Stability problems in the theory of complete fluid systems

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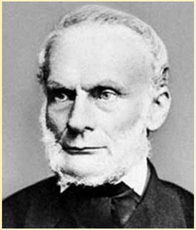
Complete fluid systems



Rudolph Clausius
[1822-1888]

Die Energie der Welt ist constant;
Die Entropie der Welt strebt einem Maximum zu

All pictures in the text thanks to wikipedia

Fluids at equilibrium

Thermodynamic state variables

mass density . . . . $\varrho = \varrho(t, x)$
absolute temperature . . . . $\vartheta = \vartheta(t, x)$

Thermodynamic functions

pressure . . . . $p = p(\varrho, \vartheta)$
internal energy . . . . $e = e(\varrho, \vartheta)$
entropy . . . . $s = s(\varrho, \vartheta)$

Gibbs' relation

$$\vartheta Ds(\varrho, \vartheta) = De(\varrho, \vartheta) + p(\varrho, \vartheta) D\left(\frac{1}{\varrho}\right)$$

Thermodynamic stability

$$\frac{\partial p(\varrho, \vartheta)}{\partial \varrho} > 0, \ \frac{\partial e(\varrho, \vartheta)}{\partial \vartheta} > 0$$

Dynamics, diffusion, transport

Macroscopic velocity

$$\mathbf{u} = \mathbf{u}(t, x), \ \frac{\mathrm{d}\mathbf{X}}{\mathrm{d}t}(t) = \mathbf{u}(t, \mathbf{X}(t)), \ \mathbf{X}(0) = x$$

Viscosity - Newton's law

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \eta \mathrm{div}_x \mathbf{u} \mathbb{I},$$

Heat conductivity - Fourier's law

$$\mathbf{q} = -\kappa \nabla_x \vartheta$$

Energetically insulated system

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0, \ \mathbf{q} \cdot \mathbf{n}|_{\partial\Omega} = 0, \ \mathbf{u} \times \mathbf{n}|_{\partial\Omega} = 0 \text{ or } (\mathbb{S} \cdot \mathbf{n}) \times \mathbf{n}|_{\partial\Omega} = 0$$

Conservation (balance) laws

Mass conservation - equation of continuity

$$\partial_t \varrho + \mathrm{div}_x (\varrho \mathbf{u}) = 0$$

Momentum equations - Newton's second law

$$\partial_t (\varrho \mathbf{u}) + \mathrm{div}_x (\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p = \mathrm{div}_x \mathbb{S}$$

Thermal energy vs. entropy balance

$$\partial_t (\varrho e) + \mathrm{div}_x (\varrho \mathbf{u} e) + \mathrm{div}_x \mathbf{q} \boxed{\geq} \mathbb{S} : \nabla_x \mathbf{u} - p \mathrm{div}_x \mathbf{u}$$

$$\partial_t (\varrho s) + \mathrm{div}_x (\varrho \mathbf{u} s) + \mathrm{div}_x \left(\frac{\mathbf{q}}{\vartheta} \right) \boxed{\geq} \frac{1}{\vartheta} \left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right)$$

Total energy balance

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left[\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) \right] \ \mathrm{d}x = 0$$

Well posedness, classical way

Jacques Hadamard
[1865 - 1963]

- **Existence.** Given problem is solvable for any choice of (admissible) data
- **Uniqueness.** Solutions are uniquely determined by the data
- **Stability.** Solutions depend continuously on the data

Well posedness, modern way

Jacques-Louis Lions [1928 - 2001]

- **Approximations.** Given problem admits an approximation scheme that is solvable analytically and, possibly, **numerically**
- **Stability.** Approximate solutions possesses uniform bounds depending solely on the data
- **Convergence.** The family of approximate solutions admits a limit representing a (generalized) solution of the given problem

Dissipation

Dissipation inequality

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left[\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) - \Theta \varrho s(\varrho, \vartheta) \right] \ \mathrm{d}x$$

$$+ \int_\Omega \frac{\Theta}{\vartheta} \left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right) \ \mathrm{d}x \leq 0$$

Ballistic free energy

$$H_\Theta(\varrho, \vartheta) = \varrho e(\varrho, \vartheta) - \Theta \varrho s(\varrho, \vartheta)$$

Relative energy

$$\mathcal{E} \left(\varrho, \vartheta, \mathbf{u} \ \middle| r, \Theta, \mathbf{U} \right)$$

$$= \int_\Omega \left[\frac{1}{2} \varrho |\mathbf{u} - \mathbf{U}|^2 + H_\Theta(\varrho, \vartheta) - \frac{\partial H_\Theta(r, \Theta)}{\partial \varrho} (\varrho - r) - H_\Theta(r, \Theta) \right] \ \mathrm{d}x$$

Dissipative solutions

Relative entropy inequality

$$\left[\mathcal{E} \left(\varrho, \vartheta, \mathbf{u} \middle| r, \Theta, \mathbf{U} \right) \right]_{t=0}^{\tau}$$

$$+ \int_0^\tau \int_\Omega \frac{\Theta}{\vartheta} \left(\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \nabla_x \vartheta}{\vartheta} \right) \, \mathrm{d}x \, \mathrm{d}t$$

$$\leq \int_0^\tau \mathcal{R}(\varrho, \vartheta, \mathbf{u}, r, \Theta, \mathbf{U}) \, \mathrm{d}t$$

