INSTITUTE of MATHEMATICS

ACADEMY of SCIENCES of the CZECH REPUBLIC

On weak solutions to a diffuse interface model of a binary mixture of compressible fluids

Eduard Feireisl

Preprint No. 41-2014

PRAHA 2014

On weak solutions to a diffuse interface model of a binary mixture of compressible fluids

Eduard Feireisl *

Institute of Mathematics of the Academy of Sciences of the Czech Republic
Žitná 25, 115 67 Praha 1, Czech Republic

Abstract

We consider the Euler-Cahn-Hilliard system proposed by Lowengrub and Truskinovsky describing the motion of a binary mixture of compressible fluids. We show that the associated initial-value problem possesses infinitely many global-in-time weak solutions for any finite energy initial data. A modification of the method of convex integration is used to prove the result.

Key words: Euler-Cahn-Hilliard system, weak solution, diffuse interface model

1 Introduction

The mathematical theory of fluid dynamics based on continuum models faces several fundamental difficulties including the problem of well-posedness of the Euler and Navier-Stokes systems on large time intervals and arbitrary initial data. The situation has become even more delicate in the light of the recent ground breaking discoveries of DeLellis and Székelyhidi [5], [6] based on the technique of the so-called *convex integration*. There is a body of evidence that the classical admissibility criteria of well-posedness of hyperbolic systems of conservation laws based on the Second law of thermodynamics fail to identify a unique weak solution. On the other hand, however, the weak solutions are the only available given the inevitable presence of discontinuities - shock waves - that may develop in a finite time regardless the smoothness of initial data.

*Eduard Feireisl acknowledges the support of the GAČR (Czech Science Foundation) project 13-00522S in the general framework of RVO: 67985840

Convex integration shows how vulnerable with respect to apparently non-physical perturbations are the weak solution at the moment we are not able to control the mechanical energy dissipation. Rather surprisingly but by the same token, the same method can be used to "construct" global in time weak solutions to a vast class of sofar open problems, among which the compressible Euler system (see Chiodaroli [2], Chiodaroli, DeLellis and Kreml [3]). In the light of these arguments, *uniqueness* rather than *existence* of solutions to problems of gas dynamics became an issue.

Very roughly indeed, the method proposed by DeLellis and Székelyhidi [5] for hyperbolic problems of the form

$$\partial_t \mathbf{v} + \mathrm{div}_x \mathbb{F}(\mathbf{v}) = 0 \tag{1.1}$$

works as follows:

- In the spirit of the seminal work of Tartar [12], the system (1.1) is understood as a linear problem
$$\partial_t \mathbf{v} + \mathrm{div}_x \mathbb{F} = 0,$$
supplemented with a (nonlinear) constitutive relation
$$\mathbb{F} = \mathbb{F}(\mathbf{v}). \tag{1.2}$$

- The constitutive relation (1.2) is replaced by an "implicit" one
$$\mathbb{F} = \mathbb{F}(\mathbf{v}) \ \Leftrightarrow |\mathbf{v}|^2 = G(\mathbf{v}, \mathbb{F}), \tag{1.3}$$
where G is a convex functional such that
$$|\mathbf{v}|^2 \leq G(\mathbf{v}, \mathbb{F}) \text{ for any admissible choice of } \mathbf{v}, \mathbb{F}. \tag{1.4}$$

- The original problem is relaxed to finding "subsolutions" satisfying the inequality (1.4), more specifically,
$$|\mathbf{v}|^2 \leq G(\mathbf{v}, \mathbb{F}) \leq \overline{E}, \tag{1.5}$$
where $\overline{E}$ is a prescribed "energy".

- The desired equality (1.3) is achieved by modulating suitable oscillations on a given subsolution. The existence of (infinitely) solutions is obtained via Baire's category argument. In addition, the solutions satisfy $|\mathbf{v}|^2 = \overline{E}$.

In this paper, we extend the method to the case when the energy $\overline{E} = \overline{E}[\mathbf{v}]$ is allowed to depend on the field $\mathbf{v}$. To be more specific, we focus on a physically motivated regularization of the Euler equations proposed in the seminal paper by Lowengrub and Truskinovsky [9]. The model describes the motion of a mixture of two immiscible compressible fluids in terms of the density $\varrho = \varrho(t,x)$, the macroscopic velocity $\mathbf{u} = \mathbf{u}(t,x)$, and the concentration difference $c = c(t,x)$, where $t \in (0,T)$ is the time and $x \in \Omega \subset R^3$ the reference Eulerian coordinate. The fluid is described by means of the standard Euler system coupled with the Cahn-Hilliard equation describing the evolution of c. The resulting system of equations reads:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{1.6}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p_0(\varrho, c) = \mathrm{div}_x \left(\varrho \nabla_x c \otimes \nabla_x c - \frac{\varrho}{2} |\nabla_x c|^2 \mathbb{I} \right), \tag{1.7}$$

$$\partial_t(\varrho c) + \mathrm{div}_x(\varrho c \mathbf{u}) = \Delta \left(\mu_0(\varrho, c) - \frac{1}{\varrho} \mathrm{div}_x \left(\varrho \nabla_x c \right) \right), \tag{1.8}$$

where

$$p_0(\varrho, c) = \varrho^2 \frac{\partial f_0(\varrho, c)}{\partial \varrho}, \ \mu_0(\varrho, c) = \frac{\partial f_0(\varrho, c)}{\partial c} \tag{1.9}$$

for a given free energy function f_0.

