NEARLY ANTIPODAL CHROMATIC NUMBER $ac'(P_n)$ OF THE PATH P_n

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(Received September 22, 2007, revised December 4, 2007)

Abstract. Chartrand et al. (2004) have given an upper bound for the nearly antipodal chromatic number $ac'(P_n)$ as $\binom{n-2}{2}+2$ for $n \geqslant 9$ and have found the exact value of $ac'(P_n)$ for $n = 5, 6, 7, 8$. Here we determine the exact values of $ac'(P_n)$ for $n \geqslant 8$. They are $2p^2 - 6p + 8$ for $n = 2p$ and $2p^2 - 4p + 6$ for $n = 2p + 1$. The exact value of the radio antipodal number $ac(P_n)$ for the path P_n of order n has been determined by Khennoufa and Togni in 2005 as $2p^2 - 2p + 3$ for $n = 2p + 1$ and $2p^2 - 4p + 5$ for $n = 2p$. Although the value of $ac(P_n)$ determined there is correct, we found a mistake in the proof of the lower bound when $n = 2p$ (Theorem 6). However, we give an easy observation which proves this lower bound.

Keywords: radio k-coloring, span, radio k-chromatic number

MSC 2000: 05C78, 05C12, 05C15

1. Introduction

Let G be a connected graph. For any positive integer k, a *radio k-coloring* of G is an assignment f of colors (positive integers) to the vertices of G such that $|f(u) - f(v)| \geqslant 1 + k - d(u, v)$ for every two distinct vertices u, v of G. The *span* $rc_k(f)$ of a radio k-coloring f of G is the maximum color assigned to a vertex of G. The *radio k-chromatic number* $rc_k(G)$ of G is the minimum span of all radio k-colorings of G. If $rc_k(f) = rc_k(G)$ for some radio k-coloring f of G then we say that f is a minimal radio k-coloring. The radio 1-chromatic number $rc_1(G)$ is then the chromatic number $\chi(G)$. If $\mathrm{diam} G = d$, the radio d-coloring of G is referred to as the *radio coloring* of G, and the radio d-chromatic number $rc_d(G)$ is called the *radio number* of G that was introduced in [1]. The radio $(d-1)$-coloring of G is referred to as the *radio antipodal coloring* of G and the radio $(d-1)$-chromatic number $ac(G)$ is called the *radio antipodal chromatic number* or simply the *antipodal number* of G.

The radio $(d-2)$-coloring is known as the *nearly antipodal coloring* of G and the radio $(d-2)$ coloring number is called the *nearly antipodal chromatic number* $ac'(G)$ of G. The radio k-coloring of a graph was defined by Chartrand et al. in [3], [4]. The radio antipodal coloring for graphs was first studied by Chartrand et al. [2], [3].

Kchikech et al. [7] have given the exact values of the radio k-chromatic number of P_n for $k \geqslant n$, which are $(n-1)k-\frac{1}{2}n(n-2)+1$ if n is even and $(n-1)k-\frac{1}{2}(n-1)^2+2$ if n is odd. Liu and Zhu [6] determined the exact value of the radio $(n-1)$-coloring number (radio number) as given below.

Theorem 1.1 ([6]). *For any integer $n \geqslant 3$,*

$$rc_{(n-1)}(P_n) = \begin{cases} 2p^2+3 & \text{if} \quad n = 2p+1, \\ 2p^2-2p+2 & \text{if} \quad n = 2p. \end{cases}$$

Next we define the symbols $\varepsilon_j, 2 \leqslant j \leqslant n$, which are used in the sequel.

Definition 1.2. For any radio k-coloring f of a path P_n and an ordering $x_1, x_2, \ldots, x_n$ of vertices of P_n with $f(x_i) \leqslant f(x_{i+1})$, $1 \leqslant i \leqslant n-1$, we define $\varepsilon_j = (f(x_j) - f(x_{j-1})) - (1 + k - d(x_j, x_{j-1}))$, $2 \leqslant j \leqslant n$. It is clear from the definition of a radio k-coloring that $\varepsilon_j \geqslant 0$, $2 \leqslant j \leqslant n$.

Khennoufa and Togni [5] determined the exact value of the radio antipodal number for the path P_n as given below.