Test functions

$$r > 0, \ \Theta > 0$$

$\mathbf{U}$ satisfying the relevant natural boundary conditions

Remainder

Remainder

$$\boxed{\mathcal{R}(\varrho, \vartheta, \mathbf{u}, r, \Theta, \mathbf{U})}$$

$$= \int_\Omega \Big(\varrho \Big(\partial_t \mathbf{U} + \mathbf{u} \cdot \nabla_x \mathbf{U} \Big) \cdot (\mathbf{U} - \mathbf{u}) + \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{U} \Big) \ \mathrm{d}x$$

$$+ \int_\Omega \Big[\Big(p(r, \Theta) - p(\varrho, \vartheta) \Big) \mathrm{div} \mathbf{U} + \frac{\varrho}{r} (\mathbf{U} - \mathbf{u}) \cdot \nabla_x p(r, \Theta) \Big] \ \mathrm{d}x$$

$$- \int_\Omega \Big(\varrho \Big(s(\varrho, \vartheta) - s(r, \Theta) \Big) \partial_t \Theta + \varrho \Big(s(\varrho, \vartheta) - s(r, \Theta) \Big) \mathbf{u} \cdot \nabla_x \Theta$$

$$+ \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta)}{\vartheta} \cdot \nabla_x \Theta \Big) \ \mathrm{d}x$$

$$+ \int_\Omega \frac{r - \varrho}{r} \Big(\partial_t p(r, \Theta) + \mathbf{U} \cdot \nabla_x p(r, \Theta) \Big) \ \mathrm{d}x$$

Weak vs. dissipative solutions

Classical (strong) solutions

Equations satisfied in the classical sense

Weak solutions

- continuity and momentum equations in the sense of distributions
- entropy (internal energy) inequality in the sense of distributions
- total energy balance

Dissipative solutions

Relative energy inequality for any trio r, Θ, $\mathbf{U}$

Properties of weak solutions

Compatibility

weak + smooth $\Rightarrow$ strong

Weak solutions with entropy inequality are dissipative

weak $\Rightarrow$ dissipative

Weak strong uniquness

Dissipative (weak) and strong solution emanating from the same initial data coincide as long as the latter exists

Global-in-time weak solutions, entropy

Pressure - density, temperature state equation

$$p(\varrho, \vartheta) = \vartheta^{5/2} P\left(\frac{\varrho}{\vartheta^{3/2}}\right) + a\vartheta^4$$

$$\varrho e(\varrho, \vartheta) = \frac{3}{2}\vartheta^{5/2} P\left(\frac{\varrho}{\vartheta^{3/2}}\right) + \frac{a}{3}\vartheta^4$$

$$\lim_{Z \to \infty} \frac{P(Z)}{Z^{5/3}} = p_\infty > 0$$

Transport coefficients

$$\underline{\mu}(1 + \vartheta^\alpha) \leq \mu(\vartheta) \leq \overline{\mu}(1 + \vartheta^\alpha), \ \ \eta(\vartheta) \leq \overline{\eta}(1 + \vartheta^\alpha), \ \alpha \in (2/5, 1],$$

$$\underline{\kappa}(1 + \vartheta^3) \leq \kappa(\vartheta) \leq \overline{\kappa}(1 + \vartheta^3)$$

Global-in-time weak solutions, internal energy

Pressure - density, temperature state equation

$$e(\varrho,\vartheta) = c_v\vartheta + H(\varrho)$$

$$p(\varrho,\vartheta) = \varrho^\gamma + \varrho\vartheta,\ \gamma > 3$$

Transport coefficients

$$\mu > 0,\ \eta \geq 0 \text{ constant}$$

$$\underline{\mu}(1+\vartheta^2) \leq \kappa(\vartheta) \leq \overline{\mu}(1+\vartheta^2)$$

Regularity criteria

Weak solutions with entropy inequality

The weak solution emanating from smooth initial data remains smooth as soon as

$$\|\nabla_x \mathbf{u}\|_{L^\infty((0,T)\times\Omega)} \leq c$$

Weak solutions with internal energy inequality

The weak solution emanating from smooth initial data remains smooth as soon as

$$\|\mathbf{u}\|_{L^\infty((0,T)\times\Omega;R^3)} + \|\mathrm{div}_x \mathbf{u}\|_{L^1(0,T;L^\infty(\Omega))} \leq c, \ \vartheta \leq \overline{\vartheta}$$

Existence theory - *a priori* bounds

Integral bounds - conservation laws

$$\sqrt{\varrho}\mathbf{u} \in L^\infty(0,T;L^2(\Omega;R^3))$$

$$\varrho \in L^\infty(0,T;L^{5/3}(\Omega))$$

$$\vartheta \in L^\infty(0,T;L^4(\Omega))$$

Gradient bounds - energy dissipation

$$\nabla_x \mathbf{u} \in L^2(0,T;L^q(\Omega;R^{3\times 3})),\ q = \frac{8}{5-\alpha}$$

$$\nabla_x \vartheta \in L^2(0,T;L^2(\Omega;R^3))$$

$$\nabla_x \log(\vartheta) \in L^2(0,T;L^2(\Omega;R^3))$$

Pressure bounds

$$p(\varrho,\vartheta)\varrho^\beta \in L^1((0,T)\times\Omega) \text{ for a certain } \beta > 0$$

