ON SUPERMAGIC REGULAR GRAPHS

Jaroslav Ivančo, Košice

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Abstract. A graph is called supermagic if it admits a labelling of the edges by pairwise different consecutive positive integers such that the sum of the labels of the edges incident with a vertex is independent of the particular vertex. Some constructions of supermagic labellings of regular graphs are described. Supermagic regular complete multipartite graphs and supermagic cubes are characterized.

Keywords: supermagic graphs, complete multipartite graphs, products of graphs

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1. Introduction

We consider finite undirected graphs without loops, multiple edges and isolated vertices. If G is a graph, then $V(G)$ and $E(G)$ stand for the vertex set and edge set of G, respectively. Cardinalities of these sets, denoted $|V(G)|$ and $|E(G)|$, are called the *order* and *size* of G.

Let a graph G and a mapping f from $E(G)$ into positive integers be given. The *index-mapping* of f is a mapping f^* from $V(G)$ into positive integers defined by

$$f^*(v) = \sum_{e\in E(G)} \eta(v,e)f(e) \quad \text{for every } v \in V(G), \tag{1}$$

where $\eta(v,e)$ is equal to 1 when e is an edge incident with a vertex v, and 0 otherwise. An injective mapping f from $E(G)$ into positive integers is called a *magic labelling* of G for the *index* λ if its index-mapping f^* satisfies

$$f^*(v) = \lambda \quad \text{for all } v \in V(G). \tag{2}$$

A magic labelling f of G is called a *supermagic labelling* of G if the set $\{f(e)\colon e \in E(G)\}$ consists of consecutive positive integers. We say that a graph G is *supermagic* (*magic*) if and only if there exists a supermagic (magic) labelling of G.

The concept of magic graphs was introduced by Sedláček [6]. The regular magic graphs are characterized in [3]. Two different characterizations of all magic graphs are given by S. Jezný, M. Trenkler [5] and R. H. Jeurissen [4].

Supermagic graphs were introduced by M. B. Stewart [8]. It is easy to see that the classical concept of a magic square of n^2 boxes corresponds to the fact that the complete bipartite graph $K_{n,n}$ is supermagic for every positive integer $n \neq 2$ (see also [8], [2]). M. B. Stewart [9] proved that the complete graph K_n is supermagic if and only if either $n \geqslant 6$ and $n \not\equiv 0 \pmod 4$, or $n = 2$. In [7] and [1], supermagic labellings of the Möbius ladders and two special classes of 4-regular graphs are constructed.

In this paper we describe some constructions of supermagic labellings of regular graphs and apply them to complete multipartite graphs and Cartesian products of circuits.

2. Conditions and constructions

Throughout the paper let $\mathfrak{S}$ and $\mathfrak{S}(d)$ denote the set of all supermagic regular graphs and the set of all supermagic d-regular graphs, respectively. Note that if f is a supermagic labelling of $G \in \mathfrak{S}$, then $f + m$, for every integer $m > -\min\{f(e)\colon e \in E(G)\}$, is a supermagic labelling of G, too. Therefore, a regular graph G is supermagic if and only if it admits a supermagic labelling $f\colon E(G) \to \{1, 2, \ldots, |E(G)|\}$. In what follows we will consider only such supermagic labellings of regular graphs. In this case, the conditions (1) and (2) require

$$\begin{aligned} |V(G)|\lambda &= \sum_{v\in V(G)} \sum_{e\in E(G)} \eta(v,e) f(e) \\ &= 2 \sum_{e\in E(G)} f(e) = \bigl(1 + |E(G)|\bigr)|E(G)|. \end{aligned}$$

Since the size of a d-regular graph satisfies $|E(G)| = \frac{d}{2}|V(G)|$, we get

$$\lambda = \tfrac{d}{2}\bigl(1 + \tfrac{d}{2}|V(G)|\bigr). \tag{3}$$

Now, we can prove the following necessary conditions for a supermagic regular graph.

Proposition 1. *Let $G \in \mathfrak{S}(d)$. Then the following statements hold:*

(i) *if $d \equiv 1 \pmod 2$, then $|V(G)| \equiv 2 \pmod 4$;*
(ii) *if $d \equiv 2 \pmod 4$ and $|V(G)| \equiv 0 \pmod 2$, then G contains no component of an odd order;*
(iii) *if $|V(G)| > 2$, then $d > 2$.*

P r o o f. Let us assume to the contrary that $d \equiv 1 \pmod 2$ and $|V(G)| \equiv 0 \pmod 4$. Then by (3), the index λ of a supermagic labelling of G is not an integer, and by (1), λ is a sum of integers, a contradiction. As the order of a regular graph of an odd degree is even, the condition (i) follows.

Suppose that $d \equiv 2 \pmod 4$, $|V(G)| \equiv 0 \pmod 2$ and G contains a component C of an odd order. Then by (3), λ is odd. Hence $\lambda|V(C)|$ is odd, too. On the other hand, by (1) and (2),

$$|V(C)|\lambda = \sum_{v\in V(C)} \sum_{e\in E(G)} \eta(v,e)f(e) = 2 \sum_{e\in E(C)} f(e),$$

a contradiction.

It is obvious that a regular graph of degree one is magic if and only if it is connected (i.e. $|V(G)| = 2$), and a 2-regular graph is never magic. □

Given graphs H and G, a *homomorphism* of H onto G is defined to be a surjective mapping $\psi\colon V(H) \to V(G)$ such that whenever u, v are adjacent in H, $\psi(u)$, $\psi(v)$ are adjacent in G. So ψ induces a mapping $\overline{\psi}\colon E(H) \to E(G)$ satisfying: if e is an edge of H with end vertices u and v, then $\overline{\psi}(e)$ is an edge of G with end vertices $\psi(u)$ and $\psi(v)$. We say that a homomorphism ψ is *harmonious* if $\overline{\psi}$ is a bijection, and *balanced* if $|\psi^{-1}(u)| = |\psi^{-1}(v)|$ for all $u, v \in V(G)$. A bijective harmonious homomorphism of H onto G is called an *isomorphism* of H onto G. If there is an isomorphism of H onto G, then we say that H is a *copy* of G. A triplet $[H, \psi, t]$ is called a *supermagic frame* of a graph G if ψ is a harmonious homomorphism of H onto G and $t\colon E(H) \to \{1, 2, \ldots, |E(H)|\}$ is an injective mapping such that $\sum_{u\in\psi^{-1}(v)} t^*(u)$ is independent of the vertex $v \in V(G)$.