Theorem 1.3 ([5]). *For any integer $n \geqslant 5$,*

$$ac(P_n) = \begin{cases} 2p^2-2p+3 & \text{if} \quad n = 2p+1, \\ 2p^2-4p+5 & \text{if} \quad n = 2p. \end{cases}$$

Although the value of $ac(P_n)$ determined in [5] is correct, we found a mistake in the proof of the lower bound when $n = 2p$ (Theorem 6 of [5]). In Theorem 6 of [5], the inequality $rc_{2p-1}(P_{2p}) \leqslant ac(P_{2p}) + (2p-1) - \varepsilon_3 - \varepsilon_{2p-1}$ is incorrect, because from Lemma 2 of [5] one gets $rc_{2p-1}(f') \leqslant rc_{2p-2}(f) + (2p-1) - \varepsilon_3 - \varepsilon_{2p-1}$, where f is a radio k-coloring of P_n and f' is a radio k'-coloring of $P_n, k' \geqslant k$, with $rc_{k'}(f') \leqslant rc_k(f) + (n-1)(k'-k) - \sum\limits_{i \in I} \min(k'-k, \varepsilon_i)$, $I = \{i_1, i_2, \ldots, i_s\} \subset \{2, 3, \ldots, n\}$ such that $i_{j+1} > i_j + 1$, $1 \leqslant j \leqslant s-1$. Since $rc_{2p-1}(P_{2p}) \leqslant rc_{2p-1}(f')$, we get $rc_{2p-1}(P_{2p}) \leqslant rc_{2p-2}(f) + (2p-1) - \varepsilon_3 - \varepsilon_{2p-1}$. We have $ac(P_{2p}) \leqslant rc_{2p-2}(f)$. The number $rc_{2p-2}(f)$ cannot be replaced by $ac(P_{2p})$ unless f is a minimal coloring. If we assume (or know) that f is a minimal coloring then there is no need of proving this theorem.

The observation below establishes the lower bound of $ac(P_{2p})$ (Theorem 6 of [5]).

Observation 1.4. *For $n = 2p$, the lower bound of the antipodal number of P_n may be proved as follows. We know that $rc_k(P_n) \geqslant rc_k(P_{n-1})$ for any positive integer k. So $ac(P_{2p}) = rc_{n-2}(P_{n=2p}) \geqslant rc_{n-2}(P_{n-1=2(p-1)+1}) = 2(p-1)^2+3 = 2p^2-4p+5$ due to Liu and Zhu* [6].

Chartrand et al. [4] have given an upper bound for the radio k-chromatic number when $1 \leqslant k \leqslant n-3$ as follows.

Theorem 1.5 ([4])**.** *For $1 \leqslant k \leqslant n-3$,*

$$rc_k(P_n) \leqslant \begin{cases} \frac{1}{2}(k^2+2k+1) & \text{if } k \text{ is odd,} \\ \frac{1}{2}(k^2+2k+2) & \text{if } k \text{ is even.} \end{cases}$$

Chartrand et al. [4] have also given a lower bound for the same number. However, from Theorem 1.3, we get the following improved lower bound for $rc_k(P_n)$, $1 \leqslant k \leqslant n-3$.

Theorem 1.6 ([7])**.** *For $1 \leqslant k \leqslant n-3$,*

$$rc_k(P_n) \geqslant \begin{cases} \frac{1}{2}(k^2+5) & \text{if } k \text{ is odd,} \\ \frac{1}{2}(k^2+6) & \text{if } k \text{ is even.} \end{cases}$$

Chartrand et al. [4] have further improved the upper bound for the particular case $k = n-3$ (i.e. for $ac'(P_n)$) as $\binom{n-2}{2}+2$ for $n \geqslant 9$ and have given the exact values of $ac'(P_n)$ for $n = 5, 6, 7, 8$ as $5, 7, 11, 16$, respectively. In this paper we give the exact value of the nearly antipodal chromatic number of P_n for $n \geqslant 8$. And consequently, the lower bound of the radio k-chromatic number of P_n for $5 \leqslant k \leqslant n-4$ is also improved.

2. Upper bound

Now we give an upper bound for $ac'(P_n)$ by defining radio $(n-3)$-colorings of P_n.

Theorem 2.1. *For any integer $n \geqslant 8$,*

$$ac'(P_n) \leqslant \begin{cases} 2p^2-6p+8 & \text{if } n = 2p, \\ 2p^2-4p+6 & \text{if } n = 2p+1. \end{cases}$$

Proof. Let P_n be the path $a_1\ a_2\ \ldots a_n$.