Although the physical relevance of such a model might be dubious because of the absence of viscous stress in the momentum equation (1.7), the system (1.7 - 1.9) can be seen either as a regularization of the Euler equations or as an inviscid limit of the associated Navier-Stokes-Cahn-Hilliard system. From the mathematical view point, the system is neither purely hyperbolic nor parabolic as the dissipation mechanism acts in a very subtle way through the coupling of the Euler and the Cahn-Hilliard systems. To the best of our knowledge, such a problem has never been studied in the framework of weak solution, in particular in the physically relevant 3-D setting.

As we show in this paper, the problem (1.6-1.8) is very close to the (inviscid) Euler system as the existence of weak solutions may be obtained by the method of convex integration. Thus, despite the presence of diffusion in (1.8), the system still admits large oscillations of the velocity field. Our method is based on a modification of DeLellis and Székelyhidi's argument, namely the energy E in (1.5) is replaced by a functional depending on the solution $\mathbf{v}$ itself provided the mapping $\mathbf{v} \mapsto E(\mathbf{v})$ is continuous or at least upper semi-continuous with respect to the topology of *uniform* convergence. The maximal regularity estimates for non-constant coefficients evolutionary equations play a crucial in our arguments. A similar method has been applied to the Euler-Fourier system in [4]. As we shall see below, the present problem is much more delicate than [4] as we have to establish compactness of the *gradient* terms appearing on the right-hand side of (1.7).

The paper is organized as follows. In Section 2, we recall the standard definition of *weak solution* to the problem (1.6 - 1.8) and state our main result. In Section 3, the system (1.6 - 1.8) is rewritten in a form suitable for application of the machinery of convex integration. Global existence of (infinitely many) weak solutions is proved in Section 3. We finish by a brief discussion on possible extensions of the theory in Section 5.

2 Weak solutions, main result

We consider the system (1.6 - 1.8) in the physically relevant 3-D setting. For the sake of simplicity, we impose the space-periodic boundary conditions. Accordingly, the physical domain $\Omega \subset R^3$ is the flat torus,

$$\Omega = \left([0,1]\Big|_{\{0,1\}} \right)^3 .$$

For technical reasons, we also impose certain restrictions on the function f_0, the latter being typically of the form

$$f_0(\varrho, c) = \alpha_1 \frac{1-c}{2} \log\left(\frac{1-c}{2}\right) + \alpha_2 \frac{1+c}{2} \log\left(\frac{1+c}{2}\right) - \beta c^2 + \log(\varrho)\left(\alpha_1 \frac{1-c}{2} + \alpha_2 \frac{1+c}{2}\right),$$

where $\alpha_1,\ \alpha_2,\ \beta > 0$ are positive constants. To avoid possible difficulties with singularities, we replace the function

$$\alpha_1 \frac{1-c}{2} \log\left(\frac{1-c}{2}\right) + \alpha_2 \frac{1+c}{2} \log\left(\frac{1+c}{2}\right) - \beta c^2$$

by a smooth (bistable) potential H. Specifically, we suppose

$$f_0(\varrho, c) = H(c) + \log(\varrho)\left(\alpha_1 \frac{1-c}{2} + \alpha_2 \frac{1+c}{2}\right),\ H \in C^2(R),\ |H''(c)| \leq \overline{H} \text{ for all } c \in R^1. \tag{2.1}$$

Remark 2.1 *We immediately see the well known problem of phase field models based on the Cahn-Hilliard equation for the phase variable, namely the concentration difference c may attain values outside its natural range $[-1,1]$. This unpleasant feature can be avoided replacing the Cahn-Hilliard by Allen-Cahn equation, cf. Blesgen [1].*

2.1 Weak solutions

We shall say that a trio $[\varrho, \mathbf{u}, c]$ is a *weak solution* to the problem (1.6 -1.8) in $(0,T) \times \Omega$ if:

- the functions ϱ, $\mathbf{u}$, c belong to the class

$$\left\{ \begin{array}{c} \varrho \in C([0,T]; C^3(\Omega)), \partial_t \varrho \in C^1([0,T] \times \Omega), \\ \\ \mathbf{u} \in L^\infty((0,T) \times \Omega; R^3) \cap C_{\rm weak}([0,T]; L^2(\Omega; R^3)),\ {\rm div}_x \mathbf{u} \in L^\infty((0,T) \times \Omega), \\ \\ c \in L^p(0,T; W^{4,p}(\Omega)),\ \partial_t c \in L^p((0,T) \times \Omega) \text{ for any } 1 \leq p < \infty; \end{array} \right\} \tag{2.2}$$

- $$\inf_{(0,T)\times\Omega} \varrho > 0; \tag{2.3}$$

- the equation of continuity (1.6) is satisfied a.a. in $(0,T)\times\Omega$;

- the momentum equation (1.7) is replaced by a family of integral identities

$$\int_0^T \int_\Omega \Big(\varrho\mathbf{u}\cdot\partial_t\varphi + \varrho\mathbf{u}\otimes\mathbf{u} : \nabla_x\varphi + p_0(\varrho,c)\mathrm{div}_x\varphi \Big)\ \mathrm{d}x\ \mathrm{d}t \tag{2.4}$$

$$= \int_0^T \int_\Omega \varrho \left(\nabla_x c \otimes \nabla_x c - \frac{1}{2}|\nabla_x c|^2\mathbb{I} \right) : \nabla_x\varphi\ \mathrm{d}x\ \mathrm{d}t$$

  for any $\varphi \in C^1((0,T)\times\Omega; R^3)$;

- the Cahn-Hilliard equation (1.8) is satisfied a.a. in $(0,T)\times\Omega$.

