GAUGE INTEGRALS AND SERIES

Charles Swartz, Las Cruces

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Abstract. This note contains a simple example which does clearly indicate the differences in the Henstock-Kurzweil, McShane and strong McShane integrals for Banach space valued functions.

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The Henstock-Kurzweil and McShane integrals are both integrals based on the concept of Riemann type partitions being refined in some way by a positive function or a gauge instead of a positive constant as is the case with the Riemann integral. The McShane integral is obtained from the Henstock-Kurzweil integral by relaxing the conditions placed on the partitions and results in an absolute integral whereas the Henstock-Kurzweil integral is a conditional integral. There seems to be no clear intuitive reason for this phenomena. This note contains a simple example which, although it offers no reason for this phenomena, does clearly indicate the difference in the two integrals.

There is a gauge type integral which is equivalent to the Bochner integral in the case of Banach space valued functions so we will include this integral in our discussion and consider Banach space valued functions.

Let X be a Banach space. The example which we present is based on series in a Banach space, and the geometry behind the example is more easily understood if presented on the unbounded interval $I = [1, \infty]$.

A partition of I is a finite collection $\{I_i\colon 1 \leqslant i \leqslant m\}$ of non-overlapping closed subintervals of I such that $I = \bigcup\limits_{i=1}^{m} I_i$.

A McShane (M) partition of I is a finite collection of pairs $\mathcal{D} = \{(t_i, I_i)\colon\ 1 \leqslant i \leqslant m\}$ such that $\{I_i\colon\ 1 \leqslant i \leqslant m\}$ is a partition of I and $t_i \in I$; the element t_i is called the tag associated with the interval I_i.

A Henstock-Kurzweil (HK) partition $\mathcal{D}$ of I is a McShane partition with the additional requirement that the tag $t_i \in I_i$ for every $i = 1, \ldots m$. If $\mathcal{D} = \{(t_i, I_i)\colon$ $1 \leqslant i \leqslant m\}$ is a McShane partition of I and $f\colon\ I \to X$, the Riemann sum of f with respect to $\mathcal{D}$ is defined to be

$$S(f, \mathcal{D}) = \sum_{i=1}^{m} f(t_i) l(I_i),$$

where $l(J)$ denotes the length of an interval J and we use the convention that $0 \cdot \infty = 0$.

A gauge γ on I is a function defined on I such that $\gamma(t)$ is an open interval containing t (an open interval at ∞ is an interval of the form $(b, \infty]$). In the case of a bounded interval gauges γ are generated by positive functions δ by setting $\gamma(t) = (t - \delta(t), t + \delta(t))$; for unbounded intervals it is more convenient to adopt our definition. An HK or M partition $\mathcal{D} = \{(t_i, I_i)\colon\ 1 \leqslant i \leqslant m\}$ is γ-fine if $I_i \subset \gamma(t_i)$ for every i.

Definition 1. A function $f\colon\ I \to X$ is Henstock-Kurzweil (HK) (McShane (M)) integrable if there exists $v \in X$ such that for every $\varepsilon > 0$ there exists a gauge γ on I such that $\|S(f, \mathcal{D}) - v\| < \varepsilon$ for every γ-fine HK (M) partition $\mathcal{D}$ of I.

The value v is called the HK (M) integral of f and is denoted by $\int_I f = \int_1^\infty f$.

[It will be clear from the context whether we are dealing with the HK or the M integral.]

For the basic properties of the scalar valued HK integral see [4], [7], [9], [14]; for the vector valued HK integral see [2], [3]. For the basic properties of the M integral see [2], [3], [5], [8], [12], [13]. We will only be using very elementary properties of these integrals.

For the case of the HK integral over a bounded interval it is known that the HK integral is more general than the Lebesgue integral and is actually equivalent to the Perron and Denjoy integrals ([4]). On the other hand the M integral is equivalent to the Lebesgue integral, and actually offers an interesting way of presenting the Lebesgue integral without prior introduction of the Lebesgue measure ([8], [14]).

The example which we will present will be based on series with values in the Banach space X. A series $\sum x_i$ with values in X is convergent if the partial sums $\left\{ \sum\limits_{i=1}^{n} x_i \right\}$ converge; the series is subseries (unconditionally) convergent if each series

$\sum x_{n_k}$ $\Big(\sum\limits_{k=1}^{\infty} x_{\pi(k)}\Big)$ converges for every subsequence $\{x_{n_k}\}$ (for every sequence $\{x_{\pi(k)}\}$ where π is a permutation of $\mathbb{N}$). For the case of Banach spaces it is known that subseries and unconditional convergence are equivalent ([1, IV.1]).