Convergence, sequential stability

Div-Curl lemma [F.Murat, L.Tartar, 1975]

Let

$$\mathbf{v}_\varepsilon \to \mathbf{v} \text{ weakly in } L^p,$$

$$\mathbf{w}_\varepsilon \to \mathbf{w} \text{ weakly in } L^q,$$

with

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} < 1.$$

Let, moreover,

$$\mathrm{div}[\mathbf{v}_\varepsilon],\ \mathrm{curl}[\mathbf{w}_\varepsilon] \text{ be precompact in } W^{-1,s}$$

Then

$$\mathbf{v}_\varepsilon \cdot \mathbf{w}_\varepsilon \to \mathbf{v} \cdot \mathbf{w} \text{ weakly in } L^r.$$

Weak sequential stability of convective terms

Ansatz for Div-Curl lemma

$$\mathbf{v}_\varepsilon = [\varrho_\varepsilon, \varrho_\varepsilon \mathbf{u}_\varepsilon], \ \mathbf{w}_\varepsilon = [u^i_\varepsilon, 0, 0, 0], \ i = 1, 2, 3$$

Aubin-Lions argument (Div-Curl lemma)

$$\varrho_\varepsilon \mathbf{u}_\varepsilon \rightharpoonup \varrho \mathbf{u}$$

$$\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon \rightharpoonup \varrho \mathbf{u} \otimes \mathbf{u}$$

$$\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon) \vartheta_\varepsilon \rightharpoonup \varrho s(\varrho, \vartheta) \vartheta$$

Pointwise convergence of temperature, I

GOAL: Use monotonicity of $s(\varrho, \vartheta)$ in ϑ to show

$$\int_0^T \int_\Omega \Big(\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon) - \varrho_\varepsilon s(\varrho_\varepsilon, \vartheta) \Big) (\vartheta_\varepsilon - \vartheta) \, {\rm d}x \, {\rm d}t \to 0$$

$$\Rightarrow$$

$$\| \vartheta_\varepsilon - \vartheta \|_{L^3} \to 0$$

STEP 1: Aubin-Lions argument (Div-Curl lemma)

$$\int_0^T \int_\Omega \varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon)(\vartheta_\varepsilon - \vartheta) \, {\rm d}x \, {\rm d}t \to 0$$

Pointwise convergence of temperature, II

STEP 2: Renormalized equation of continuity [DiPerna and P.-L. Lions, 1989]

$$\partial_t b(\varrho) + \mathrm{div}_x(b(\varrho)\mathbf{u}) + \Big(b'(\varrho)\varrho - b(\varrho)\Big)\mathrm{div}_x\mathbf{u} = 0$$

STEP 3: Aubin-Lions argument (Div-Curl lemma)

$$\overline{b(\varrho)g(\vartheta)} = \overline{b(\varrho)}\ \overline{g(\vartheta)}$$

Pointwise convergence of temperature, II

Fundamental theorem on Young measures, [J.M Ball 1989, P.Pedregal 1997]

Let $\mathbf{v}_\varepsilon : Q \subset R^N \to R^M$ be a sequence of vector fields bounded in $L^1(Q; R^M)$.
Then there exists a subsequence (not relabeled) and a family of probability measures $\{\nu_y\}_{y \in Q}$ on R^M such that:
For any Carathéodory function $\Phi = \Phi(y, Z)$, $yinQ$, $Z \in R^M$ such that

$$\Phi(\cdot, \mathbf{v}_\varepsilon) \to \overline{\Phi} \text{ weakly in } L^1(Q)$$

we have

$$\overline{\Phi}(y) = \int_{R^M} \Phi(y, Z) \, \mathrm{d}\nu_y(Z) \text{ for a.a. } y \in Q.$$

Pointwise convergence of temperature, III

STEP 4:

Since we already know from STEP 3 that

$$\nu[\varrho_\varepsilon \vartheta_\varepsilon] = \nu[\varrho_\varepsilon] \otimes \nu[\vartheta_\varepsilon],$$

Fundamental theorem yields the desired conclusion

$$\int_0^T \int_\Omega \varrho_\varepsilon s(\varrho_\varepsilon, \vartheta)(\vartheta_\varepsilon - \vartheta) \ \mathrm{d}x \ \mathrm{d}t \to 0$$

Conclusion

$$\vartheta_\varepsilon \to \vartheta \text{ a.a. on } (0, T) \times \Omega$$

Pointwise convergence of density,I

STEP 1: Renormalized equation of continuity

$$\partial_t(\varrho \log(\varrho)) + \mathrm{div}_x(\varrho \log(\varrho)\mathbf{u}) + \varrho \mathrm{div}_x \mathbf{u} = 0$$

$$\partial_t(\overline{\varrho \log(\varrho)}) + \mathrm{div}_x(\overline{\varrho \log(\varrho)}\mathbf{u}) + \overline{\varrho \mathrm{div}_x \mathbf{u}} = 0$$

