PURE EQUILIBRIUM STRATEGIES IN MULTI-UNIT AUCTIONS WITH PRIVATE VALUE BIDDERS

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Abtract

The paper examines a general class of multi-unit auctions. The class of games investigated includes uniform-price, pay-your-bid, all-pay and Vickrey auctions as special cases. The seller offers k identical units of goods and sets the minimum accepted bid. Bidders have atomless valuation distributions and they submit up to k bids. For this class, the existence of Nash equilibrium in a measurable strategy space and weakly increasing pure strategy space is proven. In many cases any equilibrium strategies can be modified in such a way that they form a pure strategy equilibrium. Properties of standard strategies in multi-unit auctions are analyzed.

Abstrakt

Studie se zabývá obecnou třídou aukcí, ve které prodávající nabízí několik jednotek daného zboží. V této obecné třídě se nachází i holandská, americká a Vickreyho aukce jako jednotlivé příklady. Prodávající nabízí k prodeji *k* s tejných jednotek zboží a určí minimální prodejní cenu za každou jednotku (nejvyšší výnos). Každý účastník aukce na straně poptávky, má spojité náhodné rozložení hodnot pro jednotky zboží a objedná nejvýše *k* jednotek daného zboží. Pro takovouto třídu aukcí je ukázáno, že Nashova rovnovážná strategie existuje, pokud účastníci aukce na straně poptávky volí strategie z prostoru měřitelných funkcí a nebo z prostoru čistých neklesajících funkcí. V mnoha případech Nashovy smíšené rovnovážné strategie mohou být upravenyč tak, že výsledné strategie tvoří Nashovy čisté rovnovážné strategie. Základní vlastnosti běžných strategií pro aukcí s více objekty jsou diskutovány.

JEL Classification: D44.
Keywords: multiple-unit auction, existence of equilibrium in discontinuous games, all-pay, pay-your-bid and uniform-price auctions, auctions with reservation price.

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1 INTRODUCTION

Until recently multi–unit auctions have received less attention than single-unit auctions in theoretical studies although Internet auctions of multiple units of goods and auctions of bundles of licences for mobile phone systems are ubiquitous. The theoretical results on multi-unit auctions are not as straightforward and powerful as those for single-unit auctions. This paper is an addendum to an enlarging literature focusing on existence of equilibrium in multi-unit auctions (see McAdams (2006), Jackson and Swinkels (2005), Athey (2001) and others).

In this paper, multi-unit auctions with independent private values and sellers' reservation prices are analyzed. I focus on a class of auctions that includes uniform-price, "Dutch", Vickrey, all-pay, and pay-your-bid auctions. I first prove the existence of equilibria for auctions in this class (which is defined by assumptions A1 - A5 detailed below).

In auctions of non-divisible goods, bidders' payoffs are not continuous. In the literature on equilibria in discontinuous games Dasgupta and Maskin (1986), Reny (1999) and Simon (1987), the class of multi-unit auctions has posed particular challenges. Lizzeri and Persico (2000), Amman and Leininger (1996) and Krishna and Morgan (1997) provide insightful analyses of classes of single-unit auctions using differential equations. This approach is difficult to extend to multi-unit auctions because we know less about systems of vector differential equations than we know about scalar differential equations. Below I therefore propose a different approach. Specifically, I prove the existence of an equilibrium in which bidders bid below their expected values. I investigate the shape of the payoff function at discontinuities that result from ties. A tie occurs when two or more submitted bids are equal but the seller, while satisfying at least one of them, can not satisfy all of the equal bids.

Consider a strategy profile of two bidders each of them submitting a tied bid with positive probability. If the profile is an equilibrium, then either a) both bidders are indifferent whether winning or losing the tie, or b) with probability 1 the tie-breaking rule makes the winner that bidder who prefers to win the tie and the loser that bidder who does not prefer to win the tie (see example 4 in Section 3 below). Otherwise, since the payoff function jumps at the discontinuity, the tied bidder has incentive to deviate from her strategy negligibly to achieve almost the same payoff as if winning or losing the tie.

In many economic applications, it is natural to use a random rule to break a tie. Clearly, by its very nature such a rule does, however, not allow to win or lose a tie with probability 1 or 0. It is, however, possible in some circumstances to find an equilibrium with a tie-breaking rule that allows to break a tie in favor of one bidder with probability 1 since then the sum of bidder's payoffs is upper-semi continuous in expectation given a bidder's value (see Dasgupta and Maskin (1986), Reny (1999) and Simon (1987)). The main idea of the paper is to consider a class of auction games such that the equilibrium profile exists when using a tie-breaking rule that allows to break a tie in favor of one bidder with probability 1 and in such a manner that in any equilibrium profile no tie occurs with positive probability. Therefore it does not matter whether the seller uses the random tie-breaking rule or not.

I demonstrate that ties do not occur with positive probability in equilibrium since every bidder (bidding below her value) is better off if she bids ever so slightly above the tie. Therefore, when searching for an equilibrium, one can a priori eliminate all strategy profiles in which ties occur with positive probability. This allows me to assume that the seller then chooses any tie-breaking rule he desires without influencing the set of bidder equilibrium strategies. The existence theorem by Reny (1999), applied to one specific tie-breaking rule, then guarantees the existence of equilibrium for any tie-breaking rule (e.g. the "random" rule that is usually considered in the literature). The approach proposed here was outlined in Bresky (1999) and uses an idea similar to Lebrun (1996). It is an alternative proof to the one used in Jackson and Swinkels (2005) who show the existence of mixed strategy equilibria for more general class of auctions applying the tie-breaking invariance on bidders equilibrium profiles. The proof provides a new approach applying result of Reny (1999) to establish an equilibrium existence in multi-unit auctions when the bidder mixes over weakly increasing strategies.

For multi-unit auctions with strictly increasing payoff function in bidder's own bid e.g. for all-pay auctions, pay-your-bid auctions, their convex combinations, bidders use pure strategies in equilibrium that are weakly increasing in bidder's own value. For all other auctions in the class of multi-unit auctions, every mixed best response can be mapped to a pure weakly increasing best response that yields the same payoff to the bidder and the same distribution of a bidder's bids. Then, opposing bidders have the same payoff for whatever strategy they use. This mapping can be made for all players simultaneously yielding an equilibrium in weakly increasing pure strategies. Therefore, if one finds a specific property of an equilibrium in weakly increasing pure strategies that depends only on the distribution of bids, then all equilibria must satisfy this property. In other words, including mixed or decreasing strategies in the set of admissible strategies complicates the analysis but does not yield different results. This result is not valid in general, e.g. if one relaxes independent distributions of values. The idea is similar to McAdams (2006) who shows that if bidder's signals are independent and private and valuations are increasing in own signal and non-decreasing in others' signals for the multi-unit uniform-price auction, then equilibrium exists.

This paper complements the literature showing the importance of better reply security as defined in Reny (1999) that generalizes to ideas of Dasgupta and Maskin (1986). Similar "purification" result is known in other games with private information including Khan and Sun (1995), Radner and Rosenthal (1982), and Milgrom and Webber (1985).

The paper is structured as follows. In Section 2 the multi-unit auction games considered here are defined. In Section 3 I show why ties can not occur in equilibrium, and why, hence, the equilibrium set does not depend on the choice of tie-breaking rule. In Section 4 I demonstrate that the auction is payoff secure as defined in Reny (1999) and that the payoff sum is upper semi-continuous with an efficient tie-breaking rule. These two conditions are sufficient for the existence of a mixed strategy equilibrium in the space of weakly increasing strategies. In Section 5, I prove the existence of an equilibrium in pure strategies that are weakly increasing. I also show that equilibria are in pure strategies only. This result requires me to restrict the analysis to auction games with continuous valuation distributions. The Appendix collects some proofs and lemmas.

2 MULTI-UNIT-AUCTION GAMES AND THEIR BASIC PROPERTIES

Consider n $(n \geq 1)$ risk-neutral bidders with continuous private valuation distributions and a seller who would like to auction k $(k \geq 1)$ identical units of goods to these bidders. The seller specifies publicly a reservation price $R \in [0, \infty)$ for his goods. In general, the seller could choose different reservation prices for different units. Simple way how to implement it is described in Jackson and Swinkels (2005). They consider reservation prices as additional bids that compeete with regular bidder bids without any effect on the existence of equilibria. Each bidder i draws values $v_i = [v_{i,1}, \cdots, v_{i,k}]$ from her private valuation distribution where $v_{i,j}$ is the value of the jth unit $(1 \leq j \leq k,\ 1 \leq i \leq n)$. Let $F_i(v_i)$ denote the cumulative distribution function of the private values $v_{i,j}$ with support on the compact set $\mathcal{V}_i \subset \times_{j=1}^{k} [0, \infty)$.[1] I focus on independent private value auctions with continuous valuation distributions. This implies that independence is required between bidders' valuation distributions but not within valuation distributions.

Assumption A1. For any $i = 1, \cdots, n$ there exists a density function $f_i(v_i)$ that is independent of the realization of opponent values v_{-i}.[2]

A bidder who wins J_i units obtains value $\sum_{j=1}^{J_i} v_{i,j}$. I assume that the marginal value from winning an additional unit weakly decreases for each bidder, i.e. $v_{i,j} \geq v_{i,j'}$ if $j < j'$. Each bidder submits k sealed bids $b_{i,1}, \cdots, b_{i,k} \in \{-\infty\} \cup [R, \infty) = \mathcal{B}$ that are denoted as b_i where bidding

[1] Symbol $\times$ means the Cartesian product.

[2] The index $-i$ represents the set of indices $\{1, \cdots, i-1, i+1, \cdots, n\}$.

$-\infty$ on a unit signals no interest on a given unit. The list of submitted bid k-tuples from all bidders is labeled as b below.

