ON MONOTONIC SOLUTIONS OF AN INTEGRAL EQUATION OF ABEL TYPE

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Abstract. We present an existence theorem for monotonic solutions of a quadratic integral equation of Abel type in $C[0,1]$. The famous Chandrasekhar's integral equation is considered as a special case. The concept of measure of noncompactness and a fixed point theorem due to Darbo are the main tools in carrying out our proof.

Keywords: quadratic integral equation, monotonic solutions, Abel, measure of noncompactness, Darbo's fixed point theorem

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1. Introduction

Quadratic integral equations have many useful applications in describing numerous events and problems of the real world. For example, quadratic integral equations are often applicable in the theory of radiative transfer, kinetic theory of gases, in the theory of neutron transport and in the traffic theory. Especially, the so-called quadratic integral equation of Chandrasekhar type can be very often encountered in many applications (cf. [6], [8], [10]–[16], [19]).

In this paper we study the nonlinear integral equation

$$x(t)=a(t)+\frac{f(t,x(t))}{\Gamma(\alpha)}\int_0^t\frac{g(k(t,s))}{(t-s)^{1-\alpha}}\,x(s)\,\mathrm{d}s,\quad t\in[0,1],\ 0<\alpha\leqslant 1. \tag{1.1}$$

Let us recall that the function $f=f(t,x)$ involved in Eq. (1.1) generates the superposition operator F defined by the formula

$$(Fx)(t)=f(t,x(t)), \tag{1.2}$$

where $x=x(t)$ is an arbitrary function defined on $[0,1]$ (cf. [1], [17]).

Using the technique associated with measures of noncompactness we show that equation (1.1) has solutions belonging to $C[0,1]$ and being nondecreasing on the interval $[0,1]$.

Our paper is motivated by [5], [8], [11] concerning quadratic integral equations of Volterra type and results in [2], [12], [20], [21] concerning Chandrasekher's integral equation and some of its generalizations.

2. Auxiliary facts and results

This section is devoted to collecting some definitions and results which will be needed further on. Assume that $(E, \|\cdot\|)$ is a real Banach space with zero element 0. Let $B(x,r)$ denote the closed ball centered at x with radius r. The symbol B_r stands for the ball $B(0,r)$.

If X is a subset of E, then $\overline{X}$ and Conv X denote the closure and convex closure of X, respectively. The symbols λX and $X+Y$ denote the usual algebraic operators on sets. Moreover, we denote by $\mathcal{M}_E$ the family of all nonempty and bounded subsets of E and by $\mathcal{N}_E$ its subfamily consisting of all relatively compact subsets.

Next we give the concept of a measure of noncompactness [9]:

Definition 1. A mapping $\mu\colon \mathcal{M}_E \to [0,+\infty)$ is said to be a measure of noncompactness in E if it satisfies the following conditions:

1) The family $\ker\mu = \{X \in \mathcal{M}_E\colon\ \mu(X)=0\}$ is nonempty and $\ker\mu \subset \mathcal{N}_E$.
2) $X \subset Y \Rightarrow \mu(X) \leqslant \mu(Y)$.
3) $\mu(\overline{X}) = \mu(\operatorname{Conv} X) = \mu(X)$.
4) $\mu(\lambda X + (1-\lambda)Y) \leqslant \lambda\mu(X) + (1-\lambda)\mu(Y)$ for $0 \leqslant \lambda \leqslant 1$.
5) If $X_n \in \mathcal{M}_E$, $X_n = \overline{X}_n$, $X_{n+1} \subset X_n$ for $n = 1,2,3,\ldots$ and $\lim\limits_{n\to\infty} \mu(X_n) = 0$ then $\bigcap\limits_{n=1}^{\infty} X_n \neq \emptyset$.

The family $\ker\mu$ described above is called the kernel of the measure of noncompactness μ.

In what follows we will work in the Banach space $C[0,1]$ consisting of all real functions defined and continuous on $[0,1]$. For convenience, we write I and $C(I)$ instead of $[0,1]$ and $C[0,1]$, respectively. The space $C(I)$ is equipped with the standard norm

$$\|x\| = \max\{|x(t)|\colon\ t \geqslant 0\}.$$

Now, we recollect the construction of the measure of noncompactness in $C(I)$ which will be used in the next section (see [6], [7]).

Let us fix a nonempty and bounded subset X of $C(I)$. For $x \in X$ and $\varepsilon \geqslant 0$ let us denote by $\omega(x, \varepsilon)$ the modulus of continuity of the function x, i.e.,

$$\omega(x, \varepsilon) = \sup\{|x(t) - x(s)|\colon\ t, s \in I,\ |t - s| \leqslant \varepsilon\}.$$

Further, let us put

$$\omega(X, \varepsilon) = \sup\{\omega(x, \varepsilon)\colon\ x \in X\}, \quad \omega_0(X) = \lim_{\varepsilon \to 0} \omega(X, \varepsilon).$$

Define

$$d(x) = \sup\{|x(s) - x(t)| - [x(s) - x(t)]\colon\ t, s \in I,\ t \leqslant s\}$$

and

$$d(X) = \sup\{d(x)\colon\ x \in X\}.$$

Observe that all functions belonging to X are nondecreasing on I if and only if $d(X) = 0$.

Now, let us define the function μ on the family $\mathcal{M}_{C(I)}$ by the formula

$$\mu(X) = \omega_0(X) + d(X).$$

The function μ is a measure of noncompactness in the space $C(I)$ [7].

We will make use of the following fixed point theorem due to Darbo [3]. To quote this theorem, we need the following definition.