Propagation of density oscillations

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(\overline{\varrho \log(\varrho)} - \varrho \log(\varrho) \right) \mathrm{d}x = -\int_\Omega \left(\overline{\varrho \mathrm{div}_x \mathbf{u}} - \varrho \mathrm{div}_x \mathbf{u} \right) \mathrm{d}x$$

Pointwise convergence of density,II

STEP 2: Effective viscous pressure [P.-L.Lions, 1998]

$$\overline{p(\varrho,\vartheta)b(\varrho)} - \overline{p(\varrho,\vartheta)}\ \overline{b(\varrho)} = \overline{[\mathcal{R}:\mathbb{S}]b(\varrho)} - [\mathcal{R}:\mathbb{S}]\overline{b(\varrho)}$$

where

$$\mathcal{R}_{i,j} \equiv \partial_{x_i}\Delta^{-1}\partial_{x_j}$$

Commutator

$$\mathcal{R}:\mathbb{S} = \boxed{\mathcal{R}:\mathbb{S} - \left(\frac{4}{3}\mu(\vartheta) + \eta(\vartheta)\right)\mathrm{div}_x\mathbf{u}} + \left(\frac{4}{3}\mu(\vartheta) + \eta(\vartheta)\right)\mathrm{div}_x\mathbf{u}$$

Commutator estimates

Commutator lemma [in the spirit of Coifman and Meyer]

Let $w \in W^{1,r}(R^N)$, $\mathbf{V} \in L^p(R^N; R^N)$ be given, where

$$1 < r < N,\ 1 < p < \infty,\ \frac{1}{r} + \frac{1}{p} - \frac{1}{N} < 1.$$

The for any s satisfying

$$\frac{1}{r} + \frac{1}{p} - \frac{1}{N} < \frac{1}{s} < 1$$

there exists $\beta > 0$ such that

$$\|\mathcal{R}[w\mathbf{V}] - w\mathcal{R}[\mathbf{V}]\|_{W^{\beta,s}(R^N,R^N)} \leq c\|w\|_{W^{1,r}}\|\mathbf{V}\|_{L^p}.$$

Pointwise convergence of density,III

STEP 3: Effective viscous pressure revisited

$$0 \leq \overline{p(\varrho,\vartheta)\varrho} - \overline{p(\varrho,\vartheta)}\varrho = (\frac{4}{3}\mu(\vartheta) + \eta(\vartheta))\Big(\overline{\varrho \mathrm{div}_x \mathbf{u}} - \varrho \mathrm{div}_x \mathbf{u}\Big)$$

yielding

$$\overline{\varrho \log(\varrho)} = \varrho \log(\varrho)$$

Conclusion

$$\varrho_\varepsilon \to \varrho \text{ a.a. in } (0,T) \times \Omega$$

Pointwise convergence of density - general case, I

STEP 1: Renormalized equation of continuity:

$$\partial_t(\varrho L_k(\varrho)) + \mathrm{div}_x(\varrho L_k(\varrho)\mathbf{u}) + T_k(\varrho)\mathrm{div}_x\mathbf{u} = 0$$

$$\partial_t(\overline{\varrho L_k(\varrho)}) + \mathrm{div}_x(\overline{\varrho L_k(\varrho)}\mathbf{u}) + \overline{T_k(\varrho)\mathrm{div}_x\mathbf{u}} = 0$$

Cut-off functions

$$T_k(\varrho) = \min\{\varrho, k\}$$

$$L_k(\varrho) = \log(\varrho),\ \ \varrho \leq k$$

Pointwise convergence of density - general case, II

Density oscillations

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(\overline{\varrho L_k(\varrho)} - \varrho L_k(\varrho)\right)\ \mathrm{d}x = \int_\Omega \left(T_k(\varrho)\mathrm{div}_x\mathbf{u} - \overline{T_k(\varrho)}\mathrm{div}_x\mathbf{u}\right)\ \mathrm{d}x$$

$$+ \int_\Omega \left(\overline{T_k(\varrho)}\mathrm{div}_x\mathbf{u} - \overline{T_k(\varrho)\mathrm{div}_x\mathbf{u}}\right)\ \mathrm{d}x$$

STEP 2: Effective viscous flux revisited

$$\overline{p(\varrho,\vartheta) T_k(\varrho)} - \overline{p(\varrho,\vartheta)}\ \overline{T_k(\varrho)}$$

$$= (\frac{4}{3}\mu(\vartheta) + \eta(\vartheta))\left(\overline{T_k(\varrho)\mathrm{div}_x\mathbf{u}} - \overline{T_k(\varrho)}\mathrm{div}_x\mathbf{u}\right)$$

Oscillations defect measure

Oscillations description

$$\sup_{k \geq 1} \left[\limsup_{\varepsilon \to 0} \int_0^T \int_\Omega | T_k(\varrho_\varepsilon) - T_k(\varrho) |^q \ {\rm d}x \ {\rm d}t \right] < \infty$$

$$q = 5/3 + 1 = 8/3$$

Convergence of the density - general case,III

STEP 3: Boundedness of oscillation deffect measure

- The limit functions ϱ, $\mathbf{u}$ satisfy the renormalized equation of continuity
-
$$\int_\Omega \left(T_k(\varrho)\mathrm{div}_x\mathbf{u} - \overline{T_k(\varrho)}\mathrm{div}_x\mathbf{u} \right) \ \mathrm{d}x \to 0 \text{ for } k \to \infty$$