If more than k bids are above or equal to the reservation price R, then the seller chooses the k highest bids as the winning bids. A tie occurs when the kth and $k+1$st highest bids are equal.[3] I assume that the seller breaks ties according to a reasonable tie-breaking rule T. Each reasonable tie-breaking rule T has to determine at least one winning bid and at least one losing bid. A typical tie-breaking rule that does the job selects randomly the winning bids from among those bids that are tied. I shall call this rule the random rule. If fewer than $k+1$ bids are above or equal to the reservation price R, then each of them wins a unit.

I denote the set of all admissible bid k-tuples of one bidder as $\mathcal{B}_i = \times_{j=1}^{k}\mathcal{B}$.[4] and the set of all admissible bid k-tuples of all bidders as $\mathcal{B}_i \times \mathcal{B}_{-i}$. Let me denote c_j as the jth highest bid by an opponent where $j = 1, \cdots, k$ and J_i as the number of bids bidder i wins when the seller sets the reservation price R and all bidders submit bids b. Then bidder i's *ex post* payoff depends on the winning values $v_{i,1}, \cdots, v_{i,J_i}$, all submitted bids b, and the reservation price R.

$$\varphi_i\left(v_i, b, R\right) = \sum_{j=1}^{J_i} \left(v_{i,j} - p_{i,j}^{w}\left(b, R\right)\right) - \sum_{j=1+J_i}^{k} p_{i,j}^{l}\left(b, R\right), \qquad (1)$$

where $p_{i,j}^{w}\left(\cdot\right)$ and $p_{i,j}^{l}\left(\cdot\right)$ are the prices paid when winning or losing the jth

[3]Other bids may be also equal to the kth and $k+1$st highest bids and, therefore, tied.

[4]Since the seller orders bids anyway, one can assume without loss of generality that bidders submit ordered k-tuples of bids $b_{i,1} \geq \cdots \geq b_{i,k}$ although the notation admits submission of k-tuples that are not ordered.

unit. I define i's *ex post* payoff from winning or losing the jth unit as $\varphi^{w}_{i,j}(v_i, b, R) = v_{i,j} - p^{w}_{i,j}(b, R)$ and $\varphi^{l}_{i,j}(v_i, b, R) = -p^{l}_{i,j}(b, R)$. As we will see presently, $-p^{l}_{i,j}(b, R)$ plays a role in all-pay auctions.

Example 1 *The following multi-unit auctions with independent private values are consistent with the setting above.*

1. *Uniform-price auction - the price for winning any unit is the $k+1$st highest bid $\bar{c} = \max(b_{i,J_i+1}, c_{k+1-J_i})$ if more than k bids above R are submitted; otherwise the price for winning a unit is R.*[5] *The price for losing a unit is* 0.

$$\varphi^{u}_{i}(v_i, b, R) = \sum_{j=1}^{J_i} [v_{i,j} - \max(R, \bar{c})], \tag{2}$$

2. *Dutch auction - the price for winning any unit is the kth highest bid $\bar{c}' = \max(b_{i,J_i}, c_{k-J_i})$ if more than k bids above R are submitted; otherwise the price for winning a unit is R. The price for losing a unit is* 0.

$$\varphi^{d}_{i}(v_i, b, R) = \sum_{j=1}^{J_i} [v_{i,j} - \max(R, \bar{c}')], \tag{3}$$

3. *Vickrey auction - the price for winning the jth unit is the $k+1-j$th highest opponent bid c_{k+1-j}, if more than $k+1-j$ opponent bids above reservation price are submitted; otherwise the price for winning a unit is R. The price for losing a unit is* 0.

$$\varphi^{v}_{i}(v_i, b, R) = \sum_{j=1}^{J_i} (v_{i,j} - \max(c_{k+1-j}, R)). \tag{4}$$

[5] In other words, the price is determined by the highest losing bid of either bidder i or her opponent.

4. *Pay-your-bid auction - the price for winning any unit is the bidder's bid. The price for losing a unit is* 0*.*

$$\varphi_i^p(v_i, b, R) = \sum_{j=1}^{J_i} [v_{i,j} - b_{i,j}]. \tag{5}$$

5. *All-pay auction - the price for both winning and losing any unit is the bidders' bid for that unit or reservation price R.*

$$\varphi_i^a(v_i, b, R) = \sum_{j=1}^{J_i} [v_{i,j} - \max(b_{i,j}, R)] - \sum_{J_i=1}^{k} \max(b_{i,j}, R). \tag{6}$$

6. *All convex combinations of payoffs of auctions 1-5 above.*

$$\varphi_i^\lambda(v_i, b_i, c, R) = \lambda_u \cdot \varphi_i^u + \lambda_d \cdot \varphi_i^d + \lambda_p \cdot \varphi_i^p + \lambda_v \cdot \varphi_i^v + \lambda_a \cdot \varphi_i^a, \tag{7}$$

$$\text{for } 0 \leq \lambda_u, \lambda_d, \lambda_p, \lambda_v, \lambda_a, \;\; \lambda_u + \lambda_d + \lambda_p + \lambda_v + \lambda_a = 1.$$

A strategy of bidder i specifies i's bids based on the information she has before the auction. This information includes her private values, the number of opponents and their valuation distributions, the seller's reservation price, and the number of units for sale. For the sake of simplicity I will write pure strategies as a function of private values only. A measurable strategy of bidder i is therefore a mapping $b_i(\cdot) : \mathcal{V}_i \rightarrow \mathcal{B}_i$ ($b_i(v_i) = [b_{i,1}(v_{i,1}, \cdots, v_{i,k}), \cdots, b_{i,k}(v_{i,1}, \cdots, v_{i,k})]$) such that $b_{i,j}(\cdot)$ is measurable function. When opponents use measurable strategies $b_{-i}(\cdot)$, then the *interim* expected payoff to a bidder i, whose values are v_i and who bids b_i, is

$$\pi_i(v_i, b_i | b_{-i}(\cdot), \mathbf{T}) = E(\varphi_i(v_i, b, R)). \tag{8}$$

Expectations are taken over opponent values v_{-i}. The probability measure of $\varphi_i(v_i, b, R)$ is induced by the opponent strategies $b_{-i}(\cdot)$, and a tie-breaking rule T for ties that occur with positive probability. In the following I will use the word *interim* payoff to denote a bidder's *interim* expected payoff after she has learned her values but before she submits her bids. In contrast, a bidder computes her *ex ante* payoff before she has learnt her values. Correspondingly I denote *interim* best response I will denote a bids b_i of bidder i whose values are v_i.

When bidder i uses her pure strategy $b_i(\cdot)$, then her *ex ante* pure strategy payoff is

$$\pi\left(b_i\left(\cdot\right)|b_{-i}\left(\cdot\right),\mathbf{T}\right)=E\left(\pi_i\left(v_i,b_i\left(v_i\right)|b_{-i}\left(\cdot\right),\mathbf{T}\right)\right), \tag{9}$$

where the expectations are taken over v_i.

The class of auctions I study is defined by the following assumptions on the *ex post* price of any unit $j=1,\cdots k$ of any bidder $i=1,\cdots n$. I shall refer to these auctions as *standard* from here on. The bid $b_{i,j}$ can win only if it is above or equal to reservation price R and $k+1-j$th opponent bid ($b_{i,j}\geq\max\left(c_{k+1-j},R\right)$). Therefore I discuss the properties of the winning prices and the ex post payoffs in this case only. Similarly, I discuss the properties of the losing prices and the ex post payoffs only for the case that the bid $b_{i,j}$ is below or equal to reservation price R and $k+1-j$th opponent bid ($b_{i,j}\leq\max\left(c_{k+1-j},R\right)$).

In this paper, I will focus on auctions with monotonic and additively separable prices.

Assumption A2. Winning and losing prices are weakly increasing in any bid j of any bidder i.[6] (Note that a bidder's payoffs are weakly increasing

[6]For any $b\in\mathcal{B}_i\times\mathcal{B}_{-i}$ and any $b'_{i,j}\in\mathcal{B}$ if $b_{i,j}\geq b'_{i,j}\geq\max(c_{k+1-j},R)$, then $p^w_{i,j}\left(b,R\right)\geq p^w_{i,j}(b'_{i,j},b_{i,-j},b_{-i},R)$ and if $b_{i,j}\leq b'_{i,j}\leq\max(c_{k+1-j},R)$, then $p^l_{i,j}\left(b,R\right)\leq$

even in other bidders' payoffs.)

Assumption A3. The auction price is additively separable, i.e. when bidder i changes her bids on some units $b_{i,p}$, $b_{i,p+1}$, $\cdots$, $b_{i,q}$ (given opponents submitted bids b_{-i}), the difference in ex post price before and after the change is the same for any of her bids on units $b_{i,1}, \cdots , b_{i,p-1}$, $b_{i,q+1}, \cdots , b_{i,k}$ if bidder i wins every tie.[7]

The assumption A3 requires that if bidder i changes her bids on some units, then strategic considerations on other units are not affected. The assumptiomn above is satisfied for auctions from example 1 only if the seller breaks the ties in the same manner before and after bidder i changes his strategies.

Assumption A4. Winning and losing prices are continuous in all submitted bids.[8] (In conjunction with additive separability, this property implies joint continuity.)

The proofs that auctions from Example 1 satisfy assumptions A2-A4 are trivial and hence omitted.

The last assumption requires that there is a weakly increasing upper bound of strategy $\bar{b}_i(\cdot)$ such that in a tie below $\bar{b}_i(\cdot)$, a bidder has an incentive to bid above the tie, and for any bid above the upper bound there is another

$p^l_{i,j}(b'_{i,j}, b_{i,-j}, b_{-i}, R)$.

[7]For any $b_{-i} \in \mathcal{B}_{-i}$, any $b_i, \hat{b}_i, b'_i, \hat{b}'_i \in \mathcal{B}_i$ if $b_{i,S} = \hat{b}'_{i,S}$, $b_{i,-S} = \hat{b}_{i,-S}$, $b'_{i,-S} = \hat{b}'_{i,-S}$, and $b'_{i,S} = \hat{b}_{i,S}$ for some $S = \{p, p+1, \cdots , q\} \subset \{1, \cdots , k\}$, then $\tilde{p}_{i,j}(b, R) - \tilde{p}_{i,j}(\hat{b}_i, b_{-i}, R) = \tilde{p}_{i,j}(b'_i, b_{-i}, R) - \tilde{p}_{i,j}(\hat{b}'_i, b_{-i}, R)$ where $\tilde{p}_{i,j}(b, R)$ is $p^w_{i,j}(b, R)$ when $b_{i,j} \geq \max(c_{k+1-j}, R)$ and $p^l_{i,j}(b, R)$ otherwise.

