HARTMAN-WINTNER TYPE CRITERIA FOR HALF-LINEAR SECOND ORDER DIFFERENTIAL EQUATIONS

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Abstract. We establish Hartman-Wintner type criteria for the half-linear second order differential equation

$$\big(r(t)\Phi(x')\big)' + c(t)\Phi(x) = 0, \quad \Phi(x) = |x|^{p-2}x,\ p > 1,$$

where this equation is viewed as a perturbation of another equation of the same form.

Keywords: half-linear differential equation, Hartman-Wintner criterion, Riccati equation, principal solution

MSC 2000: 34C10

1. Introduction

Let us consider the half-linear second order differential equation

$$(r(t)\Phi(x'))' + c(t)\Phi(x) = 0, \tag{1}$$

where $\Phi(x) := |x|^{p-1}\operatorname{sgn} x$, $p > 1$, and r, c are continuous functions, $r(t) > 0$.

This equation is a generalization of the second order Sturm-Liouville linear equation (with $p = 2$ in (1))

$$(r(t)x')' + c(t)x = 0 \tag{2}$$

and solutions of these two equations behave in many aspects very similarly. In particular, the oscillation theory extends almost verbatim from linear to half-linear equations and (1) can be classified as oscillatory or nonoscillatory according to whether

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any nontrivial solution of (1) has or does not have infinitely many zeros on any interval of the form $[T,\infty)$.

The classical Hartman-Wintner theorem for the nonoscillatory equation (2) (see, e.g. [7, p. 365]) relates the existence of the limit

$$\lim_{t\to\infty}\frac{\int^t r^{-1}(s)\left(\int^s c(\tau)\,\mathrm{d}\tau\right)\,\mathrm{d}s}{\int^t r^{-1}(s)\,\mathrm{d}s} \tag{3}$$

to the convergence of the integral $\int^\infty r^{-1}(t)w^2(t)\,\mathrm{d}t$, where w is any solution of the Riccati equation

$$w'(t)+c(t)+\frac{w^2(t)}{r(t)}=0$$

associated with (2). As a consequence of this statement we have that (2) is oscillatory provided

$$-\infty<\liminf_{t\to\infty}\frac{\int^t r^{-1}(s)\left(\int^s c(\tau)\,\mathrm{d}\tau\right)\,\mathrm{d}s}{\int^t r^{-1}(s)\,\mathrm{d}s}<\limsup_{t\to\infty}\frac{\int^t r^{-1}(s)\left(\int^s c(\tau)\,\mathrm{d}\tau\right)\,\mathrm{d}s}{\int^t r^{-1}(s)\,\mathrm{d}s}$$

or if

$$\lim_{t\to\infty}\frac{\int^t r^{-1}(s)\left(\int^s c(\tau)\,\mathrm{d}\tau\right)\,\mathrm{d}s}{\int^t r^{-1}(s)\,\mathrm{d}s}=\infty.$$

Consequently, an interesting problem is what is the oscillatory nature of (2) when the limit (3) exists and is finite. This problem was solved in [1], where it was shown that (2) (with $r(t)\equiv 1$) is oscillatory provided

$$\limsup_{t\to\infty}\frac{t}{\ln t}\left(c(\infty)-\frac{1}{t}\int_1^t\int_1^s c(\tau)\,\mathrm{d}\tau\,\mathrm{d}s\right)>\frac{1}{4},$$

where

$$c(\infty)=\lim_{t\to\infty}\frac{1}{t}\int_1^t\int_1^s c(\tau)\,\mathrm{d}\tau\,\mathrm{d}s.$$

Concerning the extension of these results to the half-linear case, the first step was made in Mirzov's paper [13] (see also [5], [14]), followed by [11], [12], where it was shown that these results naturally extend to (1). In particular, it was shown that (1) with $r\equiv 1$ is oscillatory provided

$$\lim_{t\to\infty}c_p(t)=\infty\quad\text{or}\quad-\infty<\liminf_{t\to\infty}c_p(t)<\limsup_{t\to\infty}c_p(t),$$

where

$$c_p(t)=\frac{p-1}{t^{p-1}}\int_1^t s^{p-2}\int_1^s c(\tau)\,\mathrm{d}\tau\,\mathrm{d}s. \tag{4}$$

Moreover, if $\lim_{t\to\infty} c_p(t) = c_p(\infty)$ exists and is finite and

$$\limsup_{t\to\infty} \frac{t^{p-1}}{\ln t}(c_p(\infty) - c_p(t)) > \Big(\frac{p-1}{p}\Big)^p,$$

then equation (1) is also oscillatory.