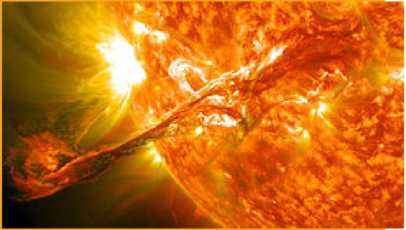
Pointwise convergence of density

$$\overline{\varrho \log(\varrho)} = \lim_{k\to\infty} \overline{\varrho L_k(\varrho)} = \lim_{k\to\infty} \varrho L_k(\varrho) = \varrho \log(\varrho)$$

$$\varrho_\varepsilon \to \varrho \text{ a.a. on } (0,T) \times \Omega$$

Turbulence or viscosity?

honey



Sun

Scaled equations

Scaling

$$X \approx \frac{X}{X_{\rm char}}$$

Mass conservation

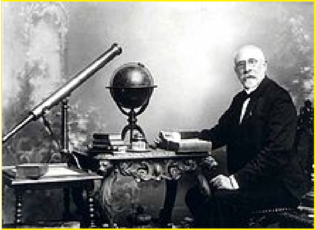
$$[{\rm Sr}]\partial_t \varrho + {\rm div}_x(\varrho \mathbf{u}) = 0$$

Momentum balance

$$[{\rm Sr}]\partial_t(\varrho \mathbf{u}) + {\rm div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{[{\rm Ro}]} \varrho \omega \times \mathbf{u} + \left[\frac{1}{{\rm Ma}^2} \right] \nabla_x p(\varrho)$$

$$= \left[\frac{1}{{\rm Re}} \right] {\rm div}_x \mathbb{S}(\nabla_x \mathbf{u}) + (\text{external forces})$$

Characteristic numbers - Strouhal number



Čeněk Strouhal
[1850-1922]

Strouhal number

$$[\mathrm{Sr}] = \frac{\mathrm{length}_{\mathrm{char}}}{\mathrm{time}_{\mathrm{char}}\mathrm{velocity}_{\mathrm{char}}}$$

Scaling by means of Strouhal number is used in the study of the long-time behavior of the fluid system, where the characteristic time is large

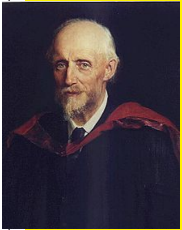
Mach number

Ernst Mach [1838-1916]

Mach number

$$[\mathrm{Ma}] = \frac{\mathrm{velocity}_{\mathrm{char}}}{\sqrt{\mathrm{pressure}_{\mathrm{char}}/\mathrm{density}_{\mathrm{char}}}}$$

Mach number is the ratio of the characteristic speed to the speed of sound in the fluid. Low Mach number limit, where, formally, the speed of sound is becoming infinite, characterizes incompressibility



Osborne Reynolds
[1842-1912]

Reynolds number

$$[\mathrm{Re}] = \frac{\mathrm{density}_{\mathrm{char}}\mathrm{velocity}_{\mathrm{char}}\mathrm{length}_{\mathrm{char}}}{\mathrm{viscosity}_{\mathrm{char}}}$$

High Reynolds number is attributed to turbulent flows, where the viscosity of the fluid is negligible

Rossby number

Carl Gustav Rossby
[1898-1957]

Rossby number

$$[\mathrm{Ro}] = \frac{\mathrm{velocity}_{\mathrm{char}}}{\omega_{\mathrm{char}}\mathrm{length}_{\mathrm{char}}}$$

Rossby number characterizes the speed of rotation of the fluid

Scaled Navier-Stokes system

Continuity equation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum equation

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \boxed{\frac{1}{\varepsilon}} \varrho \mathbf{f} \times \mathbf{u} + \boxed{\frac{1}{\varepsilon^{2m}}} \nabla_x p(\varrho)$$

$$= \boxed{\varepsilon^{\alpha}} \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}) + \boxed{\frac{1}{\varepsilon^{2n}}} \varrho \nabla_x G$$

Newtonian viscous stress

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \eta \mathrm{div}_x \mathbf{u} \mathbb{I}, \ \mu > 0$$

f-plane approximation

$$\mathbf{f} = [0, 0, 1], \ \nabla_x G = [0, 0, -1]$$

Spatial domain and boundary conditions

Infinite slab

$$\Omega = R^2 \times (0,1)$$

Complete slip boundary conditions

$$\mathbf{u} \cdot \mathbf{n} = u_3|_{\partial\Omega} = 0, \ [\mathbb{S} \cdot \mathbf{n}]_{\rm tan}|_{\partial\Omega} = 0$$

Far field conditions

$$\varrho \to \tilde{\varrho}_\varepsilon, \ \mathbf{u} \to 0 \text{ as } |x| \to \infty$$

Static density distribution

$$\nabla_x p(\tilde{\varrho}_\varepsilon) = \varepsilon^{2(m-n)} \tilde{\varrho}_\varepsilon \nabla_x G, \ m > n, \ \tilde{\varrho}_\varepsilon \to 1 \text{ as } \varepsilon \to 0$$