[8]For any $b \in \mathcal{B}_i \times \mathcal{B}_{-i}$ and any $\delta > 0$ exist $\varepsilon > 0$ such that for all $b' \in \mathcal{B}_i \times \mathcal{B}_{-i}$ within ε, if $b_{i,j} \geq \max(R, c_{k+1-j})$ and $b'_{i,j} \geq \max(R, c'_{k+1-j})$, then $|p^w_{i,j}(b, R) - p^w_{i,j}(b', R)| < \delta$; and if $\max(R, c_{k+1-j} \geq b_{i,j})$ and $\max(R, c'_{k+1-j}) \geq b'_{i,j}$, then $|p^l_{i,j}(b, R) - p^l_{i,j}(b', R)| < \delta$.

bid below the upper bound that brings at least as high a payoff as the bid above the upper bound.

ASSUMPTION A5. There is a weakly increasing upper bound strategy $\bar{b}_i(\cdot)$ such that for any values v_i and any bids b_i of bidder i and for any strategy of her opponents:

1. no bidder i has an incentive to submit jth bid above $\bar{b}_{i,j}(v_i)$,[9] and
2. the ex post winning payoff is greater than the ex post losing payoff if there is a tie below the upper bound $\bar{b}_i(\cdot)$.[10]

Specifically, in the auctions in Example 1, no bidder has an incentive to bid above her value.[11] In auctions (5) and (6) a bidder who submits her jth bid above her jth value has a nonpositive winning or losing jth unit payoff. Therefore if that bidder bids 0 instead, she is not worse off because her losing unit payoff is 0. In auctions (2), (4) and (5), a bidder who submits her jth bid above her jth value when the $k+1-j$th highest opponent bid is above her value has nonpositive winning and losing jth unit payoff. Therefore if that bidder bids her value instead, she is not worse off because her winning and losing jth unit payoff is 0. When the $k+1-j$th highest opponent bid is below the bidder's value and the bidder bids on his value instead of above the value, then she is better off. In addition, in any of these auctions, if any

[9] Consider values $v_i \in \mathcal{V}_i$ of bidder i $(i = 1, \cdots n)$, and bids $b_i \in \mathcal{B}_i$ of bidder i. Then there exists $b'_i \in \mathcal{B}_i$ where $b'_{i,j} \leq \bar{b}_{i,j}(v_i)$ for all $j = 1, \cdots, k$ such that for any opponents' bids $b_{-i} \in \mathcal{B}_{-i}$ $\varphi_i(v_i, b, R) \leq \varphi_i(v_i, b'_i, b_{-i}, R)$.

[10] For any $j = 1, \cdots k$, any v_i $(i = 1, \cdots n)$ with $v_{i,j} > \hat{v}_{i,j}$ and any $b \in \mathcal{B}_i \times \mathcal{B}_{-i}$ if $b_{i,j} = c_{k+1-j} \in [R, \bar{b}_{i,j}(v_{i,j}))$, then $v_{i,j} - p^w_{i,j}(b, R) > p^l_{i,j}(b, R)$.

[11] This assumption is an analog of No dumb bid rule in Jackson and Swinkels (2005).

bid $b_{i,j}$ below the value is tied with the $k+1-j$th highest opponent bid, then the bidder strictly prefers wining the tie to losing it. In simple auctions in Example 1, the upper bound strategy $\bar{b}_{i,j}(v_{i,j})$ that fulfills the two properties can be defined as the value $v_{i,j}$.

I discuss the last assumption in the Appendix in more detail because it is not a condition on primitives. Briefly, in lemma 14 and 15 in the Appendix I formulate two sets of conditions on the winning and losing price functions under which an auction game satisfies A5. I also illustrate these lemmas are applicable to a convex combination of simple auctions from Example 1.

I will denote the set of bidder i's measurable strategies $b_i(\cdot) : V_i \rightarrow \mathcal{B}_i$ with each jth component below or equal to $\bar{b}_{i,j}(v_i)$ as B_i and the set of these measurable strategy profiles as $B = \times_{i=1}^{n} B_i$. In Sections 3 and 4, I show the existence of equilibrium in a weakly increasing strategy space $\hat{B}_i$. The pure strategy space $\hat{B}_i$ consists of $b_i(\cdot) : \mathcal{V}_i \rightarrow \mathcal{B}_i$ such that each component of $b_{i,j}(v_i)$ is a weakly increasing function in every argument $v_{i,j'}$ (for $j, j' = 1, \cdots, k$), $\bar{b}_{i,1}(v_i) \geq b_{i,1}(v_i)$ and $\min\left(b_{i,j}(v_i), \bar{b}_{i,j}(v_i)\right) \geq b_{i,j+1}(v_i)$ for $j = 1, \cdots, k-1$. The mixed strategy space $\hat{M}_i$ is a Borel sigma algebra generated on a pure strategy space $\hat{B}_i$. The standard results of probability theory imply that $\hat{B}_i$ is a subset of B_i because any bounded weakly increasing strategy is measurable and that $\hat{B}_i$ and $\hat{M}_i$ are compact metric spaces (see Lemma 16 in the Appendix).

When opponents use mixed strategies m_{-i} from $\hat{M}_{-i}$ then we can define *interim* payoffs and *ex ante* pure strategy payoffs to bidder i. They differ from (8) and (9) only in probability measure of $\pi(v_i, b, R)$, that is induced by m_{-i} instead of $b_{-i}(\cdot)$. Player i's *ex ante* mixed strategy payoff from m_i is

$$\pi_i(m_i | m_{-i}(\cdot), T) = E(\pi_i(b_i(\cdot) | m_{-i}(\cdot), T)), \tag{10}$$

where the expectation is taken over $b_i(\cdot)$ that bidder i uses in mixed strategy m_i.[12]

Note that i's payoff is the same for a given distribution of opponent bids whether it was generated by mixed or measurable opponent strategies. I omit opponent strategies, reservation price and the tie-breaking rule in the payoff argument list to simplify notation.

3 TIES AND TIE-BREAKING RULE EQUIVALENCE

Let us now investigate the effect of tie-breaking rules on the bidder's payoff. In Section 1 I defined a tie from the seller's point of view. From the bidder's point of view, if a tie occurs with probability 0 in equilibrium, then it does not influence her payoff. When discussing ties below I consider only the ties that occur with positive probability for a given opponent's strategies.

Let me demonstrate that for a given opponent's strategy, the best response payoff for a bidder i who bids below her value must be at least as high as when every tie is broken in her favor. Assume that the seller's reservation price is 0 and consider a tie in strategies $b_{i,j}$ that occurs with positive

[12] One can decrease both winning and losing payoff by a fixed amount interpreted as information costs to each bidder for finding out unit values. It has no effect on the shape of equilibrium strategies if the costs are not so high to deter any bidder from finding out this information.

probability and that is not surely broken in favor of bidder i. If the opponent's $k+1-j$th highest bid is $b_{i,j}$ and the bidder bids $b_{i,j}+\varepsilon$ instead of $b_{i,j}$, she wins the tied unit surely at a slightly increased price. Since the bidder *ex post* prefers winning over losing the tied unit, for small ε her *ex post* and therefore *interim* payoff increases (by a jump) for a set of values of positive measure. For simplicity assume that the auction price is uniformly continuous in bids and that bidder i submits all her bids ε higher whether there is a tie or not. Then for sufficiently small ε the *ex post* and therefore *ex ante* auction price paid for any of her units changes negligibly and with negligible effect on her *ex ante* payoff. Therefore i's best response payoff cannot be worse than her payoff when winning every tie surely.

One important implication of this finding is that there are no ties in equilibrium for a reasonable tie breaking rules as long as each bidder submits her jth bid below her jth value. Since every bidder strictly prefers winning a tie over losing it and since the seller cannot satisfy all tied bids, the bidder who does not surely win every tie has incentive to increase her strategy by ε to win the tied unit. I formalize this idea in the following lemma that is proved in the Appendix.

Lemma 2 *Consider a standard multi-unit auction game and opponent measurable strategies $b_{-i}(\cdot) \in B_{-i}$ or mixed weakly increasing strategies $m_{-i} \in \hat{M}_i$. If there is a tie with positive probability so that it is not broken in the favor of bidder i with positive probability when using $b_i(\cdot) \in \hat{B}_i$, then a better response strategy $b_i^{\varepsilon}(\cdot) \in \hat{B}_i$ exists than $b_i(\cdot)$. Moreover for any measurable best response $b_i(\cdot) \in B_i$ the bidder payoff must be at least as high as if every tie that occurs with positive probability the seller breaks in bidder i's favor.*

What are the effects of a particular tie-breaking rule on the set of equilibria? Lemma 2 shows that no strategy $b_i(\cdot)$ with a tie that occurs with positive probability for given opponent strategies is the equilibrium best response. Since at any tie at least one bidder does not win the unit, that bidder deviates to win the unit. In other words, when searching for equilibrium in the set of all player profiles one can *a priori* eliminate the profiles in which the tie occurs. But no tie-breaking rule influences the payoff at these profiles without ties. Therefore choosing a particular tie-breaking rule has no influence on the best response strategy payoff and any equilibrium found for one rule T is also equilibrium for any other rule T'.