In the all above mentioned criteria, equation (1) is regarded as a perturbation of the one-term differential equation

$$(r(t)\Phi(x'))' = 0.$$

In this paper we consider equation (1) as a perturbation of a general (nonoscillatory) two-term equation

$$(r(t)\Phi(x'))' + \tilde{c}(t)\Phi(x) = 0, \tag{5}$$

i.e., (1) can be seen in the form

$$(r(t)\Phi(x')' + \tilde{c}(t)\Phi(x) + (c(t) - \tilde{c}(t))\Phi(x) = 0.$$

We will investigate oscillatory properties of (1) depending on the asymptotic behaviour of the function

$$L(t) = \frac{\int_T^t H^{-1}(s)\left(\int_T^s (c(\tau) - \tilde{c}(\tau))h^p(\tau)\,\mathrm{d}\tau\right)\,\mathrm{d}s}{\int_T^t H^{-1}(s)\,\mathrm{d}s},$$

where $H(t) = r(t)h^2(t)|h'(t)|^{p-2}$ and $h(t)$ is the so-called principal solution of the nonoscillatory equation (5). By easy computation one can find that $L(t)$ is a generalization of (3) and reduces to this function for $p = 2, r(t) \equiv 1$ and $\tilde{c} \equiv 0$ (it is well known that $h = 1$ in this case, see [5, p. 146]).

2. Preliminaries

Let x be a solution of (1). Then the function $w = r\Phi(x'/x)$ solves the Riccati equation

$$w' + c(t) + (p-1)r^{1-q}(t)|w|^q = 0, \tag{6}$$

where q is the conjugate number of p, i.e. $1/p + 1/q = 1$, and it is well known (see [2, p. 171]) that equation (1) is nonoscillatory if and only if there exists a solution of (6) on some interval of the form $[T,\infty)$.

Now we recall the half-linear version of the so-called Picone's identity (see [10] or [2, p. 172]) which, in a modified form needed in our paper, reads as follows. Let w be a solution of (6). Then for any $x \in C^1$

$$r(t)|x'|^p - c(t)|x|^p = (w(t)|x|^p)' + pr^{1-q}(t)P(r^{q-1}(t)x', \Phi(x)w(t)), \tag{7}$$

where

$$P(u,v) := \frac{|u|^p}{p} - uv + \frac{|v|^q}{q} \geqslant 0 \tag{8}$$

with the equality $P(u,v) = 0$ if and only if $v = \Phi(u)$.

Concerning the function P, we will need its quadratic estimates which are given in the next statement whose proof can be found e.g. in [6].

Lemma 1. *The function $P(u,v)$ defined in* (8) *satisfies the inequalities*

$$\begin{aligned} P(u,v) &\geqslant \tfrac{1}{2}|u|^{2-p}(v-\Phi(u))^2 \quad \text{for} \quad p \leqslant 2, \\ P(u,v) &\leqslant \tfrac{1}{2}|u|^{2-p}(v-\Phi(u))^2 \quad \text{for} \quad p \geqslant 2,\ u \neq 0. \end{aligned}$$

Furthermore, let $T > 0$ be arbitrary. There exists a constant $K = K(T) > 0$ such that

$$\begin{aligned} P(u,v) &\geqslant K|u|^{2-p}(v-\Phi(u))^2 \quad \text{for} \quad p \geqslant 2, \\ P(u,v) &\leqslant K|u|^{2-p}(v-\Phi(u))^2 \quad \text{for} \quad p \leqslant 2 \end{aligned}$$

and every $u, v \in \mathbb{R}$ satisfying $|v/\Phi(u)| \leqslant T$.

Now we derive the so-called modified Riccati equation. Let $x \in C^1$ be any function and w a solution of the Riccati equation (6). Then from Picone's identity (7) we have

$$(w|x|^p)' = r|x'|^p - c|x|^p - pr^{1-q}|x|^p P(\Phi^{-1}(w_x), w), \tag{9}$$

where $w_x = r\Phi(x'/x)$ and Φ^{-1} is the inverse function of Φ. At the same time, let h be a (positive) solution of (5) and $w_h = r\Phi(h'/h)$ the solution of the Riccati equation associated with (5). Then

$$(w_h|x|^p)' = r|x'|^p - \tilde{c}|x|^p - pr^{1-q}|x|^p P(\Phi^{-1}(w_x), w_h). \tag{10}$$

Substituting $x = h$ into (9), (10) and subtracting these equalities we get the equation (in view of the identity $P(\Phi^{-1}(w_h), w_h) = 0$)

$$((w-w_h)h^p)' + (c-\tilde{c})h^p + pr^{1-q}h^p P(\Phi^{-1}(w_h), w) = 0. \tag{11}$$

Observe that if $\tilde{c}(t) \equiv 0$ and $h(t) \equiv 1$, then (11) reduces to (6) and this is the reason why we call this equation the *modified Riccati equation*.

Further, let us recall the concept of the principal solution of the nonoscillatory equation (1) as introduced by Mirzov in [15] and later independently by Elbert and Kusano in [8]. If (1) is nonoscillatory, as mentioned at the beginning of this section, there exists a solution w of the Riccati equation (6) which is defined on some interval $[T, \infty)$. It can be shown that among all solutions of (6) there exists the *minimal* one $\tilde{w}$ (sometimes called the *distinguished* solution), minimal in the sense that any other solution of (6) satisfies the inequality $w(t) > \tilde{w}(t)$ for large t. Then the principal solution of (1) is given by the formula

$$\tilde{x} = K \exp\left\{ \int^t r^{1-q}(s)\Phi^{-1}(\tilde{w}(s))\,\mathrm{d}s \right\},$$

where K is a real nonzero constant, i.e., the principal solution $\tilde{x}$ of (1) is a solution which "produces" the minimal solution $\tilde{w} = r\Phi(\tilde{x}'/\tilde{x})$ of (6).

