^3)$ being introduced as the spaces analogous to Subsection 3.2, we have the result

Theorem 8. *Let $1 < q < \infty$ and w be an anisotropic q-weight function according to Theorem 7. Then, for every $\mathbf{f} \in \widehat{\mathbf{H}}^{-2,q}_{w,K_{\mathcal{O}}}(\mathbb{R}^3)$, and $g \in \mathcal{H}^{-1,q}_{w,0,\mathcal{K}_{\mathcal{O}}}(\mathbb{R}^3)$, there exists an unique very weak solution $\mathbf{u} \in \mathbf{L}^q_w(\mathbb{R}^3)$ to Problem (31) in the sense of Definition 3. Moreover, there exists an associated pressure functional $p \in \mathcal{H}^{-1,q}_{w,0}(\mathbb{R}^3)$ such that $\mathbf{u}, p$ solves, for all $\phi \in \widehat{\mathbf{H}}^{2,q'}_{w'}(\mathbb{R}^3)$ with $\mathcal{R}\boldsymbol{\phi} \in \mathbf{L}^{q'}_{w'}(\mathbb{R}^3)$, $\partial_3\boldsymbol{\phi} \in \mathbf{L}^{q'}_{w'}(\mathbb{R}^3)$, the equation*

$$\left\langle \mathbf{u}, S^{'}_{\omega,\nu,k}\phi \right\rangle_w - \langle p, div\ \phi\rangle_w = \langle \mathbf{f}, \phi\rangle_w\ .$$

The functions $\mathbf{u}$ *and* p *fulfill the inequality*

$$\|\mathbf{u}\|_{q,w} + \|p\|_{\mathcal{H}^{-1,q}_{w,0}} \le c\left(\|\mathbf{f}\|_{\widehat{H}^{-2,q}_{w}} + \|g\|_{\mathcal{H}^{-1,q}_{w,0}(\mathbb{R}^3)}\right). \tag{38}$$

Proof. We know that w' is an anisotropic q'-weight function, which belongs to $\tilde{A}^{+}_{q'}$, see [9].

For every $\mathbf{v} \in \mathbf{L}^{q'}_{w'}(\mathbb{R}^3)$, in accordance with Theorem 7, there exists a couple $\{\boldsymbol{\phi}, \psi\}$ in $\widehat{\mathbf{H}}^{2,q'}_{w',\sigma}(\mathbb{R}^3) \times \widehat{H}^{1,q'}_{w'}(\mathbb{R}^3)/\mathbb{R}$, strong solution to

$$S^{'}_{\omega,\nu,k}\boldsymbol{\phi} - \nabla\psi = \mathbf{v},\ \operatorname{div}\,\boldsymbol{\phi} = 0. \tag{39}$$

Let $\{\boldsymbol{\phi}_v, \psi_v\}$ be a representant of $\{\boldsymbol{\phi}, \psi\} + \mathcal{K}_{\mathcal{O}}$ which satisfies the estimate

$$\|\nabla^2\boldsymbol{\phi}_v\|_{q',w'} + \|\mathcal{R}\boldsymbol{\phi}_v\|_{q',w'} + \|\partial_3\psi_v\|_{q',w'} + \|\nabla\psi_v\|_{q',w'} \le C\|\mathbf{v}\|_{q',w'},$$

we now define $\mathbf{u} := \mathbf{u}(\mathbf{f}, g)$ setting

$$\langle \mathbf{u}, \mathbf{v}\rangle_w := \langle \mathbf{f}, \boldsymbol{\phi}_v\rangle_w + \langle g, \psi_v\rangle_w\,,$$

where $\mathbf{f} \in \widehat{\mathbf{H}}^{-2,q}_{w,K_{\mathcal{O}}}(\mathbb{R}^3)$ and $g \in \mathcal{H}^{-1,q}_{w,0,\mathcal{K}_{\mathcal{O}}}(\mathbb{R}^3)$ are given. Then one has