Theorem 3 *Consider the two standard multi-unit auction games that differ only in a tie-breaking rule* $\hat{G} =< \hat{M}_i, \varphi_i, T >_{i=1}^N$ *and* $\hat{G}' =< \hat{M}_i, \varphi_i, T' >$*. Ties occur with probability* 0 *in equilibrium and the set of equilibria of the game* $\hat{G}$ *coincides with that of* $\hat{G}'$*.*

Proof. Consider a profile m with a tie which occurs with positive probability. Assume that bidder i strictly prefers winning to losing the tie and that bidder i loses the tie with positive probability when using $b_i(\cdot) \in \hat{B}_i$. By the nature of a tie at least one bidder does not win the unit for any positive measure of pure strategies $b_i(\cdot)$ in the support of m_i. For any such $b_i(\cdot)$ there is a better strategy $b_i^{\varepsilon}(\cdot)$ by Lemma 2 and therefore m is not an equilibrium. Since the payoff in profiles without ties is independent of using T or T', the set of equilibria coincides for both $\hat{G}$ and $\hat{G}'$. ∎

In Section 4 I choose the tie-breaking rule that is the most suitable to finding the equilibrium strategy profile. This strategy profile is also the equilibrium for other tie-breaking rules according to Theorem 3. Let me illustrate that Theorem 3 is not valid for some discontinuous distributions of values.

Example 4 *Consider a single-unit first-price or single-unit second-price auction with two bidders and a zero reservation price. The first bidder has a value of* 1*, and the opponent's value is distributed uniformly on* $(1, 2]$. *If the seller uses the tie-breaking rule that surely breaks the ties in favor of the second bidder, there exists an equilibrium in which both bidders bid surely* 1. *But it is not an equilibrium if the seller breaks the tie randomly because the second bidder is strictly better off if bidding* $1 + \varepsilon$ *for a sufficiently small positive* ε. *Moreover Lebrun (1996), in his footnote* 1, *shows that no equilibrium exists for this first-price auction with the random rule.*

Please note that I do not restrict the seller in the type of information he uses for breaking ties. Therefore the game in which the seller breaks ties according to the bidder values has the same equilibria as the game with the random rule. This game does not have the same informational structure as the game that is understood as an auction in the literature because it gives the seller complete information about the bidder values if a tie occurs. But the definition of the game in Section 2 ensures that if no tie occurs the only information the seller uses for the allocation of units are the submitted bids, as is standard procedure in the auctions literature. This is utilized in Section 4.

4 THE EXISTENCE OF AN EQUILIBRIUM IN A RESTRICTED STRATEGY SPACE

In this section I show the existence of an equilibrium in an auction game with a restricted set of strategies on weakly increasing functions. One possible way of showing the existence of equilibrium in this setting used in Jackson and Swinkels (2005) is to apply the theorem by Jackson, Simon, Swinkels and Zame (2002). Roughly speaking, their theorem claims the existence of a tie-breaking rule which ensures equilibrium. In my paper I present another approach to this problem. These authors first extend the game definition to incorporate various "endogenous tie-breaking rules" and game outcomes. They ultimately, however, restrict their attention to auction games in which the tie-breaking rule is irrelevant. In a sense, I go a different but closely related route.

Since the tie-breaking rule is irrelevant as I have shown in Section 3, In my approach I apply ideas from standard existence results by Dasgupta and Maskin (1986), Reny (1999), and Simon (1987). Specifically, I construct a specific tie-breaking rule T^e called the efficient tie-breaking rule for which the existence of equilibrium is guaranteed by Reny (1999). The efficient tie-breaking rule T^e maximizes the sum of bidder *ex post* payoffs when a tie occurs. This rule thus awards the tied unit to the bidder who values the unit the most. Following this rule, the sum of bidder's payoffs is upper semi-continuous and the payoff function is "payoff secure" (Reny (1999)) in the sense of a specific bidder being able to guarantee herself almost the same payoff if her opponents change their strategies slightly. These properties are sufficient for the existence of equilibrium.

The following definition formulates a specific tie-breaking rule for the seller when a tie occurs.

Definition 5 *Let us call efficient tie-breaking rule T^e the rule in which the seller breaks any tie in favor of bidder i who gets a higher* ex post *payoff than his opponents $-i$ from winning the tied unit. If there is a tie in bids and* ex post *payoffs are equal, then the tie is broken randomly.*[13]

The efficient tie-breaking rule requires, unfortunately, the seller to have private information about the values of the bidders. This assumption does not reflect real-life auctions well and is indeed not used in the literature. Fortunately, the fact that the efficient tie-breaking rule is not implementable is not crucial because the equilibrium strategies remain in equilibrium for any tie-breaking rule, in particular the standard random one.

Definition 6 *The* ex ante *payoff function $\pi_i(m, T, R)$ is payoff secure at profile $m \in \hat{M}$ if for every $\delta > 0$, player i has a strategy m_i' that satisfies*

$$\liminf_{m''_{-i} \to m_{-i}} \pi_i(m_i', m''_{-i}, T, R) \geq \pi_i(m, T, R) - \delta. \quad (11)$$

The function $\pi_i(m, T, R)$ is payoff secure if it is payoff secure for every $m \in \hat{M}$.[14]

[13] When A1 is used, ties in values occur with probability 0 and do not influence the payoff.

[14] Since $\pi_i(m, T)$ is linear in m_i, it is sufficient to check the local payoff security for each pure strategy $b_i \in B_i$ of bidder i and any opponent mixed strategies $m_{-i} \in M_{-i}$.

Definition 6 requires that bidder i's best response payoff does not decrease by a jump even if the opponents change their strategies slightly. If the opponents increase their strategies slightly, then bidder i loses all ties that occurred before the opponents' increase in strategies and i's payoff jumps down. However, if bidder i increases his strategy by a sufficiently small ε, his payoff is negligibly smaller than the payoff from winning all ties by Lemma 2. In addition, no new tie occurs for suitable ε and the "out-of-ties" payoff is continuous in opponent bids b_{-i} by A4. Therefore the auction game payoff satisfies Definition 6. For more details see Lemma 17 in the Appendix.

Now let us examine the upper semi-continuity of the bidder's payoff sum.

Definition 7 *The standard auction payoff is summation upper semi-continuous if for any profile m and any sequence $m' \to m$ from $\hat{M}$ $\lim_{m' \to m} \sup \sum_i \pi_i(m', T, R) \leq \sum_i \pi_i(m, T, R)$.*

The theorem by Reny (1999) must be applied carefully to auctions with the random tie-breaking rule T^r because their payoff are not summation upper semi-continuous in ties. This is shown in the following example.[15]

Example 8 *Assume that there are two bidders and two units for sale ($k = 2$) in the uniform-price auction, and consider a realization of agent values and bids such that $v_{2,2} > v_{1,1}, b_{2,1} > b_{2,2} = b_{1,1} > b_{1,2} > R$ and $b_{1,1}$ is the first rejected bid. Now fix all values and bids except the second bid of bidder 2 and examine the sequence of strategies when $b^+_{2,2}$ approaches $b_{1,1}$ from above.*

[15] The auction game does not satisfy "complementarity discontinuity property" introduced by Simon (1987) that requires some player's payoff jump up whenever some other player's payoff jumps down.

Along the sequence bidder 2 *wins* $v_{2,2}$ *but at the limit (*$\lim b_{2,2}^{+} = b_{1,1}$*) the seller breaks this tie randomly between bidder* 1 *and* 2 *giving them value* $\frac{v_{1,1}+v_{2,2}}{2}$ *on average. At the limit the price is the same and the payoff sum difference is*

$$\frac{1}{2} \cdot (v_{1,1} - b_{1,1}) - \frac{1}{2} \cdot \left(v_{2,2} - \lim b_{2,2}^{+}\right) = \frac{v_{1,1} - v_{2,2}}{2} < 0. \tag{12}$$

In this case the payoff sum jumps down. If this situation occurs with positive probability, then $\sum_i \pi_i (m, T^r, R)$ *is not upper semi-continuous (see Billingsley (1968)).*

If the seller uses an efficient tie-breaking rule, then the auction payoff is summation upper semi-continuous because the payoff cannot jump down at ties and the "out-of-ties" sum of payoffs is continuous by A4.[16] For more details see Lemma 18 in the Appendix.

Now using the efficient rule, I can show that a mixed strategy equilibrium exists.

Theorem 9 *Consider a standard auction with the efficient tie-breaking rule* $\hat{G}^e = \left\langle \hat{M}_i, \varphi_i, T^e \right\rangle_{i=1}^{n}$ *k units for sale and n bidders. Then a mixed strategy equilibrium* m^* *exists in* $\hat{G}^e$*. Moreover if the bidders are symmetric, a symmetric equilibrium exists.*

Proof. Using Lemma 16 it is clear that $\hat{M}_i$ is compact metric spaces. In addition, mixed strategy payoff (10) is bounded from above because $\mathcal{V}_i$ is bounded, payoff secure and summation upper-semi continuous (see Lemma

[16] In the example 8, $T_{1,1}^{e} (b_1, b_2, v) = 0$ and $T_{2,1}^{e} (b_1, b_2, v) = 1$. Therefore the expression (12) is equal to 0 if a tie $b_{2,2} = b_{1,1}$ occurs because $v_{2,2} > v_{1,1}$.

17 and 18 in the Appendix). Then Reny (1999, Proposition 5.1 and Corollary 5.2 of the Theorem 3.1) shows that the equilibrium exists.

Symmetric bidders choose their strategies from the same strategy set $\hat{M}_i$, and i's payoff does not depend on the permutation of the opponents' strategies.[17] Since symmetry is a weaker condition than quasi-symmetry, payoff security is a weaker condition than the payoff security along the diagonal and payoff is continuous when all bidders use the same strategy from $\hat{M}_i$, one can use Proposition 5.1 and Corollary 5.3 of the theorem 4.1 in Reny (1999) to show that the game possesses a symmetric equilibrium. ∎

Theorems 3 and 9 imply the following corollary.

Corollary 10 *In a standard auction game* $\hat{G}^r = \left\langle \hat{B}_i, \varphi_i, T^r \right\rangle_{i=1}^n$ *with the random tie-breaking rule consistent, mixed equilibrium* m^* *exists.*

Corollary 10 shows that equilibrium in weakly increasing strategies exists for the random tie-breaking rule. In the next section I will show that any other measurable strategy cannot be better off. The equilibrium with random tie-breaking rule may not exist for discontinuous valuations if there is a tie that a specific bidder wins or loses with probability 1 or 0 in equilibrium with efficient tie-breaking rule as illustrated in Example 4.