Finally, we present an important subsidiary statement, whose proof can be found in [3] or [4].

Lemma 2. *Let* $\int^\infty r^{1-q}(t)\,\mathrm{d}t = \infty$. *Suppose that equation* (5) *is nonoscillatory and possesses a positive principal solution* h *such that there exists a finite limit*

$$\lim_{t\to\infty} r(t)h(t)\Phi(h'(t)) =: L > 0 \tag{12}$$

and

$$\int^\infty \frac{\mathrm{d}t}{r(t)h^2(t)(h'(t))^{p-2}} = \infty. \tag{13}$$

Further suppose that $0 \leqslant \int_t^\infty c(s)\,\mathrm{d}s < \infty$ *for large* t, (1) *is nonoscillatory and*

$$0 \leqslant \int_t^\infty (c(s) - \tilde{c}(s))h^p(s)\,\mathrm{d}s < \infty. \tag{14}$$

Then for any solution w *of the Riccati equation* (6) *corresponding to* (1) *we have*

$$\int^\infty r^{1-q}(t)h^p(t)P(\Phi^{-1}(w_h), w)\,\mathrm{d}t < \infty \quad \text{and} \quad \lim_{t\to\infty}\frac{w(t)}{w_h(t)} = 1,$$

where $w_h = r\Phi(h')/\Phi(h)$ *is the solution of the Riccati equation corresponding to* (5).

3. Hartman-Wintner type theorem

First we introduce the Hartman-Wintner type theorem, which is a completion of results published in [16]. The idea of our proof is similar to that used in [16], but for the sake of completeness and further references we include the proof.

Theorem 1. *Suppose that equations* (1) *and* (5) *are nonoscillatory and let* h *be a solution of* (5) *such that* $h'(t) \neq 0$ *for large* t *and*

$$\int^{\infty} H^{-1}(t)\,\mathrm{d}t = \infty, \quad H(t) := r(t)h^2(t)|h'(t)|^{p-2}. \tag{15}$$

Let w *be a solution of the Riccati equation* (6) *corresponding to* (1) *and* $w_h = r\Phi(h')/\Phi(h)$ *a solution of the Riccati equation corresponding to* (5) *such that*

$$\limsup_{t\to\infty} \left|\frac{w(t)}{w_h(t)}\right| < \infty. \tag{16}$$

Then for $u(t) = h^p(t)(w(t) - w_h(t))$ *and* T *sufficiently large the following statements are equivalent.*

(I) *The inequality*

$$\int_T^{\infty} \frac{u^2(t)}{H(t)}\,\mathrm{d}t < \infty \tag{17}$$

holds.

(II) *There exists a finite limit*

$$\lim_{t\to\infty} \frac{\int_T^t H^{-1}(s)\int_T^s (c(\tau) - \tilde{c}(\tau))h^p(\tau)\,\mathrm{d}\tau\,\mathrm{d}s}{\int_T^t H^{-1}(s)\,\mathrm{d}s}. \tag{18}$$

(III) *For the lower limit we have*

$$\liminf_{t\to\infty} \frac{\int_T^t H^{-1}(s)\int_T^s (c(\tau) - \tilde{c}(\tau))h^p(\tau)\,\mathrm{d}\tau\,\mathrm{d}s}{\int_T^t H^{-1}(s)\,\mathrm{d}s} > -\infty. \tag{19}$$

Proof. (I $\Rightarrow$ II): We can write (11) in the form

$$u'(t) + (c(t) - \tilde{c}(t))h^p(t) + pr^{1-q}(t)h^p(t)P(\Phi^{-1}(w_h), w) = 0.$$

Integrating from T to t we get

$$u(t) = u(T) - \int_T^t (c(s) - \tilde{c}(s))h^p(s)\,\mathrm{d}s - p\int_T^t r^{1-q}(s)h^p(s)P(\Phi^{-1}(w_h), w)\,\mathrm{d}s$$

and multiplying by H^{-1} and applying the same integration we obtain

$$\begin{aligned}\int_T^t H^{-1}(s)u(s)\,\mathrm{d}s = u(T)\int_T^t H^{-1}(s)\,\mathrm{d}s - \int_T^t H^{-1}(s)\biggl(\int_T^s (c(\tau) - \tilde{c}(\tau))h^p(\tau)\,\mathrm{d}\tau\biggr)\,\mathrm{d}s \\ - p\int_T^t H^{-1}(s)\biggl(\int_T^s r^{1-q}(\tau)h^p(\tau)P(\Phi^{-1}(w_h), w)\,\mathrm{d}\tau\biggr)\,\mathrm{d}s\end{aligned}$$