Definition 2. Let M be a nonempty subset of a Banach space E and let $\mathcal{P}\colon M \to E$ be a continuous operator which transforms bounded sets onto bounded ones. We say that $\mathcal{P}$ satisfies the Darbo condition (with a constant $k \geqslant 0$) with respect to a measure of noncompactness μ if for any bounded subset X of M we have

$$\mu(\mathcal{P}X) \leqslant k\mu(X).$$

If $\mathcal{P}$ satisfies the Darbo condition with $k < 1$ then it is called a contraction operator with respect to μ.

Theorem 1 [3]**.** *Let Q be a nonempty, bounded, closed and convex subset of the space E and let*

$$\mathcal{P}\colon\ Q \to Q$$

be a contraction with respect to the measure of noncompactness μ.

Then $\mathcal{P}$ has a fixed point in the set Q.

Remark 1 [9]. Under the assumptions of the above theorem it can be shown that the set $\operatorname{Fix}\mathcal{P}$ of fixed points of $\mathcal{P}$ belonging to Q is an element of $\ker\mu$.

3. Main theorem

In this section we will study Eq. (1.1) assuming that the following hypotheses are satisfied:

(a_1) $a\colon I \to \mathbb{R}$ is a continuous, nondecreasing and nonnegative function on I.

(a_2) $f\colon I \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists a nonnegative constant c such that

$$|f(t,x) - f(t,y)| \leqslant c|x-y|$$

for all $t \in I$ and $x, y \in \mathbb{R}$. Moreover, $f\colon I \times \mathbb{R}_+ \to \mathbb{R}_+$.

(a_3) The superposition operator F generated by the function $f(t,x)$ satisfies for any nonnegative function x the condition

$$d(Fx) \leqslant cd(x),$$

where c is the same constant as in (a_2).

(a_4) $k\colon I \times I \to \mathbb{R}$ is continuous on $I \times I$ and the function $k(t,s)$ is nondecreasing for each variable t and s, separately.

(a_5) $g\colon \operatorname{Im} k \to \mathbb{R}_+$ is a continuous nondecreasing function on the compact set $\operatorname{Im} k$.

(a_6) The inequality

$$\|a\|\Gamma(\alpha+1) + (c + mr)\|g\|r \leqslant r\Gamma(\alpha+1) \tag{3.1}$$

has a positive solution r_0 such that $c\|g\|r_0 < \Gamma(\alpha+1)$, where $m = \max\limits_{0\leqslant t\leqslant 1} f(t,0)$.

Now, we are in a position to state and prove our main result.

Theorem 2. *Let the hypotheses* (a_1)–(a_6) *be satisfied. Then Eq.* (1.1) *has at least one solution $x \in C(I)$ being nondecreasing on the interval I.*

P r o o f. Let $\mathcal{K}$ and $\mathcal{F}$ be two operators defined on the space $C(I)$ by

$$(\mathcal{K}x)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{g(k(t,s))}{(t-s)^{1-\alpha}} x(s)\,\mathrm{d}s,$$
$$(\mathcal{F}x)(t) = a(t) + f(t, x(t))(\mathcal{K}x)(t).$$

Solving Eq. (1.1) is equivalent to finding a fixed point of the operator $\mathcal{F}$ defined on the space $C(I)$.

First, we prove that $\mathcal{F}$ transforms the space $C(I)$ into itself. To do this it suffices to show that if $x \in C(I)$ then $\mathcal{K}x \in C(I)$. Fix $\varepsilon > 0$, let $x \in C(I)$ and let $t_1, t_2 \in I$

be such that $t_2 \geqslant t_1$ and $|t_2 - t_1| \leqslant \varepsilon$. Then we get

$$
\begin{aligned}
&|(\mathcal{K}x)(t_2) - (\mathcal{K}x)(t_1)| \\
&\qquad = \left|\frac{1}{\Gamma(\alpha)}\int_0^{t_2} \frac{g(k(t_2,s))}{(t_2-s)^{1-\alpha}}\, x(s)\,\mathrm{d}s - \frac{1}{\Gamma(\alpha)}\int_0^{t_1} \frac{g(k(t_1,s))}{(t_1-s)^{1-\alpha}}\, x(s)\,\mathrm{d}s\right| \\
&\qquad \leqslant \left|\frac{1}{\Gamma(\alpha)}\int_0^{t_2} \frac{g(k(t_2,s))}{(t_2-s)^{1-\alpha}}\, x(s)\,\mathrm{d}s - \frac{1}{\Gamma(\alpha)}\int_0^{t_2} \frac{g(k(t_1,s))}{(t_2-s)^{1-\alpha}}\, x(s)\,\mathrm{d}s\right| \\
&\qquad\quad + \left|\frac{1}{\Gamma(\alpha)}\int_0^{t_2} \frac{g(k(t_1,s))}{(t_2-s)^{1-\alpha}}\, x(s)\,\mathrm{d}s - \frac{1}{\Gamma(\alpha)}\int_0^{t_1} \frac{g(k(t_1,s))}{(t_2-s)^{1-\alpha}}\, x(s)\,\mathrm{d}s\right| \\
&\qquad\quad + \left|\frac{1}{\Gamma(\alpha)}\int_0^{t_1} \frac{g(k(t_1,s))}{(t_2-s)^{1-\alpha}}\, x(s)\,\mathrm{d}s - \frac{1}{\Gamma(\alpha)}\int_0^{t_1} \frac{g(k(t_1,s))}{(t_1-s)^{1-\alpha}}\, x(s)\,\mathrm{d}s\right| \\
&\qquad \leqslant \frac{1}{\Gamma(\alpha)}\int_0^{t_2} \frac{|g(k(t_2,s)) - g(k(t_1,s))|}{(t_2-s)^{1-\alpha}}\, x(s)\,\mathrm{d}s \\
&\qquad\quad + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2} \frac{|g(k(t_1,s))|}{(t_2-s)^{1-\alpha}}\, x(s)\,\mathrm{d}s \\
&\qquad\quad + \frac{1}{\Gamma(\alpha)}\int_0^{t_1} |g(k(t_1,s))|[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}]x(s)\,\mathrm{d}s.
\end{aligned}
$$