We will need one additional property of subseries convergent series.

Proposition 2. *If $\sum x_k$ is subseries convergent, then the series $\sum\limits_{k=1}^{\infty} t_k x_k$ converge uniformly for $\{t_k\} \in l^\infty$ with $\|\{t_k\}\|_\infty \leqslant 1$.*

See [13, 8.2.2] for a proof.

The series $\sum x_k$ in X is absolutely convergent if $\sum\limits_{k=1}^{\infty} \|x_k\| < \infty$. It follows easily from the completeness of X that an absolutely convergent series is subseries convergent.

We will also consider a gauge type integral which is equivalent to the Bochner integral for vector valued functions.

Definition 3. The function $f\colon I \to X$ is strongly McShane (strongly M) integrable if f is M integrable over I and for every $\varepsilon > 0$ there exists a gauge γ on I such that $\sum\limits_{i=1}^{m} \|f(t_i)l(I_i) - \int_{I_i} f\| < \varepsilon$ for every γ-fine M partition $\mathcal{D} = \{(t_i, I_i)\colon\ 1 \leqslant i \leqslant m\}$ of I.

See [10], [11], [15] for this definition. It is shown in [10], [11] that the strong M integral is equivalent to the Bochner integral (for the Bochner integral see [6], [10]). For scalar functions the M and strong M integrals coincide by Henstock's Lemma ([7], [8], [14]). That this is not the case for Banach space valued functions will be seen in Theorem 4.

Theorem 4. *Let $\{x_k\} \subset X$ be bounded and define $f\colon I \to X$ by $f = \sum \chi_{[k,k+1)} x_k$.*

(a) *f is* HK *integrable if and only if $\sum x_k$ is convergent.*

(b) *f is* M *integrable if and only if $\sum x_k$ is subseries convergent.*

(c) *f is strongly* M *integrable if and only if $\sum x_k$ is absolutely convergent.*

P r o o f. First consider the sufficiency of the 3 conditions. We begin by developing several inequalities which are used in establishing all 3 of the integrability statements and then point out the differences for the 3 integrals. Let $\varepsilon > 0$. In case (a), choose $M \in \mathbb{N}$ such that $\Big\|\sum\limits_{k=m}^{n} x_k\Big\| < \varepsilon$ for $m, n \geqslant M$; in cases (b), choose M such that $\Big\|\sum\limits_{k=M}^{\infty} s_k x_k\Big\| < \varepsilon$ when $s = \{s_k\} \in l^\infty$ and $\|s\|_\infty \leqslant 1$ (Proposition 2); in case (c), choose M such that $\sum\limits_{k=M}^{\infty} \|x_k\| < \varepsilon$ and one other condition which will be indicated when (c) is established.

Let $B > \sup\limits_k \|x_k\|$. Define a gauge γ on I as follows:

$\gamma(t) = (k, k+1)$ if $t \in (k, k+1)$,

$\gamma(k) = (k - \varepsilon/2^k B, k + \varepsilon/2^k B)$ if $k \in \mathbb{N}$ and

$\gamma(\infty) = (M, \infty]$.

Suppose that $\mathcal{D} = \{(t_i, I_i)\colon 1 \leqslant i \leqslant m\}$ is a γ-fine partition which at this point may be either an HK or an M partition. Suppose also that $t_m = \infty$ and $I_m = [b, \infty]$.

Let N be the largest integer less than or equal to b so that $N \geqslant M$.

Let $\mathcal{D}_k = \{(t_i, I_i)\colon k < t_i < k+1\}$.

We derive an inequality for $S(f, \mathcal{D}_k)$ independent of whether $\mathcal{D}$ is either an HK or an M partition. Note that $l_k =: l\Bigl(\bigcup\limits_{(t_i, I_i) \in \mathcal{D}_k} I_i\Bigr) \geqslant 1 - \varepsilon/2^k B - \varepsilon/2^{k+1} B$. Then

$$\begin{aligned} \|S(f, \mathcal{D}_k) - x_k\| &= \|x_k\|(1 - l_k) \\ &\leqslant B(\varepsilon/2^k B + \varepsilon/2^{k+1} B) < \varepsilon/2^{k-1} \quad \text{for } 1 \leqslant k < M. \end{aligned} \tag{1}$$

Since $b \in [N, N+1]$, we derive a slightly weaker inequality for $\mathcal{D}_N$.

$$\|S(f, \mathcal{D}_N) - x_N\| = \|x_N\|(1 - l_N) \leqslant \|x_N\| < \varepsilon. \tag{2}$$