and hence

$$\begin{aligned}\frac{\int_T^t H^{-1}(s)u(s)\,\mathrm{d}s}{\int_T^t H^{-1}(s)\,\mathrm{d}s} &= u(T) - \frac{\int_T^t H^{-1}(s)\left(\int_T^s (c(\tau) - \tilde{c}(\tau))h^p(\tau)\,\mathrm{d}\tau\right)\,\mathrm{d}s}{\int_T^t H^{-1}(s)\,\mathrm{d}s} \\ &\quad - p\frac{\int_T^t H^{-1}(s)\left(\int_T^s r^{1-q}(\tau)h^p(\tau)P(\Phi^{-1}(w_h), w)\,\mathrm{d}\tau\right)\,\mathrm{d}s}{\int_T^t H^{-1}(s)\,\mathrm{d}s}.\end{aligned}$$

Using the Cauchy-Schwartz inequality (suppressing the argument s in the integrated functions) we arrive at

$$0 \leqslant \frac{|\int_T^t H^{-1}u\,\mathrm{d}s|}{\int_T^t H^{-1}\,\mathrm{d}s} \leqslant \frac{[\int_T^t H^{-1}\,\mathrm{d}s]^{\frac{1}{2}}[\int_T^t H^{-1}u^2\,\mathrm{d}s]^{\frac{1}{2}}}{\int_T^t H^{-1}\,\mathrm{d}s} = \biggl(\frac{\int_T^t H^{-1}u^2\,\mathrm{d}s}{\int_T^t H^{-1}\,\mathrm{d}s}\biggr)^{\frac{1}{2}} \to 0, \quad t \to \infty.$$

From Lemma 1 we know that provided (16) holds, there exist constants K_1, K_2 such that

$$K_1\frac{u^2}{H} \leqslant r^{1-q}h^pP(\Phi^{-1}(w_h), w) \leqslant K_2\frac{u^2}{H}. \tag{20}$$

As $\int_T^\infty H^{-1}u^2\,\mathrm{d}t < \infty$, the integral $\int_T^\infty r^{1-q}h^pP(\Phi^{-1}(w_h), w)\,\mathrm{d}t$ converges too and by L'Hospital's rule we have

$$\lim_{t\to\infty} p\frac{\int_T^t H^{-1}(s)\left(\int_T^s r^{1-q}(\tau)h^p(\tau)P(\Phi^{-1}(w_h), w)\,\mathrm{d}\tau\right)\,\mathrm{d}s}{\int_T^t H^{-1}(s)\,\mathrm{d}s} < \infty.$$

Therefore,

$$\begin{aligned}&\lim_{t\to\infty}\frac{\int_T^t H^{-1}(s)(\int_T^s (c(\tau) - \tilde{c}(\tau))h^p(\tau)\,\mathrm{d}\tau)\,\mathrm{d}s}{\int_T^t H^{-1}(s)\,\mathrm{d}s} \\ &\quad = u(T) - \lim_{t\to\infty} p\frac{\int_T^t H^{-1}(s)\left(\int_T^s r^{1-q}(\tau)h^p(\tau)P(\Phi^{-1}(w_h), w)\,\mathrm{d}\tau\right)\,\mathrm{d}s}{\int_T^t H^{-1}(s)\,\mathrm{d}s} \\ &\quad = u(T) - p\int_T^\infty r^{1-q}(t)h^p(t)P(\Phi^{-1}(w_h), w)\,\mathrm{d}t < \infty.\end{aligned} \tag{21}$$

(II $\Rightarrow$ III): This implication is trivial.
(III $\Rightarrow$ I): From the first part of this proof we have

$$\frac{\int_T^t H^{-1}(s)u(s)\,\mathrm{d}s}{\int_T^t H^{-1}(s)\,\mathrm{d}s} = u(T) - \frac{\int_T^t H^{-1}(s)\left(\int_T^s (c(\tau)-\tilde{c}(\tau))h^p(\tau)\,\mathrm{d}\tau\right)\,\mathrm{d}s}{\int_T^t H^{-1}(s)\,\mathrm{d}s} - p\frac{\int_T^t H^{-1}(s)\left(\int_T^s r^{1-q}(\tau)h^p(\tau)P(\Phi^{-1}(w_h),w)\,\mathrm{d}\tau\right)\,\mathrm{d}s}{\int_T^t H^{-1}(s)\,\mathrm{d}s}.$$

The Cauchy-Schwartz inequality together with (19) and (20) implies that there exists a constant $M \in \mathbb{R}$ such that

$$-\left(\frac{\int_T^t H^{-1}(s)u^2(s)\,\mathrm{d}s}{\int_T^t H^{-1}(s)\,\mathrm{d}s}\right)^{\frac{1}{2}} \leqslant M - pK_1\frac{\int_T^t H^{-1}(s)(\int_T^s H^{-1}(\tau)u^2(\tau)\,\mathrm{d}\tau)\,\mathrm{d}s}{\int_T^t H^{-1}(s)\,\mathrm{d}s}.$$