Proposition 2. *If there is a supermagic frame of a graph G, then G is supermagic.*

P r o o f. Let us assume that $[H, \psi, t]$ is a supermagic frame of G. It can be easily seen that a mapping f given by $f(e) = t(\overline{\psi}^{-1}(e))$ for every $e \in E(G)$, is a supermagic labelling of the graph G. □

Corollary 1. *Let $H \in \mathfrak{S}$ and G be graphs. If there is a balanced harmonious homomorphism of H onto G, then $G \in \mathfrak{S}$.*

P r o o f. Suppose that ψ is a balanced harmonious homomorphism of H onto G. Since H is regular and ψ is balanced, G is regular. As $H \in \mathfrak{S}$, there is a supermagic labelling f of H. Clearly, $[H, \psi, f]$ is a supermagic frame of G. By Proposition 2, the assertion follows. □

Recall that a *δ-factor* of a graph is defined to be its δ-regular spanning subgraph. In what follows, let $\mathfrak{F}(k)$, for a positive integer $k \geqslant 2$, denote the set of all regular graphs which can be decomposed into k pairwise edge-disjoint δ-factors.

The union of two disjoint graphs G and H is denoted by $G \cup H$ and the union of $m \geqslant 1$ disjoint copies of a graph G is denoted by mG.

Corollary 2. *Let $G \in \mathfrak{S} \cap \mathfrak{F}(k)$. Then the following statements hold:*

(i) *if k is even, then $mG \in \mathfrak{S}$ for every positive integer m;*
(ii) *if k is odd, then $mG \in \mathfrak{S}$ for every odd positive integer m.*

Proof. Since $G \in \mathfrak{S}$, there is a supermagic labelling f of G for the index λ. We have $G \in \mathfrak{F}(k)$ and so there exist edge-disjoint δ-factors $F^1, F^2, \ldots, F^k$ which form a decomposition of G. Note that F^i is a factor of G, i.e. $V(F^i) = V(G)$ and $E(F^i) \subseteq E(G)$. For $i = 1, \ldots, k$ and $j = 1, \ldots, m$, let G_j, F^i_j, ξ_j and φ^i_j be a copy of G, a copy of F^i, an isomorphism of G_j onto G and an isomorphism of F^i_j onto F^i, respectively. Suppose that the graph H is a disjoint union of graphs F^i_j, i.e. $H = \bigcup\limits_{j=1}^{m} \bigcup\limits_{i=1}^{k} F^i_j$, and assume that the graph mG is a disjoint union of graphs G_j, i.e. $mG = \bigcup\limits_{j=1}^{m} G_j$. Clearly, a mapping $\psi\colon V(H) \to V(mG)$ given by $\psi(v) = \xi_j^{-1}(\varphi^i_j(v))$ when $v \in V(F^i_j)$, is a balanced harmonious homomorphism of H onto mG. Now we distinguish the following cases:

Case 1. Let k be even. Consider a mapping t_0 from $E(H)$ into positive integers given by $t_0(e) = f(\overline{\varphi}^i_j(e)) + r^i_0(j)|E(G)|$ whenever $e \in E(F^i_j)$, where

$$r^i_0(j) = \begin{cases} j-1 & \text{if } i \equiv 1 \pmod 2,\\ m-j & \text{if } i \equiv 0 \pmod 2. \end{cases}$$

Obviously, for any $e \in E(H)$, we have $1 \leqslant t_0(e) \leqslant m|E(G)| = |E(H)|$. Suppose that e_1, e_2 are edges of H satisfying $t_0(e_1) = t_0(e_2)$, i.e. $f(\overline{\varphi}^x_p(e_1)) + r^x_0(p)|E(G)| = f(\overline{\varphi}^y_q(e_2)) + r^y_0(q)|E(G)|$. Since f is an injective mapping onto the set $\{1, 2, \ldots, |E(G)|\}$, then $\overline{\varphi}^x_p(e_1) = \overline{\varphi}^y_q(e_2)$ and $r^x_0(p) = r^y_0(q)$. As φ^x_p (φ^y_q) is an isomorphism onto F^x (F^y), then $x = y$. For a fixed integer x, r^x_0 is an injective mapping and so $p = q$. Therefore, we have $\overline{\varphi}^x_p(e_1) = \overline{\varphi}^x_p(e_2)$. Hence $e_1 = e_2$. This means that t_0 is an injective mapping from $E(H)$ onto $\{1, 2, \ldots, |E(H)|\}$. Moreover, for a given vertex v of G_j, the index-mapping of t_0 satisfies

$$\begin{aligned}\sum_{u\in\psi^{-1}(v)} t_0^*(u) &= \sum_{i=1}^{k} t_0^*\big((\varphi_j^i)^{-1}(\xi_j(v))\big)\\ &= \sum_{i=1}^{k}\big(\delta|E(G)|r_0^i(j) + \sum_{e\in E(F^i)}\eta(\xi_j(v),e)f(e)\big)\\ &= \delta|E(G)|\sum_{i=1}^{k} r_0^i(j) + \sum_{e\in E(G)}\eta(\xi_j(v),e)f(e)\\ &= \delta|E(G)|(\tfrac{k}{2}(j-1)+\tfrac{k}{2}(m-j)) + f^*(\xi_j(v))\\ &= \delta|E(G)|\tfrac{k}{2}(m-1)+\lambda.\end{aligned}$$

Thus $[H,\psi,t_0]$ is a supermagic frame of the graph mG and by Proposition 2, (i) follows.