C a s e I. In this case we take $n = 2p$. Define a map f as

$$\begin{aligned} f(a_1) &= p-1, \\ f(a_{p-1-i}) &= 2p-1+i(2p-3) \quad \text{for} \quad 0 \leqslant i \leqslant p-3, \\ f(a_p) &= 2p^2-6p+8, \\ f(a_{p+1}) &= 1, \\ f(a_{2p-1-j}) &= p+1+j(2p-3) \quad \text{for} \quad 0 \leqslant j \leqslant p-3, \\ f(a_{2p}) &= 2p^2-7p+10. \end{aligned}$$

Observe that a_p gets the maximum color, i.e. $2p^2-6p+8$. Here the distance condition is verified only for vertices of the form a_{p-1-i} and a_{2p-1-j}, $0 \leqslant i, j \leqslant p-3$ as the other conditions can be checked easily.

For $0 \leqslant i, j \leqslant p-3$, $|f(a_{2p-1-j}) - f(a_{p-1-i})| = |p+1+j(2p-3) - (2p-1+i(2p-3))| = |(j-i)(2p-3) - (p-2)| \geqslant p-2+(j-i) = 1+(2p-3)-(2p-1-j-(p-1-i)) = 1+(n-3)-(2p-1-j-(p-1-i))$ if $j \leqslant i$. For $j-i=1$, $|(j-i)(2p-3)-(p-2)| = p-1 = p-2+(j-i) = 1+(n-3)-(2p-1-j-(p-1-i))$. For $j-i \geqslant 2$, $|(j-i)(2p-3)-(p-2)| \geqslant 3p-4 > 2p-5 \geqslant p-2+(j-i) = 1+(n-3)-(2p-1-j-(p-1-i))$.

Therefore the above mapping f is a radio $(n-3)$-coloring and $ac'(P_n) \leqslant 2p^2-6p+8$ for $n = 2p$.

C a s e II. In this case we take $n = 2p+1$. Define a map g as

$$\begin{aligned} g(a_1) &= p+1, \\ g(a_2) &= 2p^2-5p+7, \\ g(a_{3+i}) &= 3p+1+i(2p-1), \ 0 \leqslant i \leqslant p-4, \\ g(a_{p+j}) &= 1+j(2p-1), \ 0 \leqslant j \leqslant 1, \\ g(a_{p+2}) &= 2p^2-4p+6, \\ g(a_{p+3+l}) &= 4p+l(2p-1), \ 0 \leqslant l \leqslant p-4, \\ g(a_{2p+m}) &= p+m(2p-1), \ 0 \leqslant m \leqslant 1. \end{aligned}$$

Observe that a_{p+2} gets the maximum color, i.e. $2p^2-4p+6$. Here the distance condition is verified only for vertices of the form a_{3+i} and a_{p+3+l}, $0 \leqslant i, l \leqslant p-4$ as the other cases can be checked easily. For $0 \leqslant i, l \leqslant p-4$, $|g(a_{p+3+l}) - g(a_{3+i})| = |4p+l(2p-1) - (3p+1+i(2p-1))| = |(l-i)(2p-1)+p-1| \geqslant p-1-(l-i) = 1+(2p-2)-(p+3+l-(3+i)) = 1+(n-3)-(p+3+l-(3+i))$ if $l \geqslant i$. For $i-l=1$, $|(l-i)(2p-1)+p-1| = p = p-1-(l-i) = 1+(n-3)-(p+3+l-(3+i))$. For $i-l \geqslant 2$, $|(l-i)(2p-1)+p-1| \geqslant 3p-1 > 2p-5 \geqslant p-1-(l-i) = 1+(n-3)-(p+3+l-(3+i))$.

Therefore the above mapping g is a radio $(n-3)$-coloring and $ac'(P_n) \leqslant 2p^2-4p+6$ for $n = 2p+1$.

E x a m p l e 2.2. Here we illustrate Theorem 2.1 by giving an example of nearly antipodal colorings of P_n for $n = 8, 11, 12$.

a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
3	12	7	16	1	10	5	14

a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}
6	32	16	25	1	10	36	20	29	5	14

a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}
5	38	29	20	11	44	1	34	25	16	7	40

3. Lower bound

Here we first give a result of Liu and Zhu [6] and also a detailed proof of this which will be useful in deriving some more results in the sequel.