Suppose, by contradiction, that $\int^\infty H^{-1}(t)u^2(t)\,\mathrm{d}t = \infty$. Then by L'Hospital's rule

$$\lim_{t\to\infty}\frac{\int_T^t H^{-1}(s)(\int_T^s H^{-1}(\tau)u^2(\tau)\,\mathrm{d}\tau)\,\mathrm{d}s}{\int_T^t H^{-1}(s)\,\mathrm{d}s} = \infty$$

and

$$pK_1\frac{\int_T^t H^{-1}(s)(\int_T^s H^{-1}(\tau)u^2(\tau)\,\mathrm{d}\tau)\,\mathrm{d}s}{\int_T^t H^{-1}(s)\,\mathrm{d}s} - M \geqslant \frac{1}{2}pK_1\frac{\int_T^t H^{-1}(s)(\int_T^s H^{-1}(\tau)u^2(\tau)\,\mathrm{d}\tau)\,\mathrm{d}s}{\int_T^t H^{-1}(s)\,\mathrm{d}s}$$

for t sufficiently large, i.e.,

$$\left(\frac{\int_T^t H^{-1}(s)u^2(s)\,\mathrm{d}s}{\int_T^t H^{-1}(s)\,\mathrm{d}s}\right)^{\frac{1}{2}} \geqslant \frac{1}{2}pK_1\frac{\int_T^t H^{-1}(s)(\int_T^s H^{-1}(\tau)u^2(\tau)\,\mathrm{d}\tau)\,\mathrm{d}s}{\int_T^t H^{-1}(s)\,\mathrm{d}s}.$$

Denote $S(t) := \int_T^t H^{-1}(s)(\int_T^s H^{-1}(\tau)u^2(\tau)\,\mathrm{d}\tau)\,\mathrm{d}s$. Then

$$\left(\frac{S'(t)H(t)}{\int_T^t H^{-1}(s)\,\mathrm{d}s}\right)^{\frac{1}{2}} \geqslant \frac{1}{2}pK_1\frac{S(t)}{\int_T^t H^{-1}(s)\,\mathrm{d}s}.$$

By simple calculation we obtain

$$\frac{S'(t)}{S^2(t)} \geqslant \frac{1}{4}p^2K_1^2\frac{H^{-1}(t)}{\int_T^t H^{-1}(s)\,\mathrm{d}s}.$$

Integrating from $T_1 > T$ to t we get

$$\frac{1}{S(T_1)} > \frac{1}{S(T_1)} - \frac{1}{S(t)} \geqslant \frac{1}{4}p^2K_1^2\ln\left(\int_{T_1}^t H^{-1}(s)\,\mathrm{d}s\right) \to \infty$$

for $t \to \infty$, and this is a contradiction with the convergence of $\int^\infty H^{-1}u^2\,\mathrm{d}t$. $\square$

For easier manipulation with certain terms in the subsequent parts of this paper, let us denote

$$L(t) := \frac{\int_T^t H^{-1}(s)\big(\int_T^s (c(\tau)-\tilde c(\tau))h^p(\tau)\,\mathrm{d}\tau\big)\,\mathrm{d}s}{\int_T^t H^{-1}(s)\,\mathrm{d}s}, \quad L(\infty) := \lim_{t\to\infty} L(t).$$

Corollary 1. *Assume that the assumptions of Theorem* 1 *hold. Let either*

$$L(\infty) = \infty \quad \text{or} \quad -\infty < \liminf_{t\to\infty} L(t) < \limsup_{t\to\infty} L(t). \tag{22}$$

Then (1) *is oscillatory.*

P r o o f. Let $L(\infty) = \infty$ and suppose that (1) is nonoscillatory. Then (19) holds and by Theorem 1 the integral (17) converges for every solution u of (11) and hence the limit (18) exists as a finite number, which is a contradiction. The proof of sufficiency of the second condition in (22) is similar. □

The next theorem is the main result of this paper. It can be seen as a kind of generalization of Hartman-Wintner type criteria.

Theorem 2. *Let* $\int^\infty r^{1-q}(t)\,\mathrm{d}t = \infty$. *Suppose that equation* (5) *is nonoscillatory and let* h *be a principal solution of* (5) *such that*

$$\int^\infty H^{-1}(t)\,\mathrm{d}t = \infty, \quad \lim_{t\to\infty} r(t)h(t)\Phi(h'(t)) := M > 0,$$

where the funcion H *is defined by* (15).