C a s e 2. Let k and m be odd. Consider a mapping t_1 from $E(H)$ into positive integers given by $t_1(e) = f(\bar\varphi_j^i(e)) + r_1^i(j)|E(G)|$ whenever $e\in E(F_j^i)$, where

$$r_1^i(j) = \begin{cases} j-1 & \text{if } i\equiv 1 \pmod 2 \text{ and } i<k,\\ m-j & \text{if } i\equiv 0 \pmod 2 \text{ and } i<k-1,\\ j+\frac{m-3}{2} & \text{if } i=k-1 \text{ and } j\leqslant \frac{m+1}{2},\\ j-\frac{m+3}{2} & \text{if } i=k-1 \text{ and } j> \frac{m+1}{2},\\ m-2j+1 & \text{if } i=k \text{ and } j\leqslant \frac{m+1}{2},\\ 2m-2j+1 & \text{if } i=k \text{ and } j> \frac{m+1}{2}.\end{cases}$$

Similarly as in the case 1, it can be seen that t_1 is an injective mapping from $E(H)$ onto $\{1,2,\dots,|E(H)|\}$ and its index-mapping satisfies $\sum_{u\in\psi^{-1}(v)} t_1^*(u) = \delta|E(G)|\frac{k}{2}(m-1)+\lambda$ for any vertex v of mG. So $[H,\psi,t_1]$ is a supermagic frame of mG and by Proposition 2, (ii) follows. □

Corollary 3. *Let $H\in\mathfrak{S}(d)$ and $G\in\mathfrak{S}(d)\cap\mathfrak{F}(2)$. Then $H\cup 2G\in\mathfrak{S}(d)$.*

P r o o f. Since $H\in\mathfrak{S}(d)$ ($G\in\mathfrak{S}(d)$), there is a supermagic labelling h (g) of H (G) for the index λ_H (λ_G). We have $G\in\mathfrak{F}(2)$ and so there exist edge-disjoint δ-factors F^1, F^2 which form a decomposition of G. Evidently, $\delta=\frac{d}{2}$. For $i=1,2$ and $j=1,2$, let G_j, F_j^i, ξ_j and φ_j^i be a copy of G, a copy of F^i, an isomorphism of G_j onto G and an isomorphism of F_j^i onto F^i, respectively. Suppose that $Q = H\cup F_1^1\cup F_1^2\cup F_2^1\cup F_2^2$ and $C = H\cup G_1\cup G_2$. Clearly, a mapping $\psi\colon V(Q)\to V(C)$ given by

$$\psi(v) = \begin{cases} \xi_j^{-1}(\varphi_j^i(v)) & \text{if } v\in V(F_j^i),\\ v & \text{if } v\in V(H),\end{cases}$$

is a harmonious homomorphism of Q onto C. Now, consider a mapping t from $E(Q)$ into positive integers given by

$$t(e) = \begin{cases} g(\overline{\varphi}_i^i(e)) & \text{if } e \in E(F_i^i), \\ h(e) + |E(G)| & \text{if } e \in E(H), \\ g(\overline{\varphi}_{3-i}^i(e)) + |E(G)| + |E(H)| & \text{if } e \in E(F_{3-i}^i). \end{cases}$$

It is easy to see that t is an injective mapping from $E(Q)$ onto $\{1, 2, \ldots, |E(Q)|\}$. Moreover, we have

$$\begin{aligned} \sum_{u \in \psi^{-1}(v)} t^*(u) &= t^*\big((\varphi_j^1)^{-1}(\xi_j(v))\big) + t^*\big((\varphi_j^2)^{-1}(\xi_j(v))\big) \\ &= \delta(|E(G)| + |E(H)|) + g^*(\xi_j(v)) \\ &= \delta(|E(G)| + |E(H)|) + \lambda_G \end{aligned}$$

for any vertex v of G_j, and

$$\sum_{u \in \psi^{-1}(v)} t^*(u) = d|E(G)| + h^*(v) = d|E(G)| + \lambda_H$$

for any vertex v of H. According to (3),

$$\begin{aligned} d|E(G)| + \lambda_H &= 2\delta|E(G)| + \delta(1 + \delta|V(H)|) \\ &= 2\delta|E(G)| + \delta(1 + |E(H)|) \\ &= \delta(|E(G)| + |E(H)|) + \delta(1 + |E(G)|) \\ &= \delta(|E(G)| + |E(H)|) + \delta(1 + \delta|V(G)|) \\ &= \delta(|E(G)| + |E(H)|) + \lambda_G. \end{aligned}$$

Thus $[Q, \psi, t]$ is a supermagic frame of C and by Proposition 2, the assertion follows. □

Proposition 3. *Let $F_1, F_2, \ldots, F_k \in \mathfrak{S}$ be pairwise edge-disjoint factors which form a decomposition of a graph G. Then $G \in \mathfrak{S}$.*

Proof. Since $F_i \in \mathfrak{S}$, there is a supermagic labelling f_i of F_i for every $i = 1, \ldots, k$. Evidently, a mapping $f\colon E(G) \to \{1, \ldots, |E(G)|\}$ given by $f(e) = f_i(e) + \sum_{0<j<i} |E(F_j)|$ whenever $e \in E(F_i)$, is a supermagic labelling of G. □

The *Cartesian product* $G_1 \square G_2$ of graphs G_1, G_2 is a graph whose vertices are all ordered pairs $[v_1, v_2]$, where $v_1 \in V(G_1)$, $v_2 \in V(G_2)$, and two vertices $[v_1, v_2]$, $[u_1, u_2]$ are joined by an edge in $G_1 \square G_2$ if and only if either (a) $v_1 = u_1$ and v_2, u_2 are adjacent in G_2, or (b) v_1, u_1 are adjacent in G_1 and $v_2 = u_2$. It is easy to see that the edges of type (a) ((b)) induce a spanning subgraph F^a (F^b) of $G_1 \Box G_2$ which is isomorphic to ${}_{|V(G_1)|}G_2$ (${}_{|V(G_2)|}G_1$, respectively). F^a, F^b form a decomposition of $G_1 \Box G_2$ and so, by Proposition 3, we immediately have

Corollary 4. *Let G_1, G_2 be regular graphs satisfying ${}_{|V(G_1)|}G_2 \in \mathfrak{S}$ and ${}_{|V(G_2)|}G_1 \in \mathfrak{S}$. Then $G_1 \Box G_2 \in \mathfrak{S}$.*

The *lexicographic product* $G_1[G_2]$ of graphs G_1, G_2 is a graph whose vertices are all ordered pairs $[v_1, v_2]$, where $v_1 \in V(G_1)$, $v_2 \in V(G_2)$, and two vertices $[v_1, v_2]$, $[u_1, u_2]$ are joined by an edge in $G_1[G_2]$ if and only if either v_1, u_1 are adjacent in G_1, or $v_1 = u_1$ and v_2, u_2 are adjacent in G_2. Let us remark that isolated vertices of G_2 are allowed in this special case.