$$\begin{aligned}
|\langle \mathbf{u}, \mathbf{v}\rangle_w| &\le |\langle \mathbf{f}, \boldsymbol{\phi}_v\rangle_w + \langle g, \psi_v\rangle_w| \\
&\le \|\mathbf{f}\|_{\widehat{H}^{-2,q}_{w}}\|\nabla^2\ \boldsymbol{\phi}_v\|_{q',w'} + \|g\|_{\mathcal{H}^{-1,q}_{w,0}}\|\nabla\psi_v\|_{q',w'} \\
&\le c\left(\|\mathbf{f}\|_{\widehat{H}^{-2,q}_{w}} + \|g\|_{\mathcal{H}^{-1,q}_{w,0}}\right)\|\mathbf{v}\|_{q',w'},
\end{aligned}$$

Therefore $\mathbf{u} = \mathbf{u}(\mathbf{f}, g)$ belongs to $(\mathbf{L}^{q'}_{w'}(\mathbb{R}^3))' = \mathbf{L}^{q}_{w}(\mathbb{R}^3)$ and fulfills the stated estimate.

It remains to show that this function $\mathbf{u} = \mathbf{u}(\mathbf{f}, g)$ solves our problem. To see this, following the same arguments as for the modified Stokes system, let $\boldsymbol{\phi}, \psi$ be the test functions with $\boldsymbol{\phi} \in \widehat{\mathbf{H}}^{2,q'}_{w',\sigma}(\mathbb{R}^3)$, $\mathcal{R}\boldsymbol{\phi} \in \mathbf{L}^{q'}_{w'}(\mathbb{R}^3)$, $\partial_3\boldsymbol{\phi} \in \mathbf{L}^{q'}_{w'}(\mathbb{R}^3)$ and $\psi \in \widehat{H}^{1,q'}_{w'}(\mathbb{R}^3)$ independently. We can choose $\mathbf{v} := S^{'}_{\omega,\nu,k}\boldsymbol{\phi} - \nabla\psi \in \mathbf{L}^{q'}_{w'}(\mathbb{R}^3)$, then

$$\left\langle \mathbf{u}, S^{'}_{\omega,\nu,k}\boldsymbol{\phi} - \nabla\psi\right\rangle_w = \langle \mathbf{u}, \mathbf{v}\rangle_w = \langle \mathbf{f}, \boldsymbol{\phi}\rangle_w + \langle g, \psi\rangle_w\,.$$

Therefore $\mathbf{u}$ is a very weak solution of our problem according to the definition.

To prove its uniqueness, let $\mathbf{U} = \mathbf{U}(\mathbf{f}, g) \in \mathbf{L}^{q}_{w}(\mathbb{R}^3)$ be another very weak solution. We have

$$\left\langle \mathbf{U}, S^{'}_{\omega,\nu,k}\boldsymbol{\phi} - \nabla\psi\right\rangle_w = \langle \mathbf{f}, \boldsymbol{\phi}\rangle_w + \langle g, \psi\rangle_w \tag{40}$$

for all test functions $\boldsymbol{\phi} \in \widehat{\mathbf{H}}^{2,q'}_{w',\sigma}(\mathbb{R}^3)$ with $\mathcal{R}\phi \in \mathbf{L}^{q'}_{w'}(\mathbb{R}^3)$, $\partial_3\boldsymbol{\phi} \in \mathbf{L}^{q'}_{w'}(\mathbb{R}^3)$ and $\psi \in \widehat{H}^{1,q'}_{w'}(\mathbb{R}^3)$. Thus

$$\left\langle \mathbf{U} - \mathbf{u}, S^{'}_{\omega,\nu}\boldsymbol{\phi} - \nabla\psi\right\rangle_w \ = \ 0.$$

$S^{'}_{\omega,\nu}\boldsymbol{\phi} - \nabla\psi$ being an arbitrary element from $\mathbf{L}^{q'}_{w'}(\mathbb{R}^3)$, it follows that $\mathbf{U} - \mathbf{u} = \mathbf{0}$ in $\mathbf{L}^{q}_{w}(\mathbb{R}^3)$. □

Appendix 1

The concept of very weak solutions for Stokes or Navier–Stokes equations was introduced by Giga in 1981, see [18], by Amrouche and Girault in 1994 in a domain class $C^{1,1}$, see [4]. More recently this concept was extended by Amrouche, Rodríguez - Bellido, see [3], Galdi, Simader, Farwig, Kozono and Sohr, see e.g. [11, 17] to a setting in classical L^q-spaces. The only existence of the inhomogeneous divergence of the velocity in some L^r-spaces is required with convenient choice of exponents r and q. For the non-steady case we can refer to the work of Amann in 2000-02 in the setting of Besov spaces, see [2].