Further, let $0 \leqslant \int^\infty c(t)\,\mathrm{d}t < \infty$ *and*

$$0 \leqslant \int^\infty (c(t)-\tilde c(t))h^p(t)\,\mathrm{d}t < \infty.$$

If the limit $L(\infty) < \infty$ *exists and*

$$\limsup_{t\to\infty} \frac{\int_T^t H^{-1}(s)\,\mathrm{d}s}{\ln \int_T^t H^{-1}(s)\,\mathrm{d}s}(L(\infty)-L(t)) > \frac{1}{2q},$$

then (1) *is oscillatory.*

P r o o f. Suppose, by contradiction, that (1) is nonoscillatory. In view of Lemma 2 our assumptions ensure the existence of the finite limit

$$\lim_{t\to\infty} \frac{w(t)}{w_h(t)} = 1, \tag{23}$$

where w is a solution of the Riccati equation (6) corresponding to (1) and $w_h = r\Phi(h')/\Phi(h)$ the solution of the Riccati equation corresponding to (5). Let us investigate the behavior of the function $P(u,v)$,

$$P(u,v) = \frac{u^p}{p} - uv + \frac{v^q}{q} = u^p\Big(\frac{1}{q}\frac{v^q}{u^p} - vu^{1-p} + \frac{1}{p}\Big) = u^p Q(vu^{1-p}),$$

where $Q(x) = q^{-1}x^q - x + p^{-1} \geqslant 0$ and $Q(1) = 0$. By L'Hospital's rule (used twice) we have

$$\lim_{x\to 1} \frac{Q(x)}{(x-1)^2} = \frac{q-1}{2}.$$

Hence, for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$-\varepsilon \leqslant \frac{Q(x)}{(x-1)^2} - \frac{q-1}{2} \leqslant \varepsilon \tag{24}$$

for x satisfying $|x-1| < \delta$, and inequality (24) can be rewritten as

$$\Big(\frac{q-1}{2} - \varepsilon\Big)(x-1)^2 \leqslant Q(x) \leqslant \Big(\frac{q-1}{2} + \varepsilon\Big)(x-1)^2.$$

For $x = vu^{1-p}$ we have

$$\Big(\frac{q-1}{2} - \varepsilon\Big)(vu^{1-p}-1)^2 \leqslant Q(vu^{1-p}) \leqslant \Big(\frac{q-1}{2} + \varepsilon\Big)(vu^{1-p}-1)^2,$$

which is for $u \neq 0$ equivalent to

$$u^p\Big(\frac{q-1}{2} - \varepsilon\Big)(vu^{1-p}-1)^2 \leqslant P(u,v) \leqslant u^p\Big(\frac{q-1}{2} + \varepsilon\Big)(vu^{1-p}-1)^2.$$

By virtue of (23) there exists T_1 such that $|w/w_h - 1| < \delta$ for $t \geqslant T_1$ and hence for $u = \Phi^{-1}(w_h(t)), v = w(t)$ we have

$$w_h^q\Big(\frac{q-1}{2} - \varepsilon\Big)\Big(\frac{w}{w_h} - 1\Big)^2 \leqslant P(\Phi^{-1}(w_h), w) \leqslant w_h^q\Big(\frac{q-1}{2} + \varepsilon\Big)\Big(\frac{w}{w_h} - 1\Big)^2.$$

From the definition of w_h we get

$$\begin{aligned} h^{2p-2}(t)r^{-1}(t)(h'(t))^{2-p}\Big(\frac{q-1}{2} - \varepsilon\Big)(w(t) - w_h(t))^2 &\leqslant r^{1-q}(t)h^p(t)P(\Phi^{-1}(w_h), w) \\ &\leqslant h^{2p-2}(t)r^{-1}(t)(h'(t))^{2-p}\Big(\frac{q-1}{2} + \varepsilon\Big)(w(t) - w_h(t))^2, \end{aligned}$$

which, in terms of $u = (w - w_h)h^p$ and $H = rh^2|h'|^{p-2}$, yields

$$\Big(\frac{q-1}{2} - \varepsilon\Big)\frac{u^2(t)}{H(t)} \leqslant r^{1-q}(t)h^p(t)P(\Phi^{-1}(w_h), w) \leqslant \Big(\frac{q-1}{2} + \varepsilon\Big)\frac{u^2(t)}{H(t)}. \tag{25}$$

As (1) and (5) are nonoscillatory, the modified Riccati equation (11) holds and by its integration and using the fact that $\int^\infty r^{1-q}h^p P(\Phi^{-1}(w_h),w)<\infty$ (which follows from Lemma 2), we get

$$u(t)=u(T)-\int_T^t (c(s)-\tilde c(s))h^p(s)\,\mathrm{d}s-p\int_t^T r^{1-q}(s)h^p(s)P(\Phi^{-1}(w_h),w)\,\mathrm{d}s,$$

hence

$$\begin{aligned}u(t)=u(T)&-p\int_T^\infty r^{1-q}(t)h^p(t)P(\Phi^{-1}(w_h),w)\,\mathrm{d}t\\&+p\int_t^\infty r^{1-q}(s)h^p(s)P(\Phi^{-1}(w_h),w)\,\mathrm{d}s\\&-\int_T^t (c(s)-\tilde c(s))h^p(s)\,\mathrm{d}s.\end{aligned}$$