Theorem 3.1 [6]. *For any radio k-coloring f of a path P_n: $a_1\ a_2\ a_3 \ldots a_{n-1}\ a_n$,*

$$\sum_{i=2}^{n} d(x_i, x_{i-1}) \leqslant \begin{cases} 2p^2 - 1 & \text{if } n = 2p, \\ 2p^2 + 2p - 1 & \text{if } n = 2p + 1, \end{cases}$$

where $x_1, x_2, x_3, \ldots, x_{n-1}, x_n$ is an ordering of vertices of P_n such that $f(x_i) \leqslant f(x_{i+1})$, $1 \leqslant i \leqslant n - 1$.

P r o o f. Let $x_i = a_{\sigma(i)}$, $1 \leqslant i \leqslant n$. Then σ is a permutation of $\{1, 2, 3, \ldots, n\}$. Note that $d(x_i, x_{i-1})$ is equal to either $\sigma(i) - \sigma(i-1)$ or $\sigma(i-1) - \sigma(i)$, whichever is positive. Replacing each term $d(x_i, x_{i-1})$ in $\sum\limits_{i=2}^{n} d(x_i, x_{i-1})$ by the corresponding $\sigma(i) - \sigma(i-1)$ or $\sigma(i-1) - \sigma(i)$, whichever is positive, we obtain a summation whose entries are $\pm j$ for $j \in \{1, 2, 3, \ldots, n\}$. Altogether there are $2(n-1)$ terms of the form $\pm j$ in the summation $\sum\limits_{i=2}^{n} d(x_i, x_{i-1})$, half of them positive and half negative. To maximize the summation $\sum\limits_{i=2}^{n} d(x_i, x_{i-1})$, one needs to minimize the absolute values for negative terms while maximizing the values of positive terms. It is easy to verify that the following are the only possibilities achieving the maximum summation.

For $n = 2p$: In the summation $\sum\limits_{i=2}^{n} d(x_i, x_{i-1})$ each of $\{1, 2, 3, \ldots, p - 1\}$ occurs twice with a negative sign, each of $\{p+2, p+3, \ldots, 2p\}$ occurs twice with a positive sign, p occurs once as negative and $p + 1$ occurs once as positive. Since p and $p + 1$ occur only once in the summation, we get $\{f(x_1), f(x_{2p})\} = \{f(a_p), f(a_{p+1})\}$.

For $n = 2p + 1$:

(i) Each of the numbers $\{p+2, p+3, p+4, \ldots, 2p+1\}$ occurs twice with a positive sign, each of $\{1, 2, 3, \ldots, p-1\}$ occurs twice with a negative sign, and each of p and $p+1$ occurs once as negative. Since both p and $p+1$ occur only once in the summation, we get $\{f(x_1), f(x_{2p})\} = \{f(a_p), f(a_{p+1})\}$.

(ii) Each of the numbers $\{p+3, p+4, \ldots, 2p+1\}$ occurs twice with a positive sign, each of $\{1, 2, 3, \ldots, p\}$ occurs twice with a negative sign, and each of $p+1$ and $p+2$ occurs once as positive. Since both p and $p+2$ occur only once in the summation, we get $\{f(x_1), f(x_{2p})\} = \{f(a_{p+1}), f(a_{p+2})\}$.

Next we give the possibilities of having the distance summation $2p^2-2$ and $2p^2-3$ when $n = 2p$ is any radio k-coloring of P_n.

Lemma 3.2. *For any radio k-coloring of P_n, the following are the only possibilities for the sum $\sum\limits_{i=2}^{n} d(x_i, x_{i-1})$ to be $2p^2 - 2$ when $n = 2p$.*

(i) *In the summation each of $\{1, 2, 3, \ldots, p-1\}$ occurs twice with a negative sign, each of $\{p+1, p+3, \ldots, 2p\}$ occurs twice with a positive sign, p occurs once as negative and $p+2$ occurs once as positive. Since both p and $p+2$ occur only once in the summation, we get $\{f(x_1), f(x_{2p})\} = \{f(a_p), f(a_{p+2})\}$.*

(ii) *In the summation each of $\{1, 2, 3, \ldots, p-2, p\}$ occurs twice with a negative sign, each of $\{p+2, p+3, \ldots, 2p\}$ occurs twice with a positive sign, $p-1$ occurs once as negative and $p+1$ occurs once as positive. Since both $p-1$ and $p+1$ occur only once in the summation, we get $\{f(x_1), f(x_{2p})\} = \{f(a_{p-1}), f(a_{p+1})\}$.*