Therefore, if

$$
\omega_{g\circ k}(\varepsilon,\cdot) = \sup\{|g(k(t,s)) - g(k(\tau,s))|\colon\ t,s,\tau \in I \text{ and } |t-\tau| \leqslant \varepsilon\}
$$

then we obtain

$$
\begin{aligned}
&|(\mathcal{K}x)(t_2) - (\mathcal{K}x)(t_1)| \\
&\quad \leqslant \frac{\|x\|}{\Gamma(\alpha)}\omega_{g\circ k}(\varepsilon,\cdot)\int_0^{t_2} (t_2-s)^{\alpha-1}\,\mathrm{d}s \\
&\quad\quad + \frac{\|g\|\|x\|}{\Gamma(\alpha)}\left\{\int_0^{t_1} [(t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}]\,\mathrm{d}s + \int_{t_1}^{t_2} (t_2-s)^{\alpha-1}\,\mathrm{d}s\right\} \\
&\quad \leqslant \frac{\|x\|}{\Gamma(\alpha+1)}\omega_{g\circ k}(\varepsilon,\cdot)t_2^{\alpha} + \frac{\|g\|\|x\|}{\Gamma(\alpha+1)}[t_1^{\alpha} - t_2^{\alpha} + 2(t_2-t_1)^{\alpha}] \\
&\quad \leqslant \frac{\|x\|}{\Gamma(\alpha+1)}\omega_{g\circ k}(\varepsilon,\cdot) + \frac{2\|g\|\|x\|}{\Gamma(\alpha+1)}(t_2-t_1)^{\alpha} \\
&\quad \leqslant \frac{\|x\|}{\Gamma(\alpha+1)}\omega_{g\circ k}(\varepsilon,\cdot) + \frac{2\|g\|\|x\|}{\Gamma(\alpha+1)}\varepsilon^{\alpha}.
\end{aligned}
$$

In view of the uniform continuity of the function k on $I\times I$ we have that $\omega_k(\varepsilon,\cdot) \to 0$ as $\varepsilon \to 0$. Thus $\mathcal{K}x \in C(I)$, and consequently, $\mathcal{F}x \in C(I)$. Moreover, for each $t \in I$

we have

$$
\begin{aligned}
|(\mathcal{F}x)(t)| &\leqslant \left| a(t) + \frac{f(t,x(t))}{\Gamma(\alpha)} \int_0^t \frac{g(k(t,s))}{(t-s)^{1-\alpha}} x(s)\,\mathrm{d}s \right| \\
&\leqslant |a(t)| + \frac{1}{\Gamma(\alpha)} [|f(t,x(t)) - f(t,0)| + |f(t,0)|] \int_0^t \frac{g(k(t,s))}{(t-s)^{1-\alpha}} x(s)\,\mathrm{d}s \\
&\leqslant \|a\| + \frac{c\|x\| + m}{\Gamma(\alpha)} \|g\|\|x\| \int_0^t (t-s)^{\alpha-1}\,\mathrm{d}s \\
&= \|a\| + \frac{c\|x\| + m}{\Gamma(\alpha+1)} \|g\|\|x\|.
\end{aligned}
$$

Hence

$$
\|\mathcal{F}x\| \leqslant \|a\| + \frac{c\|x\| + m}{\Gamma(\alpha+1)} \|g\|\|x\|.
$$

Thus, if $\|x\| \leqslant r_0$ we obtain from assumption (a_6) the estimate

$$
\|\mathcal{F}x\| \leqslant \|a\| + \frac{c + mr_0}{\Gamma(\alpha+1)} \|g\| r_0 \leqslant r_0.
$$

Consequently, the operator $\mathcal{F}$ transforms the ball B_{r_0} into itself.

In what follows we will consider the operator $\mathcal{F}$ on the subset $B_{r_0}^+$ of the ball B_{r_0} defined by

$$
B_{r_0}^+ = \{x \in B_{r_0} \colon\ x(t) \geqslant 0,\ \text{for } t \in I\}.
$$

Obviously, the set $B_{r_0}^+$ is nonempty, bounded, closed and convex. In view of these facts and assumptions (a_1), (a_3) and (a_5), we deduce that $\mathcal{F}$ transforms the set $B_{r_0}^+$ into itself.