Using (21), we get in view of the definition of $L(\infty)$ and (25)

$$u(t)\geqslant L(\infty)+\Bigl(\frac{q}{2}-p\varepsilon\Bigr)\int_t^\infty \frac{u^2(s)}{H(s)}\,\mathrm{d}s-\int_T^t (c(s)-\tilde c(s))h^p(s)\,\mathrm{d}s,$$

which implies (suppressing the integration variable)

$$\begin{aligned}\int_T^t H^{-1}u\geqslant\int_T^t L(\infty)H^{-1}+\frac{q}{2}\int_T^t H^{-1}\int_s^\infty\frac{u^2}{H}-\int_T^t H^{-1}\int_T^s (c-\tilde c)h^p\\-p\varepsilon\int_T^t H^{-1}\int_s^\infty\frac{u^2}{H},\end{aligned}$$

and hence

$$\begin{aligned}&\int_T^t L(\infty)H^{-1}(s)\,\mathrm{d}s-\int_T^t H^{-1}(s)\int_T^s (c(\tau)-\tilde c(\tau))h^p(\tau)\,\mathrm{d}\tau\,\mathrm{d}s\\&\quad\leqslant\int_T^t\frac{1}{H(s)}u(s)\,\mathrm{d}s-\frac{q}{2}\int_T^t\frac{1}{H(s)}\int_s^\infty\frac{u^2(\tau)}{H(\tau)}\,\mathrm{d}\tau\,\mathrm{d}s+p\varepsilon\int_T^t\frac{1}{H(s)}\int_s^\infty\frac{u^2(\tau)}{H(\tau)}\,\mathrm{d}\tau\,\mathrm{d}s.\end{aligned}$$

Using the definition of $L(t)$ on the left-hand side and integrating by parts on the right-hand side of the last inequality, we have

$$\begin{aligned}&(L(\infty)-L(t))\int_T^t H^{-1}(s)\,\mathrm{d}s\\&\qquad\leqslant\int_T^t H^{-1}(s)u(s)\,\mathrm{d}s-\frac{q}{2}\Bigl[\int_s^\infty\frac{u^2(\tau)}{H(\tau)}\,\mathrm{d}\tau\cdot\int_T^s H^{-1}(\tau)\,\mathrm{d}\tau\Bigr]_T^t\\&\qquad-\frac{q}{2}\int_T^t\Bigl(\frac{u^2(s)}{H(s)}\int_T^s H^{-1}(\tau)\,\mathrm{d}\tau\,\mathrm{d}s\Bigr)+p\varepsilon\int_T^t H^{-1}(s)\int_s^\infty\frac{u^2(\tau)}{H(\tau)}\,\mathrm{d}\tau\,\mathrm{d}s\end{aligned}$$

and

$$
\begin{aligned}
(L(\infty)-L(t))&\int_T^t H^{-1}(s)\,\mathrm{d}s\\
&\leqslant \int_T^t \frac{H^{-1}(s)}{\int_T^s H^{-1}(\tau)\,\mathrm{d}\tau}\Bigl(u(s)\int_T^s H^{-1}(\tau)\,\mathrm{d}\tau-\frac{q}{2}\Bigl(u(s)\int_T^s H^{-1}(\tau)\,\mathrm{d}\tau\Bigr)^2\Bigr)\,\mathrm{d}s\\
&\quad-\frac{q}{2}\int_t^\infty \frac{u^2(s)}{H(s)}\,\mathrm{d}s\int_T^t H^{-1}(s)\,\mathrm{d}s+p\varepsilon\int_T^t H^{-1}(s)\int_s^\infty \frac{u^2(\tau)}{H(\tau)}\,\mathrm{d}\tau\,\mathrm{d}s
\end{aligned}
$$

and by virtue of the inequality $\alpha-\frac{1}{2}q\alpha^2\leqslant\frac{1}{2}q^{-1}$ for $\alpha=u\int^s H^{-1}$ we get

$$
\begin{aligned}
(L(\infty)-L(t))\leqslant \frac{1}{2q}\frac{\ln\int_T^t H^{-1}(s)\,\mathrm{d}s}{\int_T^t H^{-1}(s)\,\mathrm{d}s}&-\frac{q}{2}\int_t^\infty\frac{u^2(s)}{H(s)}\,\mathrm{d}s\\
&+p\varepsilon\,\frac{\int_T^t H^{-1}(s)\int_s^\infty H^{-1}(\tau)u^2(\tau)\,\mathrm{d}\tau\,\mathrm{d}s}{\int_T^t H^{-1}(s)\,\mathrm{d}s}.
\end{aligned}
$$

From Theorem 1 we obtain that $\int_t^\infty H^{-1}u^2<\infty$ and thus

$$
\limsup_{t\to\infty}\frac{\int_T^t H^{-1}(s)\,\mathrm{d}s}{\ln\int_T^t H^{-1}(s)\,\mathrm{d}s}(L(\infty)-L(t))\leqslant\frac{1}{2q}+p\varepsilon\int_t^\infty\frac{u^2(s)}{H(s)}\,\mathrm{d}s.
$$