P r o o f. Follows from the proof of Theorem 3.1. □

Lemma 3.3. *For any radio k-coloring of P_n, the following are the only possibilities for the sum $\sum\limits_{i=2}^{n} d(x_i, x_{i-1})$ to be $2p^2 - 3$ when $n = 2p$.*

(i) *In the summation each of $\{1, 2, 3, \ldots, p-1\}$ occurs twice with a negative sign, each of $\{p+1, p+2, p+4, \ldots, 2p\}$ occurs twice with a positive sign, p occurs once as negative and $p+3$ occurs once as positive. Since both p and $p+3$ occur only once in the summation, we get $\{f(x_1), f(x_{2p})\} = \{f(a_p), f(a_{p+3})\}$.*

(ii) *In the summation each of $\{1, 2, 3, \ldots, p-3, p-1, p\}$ occurs twice with a negative sign, each of $\{p+2, p+3, \ldots, 2p\}$ occurs twice with a positive sign, $p-2$ occurs once as negative and $p+1$ occurs once as positive. Since both $p-2$ and $p+1$ occur only once in the summation, we get $\{f(x_1), f(x_{2p})\} = \{f(a_{p-2}), f(a_{p+1})\}$.*

(iii) *In the summation each of $\{1, 2, 3, \ldots, p-2, p\}$ occurs twice with a negative sign, each of $\{p+1, p+3, \ldots, 2p\}$ occurs twice with a positive sign, $p-1$ occurs once as negative and $p+2$ occurs once as positive. Since both $p-1$ and $p+2$ occur only once in the summation, we get $\{f(x_1), f(x_{2p})\} = \{f(a_{p-1}), f(a_{p+2})\}$.*

(iv) *In the summation each of* $\{1+2+\ldots+p-2\}$ *occurs twice with a negative sign, each of* $\{p+2+p+3+\ldots+2p\}$ *occurs twice with a positive sign,* p *and* $p-1$ *occur once with negative sign and* $p+1$ *occurs as* $+(p+1)$ *and* $-(p+1)$*. Since both* p *and* $p-1$ *occur only once in the summation, we get* $\{f(x_1), f(x_{2p})\} = \{f(a_p), f(a_{p-1})\}$.

P r o o f. Follows from the proof of Theorem 3.1. □

Lemma 3.4. *Let* k *be an even integer and* $n = k+2 = 2p$, $n \geqslant 8$*. Let* f *be a minimal radio* k*-coloring of* P_n *and let* $x_1, x_2, x_3, \ldots, x_n$ *be as in Definition 1.2. Then* $\sum\limits_{i=2}^{n} d(x_i, x_{i+1}) = 2p^2 - 1$ *and* $\sum\limits_{i=2}^{n} \varepsilon_i = 2$.

P r o o f. For any path P_m and a radio k-coloring g we have

$$g(y_m) = (m-1)(1+k) - \sum_{i=2}^{m} d(y_i, y_{i-1}) + \sum_{i=2}^{m} \varepsilon_i + 1 \tag{1}$$

where $y_1, y_2, y_3, \ldots, y_m$ is an ordering of vertices of P_m such that $g(y_i) \leqslant g(y_{i+1})$, $1 \leqslant i \leqslant m$. Equation (1) is true because $g(y_m) = \sum\limits_{i=2}^{m} (g(y_i) - g(y_{i-1})) + g(y_1) = \sum\limits_{i=2}^{m} (k+1-d(y_i, y_{i-1}) + \varepsilon_i) + 1 = (m-1)(1+k) - \sum\limits_{i=2}^{m} d(y_i, y_{i-1}) + \sum\limits_{i=2}^{m} \varepsilon_i + 1$.

Since f is a minimal radio k-coloring of P_{k+2}, $f(x_n) = ac(P_n) = 2p^2 - 4p + 5$ by Theorem 1.3. By Theorem 3.1, the maximum possible value of $\sum\limits_{i=2}^{n} d(x_i, x_{i-1})$ is $2p^2 - 1$. Then from equation (1) we have the following possibilities:

(I) $\sum\limits_{i=2}^{k+2} d(x_i, x_{i+1}) = 2p^2 - 1$ and $\sum\limits_{i=2}^{k+2} \varepsilon_i = 2$.