Singular limits

Low Mach number

Mach number $\approx \varepsilon^m$:

$$\text{compressible } \to \text{incompressible}$$

Low Rossby number

Rossby number $\approx \varepsilon$:

$$\text{3D flow } \to \text{2D flow}$$

High Reynolds number

Reynolds number $\approx \varepsilon^{-\alpha}$:

$$\text{viscous (Navier-Stokes) } \to \text{inviscid (Euler)}$$

Low stratification

$$\frac{m}{2} > n \geq 1$$

Uniform bounds

Energy inequality

$$\int_\Omega \left[\frac{1}{2} \varrho |\mathbf{u}|^2 + \frac{1}{\varepsilon^{2m}} \left(H(\varrho) - H'(\tilde{\varrho}_\varepsilon)(\varrho - \tilde{\varrho}_\varepsilon) - H(\tilde{\varrho}_\varepsilon) \right) \right] (\tau, \cdot) \ \mathrm{d}x$$

$$+\varepsilon^\alpha \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \ \mathrm{d}x \ \mathrm{d}t$$

$$\leq \int_\Omega \left[\frac{1}{2} \varrho_{0,\varepsilon} |\mathbf{u}_{0,\varepsilon}|^2 + \frac{1}{\varepsilon^{2m}} \left(H(\varrho_{0,\varepsilon}) - H'(\tilde{\varrho}_\varepsilon)(\varrho_{0,\varepsilon} - \tilde{\varrho}_\varepsilon) - H(\tilde{\varrho}_\varepsilon) \right) \right] \ \mathrm{d}x$$

$$H(\varrho) = \varrho \int_1^\varrho \frac{p(z)}{z^2} \ \mathrm{d}z, \ p(\varrho) \approx a\varrho^\gamma, \ \gamma > \frac{3}{2}$$

Ill-prepared initial data

$$\varrho_{0,\varepsilon} = \tilde{\varrho}_\varepsilon + \varepsilon^m \varrho^{(1)}_{0,\varepsilon}, \ \varrho^{(1)}_{0,\varepsilon} \to \varrho^{(1)}_0 \text{ in } L^2(\Omega), \ \|\varrho^{(1)}_{0,\varepsilon}\|_{L^\infty} \leq c,$$

$$\mathbf{u}_{0,\varepsilon} \to \mathbf{u}_0 \text{ in } L^2(\Omega; R^3)$$

Limit system

Limit density deviation

$$\operatorname{ess}\sup_{t\in(0,T)} \|\varrho_\varepsilon(t,\cdot) - 1\|_{L^\gamma_{\rm loc}(\Omega)} \leq \varepsilon^m c$$

Limit velocity

$$\sqrt{\varrho_\varepsilon}\mathbf{u}_\varepsilon \to \mathbf{v} \left\{ \begin{array}{l} \text{weakly-(*) in } L^\infty(0,T;L^2(\Omega;R^3)), \\ \\ \boxed{\text{strongly in } L^1_{\rm loc}((0,T)\times\Omega;R^3)}, \end{array} \right.$$

Euler system

$$\partial_t \mathbf{v} + \mathbf{v}\cdot\nabla_x \mathbf{v} + \nabla_x \Pi = 0 \text{ in } (0,T)\times R^2$$

$$\mathbf{v}_0 = \mathbf{H}\left[\int_0^1 \mathbf{u}_0 \, {\rm d}x_3\right]$$

Relative entropy inequality

Relative entropy

$$\mathcal{E}_\varepsilon \left[\varrho, \mathbf{u} \middle| r, \mathbf{U} \right]$$

$$= \int_\Omega \left[\frac{1}{2} \varrho |\mathbf{u} - \mathbf{U}|^2 + \frac{1}{\varepsilon^{2m}} \Big(H(\varrho) - H'(r)(\varrho - r) - H(r) \Big) \right] \, \mathrm{d}x$$

Relative entropy inequality

$$\mathcal{E}_\varepsilon \left(\varrho, \mathbf{u} \ \middle| \ r, \mathbf{U} \right) (\tau) + \varepsilon^\alpha \int_0^\tau \int_\Omega \Big(\mathbb{S}(\nabla_x \mathbf{u}) - \mathbb{S}(\nabla_x \mathbf{U}) \Big) : \Big(\nabla_x \mathbf{u} - \nabla_x \mathbf{U} \Big) \, \mathrm{d}x \, \mathrm{d}t$$

$$\leq \mathcal{E}_\varepsilon \left(\varrho_{0,\varepsilon}, \mathbf{u}_{0,\varepsilon} \ \middle| \ r(0,\cdot), \mathbf{U}(0,\cdot) \right) + \int_0^\tau \int_\Omega \mathcal{R}(\varrho, \mathbf{u}, r, \mathbf{U}) \, \mathrm{d}x \, \mathrm{d}t$$

Test functions

$$r > 0, \ \mathbf{U} \cdot \mathbf{n}|_{\partial\Omega} = 0, \ (r - \tilde{\varrho}_\varepsilon), \ \mathbf{U} \to 0 \text{ as } |x| \to \infty$$