Remark 2.2 *As a matter of fact, the attribute "weak" is appropriate only for the momentum equation (1.7), the remaining two equations being satisfied in the strong sense.*

2.2 Main result

Our main result concerning solvability of the initial-value problem for the system (1.6-1.8) reads as follows.

Theorem 2.1 *Let the potential $f_0 = f_0(\varrho, c)$ be given by (2.1).*

Then for any choice of initial conditions

$$\varrho(0,\cdot) = \varrho_0 \in C^3(\Omega),\ \inf_\Omega \varrho_0 > 0,\ \mathbf{u}(0,\cdot) = \mathbf{u}_0 \in C^3(\Omega; R^3),\ c(0,\cdot) = c_0 \in C^2(\Omega), \tag{2.5}$$

the problem (1.6 - 1.9) admits infinitely many weak solutions in $(0,T)\times\Omega$ satisfying (2.5).

The rest of the paper is devoted to the proof of Theorem 2.1. The physical relevance of the result will be discussed in Section 5.

3 Apparatus of convex integration, oscillatory lemma

We start by rewriting the problem to fit the framework of convex integration. First we decompose

$$\varrho \mathbf{u} = \mathbf{v} + \nabla_x \Phi,\ \mathrm{div}_x \mathbf{v} = 0,\ \int_\Omega \Phi \ \mathrm{d}x = 0$$

where $\nabla_x \Phi$ may be seen as the gradient part in the Helmholtz projection. Similarly, we write

$$\varrho_0 \mathbf{u}_0 = \mathbf{v}_0 + \nabla_x \Phi_0. \tag{3.1}$$

Accordingly, the system (1.6-1.8) reads

$$\partial_t \varrho + \Delta \Phi = 0, \tag{3.2}$$

$$\partial_t \left(\mathbf{v} + \nabla_x \Phi \right) + \mathrm{div}_x \left(\frac{(\mathbf{v} + \nabla_x \Phi) \otimes (\mathbf{v} + \nabla_x \Phi)}{\varrho} \right) + \nabla_x p_0(\varrho, c) = \mathrm{div}_x \left(\varrho \nabla_x c \otimes \nabla_x c - \frac{\varrho}{2} |\nabla_x c|^2 \mathbb{I} \right), \tag{3.3}$$

$$\varrho \partial_t c + \mathbf{v} \cdot \nabla_x c + \nabla_x \Phi \cdot \nabla_x c = -\Delta \left(\frac{1}{\varrho} \mathrm{div}_x \left(\varrho \nabla_x c \right) \right) + \Delta \left(H'(c) + \frac{\alpha_2 - \alpha_1}{2} \log(\varrho) \right). \tag{3.4}$$

3.1 Density ansatz

Next, we fix the density ϱ in such a way that

$$\varrho \in C([0,T], C^3(\Omega)),\ \partial_t \varrho \in C^1([0,T] \times \Omega),$$

$$\int_\Omega \varrho(t, \cdot) \ \mathrm{d}x = \int_\Omega \varrho_0 \ \mathrm{d}x,\ \varrho(0, \cdot) = \varrho_0,\ \partial_t \varrho(0, \cdot) = -\Delta \Phi_0,\ \inf_{[0,T] \times \Omega} \varrho > 0,$$

where Φ_0 is the potential determined by (3.1).

Next, we identify the potential Φ is the unique solution of the elliptic problem

$$-\Delta \Phi(t, \cdot) = \partial_t \varrho(t, \cdot),\ \int_\Omega \Phi(t, \cdot) \ \mathrm{d}x = 0.$$

Having fixed ϱ, Φ we observe that (3.2) holds and that ϱ obviously satisfies the relevant initial condition in (2.5).

Remark 3.1 *A striking feature of the method, already exploited by Chiodaroli [2], is that the density ϱ can be chosen in an almost arbitrary way in contrast with c that will be computed on the basis of ϱ and* $\mathbf{u}$.

