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ON ALGEBRAS OF GENERALIZED LATIN SQUARES

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Abstract. The main result of this paper is the introduction of a notion of a generalized R-Latin square, which includes as a special case the standard Latin square, as well as the magic square, and also the double stochastic matrix. Further, the algebra of all generalized Latin squares over a commutative ring with identity is investigated. Moreover, some remarkable examples are added.

Keywords: ring with identity, homomorphism, one-sided ideal, two-sided ideal, module, bimodule

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1. Introduction

Latin squares were discovered by L. Euler (see [6]) in 1782. He defined them in the following way: A Latin square of order n is a square arrangement of n^2 entries of n different elements, none of them is occurring twice within each row and each column. The notion of the Latin square was originally applied to find a method of solution of different brain teasers concerning the cards or the chessboard. However, later the subject of Latin squares attracted promptly the attention of many mathematicians. A motivation and concern in the Latin squares was awoken namely by many significant results which were achieved in different branches of mathematics.

A. Cayley who investigated the multiplication tables of groups (see [3]) has found that the multiplication tables of groups are as a matter of fact special Latin squares. J. Singer in [23] investigated a class of groups which are associated with Latin squares. However, it can be shown that a Latin square need not be a multiplicative table of a group. As R. Moufang proved (see [19]) that each Latin square is a multiplicative table of a quasigroup. In addition, she found that there exists a close connection between desarguesian projective planes and nonassociative quasigroups. Owing to this fact a great deal of papers was written devoted to the investigation of the finite projective planes (see [12], [22], and as a historical source also [7]). Bipartite graphs are important in the investigation of finite projective planes. Roughly speaking, a graph $G = (V, E)$ is said to be *bipartite,* if there exists a partition $V = V_1 \cup V_2$ such that no two vertices in V_1 as well as no two vertices in V_2 are linked by an edge. We will not engage in detail in this, because of lengthy of the interpretation (for details see [5]). A remarkable application of a finite projective geometry to number theory is given in paper [22] by J. Singer. Some results of the theory of Latin squares were also used for a construction of statistical designs (see [8]). Quite recently, an application of Latin squares was found even in genetics (see [15]).

But, unfortunately, a little attention has been devoted to the investigation of the algebraic structure of the set of all Latin squares. A chief aim of this paper is to solve this problem. In order to do it, we introduce first the notion of a generalized S-Latin square, where S is a semiring. For better algebraical characterization of the set of all generalized Latin squares, we will investigate the generalized Latin squares over commutative rings with identity. Some interesting theorems will be proved and also suitable examples will be given.

In the conclusion of this section we note that an excellent survey of contemporary knowledge concerning Latin squares and their applications can be found in [5]. For the magic squares we refer to [1].

2. Generalized S-Latin squares

First of all let us point out that some kinds of generalized Latin squares are already known. For example, we mention Latin rectangles, row Latin squares, F-squares (i.e., frequency squares), incomplete Latin squares, etc. Latin cubes and hypercubes were introduced in the period 1942–45 independently by K. Kishen and R. A. Fisher (see [1], [5]). Nevertheless, for the time being no generalization of Latin squares has been introduced that would include all present kinds of generalized Latin squares including the double stochastic matrices. We attempt to solve this problem in this paper. In order to do that, we first introduce some basic notation.

Definition 1. A triple $(S, +, \cdot)$ is called a *semiring* if

(i) $(S, +)$ is an additive Abelian monoid with a zero element denoted by 0.

(ii) $(S, \cdot)$ is a multiplicative monoid with an identity denoted by e.

(iii) If $a_i \in S$, $i = 1, 2, 3$, then

$$a_1 \cdot (a_2 + a_3) = a_1 \cdot a_2 + a_1 \cdot a_3,$$
$$(a_2 + a_3) \cdot a_1 = a_2 \cdot a_1 + a_3 \cdot a_1.$$

Let us note that the above definition of a semiring differs from that introduced in [24] and [25], where $(S,+)$ and $(S,\cdot)$ are required to be semigroups only.

The set $\mathbb{R}_+$ of all nonnegative real numbers, provided with the standard operations of addition and multiplication, is a commutative semiring. An example of a noncommutative semiring $M_n(\mathbb{R}_+)$ is the set of all nonnegative $n\times n$ matrices with the usual matrix operations. It is clear that every ring with identity is a semiring. Henceforth, the symbol S will denote a commutative semiring and $M_n(S)$ will be the set of all $n\times n$ matrices over S. The set $M_n(S)$ provided with the usual matrix operations is a noncommutative semiring. We introduce now a notion which plays a crucial role in this paper.