Remainder

$$
\begin{aligned}
&\int_0^\tau \int_\Omega \mathcal{R}(\varrho, \mathbf{u}, r, \mathbf{U}) \ \mathrm{d}x \ \mathrm{d}t \\
&= \int_0^\tau \int_\Omega \varrho \left(\partial_t \mathbf{U} + \mathbf{u} \cdot \nabla_x \mathbf{U} \right) \cdot (\mathbf{U} - \mathbf{u}) \ \mathrm{d}x \ \mathrm{d}t \\
&+ \varepsilon^\alpha \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{U}) : \nabla_x (\mathbf{U} - \mathbf{u}) \ \mathrm{d}x \ \mathrm{d}t + \frac{1}{\varepsilon} \int_0^\tau \int_\Omega \varrho (\mathbf{f} \times \mathbf{u}) \cdot (\mathbf{U} - \mathbf{u}) \ \mathrm{d}x \ \mathrm{d}t \\
&+ \frac{1}{\varepsilon^{2m}} \int_0^\tau \int_\Omega \Big[(r - \varrho) \partial_t H'(r) + \nabla_x \left(H'(r) - H'(\tilde{\varrho}_\varepsilon) \right) \cdot (r \mathbf{U} - \varrho \mathbf{u}) \Big] \ \mathrm{d}x \ \mathrm{d}t \\
&- \frac{1}{\varepsilon^{2m}} \int_0^\tau \int_\Omega \mathrm{div}_x \mathbf{U} \Big(p(\varrho) - p(r) \Big) \ \mathrm{d}x \ \mathrm{d}t + \frac{1}{\varepsilon^{2n}} \int_0^\tau \int_\Omega (\varrho - r) \nabla_x G \cdot \mathbf{U} \ \mathrm{d}x \ \mathrm{d}t
\end{aligned}
$$

Reformulation

Decomposition

$$r_\varepsilon = \frac{\varrho_\varepsilon - 1}{\varepsilon^m} = q_\varepsilon + s_\varepsilon, \ \varrho_\varepsilon \mathbf{u}_\varepsilon = \mathbf{v}_\varepsilon + \mathbf{V}_\varepsilon$$

$[q_\varepsilon, \mathbf{v}_\varepsilon]$ .... non-oscillatory component

$[s_\varepsilon, \mathbf{V}_\varepsilon]$ .... oscillatory component

"Acoustic analogy" - Poincaré waves

$$\varepsilon^m \partial_t \left[\frac{\varrho_\varepsilon - 1}{\varepsilon^m} \right] + \mathrm{div}_x [\varrho_\varepsilon \mathbf{u}_\varepsilon] = 0$$

$$\varepsilon^m \partial_t [\varrho_\varepsilon \mathbf{u}_\varepsilon] + \varepsilon^{m-1} \mathbf{f} \times [\varrho_\varepsilon \mathbf{u}_\varepsilon] + \nabla_x \left[\frac{\varrho_\varepsilon - 1}{\varepsilon^m} \right] = \varepsilon \mathbf{f}_\varepsilon$$

Non-oscillatory part

$$\mathrm{div}_x \mathbf{v}_\varepsilon = 0, \ \varepsilon^{m-1} \mathbf{f} \times \mathbf{v}_\varepsilon + \nabla_x q_\varepsilon = 0$$

Test function ansatz

Density deviation

$$r = \tilde{\varrho}_\varepsilon + \varepsilon^m (q_\varepsilon + s_\varepsilon)$$

Velocity decomposition

$$\mathbf{U} = \mathbf{v}_\varepsilon + \mathbf{V}_\varepsilon$$

Initial data

$$\varrho^{(1)}_{0,\varepsilon} = (q_\varepsilon + s_\varepsilon)(0,\cdot),\ \mathbf{u}_{0,\varepsilon} = (\mathbf{v}_\varepsilon + \mathbf{V}_\varepsilon)(0,\cdot)$$

Non-oscillatory - Euler system

Diagnostic equation

$$\omega \mathbf{f} \times \mathbf{v}_\varepsilon + \nabla_x q_\varepsilon = 0, \ \omega = \varepsilon^{m-1}$$

$$\omega \mathrm{curl}\ \mathbf{v}_\varepsilon = -\Delta q_\varepsilon$$

Perturbed Euler system

$$\partial_t \left(\Delta q_\varepsilon - \omega^2 q_\varepsilon \right) - \frac{1}{\omega} \nabla^t q_\varepsilon \cdot \nabla \left(\Delta q_\varepsilon - \omega^2 q_\varepsilon \right) = 0$$

Initial data

$$\left(\Delta q_\varepsilon - \omega^2 q_\varepsilon \right)(0, \cdot) = \omega \mathrm{curl} \left[\int_0^1 \mathbf{u}_{0,\varepsilon} \ \mathrm{d}x_3 \right] - \omega^2 \int_0^1 \varrho_{0,\varepsilon} \ \mathrm{d}x_3$$

Oscillatory - vanishing part

Poincaré waves

$$\varepsilon^m \partial_t s_\varepsilon + \mathrm{div}_x \mathbf{V}_\varepsilon = 0$$

$$\varepsilon^m \partial_t \mathbf{V}_\varepsilon + \omega \mathbf{f} \times \mathbf{V}_\varepsilon + \nabla_x s_\varepsilon = 0, \ \omega = \varepsilon^{m-1}$$