(II) $\sum\limits_{i=2}^{k+2} d(x_i, x_{i+1}) = 2p^2 - 2$ and $\sum\limits_{i=2}^{k+2} \varepsilon_i = 1$.

(III) $\sum\limits_{i=2}^{k+2} d(x_i, x_{i+1}) = 2p^2 - 3$ and $\sum\limits_{i=2}^{k+2} \varepsilon_i = 0$.

Next we prove that (II) and (III) cannot occur. Let P_n: $a_1\, a_2 \ldots a_p\, a_{p+1} \ldots a_{2p-1}\, a_{2p}$. Consider the coloring f of the first $2p-1$ vertices of P_n. Let $a_{2p} = x_j$ and $\varepsilon = (f(x_{j+1}) - f(x_{j-1})) - (1 + k - d(x_{j-1}, x_{j+1}))$. Liu and Zhu [6] have given a minimal radio k-coloring of $P_{2(p-1)+1} = P_{2p-1} = P_{k+2-1} = P_{k+1}$ whose summation of distances is equal to $2(p-1)^2 + 2(p-1) - 2$ and summation of ε_i's is equal to 0 with the span $rc_k(P_{k+1}) = 2(p-1)^2 + 3 = 2p^2 - 4p + 5 = rc_k(f)$. So, from Theorem 3.1, we have the following two possibilities for the coloring f of $P_n \setminus \{a_{2p}\}$.

(i) $\Big(\sum\limits_{i=2}^{k+2} d(x_i, x_{i-1})\Big) - (d(x_j, x_{j-1}) + d(x_j, x_{j+1})) - d(x_{j-1}, x_{j+1}) = 2(p-1)^2 + 2(p-1) - 1$ and $\Big(\sum\limits_{i=2}^{k+2} \varepsilon_i\Big) - (\varepsilon_j + \varepsilon_{j+1} - \varepsilon) = 1$;

(ii) $\left(\sum_{i=2}^{k+2} d(x_i, x_{i-1})\right) - (d(x_j, x_{j-1}) + d(x_j, x_{j+1})) - d(x_{j-1}, x_{j+1}) = 2(p-1)^2 + 2(p-1) - 2$ and $\left(\sum_{i=2}^{k+2} \varepsilon_i\right) - (\varepsilon_j + \varepsilon_{j+1} - \varepsilon) = 0$.

Suppose f satisfies case II. Here $\sum_{i=2}^{k+2} d(x_i, x_{i-1}) = 2p^2 - 2$. Then the coloring f of the vertices $P_n \setminus \{a_{2p}\}$ is either of type (i) or of type (ii). Suppose it is of type (i). Then $d(x_j, x_{j+1}) + d(x_j, x_{j-1}) - d(x_{j-1}, x_{j+1}) = 2p - 1$. Let $x_{j-1} = a_r$, $x_{j+1} = a_s$, $1 \leqslant r, s < 2p$. Since $x_j = a_{2p}$, we have $d(x_j, x_{j+1}) + d(x_j, x_{j-1}) - d(x_{j-1}, x_{j+1}) = 2(2p-r)$ or $2(2p-s) = 2d(x_j, x_{j+1})$ or $2d(x_j, x_{j-1})$. Now $d(x_j, x_{j+1})$ or $2d(x_j, x_{j-1}) = \frac{1}{2}(2p-1)$ which is not possible because $\frac{1}{2}(2p-1)$ is not an integer. Suppose the coloring f of the vertices $P_n \setminus \{a_{2p}\}$ is of type (ii). Then $d(x_j, x_{j+1}) + d(x_j, x_{j-1}) - d(x_{j-1}, x_{j+1}) = 2p$. Similarly to the above, we get $d(x_j, x_{j-1})$ or $d(x_j, x_{j+1}) = p$. So x_{j-1} or $x_{j+1} = a_p$. Since $\sum_{i=2}^{k+2} \varepsilon_i = 1$, we have $\varepsilon_j + \varepsilon_{j+1} - \varepsilon = 1$. Since ε_j and ε_{j+1} appear in the summation $\sum_{i=2}^{k+2} \varepsilon_i$, we get $\varepsilon_j + \varepsilon_{j+1} = 1$. Similarly, if we consider the coloring f of the vertices $P_n \setminus \{a_1 = x_l \text{ (say)}\}$, then x_{l-1} or $x_{l+1} = a_{p+1}$ and $\varepsilon_l + \varepsilon_{l+1} = 1$. As $\sum_{i=2}^{k+2} \varepsilon_i = 1$, we have $l = j+1$ and $\varepsilon_l = \varepsilon_{j+1} = 1$ or $j = l+1$ and $\varepsilon_j = \varepsilon_{l+1} = 1$. That is, $|f(a_1) - f(a_{2p})| = 1 + k - d(a_1, a_{2p}) + 1 = 1 \Rightarrow |f(a_p) - f(a_{p+1})| = 1$, which is not possible because $k \geqslant 6$.