Corollary 5. *Let G_1, G_2 be regular graphs satisfying*

(i) $|V(G_2)| \geqslant 3$;

(ii) ${}_{|V(G_1)|}G_2 \in \mathfrak{S}$ *or G_2 is totally disconnected;*

(iii) $|V(G_2)| \equiv 0 \pmod 2$ *or* $|V(G_2)||E(G_1)| \equiv 1 \pmod 2$.

Then $G_1[G_2] \in \mathfrak{S}$.

Proof. Let n denote the order of G_2, i.e. $n = |V(G_2)|$. As $n \geqslant 3$, $K_{n,n} \in \mathfrak{S}$ (see Introduction). It is well-known that $K_{n,n}$ can be decomposed into n pairwise edge-disjoint 1-factors, i.e. $K_{n,n} \in \mathfrak{F}(n)$. Since (iii), we have ${}_{|E(G_1)|}K_{n,n} \in \mathfrak{S}$ by Corollary 2

Let D_n be a totally disconnected graph of order n. According to the definition of the lexicographic product, $G_1[G_2]$ can be decomposed into factors F_1, F_2, where F_1 is isomorphic to $G_1[D_n]$ and F_2 is isomorphic to ${}_{|V(G_1)|}G_2$. Moreover, each edge of G_1 corresponds to a subgraph of $G_1[D_n]$ which is isomorphic to $K_{n,n}$. Therefore, $G_1[D_n]$ can be decomposed into $|E(G_1)|$ pairwise edge-disjoint subgraphs isomorphic to $K_{n,n}$ and so there is a balanced harmonious homomorphism of ${}_{|E(G_1)|}K_{n,n}$ onto $G_1[D_n]$. By Corollary 1, $G_1[D_n] \in \mathfrak{S}$. Thus $F_1 \in \mathfrak{S}$ and by (ii), $F_2 \in \mathfrak{S}$. Proposition 3 implies $G_1[G_2] \in \mathfrak{S}$. □

3. Regular complete multipartite graphs

A *complete k-partite graph* is a graph whose vertices can be partitioned into $k \geqslant 2$ disjoint classes $V_1, \ldots, V_k$ such that two vertices are adjacent if and only if they belong to distinct classes. If $|V_i| = n$ for all $i = 1, \ldots, k$, then the complete k-partite graph is regular of degree $(k-1)n$ and is denoted by $K_{k[n]}$. $K_{k[1]}$ (or only K_k) is called a *complete graph*. A complete bipartite graph $K_{2[n]}$ is also denoted by $K_{n,n}$. Note that $K_{k[n]}$ can be also defined by $K_k[D_n]$, where D_n denotes the totally disconnected graph of order n.

The characterizations of supermagic complete and complete bipartite graphs (see Introduction) are extended in the following assertion.

Theorem 1. *$mK_{k[n]} \in \mathfrak{S}$ if and only if one of the following conditions is satisfied:*

(i) $n = 1$, $k = 2$, $m = 1$;
(ii) $n = 1$, $k = 5$, $m \geqslant 2$;
(iii) $n = 1$, $5 < k \equiv 1 \pmod 4$, $m \geqslant 1$;
(iv) $n = 1$, $6 \leqslant k \equiv 2 \pmod 4$, $m \equiv 1 \pmod 2$;
(v) $n = 1$, $7 \leqslant k \equiv 3 \pmod 4$, $m \equiv 1 \pmod 2$;
(vi) $n = 2$, $k \geqslant 3$, $m \geqslant 1$;
(vii) $3 \leqslant n \equiv 1 \pmod 2$, $2 \leqslant k \equiv 1 \pmod 4$, $m \geqslant 1$;
(viii) $3 \leqslant n \equiv 1 \pmod 2$, $2 \leqslant k \equiv 2 \pmod 4$, $m \equiv 1 \pmod 2$;
(ix) $3 \leqslant n \equiv 1 \pmod 2$, $2 \leqslant k \equiv 3 \pmod 4$, $m \equiv 1 \pmod 2$;
(x) $4 \leqslant n \equiv 0 \pmod 2$, $k \geqslant 2$, $m \geqslant 1$.

P r o o f. $mK_{k[n]}$ is a $(k-1)n$-regular graph which consists of m components of order kn. Moreover, in [9] Stewart proved that $K_5 \notin \mathfrak{S}$. Thus, by Proposition 1, it is easy to see that one of the conditions (i)–(x) is necessary for $mK_{k[n]} \in \mathfrak{S}$.

On the other hand, we consider the following cases.

C a s e 1. Let $n = 1$. Obviously, $K_2 \in \mathfrak{S}$. Supermagic labellings of $2K_5$, $3K_5$ and $5K_5$ are described below by giving the labels of edges v_iv_j in the upper triangles of matrices. A matrix corresponds to a component of the graph.