It seems possible to generalize Theorem 9 to discontinuous distributions of values although it requires specific restrictions specific for each multi-unit auction format so that the results by Lebrun (1996) and some of the results by McAdams (2006) when bidder's signals are independent and private and

[17] $\forall m_i \in M_i, \forall m_{-i}, m'_{-i} \in M_{-i}$ such that $\forall i' \exists i''\ m_{i'} = m'_{i''} : \pi_i(m_i, m_{-i}, T) = \pi_i(m_i, m'_{-i}, T)$ for more details see Reny (1999).

valuations are increasing in own signal and non-decreasing in others' signals can be incorporated; the proof however exceeds the scope of this paper. An intuitive argument is that in auctions 1-4 of Example 1 the bidder who bids her value is indifferent between winning or losing a tie because she has a zero payoff anyway. In the proof of Lemma 2 I ruled out ties in which a bidder submits a high bid equal to her value since it is a probability zero event for continuous distributions. In the case of discontinuous distributions one can realize that in such a tie the bidder's *ex post* payoff is continuous in her bids, and hence payoff secure. A seller who uses an efficient tie-breaking rule breaks the tie in favor of the bidder with highest payoff, thus making bidders' payoff summation upper semi-continuous.

These properties are sufficient to extend Theorem 9 for auction games with efficient tie-breaking rule when bidders' valuation distributions are not continuous (see Example 4). Unfortunately, Theorem 3 is not valid for all discontinuous valuation distributions.

One can however restrict the valuation distributions in such a way that in any mass point above the reservation price in the support of the jth value marginal distribution, the probability that $k-j$ or more opponents' values are above or equal to $v_{i,j}$ is less than 1. This implies that for any mass point value $v_{i,j}$ the bidder i has a positive probability of winning in a pay-your-bid auction because all opponents bid at most their values. Therefore it is a better response to bid $v_{i,j}-\varepsilon$ instead of $v_{i,j}$ for a bidder with value $v_{i,j}$ and sufficiently small ε. Then in equilibrium there is no tie (occurring with positive probability) that a specific bidder wins or loses with probability 1 or 0 and Corollary 10 holds (see Reny (1999) for an alternative proof). Similar restrictions are specific for other multi-unit auction games.

5 EQUILIBRIUM EXISTENCE IN MEASURABLE STRATEGY SPACE

In this section, I show first that the mixed equilibrium strategies in the space of weakly increasing strategies $\hat{M}$ identified in Section 4 are pure or can be changed to be pure. The pure strategy is the best response whenever the mixed is and yields the same distribution of bidder's bids, and hence, the opponents face the same decision-making problem. These pure strategies form an equilibrium in measurable strategy space or even if each bidder i chooses his strategy after knowing his values v_i.[18] Such a "purification" can be made in other games with private value information Khan and Sun (1995), Radner and Rosenthal (1982), and Milgrom and Webber (1985).

I then show that for any equilibrium in mixed or measurable strategies exists equilibrium in pure weakly increasing strategies with the same bid distributions. Therefore any specific property of auctions that depends on the equilibrium distribution of bids and that is valid for the set of all equilibria in pure weakly increasing strategies must be valid for all mixed equilibria. This result was discussed in McAdams (2006) for multi-unit uniform price auctions. Finally I show that, if the *ex post* auction price paid by bidder i is strictly increasing in his bids (e.g. in the case of all-pay and pay-your-bid auctions, and their convex combinations), then the equilibrium strategy is

[18] In other words, they form an equilibrium in behavioral strategies discussed in Lizzeri and Persico (2000) if one would extend behavioral strategies to multi-unit auctions.

pure and weakly increasing in bidder values. For other auction formats in the class investigated here, mixed or decreasing equilibrium strategies are possible.

Consider a mixed or measurable best response strategy in a single-unit auction and let me denote $G_i(b_i)$ the distribution of bidder i's bids and $F_i(v_i)$ the distribution of her values . Then $G_i^{-1}(F_i(\cdot))$ is the pure weakly increasing strategy generating the same distribution of bids. Moreover, it is the best response strategy. Intuitively, if a bidder i for value v_i' prefers to bid b_i' instead of $b_i < b_i'$, then the bidder's *interim* surplus difference when bidding b_i' and b_i is positive. For all v_i above v_i the sign of *interim* surplus difference when bidding b_i' and b_i must be the same because the winning and losing prices when bidding b_i' and b_i are independent of v_i, and therefore change in *interim* surplus difference is $v_i' - v_i$ times the difference in probability of winning when bidding b_i' and b_i that is not negative. Therefore, if the best response is decreasing, then the probability of winning, and winning and losing prices are the same whether the bidder bids b_i' or b_i. This implies that every *interim* best response is either unique or any bid between two best response bids is also the best response with the same probability of winning and payment.

For a multi-unit auction one can construct the pure weakly increasing best response strategy generating the same distribution of bids along one dimension. After iterating this process along different strategies one gets a strategy that is the pure weakly increasing best response strategy generating the same distribution of bids along all dimensions.

The following proposition is proven in the Appendix; it shows that instead of a mixed or nonincreasing best response a bidder can use a pure weakly increasing strategy that yields the same payoff to her and that generates the same distribution of her bids as the mixed or nonincreasing best response.

Proposition 11 *In a standard auction consider the best response* $m_i \in \hat{M}_i$ *(or* $b_i(\cdot) \in B_i$*) to any opponent equilibrium strategy* $m_{-i} \in \hat{M}_{-i}$ *(or* $b_{-i}(\cdot) \in B_{-i}$*) . Then a weakly increasing strategy* $\hat{b}_i(v_i) \in \hat{B}_i$ *exists such that*

1. $\hat{b}_i(v_i)$ *are the* interim *best response for any value* v_i *up to the measure zero set, and*

2. *the distributions of bids generated by strategies* $\hat{b}_i(\cdot)$ *and* m_i *(or* $b_i(\cdot)$*) are the same.*

Moreover if $p^w_{i,j}(\cdot)$ *strictly increases in* $b_{i,j}$*, then any two pure best response strategies* $b_i(v_i) \in \hat{B}_i$ *are the same for values that win with positive probability up to the measure zero subset and any measurable best response strategy* $b_i(\cdot) \in B_i$ *is weakly increasing for values that win with positive probability up to the measure zero subset.*

The second part of Proposition 11 says that the best response strategy is pure weakly increasing in all-pay and pay-your-bid auctions where the winning price of the jth unit strictly increases in jth unit bid. The first part of Proposition 11 says that for other *standard* auctions a mixed or nondecreasing measurable best response can be modified to be pure weakly increasing. This "purification" result is known for special cases from other studies of auctions (Engelbrecht-Wiggans and Kahn (1995a), and Lizzeri and Persico (1996)). Intuitively, if a bidder i for value $v'_{i,j}$ prefers to bid $b'_{i,j}$ instead of $b_{i,j} < b'_{i,j}$, then the bidder's *interim* surplus difference when bidding $b'_{i,j}$ and $b_{i,j}$ is positive. For all $v_{i,j}$ above $v'_{i,j}$ the sign *interim* surplus difference when bidding $b'_{i,j}$ and $b_{i,j}$ must be the same because the winning and losing prices when bidding $b'_{i,j}$ and $b_{i,j}$ are independent of $v_{i,j}$, and therefore change in *interim* surplus difference is $v_{i,j} - v'_{i,j}$ times the difference in probability of winning when bidding $b_{i,j}$ and $b'_{i,j}$ that is not negative. Therefore, if the best response is decreasing, then the probability of winning, and winning and losing prices are the same whether the bidder bids $b_{i,j}$ or $b'_{i,j}$. This implies that there exists an *interim* best response that is weakly increasing. Moreover, one can choose such an *interim* best response that preserves the distribution of bids.

An implication of Proposition 11 is that for a given valules and opponent strategies the *interim* best response bids $\hat{b}_i(v_i)$ that are weakly increasing in values exist (up to a measure zero subset of values). They form an equilibrium not only in mixed strategy space but also in the game in which each bidder knows his values v_i at first, and then selects an optimal bid list $b_i \in \mathcal{B}$ or a probability distribution over the bid set $\mathcal{B}$. Such an auction structure is more intuitive and indeed typically used in the auction literature.

Theorem 12 *In the standard multi-unit auction game, equilibrium $\hat{b}(\cdot)$ in measurable strategy space exists such that all bidders use weakly increasing strategies ($\hat{b}_i(\cdot) \in \hat{B}_i$ for $i = 1, \cdots, n$).*

Proof. Fix the equilibrium m^* in the weakly increasing strategies from Theorem 9. The weakly increasing strategy $\hat{b}_i(v_i)$ constructed in Proposition 11 is the best response for every v_i, and hence no measurable best response is better. Since strategy $\hat{b}_i(v_i)$ preserves the distribution of bids, the profile $\hat{b}_i(\cdot), \hat{m}^*_{-i}$ forms an equilibrium. After changing the opponent strategies to $\hat{b}_{-i}(\cdot)$, the weakly increasing equilibrium $\hat{b}(\cdot)$ in measurable strategy space is constructed. It is an equilibrium even if each bidder chooses her bids after knowing her values. ∎

Theorem 12 does not show that every equilibrium in weakly increasing strategies is pure. But the same steps as in the proof of this theorem can be done for any other best response. In any mixed equilibrium all bidders can change their strategies to form equilibrium in pure strategies. In any measurable equilibrium all bidders can change their strategies to form weakly increasing equilibrium. In the newly formed equilibrium no bidder faces any change in her strategic considerations and they have the same payoffs. This property is valid for all equilibria in standard auctions with the random tie-breaking rule that I summarize in the following corollary.