Next, we prove that the operator $\mathcal{F}$ is continuous on $B_{r_0}^+$. To do this, let us fix $\{x_n\}$ to be a sequence in $B_{r_0}^+$ such that $x_n \to x$. We will prove that $\mathcal{F}x_n \to \mathcal{F}x$. In fact, for each $t \in I$ we have

$$
\begin{aligned}
&|(\mathcal{F}x_n)(t) - (\mathcal{F}x)(t)| \\
&\quad = \left| \frac{f(t,x_n(t))}{\Gamma(\alpha)} \int_0^t \frac{g(k(t,s))}{(t-s)^{1-\alpha}} x_n(s)\,\mathrm{d}s - \frac{f(t,x(t))}{\Gamma(\alpha)} \int_0^t \frac{g(k(t,s))}{(t-s)^{1-\alpha}} x(s)\,\mathrm{d}s \right| \\
&\quad \leqslant \left| \frac{f(t,x_n(t))}{\Gamma(\alpha)} \int_0^t \frac{g(k(t,s))}{(t-s)^{1-\alpha}} x_n(s)\,\mathrm{d}s - \frac{f(t,x(t))}{\Gamma(\alpha)} \int_0^t \frac{g(k(t,s))}{(t-s)^{1-\alpha}} x_n(s)\,\mathrm{d}s \right| \\
&\qquad + \left| \frac{f(t,x(t))}{\Gamma(\alpha)} \int_0^t \frac{g(k(t,s))}{(t-s)^{1-\alpha}} x_n(s)\,\mathrm{d}s - \frac{f(t,x(t))}{\Gamma(\alpha)} \int_0^t \frac{g(k(t,s))}{(t-s)^{1-\alpha}} x(s)\,\mathrm{d}s \right| \\
&\quad \leqslant \frac{1}{\Gamma(\alpha)} |f(t,x_n(t)) - f(t,x(t))| \int_0^t \frac{|g(k(t,s))|}{(t-s)^{1-\alpha}} |x_n(s)|\,\mathrm{d}s \\
&\qquad + \frac{|f(t,x(t))|}{\Gamma(\alpha)} \int_0^t \frac{|g(k(t,s))|}{(t-s)^{1-\alpha}} \,|x_n(s) - x(s)|\,\mathrm{d}s.
\end{aligned}
$$

Thus

$$\|\mathcal{F}x_n - \mathcal{F}x\| \leqslant \frac{c\|g\|r_0}{\Gamma(\alpha+1)}\|x_n - x\| + \frac{\|g\|(cr_0+m)}{\Gamma(\alpha+1)}\|x_n - x\|. \tag{3.4}$$

This proves that $\mathcal{F}$ is continuous in $B_{r_0}^+$.

Now, let us take a nonempty set $X \subset B_{r_0}^+$. Fix an arbitrarily number $\varepsilon > 0$ and choose $x \in X$ and $t_1, t_2 \in I$ such that $|t_2 - t_1| \leqslant \varepsilon$. Without loss of generality we may assume that $t_2 \geqslant t_1$. Then, in view of our assumptions, we obtain