Antisymmetric acoustic propagator

$$\mathcal{B}(\omega) : \left[\begin{array}{c} s \\ \mathbf{V} \end{array} \right] \mapsto \left[\begin{array}{c} \mathrm{div}_x \mathbf{V} \\ \omega \mathbf{f} \times \mathbf{V} + \nabla_x s \end{array} \right].$$

Fourier representation

Poincaré waves

$$\varepsilon^m \partial_t \left[\begin{array}{c} s_\varepsilon(\xi, k, \omega) \\ \mathbf{V}_\varepsilon(\xi, k, \omega) \end{array} \right] = \mathrm{i}\mathcal{A}(\xi, k, \omega) \left[\begin{array}{c} s_\varepsilon(\xi, k, \omega) \\ \mathbf{V}_\varepsilon(\xi, k, \omega) \end{array} \right]$$

Hermitian matrix

$$\mathrm{i}\mathcal{B}(\omega) \approx \mathcal{A}(\xi, k, \omega) = \left[\begin{array}{cccc} 0 & \xi_1 & \xi_2 & k \\ \xi_1 & 0 & \omega \mathrm{i} & 0 \\ \xi_2 & -\omega \mathrm{i} & 0 & 0 \\ k & 0 & 0 & 0 \end{array} \right].$$

Eigenvalues

$$\lambda_{1,2}(\xi, k, \omega) = \pm \left[\frac{\omega^2 + |\xi|^2 + k^2 + \sqrt{(\omega^2 + |\xi|^2 + k^2)^2 - 4\omega^2 k^2}}{2} \right]^{1/2}$$

$$\lambda_{3,4}(\xi, k, \omega) = \pm \left[\frac{\omega^2 + |\xi|^2 + k^2 - \sqrt{(\omega^2 + |\xi|^2 + k^2)^2 - 4\omega^2 k^2}}{2} \right]^{1/2}$$

Fourier analysis

$$k \text{ fixed}, \ \psi \in C_c^\infty(0,\infty), \ 0 \leq \psi \leq 1$$

Frequency cut-off

$$Z(\tau, x_h, k, \omega) = \mathcal{F}^{-1}_{\xi \to x_h}\left[\exp\Big(\pm \mathrm{i}\lambda_j(|\xi|, k, \omega)\tau\Big)\psi(|\xi|)\hat{h}(\xi)\right], \ \tau = t/\varepsilon^m$$

$$\|Z(\tau, \cdot, k, \omega)\|_{L^\infty(R^2)}$$
$$\leq \left\|\mathcal{F}^{-1}_{\xi \to x_h}\left[\exp\Big(\pm \mathrm{i}\lambda_j(|\xi|, k, \omega)\tau\Big)\psi(|\xi|)\right]\right\|_{L^\infty(R^2)} \|h\|_{L^1(R^2)}$$

Fourier transform of radially symmetric function

$$\mathcal{F}^{-1}_{\xi \to x_h}\left[\exp\Big(\pm \mathrm{i}\lambda_j(|\xi|, k, \omega)\tau\Big)\psi(|\xi|)\right](x_h)$$
$$= \int_0^\infty \exp\Big(\pm \mathrm{i}\lambda_j(r, k, \omega)\tau\Big)\psi(r) r J_0(r|x_h|) \ \mathrm{d}r,$$

van Corput's lemma

Lemma

Let $\Lambda = \Lambda(z)$ *be a smooth function away from the origin,*

$$\partial_z \Lambda(z) \text{ monotone}, \ |\partial_z \Lambda(z)| \geq \Lambda_0 > 0$$

for all $z \in [a, b]$, $0 < a < b < \infty$. *Let* Φ *be a smooth function on* $[a, b]$. *Then*

$$\left| \int_a^b \exp\left(\mathrm{i} \Lambda(z) \tau \right) \Phi(z) \ \mathrm{d}z \right| \leq c \frac{1}{\tau \Lambda_0} \left[|\Phi(b)| + \int_a^b |\partial_z \Phi(z)| \ \mathrm{d}z \right],$$

where c *is an absolute constant* $\boxed{\textit{independent}}$ *of the specific shape* Λ *and* Φ.

Decay estimates

$L^p - L^q$ estimates

$$\|Z(\tau,\cdot,k,\omega)\|_{L^p(R^2)} \le c(\psi,p,k)\max\left\{\frac{1}{\omega\tau^{1-\beta/2}};\frac{1}{\tau^{\beta/2}}\right\}^{1-\frac{2}{p}}\|h\|_{L^{p'}(R^2)}$$

$$\text{for } p\ge 2,\ \frac{1}{p}+\frac{1}{p'}=1,\ \beta>0,\ \lambda_j\ne 0.$$

Scaling

$$\omega \approx \varepsilon^{m-1},\ \tau\approx t/\varepsilon^m$$

Dispersive decay

$$\left\|Z\left(\frac{t}{\varepsilon^m},\cdot,k,\omega\right)\right\|_{L^p(R^2)} \le c\ \varepsilon^{\frac{1}{2}-\frac{1}{p}}\max\left\{\frac{1}{t^{1-1/2m}};\frac{1}{t^{1/2m}}\right\}^{1-\frac{2}{p}}\|h\|_{L^{p'}(R^2)}$$