Definition 2. Let K be a semiring and let T be a nonempty subset of K. Then the set

$$C(T)=\{x\in K;\ xy=yx \text{ for all } y\in T\}$$

is called the *commutant of* T.

If $T_i\subset K$ for $i=1,2$ and if $T_1\subset T_2$ then $C(T_2)\subset C(T_1)$. We use now the notion of the commutant to introduce the notion of a generalized Latin square. In order to do that, we denote by H_n the $n\times n$ matrix

$$H_n=\begin{bmatrix} e & e & \dots & e\\ e & e & \dots & e\\ \vdots & \vdots & \ddots & \vdots\\ e & e & \dots & e\end{bmatrix}. \tag{1}$$

Definition 3. A generalized S-Latin square of order n is a matrix $L\in M_n(S)$ such that $L\in C(\{H_n\})$.

We denote by $L_n(S)$ the set of all generalized S-Latin squares of order n. It is clear that $H_n\in L_n(S)$. With regard to that, H_n plays a chief part in the introduction of the notion of a generalized S-Latin square; it will be called *the basic generalized S-Latin square.* If P is a permutation matrix of order n over S then, because $PH_n=H_nP$, it follows that $P\in L_n(S)$. By L^* we denote the transposed matrix to $L\in L_n(S)$.

The algebraical structure of $L_n(S)$ is given by the following obvious proposition.

Proposition 1. *A set $L_n(S)$ for $n>2$ is a noncommutative semiring.*

We now give another proposition which characterizes the generalized S-Latin squares far better.

Proposition 2. *A matrix $L=(a_{ij})\in M_n(S)$, $a_{ij}\in S$, is a generalized S-Latin square of order n if and only if there exists $s\in S$ such that*

$$LH_n = H_nL = sH_n. \tag{2}$$

P r o o f. In order for L to be a generalized S-Latin square, it has to fulfil the first equality in relation (2). Since

$$LH_n = \begin{bmatrix} \sum\limits_j a_{1j} & \sum\limits_j a_{1j} & \dots & \sum\limits_j a_{1j} \\ \sum\limits_j a_{2j} & \sum\limits_j a_{2j} & \dots & \sum\limits_j a_{2j} \\ \vdots & \vdots & \ddots & \vdots \\ \sum\limits_j a_{nj} & \sum\limits_j a_{nj} & \dots & \sum\limits_j a_{nj} \end{bmatrix},$$

$$H_nL = \begin{bmatrix} \sum\limits_i a_{i1} & \sum\limits_i a_{i2} & \dots & \sum\limits_i a_{in} \\ \sum\limits_i a_{i1} & \sum\limits_i a_{i2} & \dots & \sum\limits_i a_{in} \\ \vdots & \vdots & \ddots & \vdots \\ \sum\limits_i a_{i1} & \sum\limits_i a_{i2} & \dots & \sum\limits_i a_{in} \end{bmatrix},$$

where $i.j=1,\dots,n$, we have for $s_k=\sum\limits_{j=1}^{n} a_{kj}$ and $s'_k=\sum\limits_{i=1}^{n} a_{ik}$ that

$$LH_n = \begin{bmatrix} s_1 & s_1 & \dots & s_1 \\ s_2 & s_2 & \dots & s_2 \\ \vdots & \vdots & \ddots & \vdots \\ s_n & s_n & \dots & s_n \end{bmatrix}, \quad H_nL = \begin{bmatrix} s'_1 & s'_2 & \dots & s'_n \\ s'_1 & s'_2 & \dots & s'_n \\ \vdots & \vdots & \ddots & \vdots \\ s'_1 & s'_2 & \dots & s'_n \end{bmatrix}.$$

By (2) we get $s_i=s'_j$ for each $i,j=1,\dots,n$. Setting $s=s_i=s'_j$, we find that

$$LH_n = H_nL = sH_n.$$

Corollary ([14]). *Let L be a generalized S-Latin square of order n and suppose that* (2) *holds. Then the sums of all entries of an arbitrary row or column of L are the same and equal to $s\in S$.*

By means of Proposition 2 we introduce now a mapping $w\colon L_n(S)\to S$ by setting $w(L)=s$ provided (2) holds for $L\in M_n(S)$. The mapping w is called a *weight.* If P is a permutation matrix of order n over S then $w(P)=e$. It is clear that every double stochastic matrix is an element of $L_n(\mathbb{R}_+)$.

If $D_n(\mathbb{R}_+)$ is the set of all double stochastic matrices of order n then $D_n(\mathbb{R}_+) = w^{-1}(\{1\})$. For $n>2$ this set is a noncommutative monoid.