$$\begin{aligned}
|(\mathcal{F}x)(t_2) - (\mathcal{F}x)(t_1)| &= \bigg| a(t_2) + \frac{f(t_2, x(t_2))}{\Gamma(\alpha)} \int_0^{t_2} \frac{g(k(t_2,s))}{(t_2-s)^{1-\alpha}} x(s)\,\mathrm{d}s \\
&\quad - a(t_1) - \frac{f(t_1, x(t_1))}{\Gamma(\alpha)} \int_0^{t_1} \frac{g(t_1,s)}{(t_1-s)^{1-\alpha}} x(s)\,\mathrm{d}s \bigg| \\
&\leqslant |a(t_2) - a(t_1)| + \bigg| \frac{f(t_2, x(t_2)))}{\Gamma(\alpha)} \int_0^{t_2} \frac{g(k(t_2,s))}{(t_2-s)^{1-\alpha}} x(s)\,\mathrm{d}s \\
&\qquad - \frac{f(t_2, x(t_2))}{\Gamma(\alpha)} \int_0^{t_2} \frac{g(k(t_1,s))}{(t_2-s)^{1-\alpha}} x(s)\,\mathrm{d}s \bigg| \\
&\quad + \bigg| \frac{f(t_2, x(t_2))}{\Gamma(\alpha)} \int_0^{t_2} \frac{g(k(t_1,s))}{(t_2-s)^{1-\alpha}} x(s)\,\mathrm{d}s - \frac{f(t_2, x(t_2))}{\Gamma(\alpha)} \int_0^{t_1} \frac{g(k(t_1,s))}{(t_2-s)^{1-\alpha}} x(s)\,\mathrm{d}s \bigg| \\
&\quad + \bigg| \frac{f(t_2, x(t_2))}{\Gamma(\alpha)} \int_0^{t_1} \frac{g(k(t_1,s))}{(t_2-s)^{1-\alpha}} x(s)\,\mathrm{d}s - \frac{f(t_2, x(t_2))}{\Gamma(\alpha)} \int_0^{t_1} \frac{g(k(t_1,s))}{(t_1-s)^{1-\alpha}} x(s)\,\mathrm{d}s \bigg| \\
&\quad + \bigg| \frac{f(t_2, x(t_2))}{\Gamma(\alpha)} \int_0^{t_1} \frac{g(k(t_1,s))}{(t_1-s)^{1-\alpha}} x(s)\,\mathrm{d}s - \frac{f(t_1, x(t_1))}{\Gamma(\alpha)} \int_0^{t_1} \frac{g(k(t_1,s))}{(t_1-s)^{1-\alpha}} x(s)\,\mathrm{d}s \bigg| \\
&\leqslant \omega(a,\varepsilon) + \frac{|f(t_2, x(t_2))|}{\Gamma(\alpha)} \int_0^{t_2} \frac{|g(k(t_2,s)) - g(k(t_1,s))|}{(t_2-s)^{1-\alpha}} |x(s)|\,\mathrm{d}s \\
&\quad + \frac{|f(t_2, x(t_2))|}{\Gamma(\alpha)} \int_{t_1}^{t_2} \frac{g(|k(t_1,s))|}{(t_2-s)^{1-\alpha}} x(s)\,\mathrm{d}s \\
&\quad + \frac{|f(t_2, x(t_2))|}{\Gamma(\alpha)} \int_0^{t_1} |k(t_1,s)| \left[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\right] |x(s)|\,\mathrm{d}s \\
&\quad + \frac{|f(t_2, x(t_2)) - f(t_1, x(t_2))|}{\Gamma(\alpha)} \int_0^{t_1} \frac{|g(k(t_1,s))|}{(t_1-s)^{1-\alpha}} |x(s)|\,\mathrm{d}s \\
&\quad + \frac{|f(t_1, x(t_2)) - f(t_1, x(t_1))|}{\Gamma(\alpha)} \int_0^{t_1} \frac{|g(k(t_1,s))|}{(t_1-s)^{1-\alpha}} |x(s)|\,\mathrm{d}s \\
&\leqslant \omega(a,\varepsilon) + \frac{c\|x\|+m}{\Gamma(\alpha)} \|x\| \omega_{g\circ k}(\varepsilon,\cdot) \int_0^{t_2} (t_2-s)^{\alpha-1}\,\mathrm{d}s \\
&\quad + \frac{c\|x\|+m}{\Gamma(\alpha)} \|g\|\|x\| \bigg\{ \int_{t_1}^{t_2} (t_2-s)^{\alpha-1}\,\mathrm{d}s + \int_0^{t_1} \left[(t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}\right]\,\mathrm{d}s \bigg\} \\
&\quad + \frac{\gamma_{r_0}(\varepsilon)}{\Gamma(\alpha)} \|g\|\|x\| \int_0^{t_1} (t_1-s)^{\alpha-1}\,\mathrm{d}s + \frac{c\omega(x,\varepsilon)}{\Gamma(\alpha)} \|g\|\|x\| \int_0^{t_1} (t_1-s)^{\alpha-1}\,\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \omega(a,\varepsilon) + \frac{cr_0+m}{\Gamma(\alpha+1)} r_0 \omega_{g\circ k}(\varepsilon,\cdot) t_2^\alpha + \frac{cr_0+m}{\Gamma(\alpha+1)} \|k\| r_0 \left[t_1^\alpha - t_2^\alpha + 2(t_2-t_1)^\alpha\right] \\
&\quad + \frac{\gamma_{r_0}(\varepsilon) + c\omega(x,\varepsilon)}{\Gamma(\alpha+1)} \|g\| r_0 t_1^\alpha \\
&\leqslant \omega(a,\varepsilon) + \frac{r_0(cr_0+m)}{\Gamma(\alpha+1)} [\omega_{g\circ k}(\varepsilon,\cdot) + 2\|g\|\varepsilon^\alpha] + \frac{\|g\| r_0}{\Gamma(\alpha+1)} [\gamma_{r_0}(\varepsilon) + c\omega(x,\varepsilon)].
\end{aligned}
$$

Hence,

$$
\omega(\mathcal{F}x,\varepsilon) \leqslant \omega(a,\varepsilon) + \frac{r_0(cr_0+m)}{\Gamma(\alpha+1)} [\omega_{g\circ k}(\varepsilon,\cdot) + 2\|g\|\varepsilon^\alpha] + \frac{\|g\| r_0}{\Gamma(\alpha+1)} [\gamma_{r_0}(\varepsilon) + c\omega(x,\varepsilon)].
$$

Consequently,

$$
\omega(\mathcal{F}X,\varepsilon) \leqslant \omega(a,\varepsilon) + \frac{r_0(cr_0+m)}{\Gamma(\alpha+1)} [\omega_{g\circ k}(\varepsilon,\cdot) + 2\|g\|\varepsilon^\alpha] + \frac{\|g\| r_0}{\Gamma(\alpha+1)} [\gamma_{r_0}(\varepsilon) + c\omega(\mathrm{X},\varepsilon)],
$$

where

$$
\gamma_{r_0}(\varepsilon) = \sup\left\{|f(t_2,x) - f(t_1,x)| \colon\ t_1,t_2 \in I, x \in [0,r_0], |t_2-t_1| \leqslant \varepsilon\right\}.
$$

In view of the uniform continuity of the function $g \circ k$ on the set $I \times I$ and the continuity of the function a on I, the last inequality implies

$$
\omega_0(\mathcal{F}X) \leqslant \frac{c\|g\| r_0}{\Gamma(\alpha+1)} \omega_0(X). \tag{3.5}
$$

In what follows, fix an arbitrary $x \in X$ and $t_1, t_2 \in I$ with $t_2 > t_1$. Then, taking into account our assumptions, we have