3.2 Cahn-Hilliard equation

With ϱ, Φ fixed, we claim that the equation (3.4), supplemented with the initial condition $c(0,\cdot) = c_0$, admits a unique solution $c = c[\mathbf{v}]$ in the class specified in (2.2) for any given

$$\mathbf{v} \in L^\infty((0,T)\times\Omega;R^3)\cap C^1((0,T)\times\Omega;R^3),\ \mathrm{div}_x\mathbf{v} = 0.$$

Indeed uniqueness for a given $\mathbf{v}$ can be seen by taking the difference $c_1 - c_2$ of two possible solutions and integrating the difference of the corresponding equations multiplied on $c_1 - c_2$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \varrho|c_1-c_2|^2\ \mathrm{d}x + \int_\Omega |\Delta(c_1-c_2)|^2\ \mathrm{d}x \tag{3.5}$$

$$= \int_\Omega \frac{\nabla_x\varrho}{\varrho}\cdot\nabla_x(c_1-c_2)\Delta(c_1-c_2)\ \mathrm{d}x + \int_\Omega \left(H'(c_1)-H'(c_2)\right)\Delta(c_1-c_2)\ \mathrm{d}x,$$

where, by interpolation,

$$\int_\Omega \frac{\nabla_x\varrho}{\varrho}\cdot\nabla_x(c_1-c_2)\Delta(c_1-c_2)\ \mathrm{d}x \le C_1\|\nabla_x c_1 - \nabla_x c_2\|_{L^2(\Omega;R^3)}\|\Delta c_1 - \Delta c_2\|_{L^2(\Omega)}$$

$$\le C_2\|c_1-c_2\|^{1/2}_{L^2(\Omega)}\|\Delta c_1-\Delta c_2\|^{3/2}_{L^2(\Omega)} \le \frac{1}{2}\|\Delta c_1-\Delta c_2\|^2_{L^2(\Omega)} + C_3\|c_1-c_2\|^2_{L^2(\Omega)}.$$

Moreover, by virtue of the hypothesis (2.1), the function H' is globally Lipschitz and uniqueness follows (3.5) by Gronwall's argument.

Remark 3.2 *This is the only point in the proof where we need the hypothesis (2.1).*

It is worth noting that (3.5) is in fact *independent* of the field $\mathbf{v}$. In particular, using the same argument we deduce that

$$\sup_{t\in[0,T]}\|c(t,\cdot)\|_{L^2(\Omega)} + \|c\|_{L^2(0,T;W^{2,2}(\Omega))} \le C, \tag{3.6}$$

where the bound is independent of $\mathbf{v}$.

The proof of *existence* of c for any given $\mathbf{v}$ is a routine matter and relies on the available *a priori* estimates. These will be discussed in Section 4.1 below.

3.3 Velocity equation

In the light of the previous discussion, the problem (3.2-3.4) reduces to finding a bounded measurable vector field $\mathbf{v} = \mathbf{v}(t,x)$ such that

$$\mathbf{v}(0,\cdot) = \mathbf{v}_0,\ \mathrm{div}_x \mathbf{v} = 0, \tag{3.7}$$

$$\partial_t \mathbf{v} + \mathrm{div}_x \left(\frac{(\mathbf{v} + \nabla_x \Phi) \otimes (\mathbf{v} + \nabla_x \Phi)}{\varrho} - \frac{1}{3} \frac{|\mathbf{v} + \nabla_x \Phi|^2}{\varrho} \mathbb{I} - \varrho \left(\nabla_x c[\mathbf{v}] \otimes \nabla_x c[\mathbf{v}] - \frac{1}{3} |\nabla_x c[\mathbf{v}]|^2 \right) \right) = 0, \tag{3.8}$$

$$\frac{1}{2} \frac{|\mathbf{v} + \nabla_x \Phi|^2}{\varrho} = \overline{E}[\mathbf{v}] \equiv \Lambda(t) - \frac{3}{2} \left(\frac{1}{6} |\nabla_x c[\mathbf{v}]|^2 + p_0(\varrho, c[\mathbf{v}]) + \partial_t \nabla_x \Phi \right), \tag{3.9}$$

where $c = c[\mathbf{v}]$ is the unique solution of the Cahn-Hilliard equation discussed in the preceding section and $\Lambda = \Lambda(t)$ is a suitable spatially homogeneous function to be chosen below.

3.4 Subsolutions

Let $R^{3\times 3}_{0,\mathrm{sym}}$ denote the space of all traceless symmetric matrices in R^3, with $\lambda_{\max}[\mathbb{A}]$ denoting the maximal eigenvalue of a symmetric matrix $\mathbb{A}$.

Following the strategy of DeLellis and Székelyhidi [5] , we introduce the set of subsolutions:

$$X_0 = \left\{ \mathbf{w} \ \Big| \ \mathbf{w} \in L^\infty((0,T) \times \Omega, R^3) \cap C^1((0,T) \times \Omega, R^3) \cap C_{\mathrm{weak}}(0,T; L^2(\Omega; R^3)), \right.$$

$$\mathbf{w}(0,\cdot) = \mathbf{v}_0,\ \mathrm{div}_x \mathbf{w} = 0 \text{ in } (0,T) \times \Omega,$$

$$\partial_t \mathbf{w} + \mathrm{div}_x \mathbb{V} = 0 \text{ in } (0,T) \times \Omega \text{ for some } \mathbb{V} \in C^1((0,T) \times \Omega; R^{3\times 3}_{0,\mathrm{sym}}),$$

$$\frac{3}{2} \lambda_{\max} \left[\frac{(\mathbf{w} + \nabla_x \Phi) \otimes (\mathbf{w} + \nabla_x \Phi)}{\varrho} - \mathbb{V} - \varrho \left(\nabla_x c[\mathbf{w}] \otimes \nabla_x c[\mathbf{w}] - \frac{1}{3} |\nabla_x c[\mathbf{w}]|^2 \mathbb{I} \right) \right] < \overline{E}(\mathbf{w}) - \delta$$