Corollary 13 *Consider the standard multi-unit auction with a random tie-breaking rule. Then any mixed weakly increasing equilibrium strategy $m_i \in \hat{M}_i$ can be rearranged to be pure, and any measurable equilibrium strategy $b_i(\cdot) \in B_i$ can be rearranged to be weakly increasing in such a way that the bid distribution is not changed. Moreover if $p^w_{i,j}(\cdot)$ increases in $b_{i,j}$, then every mixed best response below $\bar{b}(v_i)$ is pure on values that win with positive probability and every measurable best response below $\bar{b}(v_i)$ is weakly increasing on values that win with positive probability.*

An implication of this feature is that any possible equilibrium bid distribution is generated by weakly increasing equilibrium strategies. That is why the results of studies that focus only on pure weakly increasing equilibrium strategies in auctions are valid also for equilibria in other strategy spaces.

6 CONCLUSION

In this paper I have shown that in multi-unit auction games the discontinuities in payoffs are of a special form that allow to prove the existence of equilibrium in a straightforward manner. The key argument is that each bidder can bid in such a way that she wins any tie. Of course, the seller in an auction can not give units to all tied bidders. Since bidders are aware of the relevant tie-breaking rule, they adopt strategies that guarantee that no tie occurs in equilibrium with positive probability. Therefore, the equilibrium strategies in auction games secure payoffs as high as when winning all ties surely even if opponents change their strategies slightly.

For continuous distributions of values the tie-breaking rule that the seller uses does not matter. Even the efficient tie-breaking rule will not affect the set of equilibrium strategies. The game with this rule is payoff secure and summation upper semi-continuous and an equilibrium in weakly increasing strategies exists. Since the distributions of values are independent Proposition 11 guarantees that for any equilibrium strategy a bidder can use a pure weakly increasing best response strategy having the same payoff and distribution of bids. This allows to "purify" the equilibrium.

The set of weakly increasing equilibria is a representative subset of all equilibria for standard auctions with continuous distributions of values. This means that to study all possible equilibrium bid distributions one can restrict one's attention to weakly increasing strategy spaces. The fact that the existence of equilibrium is independent of choosing a specific tie-breaking rule that allows one to eliminate strategies with ties seems to be fairly intuitive and applicable to other games that are payoff secure. The auction game with an efficient tie-breaking rule is a special case of the "augmented" auction game discussed in Lebrun (1996). and Jackson, Simon, Swinkels and Zame (2002).

In this paper I present a unified approach to characterizing the equilibrium set of a large class of private value multi-unit auctions (cf. Lizzeri and Persico (2000) in single unit auctions). The approach proposed here allows us to understand exactly what assumptions are necessary to obtain an equilibrium strategy with the desirable features for private value auctions and even for some unusual combinations of auctions satisfying assumptions A1-A5.

One more challenging question is of how to extend the existence result presented in Theorem 9 to auctions with common value elements. Real-life auctions tend to have features of private and common values. Although in common value auctions a bidder does not know her value exactly, she typically prefers winning to losing a tie. By reasoning similar to the one applied in this article one might conjecture that the bidder can always bid in such a way as to assure herself of winning the tie implying no tie in equilibrium.

7 APPENDIX

Let me discuss the conditions on primitives that are sufficient for the existence of a specific upper bound $\bar{b}_{i,j}(v_i)$ used in assumption A5 in Section 2. The following two lemmas formulate simple conditions on winning and losing price functions that guarantee the existence of an upper bound $\bar{b}_{i,j}(v_i)$ and that are applicable to all auctions in example 1.

Lemma 14 *Let us assume that there exists strictly increasing continuous strategy* $\tilde{b}_i(\cdot) \in B_i$ *such that if* $b_{i,j}$ *is tied, then* $p^w_{i,j}(b) - p^l_{i,j}(b) = \tilde{b}^{-1}_{i,j}(b_i)$. *Then* $\tilde{b}_i(\cdot)$ *is an upper bound satisfying A5.*

Proof. First realize that for every v_i and bid $b_i \in \mathcal{B}_i$ below $\tilde{b}_i(v_i)$, each bidder strictly prefers winning to losing a tie because in ties $p^w_{i,j}(b) - p^l_{i,j}(b) = \tilde{b}^{-1}_{i,j}(b_i) < \tilde{b}^{-1}_{i,j}\left(\tilde{b}_i(v_i)\right) = v_{i,j}$ where the inequality is valid because $\tilde{b}_{i,j}(\cdot)$ is strictly increasing. Therefore $v_{i,j} - p^w_{i,j}(b) > -p^l_{i,j}(b)$ and the second point of A5 is satisfied.

Let us fix v_i and b_i where b_i exceeds $\tilde{b}_i(v_i)$ in at least one component. We can then show that a new bid vector $\tilde{b}_i = \left[\min\left(b_{i,j}, \tilde{b}_{i,j}(v_{i,j})\right)\right]_{j=1}^{k}$ makes the bidder at least as good as b_i. A bid k-tuple $\tilde{b}_i$ is an element of feasible bid set $\mathcal{B}_i$ because b_i and $\tilde{b}_i(v_i)$ are elements of $\mathcal{B}_i$. I will proceed by induction in a bid component showing that in each step to bid $\left[b_{i,1}, \cdots, b_{i,j}, \tilde{b}_{i,j+1}, \ldots, \tilde{b}_{i,k}\right]$ is at least as good as to bid $\left[b_{i,1}, \cdots, b_{i,j}, \tilde{b}_{i,j+1}, \ldots, \tilde{b}_{i,k}\right]$. Let me start with $j = k$. It is oblivious that a bidder's payoff is the same when $b_{i,j} = \tilde{b}_{i,j}$. Let me focus on the opposite case when $b_{i,j} > \tilde{b}_{i,j}$. If both $\tilde{b}_{i,j}$ and $b_{i,j}$ either win or lose jth unit, then the bidder is not worse off because decreasing the jth unit bid weakly decreases the price by A2, and hence, bidding

$b_{i,j}$ is at least as good as bidding $\tilde{b}_{i,j}$. Otherwise $b_{i,j}$ wins a unit but $\tilde{b}_{i,j}$ does not. In this case there exist a tied bid $b'_{i,j} \in \left[\tilde{b}_{i,j}, b_{i,j}\right]$ and I denote $\tilde{b}_i^j = \left[b_{i,1}, \cdots, b_{i,j-1}, b'_{i,j}, \tilde{b}_{i,j+1}, \ldots, \tilde{b}_{i,k}\right]$. In such a tie, bidder i weakly prefers to lose than to win because $p^w_{i,j}\left(\tilde{b}_i^j, b_{-i}\right) - p^l_{i,j}\left(\tilde{b}_i^j, b_{-i}\right) = \tilde{b}^{-1}_{i,j}\left(\tilde{b}_i^j, b_{-i}\right) \geq \tilde{b}^{-1}_{i,j}\left(b_{i,1}, \cdots, \tilde{b}_{i,j}, \ldots, \tilde{b}_{i,k}\right) = v_{i,j}$. By assumption A2 the following chain of inequalities can be derived

$$-p^l_{i,j}\left(\tilde{b}_{i,j}, \tilde{b}^j_{i,-j}, b_{-i}\right) \geq -p^l_{i,j}\left(\tilde{b}^j_i, b_{-i}\right) \geq v_{i,j} - p^w_{i,j}\left(\tilde{b}^j_i, b_{-i}\right) \geq v_{i,j} - p^w_{i,j}\left(b_{i,j}, \tilde{b}^j_{i,-j}, b_{-i}\right)$$

This chain says that after decreasing her bid to $\tilde{b}_{i,j}$ the losing price makes the bidder at least as well off as if bidding $b'_{i,j}$ and that after increasing her bid to $b_{i,j}$ the winning price does not make her better off than if bidding $b'_{i,j}$. Repeating induction steps until $j = 0$ one concludes that bidding $\tilde{b}_i$ is at least as good as bidding b_i. In other words, no bidder has an incentive to bid above $\tilde{b}_{i,j}(v_{i,j})$. By implication $\tilde{b}_{i,j}(v_{i,j})$ satisfies the second point of A5. ∎

For the linear combination of auctions in example 1 the winning price in ties is $p^w_{i,j}(b) = b_{i,j}$ and the losing price in ties is $p^l_{i,j}(b) = \lambda_a \cdot b_{i,j}$. Therefore if $0 \leq \lambda_a < 1$, then a strictly increasing continuous strategy to that lemma 14 is applicable is $[(1-\lambda_a) \cdot b_{i,j}]_{j=1}^k$ .For all-pay auctions when $\lambda_a = 1$, winning and losing prices are equal in ties $p^w_{i,j}(b) = b_{i,j} = p^l_{i,j}(b)$, and therefore $\tilde{b}_{i,j}(v_{i,j}) = 0$ which is not strictly increasing. For such an auction the following lemma is applicable.

Lemma 15 *Let us assume that there exists a strictly increasing continuous strategy $\tilde{b}_i(\cdot) \in B_i$ and $K \geq 0$ with the following properties:*

1. $p^w_{i,j}(b) - p^l_{i,j}(b) \leq \tilde{b}^{-1}_{i,j}(b_i)$ *if* $b_{i,j}$ *is tied,*

2. $v_{i,j}+K\leq p^{w}_{i,j}\left(\tilde{b}_{i,j}\left(v_{i,j}\right),b_{i,-j},b_{-i}\right)$ *and* $K\leq p^{l}_{i,j}\left(\tilde{b}_{i,j}\left(v_{i,j}\right),b_{i,-j},b_{-i}\right)$,

3. $p^{w}_{i,j}\left(0,b_{i,-j},b_{-i}\right)\leq K$ *and* $p^{l}_{i,j}\left(0,b_{i,-j},b_{-i}\right)\leq K$, *and*

4. *For any submitted bids of all bidders and any pair of equal bids of bidder i that both either win or lose, the price of the unit with the higher index value is lower or equal to the price of the unit with the lower index value; i.e. winning and losing prices are weakly decreasing in the unit number.* [19]

Then $\tilde{b}_i(\cdot)$ *is an upper bound satisfying A*5.