As $\lim\limits_{t\to\infty} w/w_h=1$, ε and also the last term of the above inequality are arbitrarily small and we have a contradiction with our assumption. □

Corollary 2. *Let $r(t)\equiv1$, $\tilde c=\tilde\gamma/t^p$ where $\tilde\gamma=((p-1)/p)^p$, i.e., (5) is the generalized Euler equation with the critical coefficient*

$$
(\Phi(y'))'+\frac{\tilde\gamma}{t^p}\Phi(y)=0. \tag{26}
$$

Let $\int_t^\infty c(s)\,\mathrm{d}s\geqslant0$ for large t and

$$
0\leqslant\int_t^\infty\Bigl(c(s)-\frac{\tilde\gamma}{s^p}\Bigr)s^{p-1}(s)\,\mathrm{d}s<\infty.
$$

If, for T sufficiently large, the limit

$$
L(\infty)=\lim_{t\to\infty}\frac{\int_T^t s^{-1}\int_T^s(c-\tilde\gamma/\tau^p)\,\tau^{p-1}\,\mathrm{d}\tau\,\mathrm{d}s}{\ln|t/T|}<\infty
$$

exists and

$$\limsup_{t\to\infty} \frac{\ln|t/T|}{\ln\ln|t/T|}\left(L(\infty) - \frac{\int_T^t s^{-1}\int_T^s (c-\tilde{\gamma}/\tau^p)\,\tau^{p-1}\,\mathrm{d}\tau\,\mathrm{d}s}{\ln|t/T|}\right) > \frac{1}{2q},$$

then (1) *is oscillatory.*

P r o o f. The function $h(t) = t^{(p-1)/p}$ is the principal solution of (26) (see [9]),

$$\lim_{t\to\infty} h(t)\Phi(h'(t)) = \lim_{t\to\infty} t^{(p-1)/p}\Big(\frac{p-1}{p}t^{-1/p}\Big)^{p-1} = \Big(\frac{p-1}{p}\Big)^{p-1}$$

and

$$\int^{\infty} \frac{\mathrm{d}t}{h^2(t)(h'(t))^{p-2}} = \Big(\frac{p}{p-1}\Big)^{p-2}\int^{\infty}\frac{\mathrm{d}t}{t} = \infty.$$

The statement follows from Theorem 2. □

References

[1] *T. Chantladre, N. Kandelaki, A. Lomtatidze*: Oscillation and nonoscillation criteria for a second order linear equation. Georgian Math. J. *6* (1999), 401–414.

[2] *O. Došlý*: Half-Linear Differential Equations (A. Cañada, P. Drábek, A. Fonda, eds.). Handbook of Differential Equations: Ordinary Differential Equations, Vol. I, Elsevier, Amsterdam, 2004, pp. 161–357.

[3] *O. Došlý, A. Lomtatidze*: Oscillation and nonoscillation criteria for half-linear second order differential equations. To appear in Hiroshima Math. J.

[4] *O. Došlý, Z. Pátíková*: Hille-Wintner type comparison criteria for half-linear second order differential equations. Arch. Math., Brno *42* (2006), 185–194.

[5] *O. Došlý, P. Řehák*: Half-Linear Differential Equations. North Holland Mathematics Studies 202, Elsevier, Amsterdam, 2005.

[6] *O. Došlý, J. Řezníčková*: Regular half-linear second order differential equations. Arch. Math. *39* (2003), 233–245.

[7] *P. Hartman*: Ordinary Differential Equations. SIAM, Philadelphia, 2002.

[8] *Á. Elbert, T. Kusano*: Principal solutions of nonoscillatory half-linear differential equations. Adv. Math. Sci. Appl. *18* (1998), 745–759.

[9] *Á. Elbert, A. Schneider*: Perturbations of the half-linear Euler differential equation. Result. Math. *37* (2000), 56–83.

[10] *J. Jaroš, T. Kusano*: A Picone type identity for half-linear differential equations. Acta Math. Univ. Comenianea *68* (1999), 137–151.

[11] *N. Kandelaki, A. Lomtatidze, D. Ugulava*: On oscillation and nonoscillation of a second order half-linear equation. Georgian Math. J. *2* (2000), 329–346.

[12] *H. J. Li, C. C. Yeh*: Oscillations of half-linear second order differential equations. Hiroshima Math. J. *25* (1995), 585–594.

[13] *J. D. Mirzov*: Analogue of the Hartman theorem. Diff. Urav. *25* (1989), 216–222. (In Russian.)

[14] *J. D. Mirzov*: Asymptotic Properties of Solutions of Systems of Nonlinear Nonautonomous Ordinary Differential Equations. Masaryk University Press, Brno, 2004.

[15] *J. D. Mirzov*: Principal and nonprincipal solutions of a nonoscillatory system. Tbiliss. Gos. Univ. Inst. Prikl. Mat. Trudy *31* (1988), 100–117.
[16] *J. Řezníčková*: Half-linear Hartman-Wintner theorems. Stud. Univ. Žilina Math. Ser. *15* (2002), 56–66.

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