$$\left. \text{in } (0,T) \times \Omega \text{ for some } \delta > 0 \right\}.$$

Note that

$$\frac{1}{2} \frac{|\mathbf{w} + \nabla_x \Phi|^2}{\varrho} \tag{3.10}$$

$$\leq \frac{3}{2} \lambda_{\max} \left[\frac{(\mathbf{w} + \nabla_x \Phi) \otimes (\mathbf{w} + \nabla_x \Phi)}{\varrho} - \mathbb{V} - \varrho \left(\nabla_x c[\mathbf{w}] \otimes \nabla_x c[\mathbf{w}] - \frac{1}{3} |\nabla_x c[\mathbf{w}]|^2 \mathbb{I} \right) \right] \text{ for any } \mathbf{w},\ \mathbb{V},$$

whereas the equality holds only if

$$\mathbb{V} + \varrho \left(\nabla_x c[\mathbf{w}] \otimes \nabla_x c[\mathbf{w}] - \frac{1}{3} |\nabla_x c[\mathbf{w}]|^2 \mathbb{I} \right) = \frac{(\mathbf{w} + \nabla_x \Phi) \otimes (\mathbf{w} + \nabla_x \Phi)}{\varrho} - \frac{1}{3} \frac{|\mathbf{w} + \nabla_x \Phi|^2}{\varrho} \mathbb{I}. \tag{3.11}$$

In contrast with [5], the function $\overline{E}$ is allowed to depend on $\mathbf{w}$.

The last ingredient we need to run the convex integration machinery is the following oscillatory lemma, see [8, Lemma 3.1]:

Lemma 3.1 *Let $U \subset R \times R^3$ be a bounded open set. Suppose that*

$$\mathbf{g} \in C(U; R^3),\ \mathbb{W} \in C(U; R^{3\times 3}_{\mathrm{sym},0}),\ e, r \in C(U),\ r > 0,\ e \leq \overline{e} \text{ in } U$$

are given such that

$$\frac{3}{2}\lambda_{\max}\left[\frac{\mathbf{g}\otimes\mathbf{g}}{r} - \mathbb{W}\right] < e \text{ in } U.$$

Then there exist sequences

$$\mathbf{w}_n \in C^\infty_c(U; R^3),\ \mathbb{V}_n \in C^\infty_c(U; R^{3\times 3}_{\mathrm{sym},0}),\ n = 0, 1, \ldots$$

such that

$$\partial_t \mathbf{w}_n + \mathrm{div}_x \mathbb{V}_n = 0,\ \mathrm{div}_x \mathbf{w}_n = 0 \text{ in } R^3,$$

$$\frac{3}{2}\lambda_{\max}\left[\frac{(\mathbf{g}+\mathbf{w}_n)\otimes(\mathbf{g}+\mathbf{w}_n)}{r} - (\mathbb{W}+\mathbb{V}_n)\right] < e \text{ in } U,$$

and

$$\mathbf{w}_n \to 0 \text{ weakly in } L^2(U; R^3),\ \liminf_{n\to\infty}\int_U \frac{|\mathbf{w}_n|^2}{r}\ \mathrm{d}x\ \mathrm{d}t \geq c(\overline{e})\int_U \left(e - \frac{1}{2}\frac{|\mathbf{g}|^2}{r}\right)^2\ \mathrm{d}x\ \mathrm{d}t.$$

This lemma is a "singular" variant of similar results proved by Chiodaroli [2], and DeLellis, Székelyhidi [5]. We point out that the functions $\mathbf{g}$, $\mathbb{W}$ are continuous but not necessarily bounded on the open set U, and, similarly, r need not be bounded below away from zero. Note, however, that such generality is not really needed in the present paper.

4 Existence of infinitely many solutions

Having collected all the necessary material, we are ready to prove Theorem 2.1. To begin, fix the function $\Lambda(t)$ in $\overline{E}$ (cf. (3.9)) large enough for the space X_0 to be non-empty. As a matter of fact, we can do it is such a way that the cardinality of X_0 is infinite.

4.1 Boundedness of the set of subsolutions

The first necessary step is to show that the set X_0 is bounded in $L^\infty((0,T)\times\Omega;R^3)$. Note that this is not completely obvious as we only know from (3.10) that

$$|\mathbf{w}| \leq C\left(1+|c[\mathbf{w}]|+|\nabla_x c[\mathbf{w}]|\right) \text{ pointwise in the open set } (0,T)\times\Omega. \tag{4.1}$$

Thus, in order to conclude that (4.1) yields a uniform bound on $\mathbf{w}$ we have to derive bounds for c, $\nabla_x c$ in terms of $\mathbf{w}$.

In view of (3.6) and the standard Sobolev embedding $W^{2,2}(\Omega)\hookrightarrow W^{1,6}(\Omega)$ we already have

$$\mathbf{w} \text{ bounded in } L^2(0,T;L^6(\Omega;R^3)), \tag{4.2}$$

uniformly for all $\mathbf{w}\in X_0$. Consequently, equation (3.4) for $c=c[\mathbf{w}]$ reads

$$\partial_t c + \frac{1}{\varrho}\Delta\left(\frac{1}{\varrho}\mathrm{div}_x\left(\varrho\nabla_x c\right)\right) = -\frac{1}{\varrho}\mathbf{w}\cdot\nabla_x c + h,$$

where h is bounded in $L^2((0,T)\times\Omega)$.