Proof. First realize that for every v_i and bid $b_i\in\mathcal{B}_i$ below $\tilde{b}_i(v_i)$ each bidder strictly prefers winning to losing a tie because in ties $p^{w}_{i,j}(b)-p^{l}_{i,j}(b)\leq\tilde{b}^{-1}_{i,j}(b_i)<\tilde{b}^{-1}_{i,j}\left(\tilde{b}_i(v_i)\right)=v_{i,j}$ where the second inequality is holds because $\tilde{b}_{i,j}(\cdot)$ is strictly increasing. Therefore $v_{i,j}-p^{w}_{i,j}(b)>-p^{l}_{i,j}(b)$ and the second point of A5 is satisfied.

Let me construct a sequence of bid k-tuples $\tilde{b}^{m}_{i}=\left[\tilde{b}^{m}_{i,j}\right]^{k}_{j=1}$ with initial $\tilde{b}^{0}_{i}=b_i$. Then if $\tilde{b}^{m}_{i,j}\leq\tilde{b}_{i,j}(v_i)$ for all j the iteration process stops and $\tilde{b}^{m}_{i}$ is at least as good as b_i and $\tilde{b}_{i,j}(v_{i,j})$ satisfies the second point of A5; otherwise I take the highest unit number j for that a bidder submits a bid above $\tilde{b}_{i,j}(v_i)$ ($\tilde{b}^{m}_{i,j}>\tilde{b}_{i,j}(v_i)$) and show that if a bidder bids $\min(0,b_{i,k})$ instead of $\tilde{b}^{m}_{i,j}$, then she is not worse off. In the new bid k-tuple bids $\tilde{b}^{m}_{i,j+1},\cdots,\tilde{b}^{m}_{i,k}$ correspond to units $v_{i,j},\cdots,v_{i,k-1}$; i.e. $\tilde{b}^{m+1}_{i}=\left[\tilde{b}^{m}_{i,1},\cdots,\tilde{b}^{m}_{i,j-1},\tilde{b}^{m}_{i,j+1},\cdots,\tilde{b}^{m}_{i,k},\min(0,b_{i,k})\right]$.

Let me decompose a change from $\tilde{b}^{m}_{i}$ to $\tilde{b}^{m+1}_{i}$ into the following steps

[19] For any $b\in B_i\times B_{-i}$ and any $j'=1,\cdots,j$ if $b_{i,j'}=b_{i,j}\geq\max(c_{k+1-j},R)$, then $p^{w}_{i,j'}(b)\leq p^{w}_{i,j'+1}(b)\leq\cdots\leq p^{w}_{i,j}(b)$ and if $b_{i,j'}=b_{i,j}\leq\max(c_{k+1-j'},R)$, then $p^{l}_{i,j'}(b)\leq p^{l}_{i,j'+1}(b)\leq\cdots\leq p^{l}_{i,j}(b)$.

$$\tilde{b}_i^m = \left[\tilde{b}_{i,1}^m, \cdots, \tilde{b}_{i,j}^m, \cdots, \tilde{b}_{i,k}^m\right]$$
$$\left[\tilde{b}_{i,1}^m, \cdots, \tilde{b}_{i,j-1}^m, \tilde{b}_{i,j+1}^m, \tilde{b}_{i,j+1}^m, \tilde{b}_{i,j+2}^m \cdots, \tilde{b}_{i,k}^m\right]$$
$$\vdots$$
$$\left[\tilde{b}_{i,1}^m, \cdots, \tilde{b}_{i,j-1}^m, \tilde{b}_{i,j+1}^m, \cdots, \tilde{b}_{i,j+s}^m, \tilde{b}_{i,j+s}^m, \cdots, \tilde{b}_{i,k}^m\right]$$
$$\vdots$$
$$\left[\tilde{b}_{i,1}^m, \cdots, \tilde{b}_{i,j-1}^m, \tilde{b}_{i,j+1}^m, \cdots, \tilde{b}_{i,k-1}^m, \tilde{b}_{i,k}^m, \tilde{b}_{i,k}^m\right]$$
$$\tilde{b}_i^{m+1} = \left[\tilde{b}_{i,1}^m, \cdots, \tilde{b}_{i,j-1}^m, \tilde{b}_{i,j+1}^m, \cdots, \tilde{b}_{i,k}^m, \min(0, b_{i,k})\right]$$

Along the k-tuple bid sequence above, a bid on the $j+s-1$st unit drops from $\tilde{b}_{i,j+s-1}^m$ to $\tilde{b}_{i,j+s}^m$ for $s = 1, \cdots, k-j$ and in the last step $\tilde{b}_{i,k}^m$ drops to $\min(0, b_{i,k})$. Using the 4th property of the preceding lemma, the fact that the marginal value from winning an additional unit weakly decreases ($v_{i,j} \geq v_{i,j'}$ if $j < j'$), and A2, it is obvious that if the bidder bids $\tilde{b}_{i,j+s}^m$ on the $j+s-1$st unit, she is not worse off than if she bids $\tilde{b}_{i,j+s}^m$ on the $j+s$th unit because $c_{k+2-j-s} \geq c_{k+1-j-s}$. Moreover, the payoff from bidding $\tilde{b}_{i,j}^m$ on the jth unit yields a payoff less than $-K$ by the second property of the preceding lemma. By the 3. property of the preceding lemma, this payoff is at least as good as bidding $\min(0, b_{i,k})$ on the kth unit. Therefore bidding $\tilde{b}_i^{m+1}$ is at least as good as bidding $\tilde{b}_i^m$ which completes the proof. ∎

For all-pay auction combinations with pay-your-bid, uniform-price and Dutch auctions of example 1 ($\lambda_a + \lambda_p + \lambda_u + \lambda_d = 1$) lemma 15 is applicable for $K = 0$ and $\tilde{b}_{i,j}(v_i) = v_{i,j}$. The Vickrey auction cannot be included because the 4th property of the preceding lemma is too restrictive.

Lemma 16 *Pure strategy spaces $\hat{B}_i$ and $\hat{M}_i$ are compact metric spaces.*

Proof. It is a standard result of probability theory that the set of probability distribution functions $\mathcal{P}_i = \{\omega_i : \mathcal{V}_i \to [0,1]\}$ is a compact metric space in the topology of almost everywhere pointwise convergence for any $i = 1, \cdots, n$. There is a one-to-one mapping from $\mathcal{P}_i$ to the set of weakly increasing functions $\hat{B}_i = \{b_i : \mathcal{V}_i \to [0, \bar{v}_i]\}$ and, therefore, it is a compact metric space in which two strategies are considered to be identical if they are equal almost everywhere with respect to Lebesgue measure similarly as in L_p spaces. From Billingsly (1968) we know that if $\hat{B}_i$ is a compact metric space, then the set $\hat{M}_i$ of Borel probability measures on $\hat{B}_i$ is a compact metric space. ∎

Proof of Lemma 2. Consider a profile $b_i(\cdot), m_{-i}$ or $b_i(\cdot), b_{-i}$ in which a tie that the bidder loses with positive probability occurs with positive probability. Let me denote any bid for which a tie occurs as $b_{i,j}$ (for some $i = 1, \cdots, n$ and $j = 1, \cdots k$). For simplicity let me at first analyze the case that $b_{i,j-1} > b_{i,j} > b_{i,j+1}$ in the tie.

Consider the set of values for which the bidder i is indifferent between winning and losing a tie when she bids $b_{i,j}$, and the tie is not broken in her favor surely. Since $b_i(\cdot) \in \hat{B}_i$, then the set of such values is empty or has measure zero because the distribution of values is continuous by A1 and the bidder strictly prefers winning to losing for bids strictly below strictly increasing upper bound $\bar{b}_{i,j}(\cdot)$ by the second point of A5, and, therefore, I can neglect it.

Consider the set of values v_i for which the bidder strictly prefers winning the tied unit to losing it when she bids $b_{i,j}$, and the tie is not broken in her favor surely. If instead the tie is surely broken in her favor for any v_i from this set, her *ex ante* payoff jumps up. The bidder who increases her bids on jth unit by a sufficiently small ε is better off because she wins the tied unit surely and her *ex post* payoff (1) is negligibly smaller than if winning the unit at the tie by A4 and A2 and, hence, $b_{i,j} + \varepsilon = b_{i,j}^{\varepsilon}(v_i)$ is a better response than $b_{i,j}$. Since the bidder uses strategy $b_i(\cdot)$ from $\hat{B}_i$, then for any such v_i there exists sufficiently small ε such that $b_{i,j} + \varepsilon \leq \bar{b}_{i,j}(\cdot)$ because $\bar{b}_{i,j}(\cdot)$ is strictly increasing by A5. Moreover the bidder may be forced to increase her bids for other values for that she submits jth bid in the interval $(b_{i,j}, b_{i,j} + \varepsilon)$. But for sufficiently small ε the set of these values is arbitrarily small having negligible contribution on the *ex ante* payoff by A2 and A1 and, hence $b_i^{\varepsilon}(\cdot) \in \hat{B}_i$.

If it happens that $b_{i,j} = b_{i,j+1} = \cdots = b_{i,j'}$ in the tie, then the bidder can also increase her bids on $j + 1$st, $\cdots$, j'th units in the interval $(b_{i,j}, b_{i,j} + \varepsilon)$. Using the same arguments as for the jth unit bid, the payoff either jumps up (if some of the bids $b_{i,j} = b_{i,j+1} = \cdots = b_{i,j'}$ are tied) or it has negligible effect for sufficiently small ε on the bidder's payoff by A5, A4, A2 and A1. Moreover $b_i^{\varepsilon}(\cdot)$ is still in $\hat{B}_i$ because $\bar{b}_{i,j}(\cdot) \leq \bar{b}_{i,j+1}(\cdot) \leq \cdots \leq \bar{b}_{i,j'}(\cdot)$ by A5.