We recall that, in general, a very weak solution have no differentiability, neither with respect to the space variable nor the time one, and no finite energy. So, a very weak solution is not necessarily a weak solution. But in case of Serrin's class of uniqueness the very weak solution will coincide with a weak solution.

The notion of very weak solution has been generalized by Schumacher see [29, 30] to a setting in weighted Lebesque and Bessel potential spaces using arbitrary Muckenhoupt weights. Further, a very weak solution may be generalized so that traces on the boundary can be defined, see [29].

Let us mention that during last years the problem of flow around rotating body was studied by several authors and we will just refer some of them see [7, 15, 8, 9, 20, 22, 27, 16, 25, 6]

Let us explain the problem which arise from the motion of fluid around a rotating body. Let $D(t) \subset \mathbb{R}^3$ be given, the time-dependent exterior domain past a rotating body. We assume that $D(t)$ is filled with a viscous incompressible fluid modelled by the Navier-Stokes equations with the velocity $\mathbf{v}_\infty$ at infinity. So, given the coefficient of viscosity $\nu > 0$ and an external force $\tilde{\mathbf{f}} := \tilde{\mathbf{f}}(\mathbf{y}, t)$, we are looking for the velocity $\tilde{\mathbf{v}} := \tilde{\mathbf{v}}(\mathbf{y}, t)$ and the pressure $\tilde{q} := \tilde{q}(\mathbf{y}, t)$ solving the nonlinear system

$$\begin{aligned} \tilde{\mathbf{v}}_t - \nu\Delta\tilde{\mathbf{v}} + (\tilde{\mathbf{v}}\cdot\nabla)\tilde{\mathbf{v}} + \nabla\tilde{q} &= \tilde{\mathbf{f}} && \text{in } D(t),\ t>0,\\ \operatorname{div}\,\tilde{\mathbf{v}} &= 0 && \text{in } D(t),\ t>0,\\ \tilde{\mathbf{v}}(\mathbf{y},t) &= \boldsymbol{\omega}\wedge\mathbf{y} && \text{on } \partial D(t),\ t>0,\\ \tilde{\mathbf{v}}(\mathbf{y},t) &\to \mathbf{v}_\infty && \text{as } |\mathbf{y}|\to\infty\,. \end{aligned} \tag{41}$$

Due to the rotation of the body with the angular velocity $\boldsymbol{\omega}$, we have

$$D(t) = O_\omega(t)D,$$

where $D \subset \mathbb{R}^3$ is a fixed exterior domain and $O_\omega(t)$ denotes the orthogonal matrix

$$O_\omega(t) = \begin{pmatrix} \cos|\boldsymbol{\omega}|t & -\sin|\boldsymbol{\omega}|t & 0\\ \sin|\boldsymbol{\omega}|t & \cos|\boldsymbol{\omega}|t & 0\\ 0 & 0 & 1 \end{pmatrix} \tag{42}$$

After the change of variables $\mathbf{x} := O_\omega(t)^T\mathbf{y}$ and passing to the new functions $\mathbf{u}(\mathbf{x}, t) := O_\omega^T\tilde{\mathbf{v}}(\mathbf{y}, t) - \mathbf{v}_\infty$ and $p(\mathbf{x}, t) := \tilde{q}(\mathbf{y}, t)$, as well as to the force term