Since ϱ is smooth and bounded below away from zero, the standard parabolic theory yields the estimate

$$\begin{aligned}\|\partial_t c\|_{L^2((0,T)\times\Omega)} + \|c\|_{L^2(0,T;W^{4,2}(\Omega))} &\leq C\left(1+\|\mathbf{w}\cdot\nabla_x c\|_{L^2((0,T)\times\Omega)}\right)\\ &\leq C\left(1+\|\mathbf{w}\|_{L^2(0,T;L^6(\Omega;R^3))}\|\nabla_x c\|_{L^\infty(0,T;L^3(\Omega;R^3))}\right).\end{aligned}$$

Now,

$$\|c\|_{L^\infty(0,T;W^{2,2}(\Omega))} \leq C\left(\|\partial_t c\|_{L^2((0,T)\times\Omega)} + \|c\|_{L^2(0,T;W^{4,2}(\Omega))}\right),$$

while, by interpolation,

$$\|\nabla_x c\|_{L^3(\Omega;R^3)} \leq C\|c\|^{\alpha}_{W^{2,2}(\Omega)}\|c\|^{1-\alpha}_{L^2(\Omega)} \text{ for a certain } 0<\alpha<1.$$

Thus, combining (4.2) with the previous estimates, we may infer that

$$\|\partial_t c\|_{L^2((0,T)\times\Omega)} + \|c\|_{L^2(0,T;W^{4,2}(\Omega))} \leq C,$$

in particular,

$$\sup_{t\in(0,T)}\|\nabla_x c\|_{L^6(\Omega;R^3)} \leq C;$$

whence, going back to (4.1),

$$\|\mathbf{w}\|_{L^\infty(0,T;L^6(\Omega))} \leq C$$

uniformly for $\mathbf{w} \in X_0$.

We conclude by a bootstrap argument. We already know that

$$\partial_t c + \frac{1}{\varrho}\Delta\left(\frac{1}{\varrho}\mathrm{div}_x\left(\varrho\nabla_x c\right)\right) = h,$$

where h is bounded in $L^\infty(0,T;L^2(\Omega))$. Since the elliptic generator has smooth coefficients generated by ϱ, we can evoke the general theory of $L^p - L^q$ estimates, specifically the version of Denk, Hieber, Prüss [7, Theorem 2.3] to deduce that

$$\|\partial_t c\|_{L^q(0,T;L^2(\Omega)} + \|c\|_{L^q(0,T;W^{4,2}(\Omega))} \leq C \text{ for any } 1 \leq q < \infty,$$

in particular,

$$|\nabla_x c| \leq C \text{ in } (0,T)\times\Omega,$$

yielding the desired bound

$$\|\mathbf{w}\|_{L^\infty((0,T)\times\Omega;R^3)} \leq C \tag{4.3}$$

uniformly for $\mathbf{w} \in X_0$.

With (4.3) at hand, it is a routine matter to observe that c belongs to the regularity class specified in (2.2). Moreover, we have the following compactness property

$$c[\mathbf{w}_n] \to c[\mathbf{w}],\ \nabla_x c[\mathbf{w}_n] \to \nabla_x[\mathbf{w}] \text{ uniformly in } [0,T]\times\Omega$$

$$\text{whenever} \tag{4.4}$$

$$\mathbf{w}_n \to \mathbf{w} \text{ weakly-(*) in } L^\infty((0,T)\times\Omega;R^3).$$

4.2 Proof of Theorem 2.1

Following the arguments of DeLellis and Székelyhidi [5], we introduce a functional

$$I[\mathbf{w}] = \int_0^T \int_\Omega \left(\frac{1}{2}\frac{|\mathbf{w} + \nabla_x\Phi|^2}{\varrho} - \overline{E}[\mathbf{w}]\right)\ \mathrm{d}x\ \mathrm{d}t.$$

In accordance with (3.10), $I[\mathbf{w}] < 0$ for any $\mathbf{w} \in X_0$, and, by virtue of (4.3), (4.4),

$$I : \overline{X}_0 \to [0,\infty)$$

is a convex (lower semi-continuous) functional on the closure $\overline{X}_0$ of X_0 with respect to the topology induced by $C_{\mathrm{weak}}([0,T];L^2(\Omega;R^3))$. As a consequence of Baire's category argument, the set of points of continuity of I in $\overline{X}_0$ has infinite cardinality.

Our ultimate goal is to show that all points of continuity $\mathbf{v}$ of X_0 satisfy (3.9) a.a. in $(0,T)\times\Omega$, or, equivalently,

$$I[\mathbf{v}] = 0.$$

In view of (3.10), (3.11) and the definition of X_0, this yields (3.7), (3.8) completing the proof of Theorem 2.1.