Suppose f satisfies case III. Here $\sum_{i=2}^{k+2} d(x_i, x_{i-1}) = 2p^2 - 3$. By Lemma 3.3, $\{f(a_{p-2}), f(a_{p+1})\} = \{0, 2p^2 - 4p + 5\}$ or $\{f(a_p), f(a_{p+3})\} = \{0, 2p^2 - 4p + 5\}$ or $\{f(a_{p-1}), f(a_{p+2})\}$ or $\{f(a_p), f(a_{p-1})\} = \{0, 2p^2 - 4p + 5\}$. The coloring of $P_n \setminus \{a_1\}$ is either of type (i) or of type (ii). Suppose it is of type (i). That is, the distance summation is $2(p-1)^2 + 2(p-1) - 1$. So by Theorem 3.1, either of the sets $\{f(a_p), f(a_{p+1})\}$, $\{f(a_{p+1}), f(a_{p+2})\}$ is equal to $\{0, 2p^2 - 4p + 5\}$, which is not possible because to repeat the color 0 (or $2p^2 - 4p + 5$) its distance from the previous 0 (or $2p^2 - 4p + 5$) color must be at least $2p - 1$. Suppose the coloring $P_n \setminus \{a_1\}$ is of type (ii). Then $d(x_l, x_{l-1}) + d(x_l, x_{l+1}) - d(x_{l-1}, x_{l+1}) = 2p - 1 \Rightarrow 2d(x_l, x_{l-1})$ or $d(x_l, x_{l+1}) = 2p - 1 \Rightarrow d(x_l, x_{l-1})$ or $d(x_l, x_{l+1}) = \frac{1}{2}(2p-1)$, which is not possible.

Theorem 3.5. *For any integer* $n \geqslant 8$,

$$ac'(P_n) \geqslant \begin{cases} 2p^2 - 6p + 8 & \text{if } n = 2p, \\ 2p^2 - 4p + 6 & \text{if } n = 2p + 1. \end{cases}$$

P r o o f. C a s e I. k odd and $n = k + 3 = 2p$.

Let f be any radio k-coloring of the path P_n: $a_1\ a_2\ a_3\ \ldots\ a_{n-1}\ a_{n=2p}$ with span

$2p^2-6p+7$. Let $x_1, x_2, x_3, \ldots, x_{n-1}, x_n$ be the ordering of the vertices such that $f(x_i) \leqslant f(x_{i+1})$, $1 \leqslant i \leqslant n-1$. Consider the coloring f of the first $2(p-1)$ vertices of P_n. Liu and Zhu [6] have given a minimal radio k-coloring of $P_{k+1=2(p-1)}$ with $\sum\limits_{i=2}^{2(p-1)} d(x_i, x_{i-1}) = 2(p-1)^2 - 1$ and $\sum\limits_{i=2}^{2(p-1)} \varepsilon_i = 0$ (where the span is $2(p-1)^2 - 2(p-1) + 2 = 2p^2 - 6p + 6$). If the span of the coloring f of $P_n \setminus \{a_{2p-1}, a_{2p}\}$ is $2p^2-6p+7$, then from Theorem 3.1 we have the following two possibilities for the coloring f of $P_n \setminus \{a_{2p-1}, a_{2p}\}$:

(i) $\sum\limits_{i=2}^{2(p-1)} d(x_i, x_{i-1}) = 2(p-1)^2 - 1$ and $\sum\limits_{i=2}^{2(p-1)} \varepsilon_i = 1$,

(ii) $\sum\limits_{i=2}^{2(p-1)} d(x_i, x_{i-1}) = 2(p-1)^2 - 2$ and $\sum\limits_{i=2}^{2(p-1)} \varepsilon_i = 0$.