$2K_5$:

v_1	5	10	18	9
v_2		7	17	13
v_3			6	19
v_4				1
	v_2	v_3	v_4	v_5

v_6	16	11	3	12
v_7		14	4	8
v_8			15	2
v_9				20
	v_7	v_8	v_9	v_{10}

$3K_5$:

3	4	26	29
	25	7	27
		28	5
			1

6	16	23	17
	11	15	30
		22	13
			2

20	21	12	9
	14	18	10
		8	19
			24

$5K_5$:

3	4	46	49
	45	7	47
		48	5
			1

6	9	37	50
	38	19	39
		44	11
			2

10	14	36	42
	35	17	40
		41	12
			8

16	27	33	26
	21	22	43
		34	20
			13

25	28	31	18
	30	24	23
		15	29
			32

The graph K_5 can be decomposed into two pairwise edge-disjoint Hamiltonian circuits and so $2K_5 \in \mathfrak{F}(2)$. Then, by Corollary 2, $mK_5 \in \mathfrak{S}$ for every even positive integer m. Finally, if $7 \leqslant m \equiv 1 \pmod 2$, then either $m = 4p+3$ or $m = 4p+5$ for some positive integer p. Thus mK_5 is isomorphic to either $3K_5 \cup 2(2p)K_5$ or $5K_5 \cup 2(2p)K_5$. Since $3K_5$, $5K_5$ belong to $\mathfrak{S}(4)$ and $2pK_5 \in \mathfrak{S}(4) \cap \mathfrak{F}(2)$, we have $mK_5 \in \mathfrak{S}$ by Corollary 3.

Now assume that one of (iii)–(v) is satisfied. Then $6 \leqslant k \not\equiv 0 \pmod 4$. Stewart [9] proved that $K_k \in \mathfrak{S}$ in this case. Since for k even (odd) K_k can be decomposed into $k-1$ ($\frac{k-1}{2}$) pairwise edge-disjoint 1-factors (Hamiltonian circuits), K_k belongs to $\mathfrak{F}(k-1)$ ($\mathfrak{F}(\frac{k-1}{2})$, respectively). According to Corollary 2, $mK_k \in \mathfrak{S}$.

Case 2. Let $n = 2$. Assume to the contrary that $k \geqslant 3$ is the minimum integer such that $mK_{k[2]} \notin \mathfrak{S}$ for some positive integer m. Denote the vertices of $K_{k[2]}$ by $v_1, v_2, \dots, v_{2k}$ in such a way that v_{2i-1} and v_{2i} are non-adjacent vertices for all $i = 1, \dots, k$. Then the set of edges $\{v_iv_j\colon\ j-i \equiv p \pmod 2\}$, $p \in \{0,1\}$, induces a $(k-1)$-factor F^p of $K_{k[2]}$. F^0, F^1 form a decomposition of $K_{k[2]}$ and so $K_{k[2]} \in \mathfrak{F}(2)$. Since $mK_{k[2]} \notin \mathfrak{S}$, according to Corollary 2, we have $K_{k[2]} \notin \mathfrak{S}$. Moreover, $k > 4$, because $K_{3[2]}$ and $K_{4[2]}$ admit supermagic labellings which are described below by giving the labels of edges v_iv_j (and –, if v_i, v_j are non-adjacent) in the upper triangles of matrices.

$K_{3[2]}$:

	v_2	v_3	v_4	v_5	v_6
v_1	–	1	4	9	12
v_2		11	10	2	3
v_3			–	8	6
v_4				7	5
v_5					–

$K_{4[2]}$:

–	24	1	23	2	22	3
	13	12	14	11	15	10
		–	5	8	4	21
			20	17	9	16
				–	6	7
					19	18
						–

Suppose that k is odd, i.e. $k = 2q+1$. As $4 \leqslant 2q < k$, $K_{2q[2]} \in \mathfrak{S}$. Let H be a subgraph of $K_{k[2]}$ induced by $\{v_1, \dots, v_{4q}\}$. H is isomorphic to $K_{2q[2]}$ and so there is a supermagic labelling t of H for the index λ. Consider a mapping $f\colon\ E(K_{k[2]}) \to \{1, 2, \dots, 4kq\}$ given by

$$f(e) = \begin{cases} t(e) + 4q & \text{if } e \in E(H), \\ i & \text{if } e = v_iv_{2k-1} \text{ where } 1 \leqslant i \leqslant q \text{ or } 3q < i \leqslant 4q, \\ i & \text{if } e = v_iv_{2k} \text{ where } q < i \leqslant 3q, \\ 4kq + 1 - i & \text{if } e = v_iv_{2k-1} \text{ where } q < i \leqslant 3q, \\ 4kq + 1 - i & \text{if } e = v_iv_{2k} \text{ where } 1 \leqslant i \leqslant q \text{ or } 3q < i \leqslant 4q. \end{cases}$$

Obviously, f is an injective mapping from $E(K_{k[2]})$ onto $\{1, 2, \ldots, |E(K_{k[2]})|\}$. Moreover, we have

$$\begin{aligned} f^*(v_{2k-1}) &= \sum_{i=1}^{q} i + \sum_{i=q+1}^{3q} (4kq + 1 - i) + \sum_{i=3q+1}^{4q} i \\ &= \tfrac{1}{2}q(q+1) + 2q(4kq+1) - q(4q+1) + \tfrac{1}{2}q(7q+1) \\ &= 2q(4kq+1) = 2q(4(2q+1)q+1) = 16q^3 + 8q^2 + 2q, \end{aligned}$$

similarly,

$$f^*(v_{2k}) = 16q^3 + 8q^2 + 2q$$

and

$$\begin{aligned} f^*(v_j) &= t^*(v_j) + 4q(2q-1)2 + j + (4kq + 1 - j) \\ &= \lambda + 16q^2 + 4q(k-2) + 1 \end{aligned}$$

for all $j = 1, \ldots, 4q$. According to (3),

$$\begin{aligned} \lambda + 16q^2 + 4q(k-2) + 1 &= (2q-1)(1 + (2q-1)4q) + 16q^2 + 4q(2q-1) + 1 \\ &= 16q^3 + 8q^2 + 2q. \end{aligned}$$

Therefore, f is a supermagic labelling of $K_{k[2]}$, a contradiction to $K_{k[2]} \notin \mathfrak{S}$.

Suppose that k is even, i.e. $k = 2q$. It is easy to see that $K_{k[2]} = K_2[K_{q[2]}]$. As $3 \leqslant q < k$, $2K_{q[2]} \in \mathfrak{S}$. By Corollary 5, $K_{k[2]} \in \mathfrak{S}$, which is again a contradiction.