(3.6)

$$
\begin{aligned}
&|(\mathcal{F}x)(t_2) - (\mathcal{F}x)(t_1)| - [(\mathcal{F}x)(t_2) - (\mathcal{F}x)(t_1)] \\
&= \left| a(t_2) + \frac{f(t_2,x(t_2))}{\Gamma(\alpha)} \int_0^{t_2} \frac{g(k(t_2,s))}{(t_2-s)^{1-\alpha}}\, x(s)\,\mathrm{d}s \right. \\
&\qquad\qquad \left. - a(t_1) - \frac{f(t_1,x(t_1))}{\Gamma(\alpha)} \int_0^{t_1} \frac{g(k(t_1,s))}{(t_1-s)^{1-\alpha}}\, x(s)\,\mathrm{d}s \right| \\
&\quad - \left[a(t_2) + \frac{f(t_2,x(t_2))}{\Gamma(\alpha)} \int_0^{t_2} \frac{g(k(t_2,s))}{(t_2-s)^{1-\alpha}}\, x(s)\,\mathrm{d}s \right. \\
&\qquad\qquad \left. - a(t_1) - \frac{f(t_1,x(t_1))}{\Gamma(\alpha)} \int_0^{t_1} \frac{g(k(t_1,s))}{(t_1-s)^{1-\alpha}}\, x(s)\,\mathrm{d}s \right] \\
&\leqslant \{|a(t_2) - a(t_1)| - [a(t_2) - a(t_1)]\} \\
&\quad + \left| \frac{f(t_2,x(t_2))}{\Gamma(\alpha)} \int_0^{t_2} \frac{g(k(t_2,s))}{(t_2-s)^{1-\alpha}}\, x(s)\,\mathrm{d}s - \frac{f(t_1,x(t_1))}{\Gamma(\alpha)} \int_0^{t_2} \frac{g(k(t_2,s))}{(t_2-s)^{1-\alpha}}\, x(s)\,\mathrm{d}s \right| \\
&\quad + \left| \frac{f(t_1,x(t_1))}{\Gamma(\alpha)} \int_0^{t_2} \frac{g(k(t_2,s))}{(t_2-s)^{1-\alpha}}\, x(s)\,\mathrm{d}s - \frac{f(t_1,x(t_1))}{\Gamma(\alpha)} \int_0^{t_1} \frac{g(k(t_1,s))}{(t_1-s)^{1-\alpha}}\, x(s)\,\mathrm{d}s \right|
\end{aligned}
$$

$$
\begin{aligned}
&-\Bigg\{\Bigg[\frac{f(t_2,x(t_2))}{\Gamma(\alpha)}\int_0^{t_2}\frac{g(k(t_2,s))}{(t_2-s)^{1-\alpha}}\,x(s)\,\mathrm{d}s-\frac{f(t_1,x(t_1))}{\Gamma(\alpha)}\int_0^{t_2}\frac{g(k(t_2,s))}{(t_2-s)^{1-\alpha}}\,x(s)\,\mathrm{d}s\Bigg]\\
&+\Bigg[\frac{f(t_1,x(t_1))}{\Gamma(\alpha)}\int_0^{t_2}\frac{g(k(t_2,s))}{(t_2-s)^{1-\alpha}}\,x(s)\,\mathrm{d}s-\frac{f(t_1,x(t_1))}{\Gamma(\alpha)}\int_0^{t_1}\frac{g(k(t_1,s))}{(t_1-s)^{1-\alpha}}\,x(s)\,\mathrm{d}s\Bigg]\Bigg\}\\
\leqslant&\ \{|f(t_2,x(t_2))-f(t_1,x(t_1))|-[f(t_2,x(t_2))-f(t_1,x(t_1))]\}\\
&\qquad\times\frac{1}{\Gamma(\alpha)}\int_0^{t_2}\frac{g(k(t_2,s))}{(t_2-s)^{1-\alpha}}\,x(s)\,\mathrm{d}s\\
&+\frac{f(t_1,x(t_1))}{\Gamma(\alpha)}\Bigg\{\Bigg|\int_0^{t_2}\frac{g(k(t_2,s))}{(t_2-s)^{1-\alpha}}\,x(s)\,\mathrm{d}s-\int_0^{t_1}\frac{g(k(t_1,s))}{(t_1-s)^{1-\alpha}}\,x(s)\,\mathrm{d}s\Bigg|\\
&\qquad-\Bigg[\int_0^{t_2}\frac{g(k(t_2,s))}{(t_2-s)^{1-\alpha}}\,x(s)\,\mathrm{d}s-\int_0^{t_1}\frac{g(k(t_1,s))}{(t_1-s)^{1-\alpha}}\,x(s)\,\mathrm{d}s\Bigg]\Bigg\}.
\end{aligned}
$$

Now, we will prove that

$$\int_0^{t_2}\frac{g(k(t_2,s))}{(t_2-s)^{1-\alpha}}\,x(s)\,\mathrm{d}s-\int_0^{t_1}\frac{g(k(t_1,s))}{(t_1-s)^{1-\alpha}}\,x(s)\,\mathrm{d}s\geqslant 0.$$

In fact, we have

$$
\begin{aligned}
&\int_0^{t_2}\frac{g(k(t_2,s))}{(t_2-s)^{1-\alpha}}x(s)\,\mathrm{d}s-\int_0^{t_1}\frac{g(k(t_1,s))}{(t_1-s)^{1-\alpha}}x(s)\,\mathrm{d}s\\
&\quad=\int_0^{t_2}\frac{g(k(t_2,s))}{(t_2-s)^{1-\alpha}}\,x(s)\,\mathrm{d}s-\int_0^{t_2}\frac{g(k(t_1,s))}{(t_2-s)^{1-\alpha}}\,x(s)\,\mathrm{d}s\\
&\qquad+\int_0^{t_2}\frac{g(k(t_1,s))}{(t_2-s)^{1-\alpha}}\,x(s)\,\mathrm{d}s-\int_0^{t_1}\frac{g(k(t_1,s))}{(t_2-s)^{1-\alpha}}\,x(s)\,\mathrm{d}s\\
&\qquad+\int_0^{t_1}\frac{g(k(t_1,s))}{(t_2-s)^{1-\alpha}}\,x(s)\,\mathrm{d}s-\int_0^{t_1}\frac{g(k(t_1,s))}{(t_1-s)^{1-\alpha}}\,x(s)\,\mathrm{d}s\\
&\quad=\int_0^{t_2}\frac{(g(k(t_2,s))-g(k(t_1,s)))}{(t_2-s)^{1-\alpha}}x(s)\,\mathrm{d}s+\int_{t_1}^{t_2}\frac{g(k(t_1,s))}{(t_2-s)^{1-\alpha}}\,x(s)\,\mathrm{d}s\\
&\qquad+\int_0^{t_1}g(k(t_1,s))[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]x(s)\,\mathrm{d}s.
\end{aligned}
$$