Let $\mathcal{D}_{\mathbb{N}} = \{(t_i, I_i); t_i \in \mathbb{N}\}$. Then

$$\|S(f, \mathcal{D}_{\mathbb{N}})\| = \Bigl\| \sum_{k=1}^{N+1} x_k \sum_{t_i = k} l(I_i) \Bigr\| \leqslant \sum_{k=1}^{N+1} \|x_k\| l(\gamma(k)) \leqslant \sum_{k=1}^{N+1} \varepsilon/2^{k-1} < 2\varepsilon. \tag{3}$$

Finally, let $\mathcal{D}_\infty = \{(t_i, I_i)\colon t_i = \infty\}$. Then

$$\|S(f, \mathcal{D}_\infty)\| = 0. \tag{4}$$

In case $\mathcal{D}$ is either an HK or an M partition,

$$S(f, \mathcal{D}) - \sum_{k=1}^{\infty} x_k = \sum_{k=1}^{N} S(f, \mathcal{D}_k) + S(f, \mathcal{D}_{\mathbb{N}}) + S(f, \mathcal{D}_\infty) - \sum_{k=1}^{\infty} x_k. \tag{5}$$

Now consider case (a) so $\mathcal{D}$ is an HK partition and inequality (1) will hold for $1 \leqslant k < N$ and $\mathcal{D}_\infty = \{(\infty, [b, \infty])\}$. From (5) and (1)–(4) and the choice of M,

$$\begin{aligned} \Bigl\| S(f, \mathcal{D}) - \sum_{k=1}^{\infty} x_k \Bigr\| &\leqslant \Bigl\| \sum_{k=1}^{N-1} S(f, \mathcal{D}_k) - x_k \Bigr\| + \|S(f, \mathcal{D}_N) - x_N\| + \|S(f, \mathcal{D}_{\mathbb{N}})\| \\ &\quad + \Bigl\| \sum_{k=N+1}^{\infty} x_k \Bigr\| < \sum_{k=1}^{N-1} \varepsilon/2^{k-1} + \varepsilon + 2\varepsilon + \varepsilon < 6\varepsilon. \end{aligned}$$

It follows that f is HK integrable with integral $\int_I f = \sum_{k=1}^{\infty} x_k$.

Next, consider case (b) so that $\mathcal{D}$ is now an M partition.

In this case the subintervals in $(M, b]$ can have ∞ as their tag so the situation is quite different from case (a); this points out the difference in the 2 integrals.

For $M \leqslant k \leqslant N$, $S(f, \mathcal{D}_k) - x_k = x_k(1 - l_k)$ and l_k may equal 0 so from (5) and (1)–(4) and the choice of M, we have

$$\begin{aligned}
(6) \qquad \Bigl\| S(f,\mathcal{D}) - \sum_{k=1}^{\infty} x_k \Bigr\| &\leqslant \sum_{k=1}^{M-1} \|S(f,\mathcal{D}_k) - x_k\| + \Bigl\| \sum_{k=M}^{N} x_k(1-l_k) \Bigr\| \\
&\quad + \|S(f,\mathcal{D}_{\mathbb{N}})\| + \Bigl\| \sum_{k=N+1}^{\infty} x_k \Bigr\| \\
&\leqslant \sum_{k=1}^{M-1} \varepsilon/2^{k-1} + \Bigl\| \sum_{k=M}^{N} x_k(1-l_k) \Bigr\| + 2\varepsilon + \varepsilon \\
&< 5\varepsilon + \Bigl\| \sum_{k=M}^{N} x_k(1-l_k) \Bigr\| < 6\varepsilon
\end{aligned}$$

since $0 < l_k \leqslant 1$ (Proposition 2). It now follows that f is M integrable.

Finally, consider case (c). In this case we impose an additional condition on the choice of M; M is chosen such that $\| \int_a^{\infty} f \| < \varepsilon$. This is possible since f is M integrable by (b) (Theorem 11 of [12]). It suffices to show that

$$(7) \qquad \sum_{k=1}^{m} \Bigl\| f(t_k) l(I_k) - \int_{I_k} f \Bigr\| \leqslant 6\varepsilon.$$

For $t_i \in (k, k+1)$, $f(t_i) l(I_i) - \int_{I_i} f = 0$.