Arguing by contradiction, we suppose that $\mathbf{v}\in\overline{X}_0$ is a point of continuity of I such that

$$I[\mathbf{v}] < 0.$$

Since I is continuous at $\mathbf{v}$, there exists a sequence $\mathbf{v}_m\in X_0$ (with the associated fluxes $\mathbb{V}_m$) such that

$$\mathbf{v}_m \to \mathbf{v} \text{ in } C_{\rm weak}([0,T];L^2(\Omega;R^3)),\ I[\mathbf{v}_m]\to I[\mathbf{v}].$$

In agreement with the definition of X_0 we have

$$\frac{3}{2}\lambda_{\max}\left[\frac{(\mathbf{v}_m+\nabla_x\Phi)\otimes(\mathbf{v}_m+\nabla_x\Phi)}{\varrho} - \mathbb{V}_m - \varrho\left(\nabla_x c[\mathbf{v}_m]\otimes\nabla_x c[\mathbf{v}_m] - \frac{1}{3}|\nabla_x c[\mathbf{v}_m]|^2\mathbb{I}\right)\right]$$

$$< \overline{E}(\mathbf{v}_m) - \delta_m = \Lambda(t) - \frac{3}{2}\left(\frac{1}{6}|\nabla_x c[\mathbf{v}_m]|^2 + p_0(\varrho, c[\mathbf{v}_m]) + \partial_t\nabla_x\Phi\right) - \delta_m,\ \delta_m\to 0.$$

Fixing m we apply Lemma 3.1 choosing

$$U=(0,T)\times\Omega,\ e=\overline{E}[\mathbf{v}_m]-\frac{\delta_m}{2},\ r=\varrho,$$

$$\mathbf{g}=\mathbf{w}_m+\nabla_x\Phi,\ \mathbb{W}=\mathbb{V}_m+\varrho\left(\nabla_x c[\mathbf{v}_m]\otimes\nabla_x c[\mathbf{v}_m]-\frac{1}{3}|\nabla_x c[\mathbf{v}_m]|^2\mathbb{I}\right).$$

Note that, by virtue of the uniform bound (4.3), the energies $E[\mathbf{v}_m]$ are bounded uniformly in X_0. We consider the sequence

$$\mathbf{v}_{m,n}=\mathbf{v}_m+\mathbf{w}_{m,n},\ \text{where } \{\mathbf{w}_{m,n}\}_{n=1}^\infty \text{ is the sequence contructed in Lemma 3.1.}$$

Clearly, $\mathbf{v}_{m,n}$ and the associated fluxes $\mathbb{V}_{m,n}=\mathbb{V}_m+\mathbb{V}_{m,n}$ satisfy

$$\partial_t\mathbf{v}_{m,n}+{\rm div}_x\mathbb{V}_{m,n}=0,\ {\rm div}_x\mathbf{v}_{m,n}=0,\ \mathbf{v}_{m,n}(0,\cdot)=\mathbf{v}_0.$$

Moreover, in accordance with the conclusion of Lemma 3.1,

$$\frac{3}{2}\lambda_{\max}\left[\frac{(\mathbf{v}_{m,n}+\nabla_x\Phi)\otimes(\mathbf{v}_{m,n}+\nabla_x\Phi)}{\varrho} - \mathbb{V}_{m,n} - \varrho\left(\nabla_x c[\mathbf{v}_m]\otimes\nabla_x c[\mathbf{v}_m] - \frac{1}{3}|\nabla_x c[\mathbf{v}_m]|^2\mathbb{I}\right)\right]$$

$$< \overline{E}(\mathbf{v}_m) - \delta_m.$$

Consequently, by virtue of (4.4), we may assume that for all $n \geq n(m)$ large enough, we have $\mathbf{v}_{m,n} \in X_0$, and, extracting a suitable diagonal subsequence, we may suppose

$$\mathbf{v}_{m,n(m)} \to \mathbf{v} \text{ in } C_{\rm weak}([0,T]; L^2(\Omega; R^3)), \text{ in particular } I[\mathbf{v}_{m,n(m)}] \to I[\mathbf{v}] \text{ as } m \to \infty. \tag{4.5}$$

Finally, evoking again the conclusion of Lemma 3.1, we observe that the sequence $\mathbf{v}_{m,n(m)}$ can be taken in such a way that

$$\liminf_{m\to\infty} I[\mathbf{v}_{m,n(m)}] = \liminf_{m\to\infty} \int_0^T \int_\Omega \left(\frac{1}{2} \frac{|\mathbf{v}_m + \mathbf{w}_{m,n(m)} + \nabla_x \Phi|^2}{\varrho} - \overline{E}[\mathbf{w}_{m,n(m)}] \right) \ {\rm d}x \ {\rm d}t$$

$$= \lim_{m\to\infty} \int_0^T \int_\Omega \left(\frac{1}{2} \frac{|\mathbf{v}_m + \nabla_x \Phi|^2}{\varrho} - \overline{E}[\mathbf{w}_{m,n(m)}] \right) \ {\rm d}x \ {\rm d}t + \liminf_{m\to\infty} \int_0^T \int_\Omega \frac{1}{2} \frac{|\mathbf{w}_{m,n(m)}|^2}{\varrho} \ {\rm d}x \ {\rm d}t$$

$$\geq I[\mathbf{v}] + C \liminf_{m\to\infty} \int_0^T \int_\Omega \left(E[\mathbf{v}_m] - \delta_m - \frac{1}{2} \frac{|\mathbf{w}_m + \nabla_x \Phi|^2}{\varrho} \right)^2 \ {\rm d}x \ {\rm d}t = I[\mathbf{v}] + C(T, |\Omega|)|I[\mathbf{v}]|^2,$$

which is compatible with (4.5) only if $I[\mathbf{v}] = 0$.