In other words for all ties that occur with positive probability in a strategy profile each bidder strictly prefers to increase her strategy, she does so and as $\varepsilon \to 0$ she gains the payoff arbitrarily close to the payoff when breaking every tie into her favor.

To finish Lemma 2 it is necessary to check that if any tie occurs with positive probability there is an ε for which no new tie occurs after the bidder changes the tied strategy by ε. By the standard results of probability theory each bidder places *ex ante* positive probability on at most countably many bids. Then there is at most countably many ties in any original strategy profile and for at most countably many ε a tie occurs in the new profile after one bidder increases his strategy by ε. ∎

Lemma 17 *The payoff in an auction game with any tie-breaking rule consistent with* ex post *payoff (1) and assumptions A1, A4, A6 and A5 is locally payoff secure (see definition 6).*

Proof. Consider any $b_i(\cdot) \in m_i$ in any profile $m \in \hat{M}$. Let me define $b_i^{\varepsilon}(\cdot)$ to be $b_i(\cdot)$ if no tie occurs when using $b_i(\cdot), m_{-i}$ and the strategy constructed in lemma 2 otherwise. Then $\pi_i(b_i^{\varepsilon}(\cdot), m_{-i}) \geq \pi_i(b_i(\cdot), m_{-i})$ and no tie occurs when using $b_i^{\varepsilon}(\cdot)$, m_{-i} whether there is a tie in $b_i(\cdot)$, m_{-i} or not.

Using the standard results of probability theory (e.g. in Billingsley (1968)) for any sequence $m''_{-i} \to m_{-i}$ $\lim_{m''_{-i} \to m_{-i}} \pi_i(b_i^{\varepsilon}(\cdot), m''_{-i}) = \pi_i(b_i^{\varepsilon}(\cdot), m_{-i})$ because the *ex post* "out-of-ties" payoff is continuous in opponent strategies by A4.

Combining the two findings above, (11) is valid for any pure strategy $b_i(\cdot) \in m_i$, and, therefore, for any mixed strategy m_i (see footnote 14). ∎

Lemma 18 *An auction game with the efficient tie-breaking rule satisfying assumptions A1, A4 and A5 is summation upper-semi continuous (see definition 7).*

Proof. Consider any realization of values $v = [v_i]_{i=1}^{n}$ of all bidders. Using their chosen strategy the bidders submit bids b and receive total payoff $\sum_i \pi_i(v_i, b, R)$. At first I show that for any sequence of bids $b' \to b$ the *ex post* payoff sum is upper-semi continuous.

If the set of winning and losing bids in the limit of b' is the same as in b, the allocation of units is the same and $\sum_i \pi_i (v_i, b, R)$ is continuous because the *ex post* price is continuous by A4. If the set of winning and losing bids in the limit of b' is not the same as in b, units might be allocated to different bidders (see Example 8). But the efficient tie–breaking rule maximizes the payoff sum at b and therefore it cannot be below the payoff sum in the limit of b' by A4.

If the bidder uses mixed strategies, and one takes expectations over all realizations of values v (see Billingsley (1968)), the upper-semi continuity of the sum of bidder payoffs is preserved. For an alternative proof see Wagner (1977). ∎

Proof of the proposition 11. For any measurable strategy $b_i (\cdot)$ for all v_i (up to a measure 0 subset), $b_i (v_i)$ must be the pointwise best response $b_i (v_i)$ can be improved upon. Similarly for any pure strategy $b_i (\cdot)$ from the support of the mixed equilibrium strategy m_i (up to a measure 0 subset) for all v_i (up to a measure 0 subset), $b_i (v_i)$ must be the pointwise best response otherwise m_i can be improved upon. In addition, no tie occurs with positive probability by lemma 2 when using $b_i (\cdot)$. Therefore the pointwise payoff from $b_i (v_i)$ is independent of the way the seller breaks the ties for all v_i (up to a measure 0 subset). Without a loss of generality I will assume that bidder i wins every tie surely. Then his pointwise payoff (8) given the opponent strategies and reservation bid R is

$$\begin{aligned}\pi_i\left(v_i,b_i\right) &= \sum_{1\le j\le k} E\left(\left(v_{i,j}-p^w_{i,j}\left(b\right)\right)\cdot I\left(b_{i,j}\ge \max\left(c_{k+1-j},R\right)\right)\right) - \\ &\quad \sum_{1\le j\le k} E\left(p^l_{i,j}\left(b\right)\cdot I\left(\max\left(c_{k+1-j},R\right)>b_{i,j}\right)\right). \qquad (13)\end{aligned}$$

At first let me show that for any two values v_i and v_i' from $\mathcal{V}_i$ with $v_{i,j}\ge v_{i,j}'$ for some $j=1,\cdots,k$ and their pointwise best response bids b_i and b_i' from $\mathcal{B}_i$ with $b_{i,j}<b_{i,j}'$, the probabilities of winning of jth unit are the same for any jth bid between $b_{i,j}$ and $b_{i,j}'$.

Consider any j such that $b_{i,j}<b_{i,j}'$ and let p and q $(1\le p\le j\le q)$ be the smallest and largest j such that $b_{i,p}<b_{i,p}',\cdots\cdots,b_{i,j}<b_{i,j}',\cdots,b_{i,q}<b_{i,q}'$. The following bids $\hat{b}_i=\left(b_{i,1},\cdots,b_{i,p-1},b_{i,p}',\cdots,b_{i,q}',b_{i,q+1},\cdots,b_{i,k}\right)$ and $\hat{b}_i'=\left(b_{i,1}',\cdots,b_{i,p-1}',b_{i,p},\cdots,b_{i,q},b_{i,q+1}',\cdots,b_{i,k}'\right)$ have weakly increasing components and hence are elements of $\mathcal{B}_i$ because b_i and b_i' are elements of $\mathcal{B}_i$ and because of the following inequality strings $b_{i,p-1}\ge b_{i,p-1}'\ge b_{i,p}'$, $b_{i,q}'>b_{i,q}\ge b_{i,q+1}$, $b_{i,p-1}'\ge b_{i,p}'>b_{i,p}$ and $b_{i,q}\ge b_{i,q+1}\ge b_{i,q+1}'$.

Then the following best response *ex ante* inequalities are true

$$\pi_i(v_i,b_i)\ge\pi_i(v_i,\hat{b}_i) \text{ and } \pi_i(v_i',b_i')\ge\pi_i(v_i',\hat{b}_i'). \qquad (14)$$

After summing (14), substituting (13) and using A3 one gets,

$$\sum_{p\le j'\le q}\left(v_{i,j'}'-v_{i,j'}\right)\cdot\left(P_{j'}\left(b_{i,j'}\right)-P_{j'}\left(b_{i,j'}'\right)\right)\ge 0, \qquad (15)$$

where $P_{j'}\left(b\right)$ is the probability that the bidder i wins j'th unit when he submits bid b on that unit above R.

Since the probability of winning a unit $P_{j'}\left(b\right)$ is weakly increasing in i's j'th bid and $b_{i,j'}>b_{i,j'}'$ for $j'=p,\cdots,q$ by definition, the terms $P_{j'}\left(b_{i,j'}\right)-$

$P_{j'}\left(b'_{i,j'}\right) \geq 0$ in the above sum. Since $v'_{i,j'} - v_{i,j'} \leq 0$ by assumption it must be that

$$P_{j'}\left(b_{i,j'}\right) - P_{j'}\left(b'_{i,j'}\right) = E\left(I\left(b_{i,j} \leq c_{k+1-j} < b'_{i,j}\right)\right) = 0 \qquad (16)$$

whenever $v'_{i,j'} - v_{i,j'} < 0$ for $j' = 1, \cdots, k$ that shows that the probabilities of winning are the same. In addition, since b'_i is the best response and winning and losing prices are weakly increasing in bids, it must be that prices for any bids $\hat{b}_i$ where $b'_{i,j} \geq \hat{b}_{i,j} \geq \hat{b}'_{i,j}$ are the same. This implies that for all $\hat{b}_i$ the pointwise payoff $\pi_i\left(v'_i, \hat{b}_i\right)$ is the same. This result is similar to McAdams (2006) on allocation and interim expected payment equivalence of multi-unit uniform-price auction.

It implies that the best response is not pure weakly increasing only in the regions where no $k+1-j$th opponent highest bid falls $\left(E\left(I\left(b_{i,j} \leq c_{k+1-j} < b'_{i,j}\right)\right) = 0\right)$. But when $p^w_{i,j}(\cdot)$ strictly increases in $b_{i,j}$, then any bid in this region above $b_{i,j}$ is worse than bidding $b_{i,j}$ and hence even in this region the strategy is pure weakly increasing. That completes the proof of the last part proposition *11.* It remains to check that when $p^w_{i,j}(\cdot)$ is constant for bids in the interval $\left[b_{i,j}, b'_{i,j}\right]$ the iteration process described above the proposition *11* converges to the pure weakly increasing best response strategy generating the same distribution of bids.

In the beginning part of the proposition I define the strategy such that for some j and any $v_{i,-j}$ I define the strategy $\hat{b}_i(\cdot)$ which is weakly increasing in $v_{i,j}$ and generates the same distribution of bids as m_i;

$$P\left(\hat{b}_{i,j}(v_i) < b_{i,j} | v_{i,-j}\right) = P\left(m_{i,j} < b_{i,j} | v_{i,-j}\right). \qquad (17)$$

For every $v_{i,-j}$ it is a kind of "average" of all pure strategies $b_i(\cdot)$ in support of m_i. Therefore by the standard result of probability theory and Fubini's theorem $\hat{b}_i(\cdot) \in \hat{B}_i$ and $\hat{b}_i(\cdot)$ and m_i generate the same distribution of bids. Moreover $\hat{b}_i(\cdot)$ is within the two strategies in the support of m_i and therefore it is the best response. Moreover following the steps of McAdams (2006) one can conclude that the iteration process converges the strategy converges to the pure weakly increasing best response strategy generating the same distribution of bids as the original mixed strategy m_i. Similar results can be obtained for measurable best response strategy. ∎

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