Since $k(t,s)$ is nondecreasing with respect to t, we have that $k(t_2,s)\geqslant k(t_1,s)$; moreover, g is nondecreasing, hence $g(k(t_2,s))\geqslant g(k(t_1,s))$, and therefore

$$\int_0^{t_2}\frac{(g(k(t_2,s))-g(k(t_1,s)))}{(t_2-s)^{1-\alpha}}\,x(s)\,\mathrm{d}s\geqslant 0. \tag{3.7}$$

On the other hand, since the term $(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}$ is negative for $0\leqslant s<t_1$, we have

$$\begin{aligned}(3.8)\quad &\int_0^{t_1} g(k(t_1,s))[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]x(s)\,\mathrm{d}s+\int_{t_1}^{t_2}\frac{g(k(t_1,s))}{(t_2-s)^{1-\alpha}}x(s)\,\mathrm{d}s\\ &\geqslant \int_0^{t_1} g(k(t_1,t_1))[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]x(s)\,\mathrm{d}s+\int_{t_1}^{t_2}\frac{g(k(t_1,t_1))}{(t_2-s)^{1-\alpha}}x(s)\,\mathrm{d}s\\ &= g(k(t_1,t_1))x(s)\left[\int_0^{t_2}\frac{\mathrm{d}s}{(t_2-s)^{1-\alpha}}-\int_0^{t_1}\frac{\mathrm{d}s}{(t_1-s)^{1-\alpha}}\right]\\ &= g(k(t_1,t_1))\,\frac{t_2^\alpha-t_1^\alpha}{\alpha}\,x(s)\geqslant 0.\end{aligned}$$

Finally, (3.7) and (3.8) imply

$$\int_0^{t_2}\frac{g(k(t_2,s))}{(t_2-s)^{1-\alpha}}x(s)\,\mathrm{d}s-\int_0^{t_1}\frac{g(k(t_1,s))}{(t_1-s)^{1-\alpha}}x(s)\,\mathrm{d}s\geqslant 0.$$

This together with (3.6) yields

$$\begin{aligned}&|(\mathcal{F}x)(t_2)-(\mathcal{F}x)(t_1)|-[(\mathcal{F}x)(t_2)-(\mathcal{F}x)(t_1)]\\ &\qquad=\{|f(t_2,x(t_2))-f(t_1,x(t_1))|-[f(t_2,x(t_2))-f(t_1,x(t_1))]\}\\ &\qquad\quad\times\frac{1}{\Gamma(\alpha)}\int_0^{t_2}\frac{g(k(t_2,s))}{(t_2-s)^{1-\alpha}}x(s)\,\mathrm{d}s\leqslant\frac{\|g\|r_0}{\Gamma(\alpha+1)}d(Fx).\end{aligned}$$

The above estimate implies

$$d(\mathcal{F}x)\leqslant\frac{\|g\|r_0}{\Gamma(\alpha+1)}d(Fx).$$

Therefore,

$$d(\mathcal{F}x)\leqslant\frac{c\|g\|r_0}{\Gamma(\alpha+1)}d(x)$$

and consequently,

$$d(\mathcal{F}X)\leqslant\frac{c\|g\|r_0}{\Gamma(\alpha+1)}d(X). \tag{3.9}$$

Finally, from (3.5) and (3.9) and the definition of the measure of noncompactness μ, we obtain

$$\mu(\mathcal{F}X)\leqslant\frac{c\|g\|r_0}{\Gamma(\alpha+1)}\mu(X).$$

Now, the above obtained inequality together with the fact that $c\|g\|r_0<\Gamma(\alpha+1)$ enables us to apply Theorem 1, hence Eq. (1.1) has at least one solution $x\in C(I)$. This completes the proof.

4. Examples

E x a m p l e 1. If $f(t,x)=1$ and $g(u)=u$, then Eq. (1.1) becomes the well-known linear Abel integral equation of the second kind

$$x(t)=a(t)+\int_0^t \frac{k(t,s)}{(t-s)^{1-\alpha}}x(s)\,\mathrm{d}s. \tag{4.10}$$

Abel integral equations have applications in many fields of physics and experimental sciences. For example, problems in mechanics, spectroscopy, scattering theory, elasticity theory and plasma physics often lead to such equations, [18].