$\mathbf{f}(\mathbf{x},t) := O_\omega(t)^T \tilde{\mathbf{f}}(\mathbf{y},t)$, we arrive at the modified Navier–Stokes system

$$\begin{aligned} \mathbf{u}_t - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} - ((\boldsymbol{\omega}\wedge\mathbf{x})\cdot\nabla)\mathbf{u} & \\ +(O_\omega(t)^T\mathbf{v}_\infty\cdot\nabla)\mathbf{u} + \boldsymbol{\omega}\wedge\mathbf{u} + \nabla p = \mathbf{f} & \qquad \text{in } D,\ t>0, \\ \text{div } \mathbf{u} = 0 & \qquad \text{in } D,\ t>0, \\ \mathbf{u}(\mathbf{x},t) + O_\omega(t)^T\mathbf{v}_\infty = \boldsymbol{\omega}\wedge\mathbf{x} & \qquad \text{on } \partial D,\ t>0, \\ \mathbf{u}(\mathbf{x},t)\to 0 & \qquad \text{as } |\mathbf{x}|\to\infty\,. \end{aligned} \tag{43}$$

Note that, because of the new coordinate system attached to the rotating body, equation $(43)_{1,2}$ contains three new terms, the classical Coriolis force term $\boldsymbol{\omega}\wedge\mathbf{u}$ (up to a multiplicative constant) and the terms $((\boldsymbol{\omega}\wedge\mathbf{x})\cdot\nabla)\mathbf{u}$ and $(O_\omega(t)^T\mathbf{v}_\infty\cdot\nabla)\mathbf{u}$ which are not subordinate to the Laplacian of $\mathbf{u}$ in unbounded domains (even with small values of $|\boldsymbol{\omega}|$).
Note that stationary solutions to Problem (43) correspond to time-periodic solutions to the original problem (41) (see e.g. [20] and the cited references there). Linearizing (43) in $\mathbf{u}$ at $\mathbf{u}=\mathbf{0}$ we arrive in the steady situation at the modified Oseen system of the introduction, with the inhomogeneous boundary condition on ∂D.

Appendix 2

The next Lemma is related to a weighted Poincaré inequality.

A mapping $C(.) : A_q \mapsto \mathbb{R}_+$ is called A_q-consistently increasing, if, for all $c\in\mathbb{R}_+$, $\sup\,\{C(\omega) : \omega\in A_q, A_q(\omega)\le c\} < \infty$.
It is called A_q-consistently decreasing, if $\frac{1}{C(.)}$ is A_q -consistently increasing.

Lemma 4. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$, and $w\in A_q$, $q\in(1,\infty)$. Suppose $h(.)$ is a continuous seminorm on $H^{1,1}(\Omega)$ such that $h(C)=0$ implies $C=0$ for constant functions C. There exists an A_q-consistently increasing constant function C such that*

$$\|u\|_{q,w} \le C\|\nabla u\|_{q,w} \textit{ for every } u\in\widehat{H}^{1,q}_w(\Omega) \textit{ with } h(u)=0 \textit{ or } u\in\widehat{H}^{1,q}_{w,0}(\Omega). \tag{44}$$

Proof. This result has been proved by Fröhlich in [12]. The following property is also established : There exists an A_q-consistently constant function C such that $|||u|||_{2,q,w} \le c\|\nabla^2 u\|_{q,w}$, for every $u\in\widehat{H}^{2,q}_w(\Omega)$ with $u|_\Gamma = 0$ where Γ is a connected component of $\partial\Omega$. □

Concluding remarks.

For both rotating Stokes and Oseen problems, a rigourous mathematical treatment is done here by choosing appropriate weighted $\mathbf{L}^q$-spaces and using the concept of very weak solution (definitions in Sections 3.2 and 4.3 with classical integral formulations). The duality arguments rely on the existence theory of the respective strong solutions, based on Littlewood-Paley decomposition.

This leads to the Muckenhaupt class of $A_{q/2}$ weights. Moreover, for the Oseen operator, the natural anisotropic weights do not belong to A_q: it turned out to be necessary that the weights are also Muckenhaupt type in the direction of the axis of rotation.

A second group of problems concerns the rotating Oseen and Navier-Stokes systems in exterior domains: we have to investigate their very weak solutions. For this purpose, and to get the important estimates, we need continuous embedding properties (generalized weighted results). A forthcoming paper will collect our results [23].

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