Case 3. Let $n \geqslant 3$. It is easy to see that $mK_{k[n]} = (mK_k)[D_n]$, where D_n is the totally disconnected graph of order n. Corollary 5 implies $mK_{k[n]} \in \mathfrak{S}$ for each of the conditions (viii)–(x). Therefore, suppose that (vii) is satisfied. Then $k \equiv 1 \pmod 4$, i.e. $k = 4q+1$. K_k can be decomposed into $2q$ Hamiltonian circuits. Hence $K_k[D_n]$ can be decomposed into $2q$ pairwise edge-disjoint factors isomorphic to $C_k[D_n]$, where C_k denotes a circuit of length k. According to Corollary 5, $C_k[D_n] \in \mathfrak{S}$ and by Proposition 3, $K_k[D_n] \in \mathfrak{S}$. Moreover, $K_k[D_n] \in \mathfrak{F}(2q)$ and so Corollary 2 implies $mK_k[D_n] \in \mathfrak{S}$. □

Combining Theorem 1 and Corollary 4 we obtain sufficient conditions for the Cartesian product $K_{k[n]} \square K_{p[q]}$ to be supermagic. For illustration we present only the following

Corollary 6.

(i) *Let $k \geqslant 5$ and $p \geqslant 5$ be odd integers. Then $K_k \square K_p \in \mathfrak{S}$.*
(ii) *Let $n \geqslant 4$ and $q \geqslant 4$ be even integers. Then $K_{k[n]} \square K_{p[q]} \in \mathfrak{S}$.*

The *line graph* $L(G)$ of a graph G is a graph with the vertex set $V(L(G)) = E(G)$, where $e, e' \in E(G)$ are adjacent in $L(G)$ whenever they have a common end vertex in G. Note that all edges of a graph G incident with a vertex v induce a subgraph $K(v)$ of $L(G)$, which is isomorphic to a complete graph of order $\deg(v)$. Subgraphs $K(v)$, for all $v \in V(G)$, form a decomposition of $L(G)$, where any edge of G belongs to precisely two distinct subgraphs. Therefore, there is a balanced harmonious homomorphism of $\bigcup_{v \in V(G)} K(v)$ onto $L(G)$. Combining Corollary 1 and Theorem 1 we immediately obtain

Corollary 7. *Let G be a d-regular graph, where $d \geqslant 5$. If either $d \equiv 2 \pmod 4$ and $|V(G)| \equiv 1 \pmod 2$, or $d \equiv 1 \pmod 4$, then $L(G) \in \mathfrak{S}$.*

4. Cartesian products of circuits

In this section we deal with supermagic labellings of the Cartesian products of circuits. The circuit of order n is denoted by C_n.

Theorem 2. *$C_n \square C_n \in \mathfrak{S}$ for any integer $n \geqslant 3$.*

Proof. Denote the vertices of C_n by $v_1, v_2, \ldots, v_n$ in such a way that its edges are $v_i v_{i+1}$ for $i = 1, \ldots, n$, the subscripts being taken modulo n. Let G_i be the subgraph of $C_n \square C_n$ induced by $\{[v_i, v_j]\colon j = 1, \ldots, n\}$ for $i = 1, \ldots, n$ and by $\{[v_j, v_{2n+1-i}]\colon j = 1, \ldots, n\}$ for $i = n+1, \ldots, 2n$. Obviously, $G_1, \ldots, G_{2n}$ form a decomposition of $C_n \square C_n$ into pairwise edge-disjoint circuits.

For every $i \in \{1, \ldots, 2n\}$ let H_i be a circuit with the vertex set $\{u^i_j\colon j = 0, \ldots, n-1\}$ and let ψ_i be an isomorphism of H_i onto G_i such that

$$\psi_i(u^i_j) = \begin{cases} [v_i, v_{i+j}] & \text{if } i \leqslant n, \\ [v_{2n+1-i+j}, v_{2n+1-i}] & \text{if } i > n, \end{cases}$$

the subscripts being taken modulo n. Put $H = \bigcup_{i=1}^{2n} H_i$. Then the mapping ψ from $V(H)$ into $V(C_n \square C_n)$ given by $\psi(u^i_j) = \psi_i(u^i_j)$, is a harmonious homomorphism of H onto $C_n \square C_n$. Moreover,

$$\begin{aligned} \psi^{-1}([v_r, v_r]) &= \{u^r_0, u^{2n+1-r}_0\}, \\ \psi^{-1}([v_r, v_{r+k}]) &= \{u^r_k, u^{2n+1-r}_{n-k}\}, \\ \psi^{-1}([v_r, v_{r-k}]) &= \{u^r_{n-k}, u^{2n+1-r}_k\}. \end{aligned}$$

Consider the mapping $t\colon E(H) \to \{1, 2, \ldots, 2n^2\}$ given by

$$t(u_j^i u_{j+1}^i) = \begin{cases} 2jn + i & \text{if } j \equiv 0 \pmod 2,\\ 1 + 2(j+1)n - i & \text{if } j \equiv 1 \pmod 2. \end{cases}$$

Clearly, t is a bijective mapping and its index-mapping satisfies

$$t^*(u_j^i) = 4jn + 1 \text{ for } j \neq 0,$$
$$t^*(u_0^i) = 2n^2 + 1 \text{ for } n \equiv 0 \pmod 2,$$
$$t^*(u_0^i) = 2(n-1)n + 2i \text{ for } n \equiv 1 \pmod 2.$$

Hence

$$\sum_{u \in \psi^{-1}([v_r, v_{r\pm k}])} t^*(u) = 4kn + 1 + 4(n-k)n + 1 = 4n^2 + 2,$$
$$\sum_{u \in \psi^{-1}([v_r, v_r])} t^*(u) = 2(2n^2 + 1) = 4n^2 + 2$$

for $n \equiv 0 \pmod 2$ and

$$\sum_{u \in \psi^{-1}([v_r, v_r])} t^*(u) = 2(n-1)n + 2r + 2(n-1)n + 2(2n + 1 - r) = 4n^2 + 2$$

for $n \equiv 1 \pmod 2$. Therefore, $[H, \psi, t]$ is a supermagic frame of $C_n \square C_n$ and by Proposition 2, $C_n \square C_n \in \mathfrak{S}$. □

The following two assertions exploit the structure of the labellings of $C_3 \square C_3$ and $C_4 \square C_4$ described above.