E x a m p l e 2. If $\alpha=1$, $f(t,x)=x$ and $g(u)=u$, then Eq. (1.1) is the well-known quadratic integral equation of Volterra type

$$x(t)=a(t)+x(t)\int_0^t k(t,s)x(s)\,\mathrm{d}s. \tag{4.11}$$

In the case $a(t)=1$ and $k(t,s)=t(t+s)^{-1}\varphi(s)$, Eq. (4.11) takes the form

$$x(t)=1+x(t)\int_0^t \frac{t}{t+s}\,\varphi(s)x(s)\,\mathrm{d}s. \tag{4.12}$$

Eq. (4.12) is a Volterra counterpart of the famous quadratic integral equation of Chandrasekhar type

$$x(t)=1+x(t)\int_0^1 \frac{t}{t+s}\,\varphi(s)x(s)\,\mathrm{d}s. \tag{4.13}$$

Eq. (4.13) has been considered in many papers and monographs (cf. [2], [8], [12], [19] for instance). Note that in order to apply our technique to Eq. (4.12) we have to impose an additional condition that the characteristic function φ is continuous nondecreasing and satisfies $\varphi(0)=0$. This condition will ensure that the kernel $k(t,s)$ defined by

$$k(t,s)=\begin{cases} 0, & s=0,\ t\geqslant 0\\ \dfrac{t}{t+s}\,\varphi(s), & s\neq 0,\ t\geqslant 0\end{cases}$$

is continuous on $I\times I$ in accordance with assumption (a_4).

E x a m p l e 3. If $g(u)=u$, then Eq. (1.1) takes the form

$$x(t)=a(t)+\frac{f(t,x(t))}{\Gamma(\alpha)}\int_0^t \frac{k(t,s)}{(t-s)^{1-\alpha}}x(s)\,\mathrm{d}s. \tag{4.14}$$

This equation is a special case of the quadratic integral equation of fractional order

$$x(t)=a(t)+\frac{f(t,x(t))}{\Gamma(\alpha)}\int_0^t\frac{u(s,x(s))}{(t-s)^{1-\alpha}}\,\mathrm{d}s \tag{4.15}$$

studied by Banaś and Rzepka in [4]. The authors in [4] used a simpler measure of noncompactness than that we used here. Also, in our assumptions, we relaxed the condition in [4] that the function f is nondecreasing with respect to each of the variables separately.

Example 4. Consider the quadratic integral equation of Abel type

$$x(t)=t^2+\frac{tx(t)}{2(1+t^2)\Gamma(\frac12)}\int_0^t\frac{\ln(1+\sqrt{t+s})}{\sqrt{t-s}}x(s)\,\mathrm{d}s. \tag{4.16}$$

In this example we have $a(t)=t^2$ and this function satisfies assumption (a_1) and $\|a\|=1$. Here $k(t,s)=\sqrt{t+s}$ and this function satisfies assumption (a_4). Let $g\colon [0,\sqrt2]\to\mathbb{R}_+$ be given by $g(y)=\ln(1+y)$, then g satisfies assumption (a_5) with $\|g\|=\ln(1+\sqrt2)$. Also, $f(t,x)=\frac12tx(1+t^2)^{-1}$ and it satisfies assumption (a_2) since $f\colon I\times\mathbb{R}_+\to\mathbb{R}_+$ and

$$|f(t,x)-f(t,y)|\leqslant\tfrac14|x-y|$$

for all $x,y\in\mathbb{R}$ and $t\in I$. Moreover, the function f satisfies assumption (a_3). Indeed, taking an arbitrary nonnegative function $x\in C(I)$ and $t_1,t_2\in I$ such that $t_2\geqslant t_1$, we obtain

$$\begin{aligned}
&|(Fx)(t_2)-(Fx)(t_1)|-[(Fx)(t_2)-(Fx)(t_1)]\\
&\quad=|f(t_2,x(t_2))-f(t_1,x(t_1))|-[f(t_2,x(t_2))-f(t_1,x(t_1))]\\
&\quad=\Big|\frac{t_2}{2(1+t_2^2)}x(t_2)-\frac{t_1}{2(1+t_1^2)}x(t_1)\Big|\\
&\qquad-\Big[\frac{t_2}{2(1+t_2^2)}x(t_2)-\frac{t_1}{2(1+t_1^2)}x(t_1)\Big]\\
&\quad\leqslant\frac{t_2}{2(1+t_2^2)}\,|x(t_2)-x(t_1)|+\Big|\frac{t_2}{2(1+t_2^2)}-\frac{t_1}{2(1+t_1^2)}\Big|x(t_1)\\
&\qquad-\frac{t_2}{2(1+t_2^2)}\,[x(t_2)-x(t_1)]-\Big[\frac{t_2}{2(1+t_2^2)}-\frac{t_1}{2(1+t_1^2)}\Big]x(t_1)\\
&\quad\leqslant\frac{t_2}{2(1+t_2^2)}\,\{|x(t_2)-x(t_1)|-[x(t_2)-x(t_1)]\}\\
&\quad\leqslant\frac{t_2}{2(1+t_2^2)}d(x)\leqslant\frac14d(x).
\end{aligned}$$

In this case inequality (3.1) has the form ($\alpha=\frac12$, $c=\frac14$, $m=0$)

$$\Gamma\Big(\frac32\Big)+\frac14\ln(1+\sqrt2)r\leqslant r\Gamma\Big(\frac32\Big)$$

and this admits

$$r_0 = \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}) - \ln\sqrt{1+\sqrt{2}}}$$

as a positive solution since $\Gamma(\frac{1}{2}) \simeq 1.77245$ and $\ln\sqrt{1+\sqrt{2}} \simeq 0.440687$. Moreover,

$$c\|g\|r_0 = \frac{\ln\sqrt{1+\sqrt{2}}}{\Gamma(\frac{1}{2}) - \ln\sqrt{1+\sqrt{2}}}\Gamma\Big(\frac{3}{2}\Big) < \Gamma\Big(\frac{3}{2}\Big).$$

Theorem 2 guarantees that equation (4.16) has a nondecreasing solution.

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