For $t_i = k \in \mathbb{N}$,

$$\begin{aligned}
\sum_{t_i = k} \Bigl\| f(t_i) l(I_i) - \int_{I_i} f \Bigr\| &= \sum_{t_i = k} \Bigl\| x_k\bigl(l(I_i \cap (k-1,k)) + l(I_i \cap [k,k+1))\bigr) \\
&\quad - \int_{I_i \cap (k-1,k)} f - \int_{I_i \cap [k,k+1)} f \Bigr\| \\
&= \sum_{t_i = k} \|(x_k - x_{k-1})\| l(I_i \cap (k-1,k)) \\
&\leqslant \|x_k - x_{k-1}\| l(\gamma(k)) \leqslant 2B\varepsilon/2^{k-1}B = \varepsilon/2^{k-2}.
\end{aligned}$$

For $t_i = \infty$, $I \subset [M, b]$, let $J = \bigcup_{t_i=\infty, I_i \subset [M,b]} I_i$. Then

$$\left\| \int_J f \right\| = \left\| \sum_{k=M}^{N-1} \int_{J\cap[k,k+1]} f + \int_{J\cap[N,b]} f \right\|$$
$$= \left\| \sum_{k=M}^{N-1} x_k l(J \cap [k, k+1]) + x_N l([N, b]) \right\| \leqslant \sum_{k=M}^{N} \|x_k\| < \varepsilon.$$

For $t_m = \infty, I_m = [b, \infty]$, by choice of M, $\|f(t_m)l(I_m) - \int_{I_m} f\| < \varepsilon$. Hence,

$$\sum_{k=1}^{m} \left\| f(t_k)l(I_k) - \int_{I_k} f \right\| = \sum_{k=1}^{\infty} \sum_{t_i \in (k,k+1)} \left\| f(t_i)l(I_i) - \int_{I_i} f \right\|$$
$$+ \sum_{k=1}^{\infty} \sum_{t_i=k} \left\| f(t_i)l(I_i) - \int_{I_i} f \right\| + \sum_{t_i=\infty, I_i \subset [M,b]} \left\| f(t_i)l(I_i) - \int_{I_i} f \right\|$$
$$+ \left\| f(t_m)l(I_m) - \int_{I_m} f \right\| \leqslant \sum_{k=1}^{\infty} \varepsilon/2^{k-2} + \varepsilon + \varepsilon \leqslant 6\varepsilon.$$

We now consider the necessity of the 3 conditions.

For (a) since there are no improper integrals for the HK integral ([14, 4.4]), $\int_1^\infty f = \lim_n \int_1^n f = \lim_n \sum_{k=1}^{n} x_k = \sum_{k=1}^{\infty} x_k$.

For (b), if f is M integrable, then the indefinite integral of f is countably additive ([7, 3.11.11], [14, 6.2]). Thus,

$$\sum_{k=1}^{\infty} \int_k^{k+1} f = \sum_{k=1}^{\infty} x_k$$

and since any rearrangement of $\{[k, k+1]\}$ satisfies the same condition, the series $\sum x_k$ is unconditionally convergent and, therefore, subseries convergent.

Finally, for (c), if f is strongly M integrable, it follows that the scalar function $\|f(\cdot)\|$ is M integrable. So, by the countable additivity of the indefinite integral $\int \|f(\cdot)\|$, we have

$$\sum_{k=1}^{\infty} \int_k^{k+1} \|f(\cdot)\| = \sum_{k=1}^{\infty} \|x_k\| = \int_1^{\infty} \|f(\cdot)\|,$$

and the series $\sum x_k$ is absolutely convergent.

R e m a r k 5. If one is only interested in the scalar version of Theorem 4, the conditions (b) and (c) are, of course, equivalent and the proof of (c) is much simpler in this case. In (6), the term $\left\| \sum\limits_{k=M}^{N} x_k(1-l_k) \right\|$ can be replaced by $\sum\limits_{k=M}^{N} |x_k|$ and the argument goes through.

The sufficiency of condition (b) has also been considered by Gordon ([5, Theorem 15]) where he uses a different characterization of subseries convergence and different methods.

Let $HK(X)$ $(M(X), SM(X))$ be the space of all X-valued HK integrable (M integrable, strongly M integrable) functions defined on I. We have

Corollary 6. *For any X, $M(X) \subsetneq HK(X)$. For any infinite dimensional X, $SM(X) \subsetneq M(X)$.*

P r o o f. For any $X \neq \{0\}$, there is a convergent series which is not subseries convergent $\left[x \neq 0, \sum(-1)^k x/k\right]$. If X is infinite dimensional, by the Dvoretsky-Rogers Theorem ([1]), there is a series which is subseries convergent, but not absolutely convergent. $\square$

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Author's address: *Charles Swartz*, New Mexico State University, Department of Mathematical Sciences, P.O. Box 30001, Las Cruces, New Mexico 88003-8001, USA, e-mail: `cswartz@nmsu.edu`.