Then by Theorem 3.1 and Lemma 3.2,

$$\begin{aligned} &\{f(a_{p-1}), f(a_p)\} \text{ or } \{f(a_{p-1}), f(a_{p+1})\} \text{ or } \{f(a_{p-2}), f(a_{p-1})\} \\ &\qquad = \{0, 2p^2-6p+7\}. \end{aligned} \tag{3}$$

Similarly, if we consider the coloring f of $P_n \setminus \{a_1, a_2\}$ and if the span of the coloring of $P_n \setminus \{a_1, a_2\}$ is $2p^2-6p+7$, we get

$$\begin{aligned} &\{f(a_{p+1}), f(a_{p+2})\} \text{ or } \{f(a_{p+1}), f(a_{p+3})\} \text{ or } \{f(a_p), f(a_{p+2})\} \\ &\qquad = \{0, 2p^2-6p+7\}. \end{aligned} \tag{4}$$

Equations (2) and (3) lead to a contradiction because the color 0 or $2p^2-6p+7$ can be repeated only if its distance is at least $2p-2$. Suppose the span of the coloring of $P_n \setminus \{a_1, a_2\}$ is $2p^2-6p+6$ and the span of the coloring of $P_n \setminus \{a_{2p-1}, a_{2p}\}$ is $2p^2-6p+7$. If the span of the coloring of $P_n \setminus \{a_1, a_2\}$ is $2p^2-6p+6$, then $\{f(a_{p+1}), f(a_{p+2})\} = \{0, 2p^2-6p+6\}$. If the span of the coloring of $P_n \setminus \{a_{2p-1}, a_{2p}\}$ is $2p^2-6p+7$, then $\{f(a_{p-1}), f(a_p)\}$ or $\{f(a_{p-1}), f(a_{p+1})\}$ or $\{f(a_{p-2}), f(a_{p-1})\} = \{0, 2p^2-6p+7\}$, which is not possible as the colors $2p^2-6p+6$ and $2p^2-6p+7$ are at least $2p-3$ apart and the color 0 can be repeated if its distance from the previous 0 color is at least $2p-2$. Similarly we get a contradiction when the span of the coloring of $P_n \setminus \{a_1, a_2\}$ is $2p^2-6p+7$ and the span of the coloring of $P_n \setminus \{a_{2p-1}, a_{2p}\}$ is $2p^2-6p+6$.

Case II. k even and $n = k+3 = 2p+1$.

Let f be any radio k-coloring of the path P_n: $a_1\ a_2\ a_3\ \ldots\ a_{n-1}\ a_{n=2p+1}$ with span $2p^2-4p+5$. Let $x_1, x_2, x_3, \ldots, x_{n-1}, x_n$ be the ordering of the vertices such that $f(x_i) \leqslant f(x_{i+1})$, $1 \leqslant i \leqslant n-1$. Consider the coloring of the first $2p$ vertices of P_n. Then by Lemma 3.4, this coloring has the sum of distances equal to $2p^2-1$ and the sum of ε_i's is equal to 2. So by Theorem 3.1 we have $\{f(a_p), f(a_{p+1})\} =$

$\{0, 2p^2 - 4p + 5\}$. Similarly, if we consider the coloring of $P_n \setminus \{a_1\}$ then we get $\{f(a_{p+1}), f(a_{p+2})\} = \{0, 2p^2 - 4p + 5\}$. This is a contradiction.

From Theorem 2.1 and Theorem 3.5 we summarize the main result of this paper.

Theorem 3.6. *For any integer $n \geqslant 8$,*

$$ac'(P_n) = \begin{cases} 2p^2 - 6p + 8 & \text{if } n = 2p, \\ 2p^2 - 4p + 6 & \text{if } n = 2p + 1. \end{cases}$$

Observation 3.7. *For any positive integers m and n with $m < n$ one gets that $rc_k(P_m) \leqslant rc_k(P_n)$. Therefore by Theorem 3.6 we have for $n = k + i$, $i \geqslant 4$, that $rc_{n-i}(P_{(n-i)+3})$ is $2p^2-6p+8$ for $(n-i)+3 = 2p$ and $2p^2-4p+6$ for $(n-i)+3 = 2p+1$. Since $(n-i)+3 < n$, we get the following lower bound for $rc_k(P_n)$ with $n = k + i$, $i \geqslant 4$:*

$$rc_k(P_n) \geqslant \begin{cases} 2p^2 - 6p + 8 & \text{if } n = 2p - 3 + i, \\ 2p^2 - 4p + 6 & \text{if } n = 2p - 2 + i. \end{cases}$$

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