Proposition 4. *Let G be a 3-regular graph containing a 1-factor. Then $L(G) \in \mathfrak{S}$.*

Proof. Let F_1 be a 1-factor of G. Put $p = |E(F_1)|$. Denote the vertices of G by $v_1, v_2, \ldots, v_{2p}$ in such a way that $E(F_1) = \{v_i v_{2p+1-i}\colon i = 1, \ldots, p\}$. The set $E(G) - E(F_1)$ induces a 2-factor F_2 of G. Clearly, there is a permutation α of $\{1, 2, \ldots, 2p\}$ such that $E(F_2) = \{v_i v_{\alpha(i)}\colon i = 1, \ldots, 2p\}$.

Let T_i be the complete graph with the vertex set $\{x_i, y_i, z_i\}$, for $i = 1, 2, \ldots, 2p$. Let H be the disjoint union of T_i, i.e. $H = \bigcup_{i=1}^{2p} T_i$. Then the mapping $\psi\colon V(H) \to E(G)$ given by $\psi(x_i) = v_i v_{2p+1-i}$, $\psi(y_i) = v_i v_{\alpha(i)}$, $\psi(z_i) = v_i v_{\alpha^{-1}(i)}$, is a harmonious homomorphism of H onto $L(G)$.

Consider the mapping $t\colon E(H) \to \{1, 2, \ldots, 6p\}$ given by

$$t(e) = \begin{cases} i & \text{if } e = x_i y_i,\\ 4p + 1 - i & \text{if } e = y_i z_i,\\ 4p + i & \text{if } e = z_i x_i. \end{cases}$$

Obviously, t is an injective mapping and its index-mapping satisfies

$$t^*(x_i) = 4p + 2i,$$
$$t^*(y_i) = 4p + 1,$$
$$t^*(z_i) = 8p + 1$$

for all $i = 1, 2, \ldots, 2p$. Hence,

$$\begin{aligned}\sum_{u\in\psi^{-1}(e)} t^*(u) &= \sum_{u\in\psi^{-1}(v_j v_{2p+1-j})} t^*(u)\\ &= t^*(x_j) + t^*(x_{2p+1-j})\\ &= 4p + 2j + 4p + 2(2p + 1 - j) = 12p + 2\end{aligned}$$

for $e \in E(F_1)$, and

$$\begin{aligned}\sum_{u\in\psi^{-1}(e)} t^*(u) &= \sum_{u\in\psi^{-1}(v_j v_{\alpha(j)})} t^*(u)\\ &= t^*(y_j) + t^*(z_{\alpha^{-1}(j)})\\ &= 4p + 1 + 8p + 1 = 12p + 2\end{aligned}$$

for $e \in E(F_2)$. Therefore, $[H, \psi, t]$ is a supermagic frame of $L(G)$ and by Proposition 2, $L(G) \in \mathfrak{S}$. □

Proposition 5. *Let G be a bipartite 4-regular graph which can be decomposed into pairwise edge-disjoint subgraphs isomorphic to C_4. Then $G \in \mathfrak{S}$.*

Proof. Suppose that V_1, V_2 are parts of G and $G_1, \ldots, G_k$ are pairwise edge-disjoint subgraphs of G isomorphic to C_4. Let F be a graph with the vertex set $V(F) = V_2$, where $u, v \in V_2$ are joined by an edge in F whenever $\{u, v\} \subseteq V(G_i)$ for some $i \in \{1, \ldots, k\}$ (multiple edges are allowed in this special case). Clearly, F is a 2-regular graph and so there is a permutation α of V_2 such that $E(F) = \{v\alpha(v) \colon v \in V_2\}$.

For every $i = 1, \ldots, k$ let H_i be the circuit with the vertex set $\{w_i, x_i, y_i, z_i\}$ and the edge set $\{w_i x_i, x_i y_i, y_i z_i, z_i w_i\}$. Then there is an isomorphism ψ_i of H_i onto G_i such that $\alpha(\psi_i(x_i)) = \psi_i(z_i)$. Put $H = \bigcup_{i=1}^{k} H_i$. The mapping $\psi \colon V(H) \to V(G)$ given by $\psi(v) = \psi_i(v)$ when $v \in V(H_i)$, is a harmonious homomorphism of H onto G. Note that, for all $i = 1, \ldots, k$, $\psi(w_i) \in V_1$, $\psi(x_i) \in V_2$, $\psi(y_i) \in V_1$, $\psi(z_i) \in V_2$ for $u \in V_1$, $|\psi^{-1}(u)| = 2$ and for $v \in V_2$, $\psi^{-1}(v) = \{x_r, z_s\}$, where $\psi(x_r)\alpha(\psi(x_r))$ and $\alpha^{-1}(\psi(z_s))\psi(z_s)$ are edges of F incident with v.

Consider the mapping $t\colon\ E(H) \to \{1, 2, \ldots, 4k\}$ given by

$$t(e) = \begin{cases} i & \text{if } e = w_i x_i, \\ 2k+1-i & \text{if } e = x_i y_i, \\ 2k+i & \text{if } e = y_i z_i, \\ 4k+1-i & \text{if } e = z_i w_i. \end{cases}$$

Obviously, t is a bijection and its index-mapping satisfies

$$\begin{aligned} t^*(w_i) = t^*(y_i) &= 4k+1, \\ t^*(x_i) &= 2k+1, \\ t^*(z_i) &= 6k+1. \end{aligned}$$

Hence

$$\sum_{u \in \psi^{-1}(v)} t^*(u) = 2(4k+1)$$

for $v \in V_1$ and

$$\sum_{u \in \psi^{-1}(v)} t^*(u) = (2k+1) + (6k+1) = 2(4k+1)$$

for $v \in V_2$. Therefore, $[H, \psi, t]$ is a supermagic frame of G and by Proposition 2, $G \in \mathfrak{S}$. □

As the Cartesian product of even circuits is a bipartite 4-regular graph which can be decomposed into pairwise edge-disjoint subgraphs isomorphic to C_4, by Proposition 5, we immediately have

Theorem 3. *Let $n \geqslant 2$, $k \geqslant 2$ be integers. Then $C_{2n} \square C_{2k} \in \mathfrak{S}$.*

This result suggests a conjecture:

Conjecture. $C_n \square C_k \in \mathfrak{S}$ for all $n, k \geqslant 3$.