We have proved Theorem 2.1.

5 Discussion

Apparently, at least some if not all solutions the existence of which is claimed in Theorem 2.1 are not physically admissible. Similarly to their counterparts constructed by DeLellis and Székelyhidi [5], Scheffer [10], Shnirelman [11] for the incompressible Euler system, they violate the First law of thermodynamics in the sense that the total energy of the system,

$$\mathcal{E}_{\rm tot}(\varrho, \mathbf{u}, c) \equiv \int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \frac{1}{2} \varrho |\nabla_x c|^2 + \varrho f_0(\varrho, c) \right) \ {\rm d}x$$

experiences a positive jump at the initial instant $t = 0$,

$$\liminf_{t\to 0+} \mathcal{E}_{\rm tot}(\varrho, \mathbf{u}, c)(t) > \mathcal{E}_{\rm tot}(\varrho_0, \mathbf{u}_0, c_0).$$

On the other, however, the method of the present paper can be used, exactly as in [4], to produce the following result:

Theorem 5.1 *Under the hypotheses of Theorem 2.1, suppose we are given $T > 0$ and the initial data*

$$\varrho(0,\cdot) = \varrho_0 \in C^3(\Omega), \ \inf_{\Omega} \varrho_0 > 0, \ c(0,\cdot) = c_0 \in C^2(\Omega). \tag{5.1}$$

Then there exists $\mathbf{u}_0 \in L^\infty(\Omega; R^3)$ such that the initial-value problem (1.6-1.8), (5.1),

$$\mathbf{u}(0,\cdot) = \mathbf{u}_0$$

admits infinitely many weak solutions in $(0,T) \times \Omega$ satisfying the energy inequality

$$\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \frac{1}{2}\varrho|\nabla_x c|^2 + \varrho f_0(\varrho, c) \right)(t,\cdot)\ \mathrm{d}x \leq \int_\Omega \left(\frac{1}{2}\varrho_0|\mathbf{u}_0|^2 + \frac{1}{2}\varrho_0|\nabla_x c_0|^2 + \varrho_0 f_0(\varrho_0, c_0) \right)\ \mathrm{d}x$$

for a.a. $t \in (0,T)$.

With the tools developed in this paper, the proof of Theorem 5.1 can be carried over in the same way as [4, Theorem 4.2].

It would be interesting to identify the class of the initial data $\mathbf{u}_0$ for which Theorem 5.1 applies, in particular, whether or not such data may be attained by a regular solution in a finite time. The recent examples concerning solutions of the Riemann problem for the compressible Euler system indicate that it might be possible, see Chiodaroli, DeLellis, and Kreml [3].

References

[1] T. Blesgen. A generalization of the Navier-Stokes equations to two-phase flow. *J. Phys. D Appl. Phys.*, **32**:1119–1123, 1999.

[2] E. Chiodaroli. A counterexample to well-posedness of entropy solutions to the compressible Euler system. 2012. Preprint.

[3] E. Chiodaroli, C. DeLellis, and O. Kreml. Global ill-posedness of the isentropic system of gas dynamics. 2012. Preprint.

[4] E. Chiodaroli, E. Feireisl, and O. Kreml. On the weak solutions to the equations of a compressible heat conducting gas. *Annal. Inst. Poincaré, Anal. Nonlinear.*, 2014. To appear.

[5] C. De Lellis and L. Székelyhidi, Jr. On admissibility criteria for weak solutions of the Euler equations. *Arch. Ration. Mech. Anal.*, **195**(1):225–260, 2010.

[6] C. De Lellis and L. Székelyhidi, Jr. The *h*-principle and the equations of fluid dynamics. *Bull. Amer. Math. Soc. (N.S.)*, **49**(3):347–375, 2012.

[7] R. Denk, M. Hieber, and J. Prüss. Optimal $L^p - L^q$-estimates for parabolic boundary value problems with inhomogenous data. *Math. Z.*, **257**:193–224, 2007.

[8] D. Donatelli, E. Feireisl, and P. Marcati. Well/ill posedness for the Euler-Korteweg-Poisson system and related problems. *Commun. Partial Differential Equations*, 2014. Submitted.

[9] J. Lowengrub and L. Truskinovsky. Quasi-incompressible Cahn-Hilliard fluids and topological transitions. *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.*, 454(1978):2617–2654, 1998.

[10] V. Scheffer. An inviscid flow with compact support in space-time. *J. Geom. Anal.*, **3**(4):343–401, 1993.

[11] A. Shnirelman. Weak solutions of incompressible Euler equations. In *Handbook of mathematical fluid dynamics, Vol. II*, pages 87–116. North-Holland, Amsterdam, 2003.

[12] L. Tartar. Compensated compactness and applications to partial differential equations. *Nonlinear Anal. and Mech., Heriot-Watt Sympos., L.J. Knopps editor, Research Notes in Math 39, Pitman, Boston*, pages 136–211, 1975.