The graph Q_n of the n-dimensional cube can be defined by induction as follows:

$$Q_1 = K_2 \quad \text{and} \quad Q_{k+1} = Q_k \square K_2 \ \text{ for any positive integer } k.$$

We conclude this paper with a characterization of supermagic cubes Q_n, but first we prove the following auxiliary result.

Lemma 1. $C_4 \square C_4 \square C_4 \in \mathfrak{S}$.

Proof. Put $G = C_4 \square C_4$. According to Theorem 2, there is a supermagic labelling g of G for the index $\lambda = 66$. Evidently, there exist edge-disjoint 1-factors F^1, F^2, F^3, F^4 which form a decomposition of G. Now, consider the graph $G \square C_4$. Denote the vertices of C_4 by x_1, x_2, x_3, x_4 in such a way that its edge set is $\{x_1x_2, x_2x_3, x_3x_4, x_4x_1\}$. For $j = 1, \ldots, 4$ let G_j be the subgraph of $G \square C_4$ induced by $\{[v, x_j]\colon\ v \in V(G)\}$. Define a mapping $t\colon\ E(G_1 \cup \ldots \cup G_4) \to \{1, 2, \ldots, 128\}$ by

$$t([u, x_j][v, x_j]) = g(uv) + a_{i,j}$$

if $uv \in E(F^i)$, where

$$(a_{i,j}) = \begin{pmatrix} 64 & 32 & 96 & 0 \\ 32 & 96 & 0 & 64 \\ 64 & 32 & 96 & 0 \\ 32 & 64 & 0 & 96 \end{pmatrix}.$$

It is not difficult to check that t is a bijection. The index-mapping of t satisfies

$$t^*([v, x_j]) = g^*(v) + \sum_{i=1}^{4} a_{i,j} = \lambda + \sum_{i=1}^{4} a_{i,j}$$

for every $v \in V(G)$. Thus,

$$\begin{aligned} t^*([v, x_1]) = t^*([v, x_3]) &= 258, \\ t^*([v, x_2]) &= 290, \\ t^*([v, x_4]) &= 226. \end{aligned}$$

Denote the vertices of G by $v_1, v_2, \ldots, v_{16}$. For $i = 1, \ldots, 16$ let C^i be the subgraph of $G \square C_4$ induced by $\{[v_i, x_j]\colon\ j = 1, \ldots, 4\}$. Define the mapping $h\colon\ E(C^1 \cup \ldots \cup C^{16}) \to \{129, 130, \ldots, 192\}$ by

$$h(e) = \begin{cases} 128 + i & \text{if } e = [v_i, x_1][v_i, x_2], \\ 161 - i & \text{if } e = [v_i, x_2][v_i, x_3], \\ 160 + i & \text{if } e = [v_i, x_3][v_i, x_4], \\ 193 - i & \text{if } e = [v_i, x_4][v_i, x_1]. \end{cases}$$

It is easy to check that h is a bijection and its index-mapping satisfies

$$\begin{aligned} h^*([v_i, x_1]) = h^*([v_i, x_3]) &= 321, \\ h^*([v_i, x_2]) &= 289, \\ h^*([v_i, x_4]) &= 353 \end{aligned}$$

for all $i = 1, \ldots, 16$.

The mapping $f\colon E(G\Box C_4) \to \{1, 2, \ldots, 192\}$ given by

$$f(e) = \begin{cases} t(e) & \text{if } e \in E(G_1 \cup \ldots \cup G_4), \\ h(e) & \text{if } e \in E(C^1 \cup \ldots \cup C^{16}) \end{cases}$$

is a bijection satisfying $f^*([v_i, x_j]) = t^*([v_i, x_j]) + h^*([v_i, x_j]) = 579$ for all $i = 1, \ldots, 16$, $j = 1, \ldots, 4$, i.e. f is a supermagic labelling of $G\Box C_4$. □

Theorem 4. $Q_n \in \mathfrak{S}$ *if and only if either* $n = 1$ *or* $4 \leqslant n \equiv 0 \pmod 2$.

Proof. Q_n is a connected n-regular graph of order 2^n. Thus, Proposition 1 implies the necessary condition for $Q_n \in \mathfrak{S}$.

On the other hand, obviously $Q_1 \in \mathfrak{S}$. It is easy to see that Q_4 (Q_6) is isomorphic to $C_4\Box C_4$ ($C_4\Box C_4\Box C_4$) and so, by Theorem 2 (Lemma 1), $Q_4 \in \mathfrak{S}$ ($Q_6 \in \mathfrak{S}$, respectively).

Suppose that $Q_{2k} \in \mathfrak{S}$ for an integer $k \geqslant 2$. Since $Q_{2k} \in \mathfrak{F}(2k)$, we have $16Q_{2k} \in \mathfrak{S}$ by Corollary 2. Similarly, $2^{2k}Q_4 \in \mathfrak{S}$. According to Corollary 4, $Q_{2k}\Box Q_4 \in \mathfrak{S}$. As Q_{2k+4} is isomorphic to $Q_{2k}\Box Q_4$, $Q_{2k+4} \in \mathfrak{S}$. By induction, $Q_n \in \mathfrak{S}$ for any even integer $n \geqslant 4$. □

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Author's address: *Jaroslav Ivančo*, Department of Geometry and Algebra, P. J. Šafárik University, 041 54 Košice, Jesenná 5, Slovakia, e-mail: ivanco@duro.upjs.sk.