R e m a r k. The characterization of a process of a point mutation of RNA (ribonucleic acid) can be described by a symmetric doubly stochastic matrix of order 4:

$$P = \begin{bmatrix} p & p_1 & p_2 & p_2 \\ p_1 & p & p_2 & p_2 \\ p_2 & p_2 & p & p_1 \\ p_2 & p_2 & p_1 & p \end{bmatrix},$$

where $0 < p_2 < p_1 \ll p < 1$ and $p + p_1 + 2p_2 = 1$ (see [15]). Then we have $P \in D_4(\mathbb{R}_+) \subset L_4(\mathbb{R}_+)$.

Let us recall the well-known Birkhoff theorem which states that every double stochastic matrix is an element of the convex hull of m permutation matrices of order n when $m \leqslant (n-1)^2 + 1$ (see [2], [17], [18]). It has applications in stochastic genetics.

Note that the Birkhoff theorem has been generalized by M. Hall (see [9]) to nonnegative matrices. As an illustration we introduce the following example.

E x a m p l e 1. Given the Heavenly turtle magic square

$$M = \begin{bmatrix} 4 & 9 & 2 \\ 3 & 5 & 7 \\ 8 & 1 & 6 \end{bmatrix},$$

then $M \in L_3(\mathbb{N}_+)$, where $\mathbb{N}_+ = \mathbb{N} \cup \{0\}$. We see that M is a linear combination of five permutation matrices:

$$M = 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + 3\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 6\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} + 2\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

It seems that a statement similar to the Birkhoff theorem holds also for any generalized S-Latin square, where permutation matrices are replaced by permutation matrices over S.

3. Algebras of generalized R-Latin squares

This section is devoted to the investigation of the generalized Latin squares over a ring. First, we give some remarks. A reference on the generalized R-Latin squares was already given in [14]. As was formerly said a ring with identity is a special kind of a semiring which in addition has opposite elements. Owing to this property, the set of all generalized Latin squares over a ring is more interesting than $L_n(S)$ and also has many applications. Among them we can mention its utilizability in the planning of experiments in the operational research, in agriculture and lately also in genetics (see e.g. [9], [15]).

Definition 4 (see [20]). Let R be a commutative ring with identity element e. An R-*algebra* is an (R,R)-bimodule A which is also a ring satisfying the conditions

(i) $(r_1r_2)a = r_1(r_2a) = (r_1a)r_2$, $r_1, r_2 \in R$, $a \in A$,

(ii) $ea = ae = a$, $e \in R$, $a \in A$.

A ring R is an R-algebra, because it fulfils the conditions of the preceding definition. Denote by $M_n(R)$ the set of all n-square matrices over R. Then it is clear that $M_n(R)$ is a noncommutative R-algebra with identity (with respect to the usual matrix operations).

A subset A_1 of an R-algebra is said to be an R-*subalgebra* of A if $0, e \in A_1$ and if A_1 is closed with respect to all operations of A. The notion of a homomorphism is defined in the standard way.

Definition 5. Let A be an R-algebra with identity. A subset I of A is called a *two sided ideal of* A if I is a subbimodule of A, $AI \subset I$, and $IA \subset I$.

Let A, B be R-algebras. If h is a homomorphism of A on B, then the set $\operatorname{Ker} h = h^{-1}\{0\}$ is called the *kernel* of h. The set $\operatorname{Ker} h$ is a two-sided ideal of A.

Furthermore, the symbol R will denote a commutative and associative ring with identity element e.

Definition 6. Let A be an R-algebra with identity. An *involution* is an automorphism $*\colon\ A \to A$ such that

(i) $(x^*)^* = x$, $(x+y)^* = x^* + y^*$, $(xy)^* = y^*x^*$, $x, y \in A$,

(ii) $(rx)^* = rx^*$, $r \in R$, $x \in A$.

An R-algebra which has an involution is called an *algebra with involution.* The R-algebra $M_n(R)$ is an algebra with involution, where M^* is the transposed matrix to M.

The notion of a generalized Latin square over a ring R (in short a generalized R-Latin square) is defined analogously to Definition 3 (namely as an element of a commutant of H_n).

Note that equality (2) holds also for generalized R-Latin squares. The set of all generalized R-Latin squares of order n will be denoted by $L_n(R)$.

Theorem 1. *The set $L_n(R)$ is an R-algebra with involution which is noncommutative for $n > 2$.*

P r o o f. It is essentially the same as that for Proposition 1. We show only that for $L \in L_n(R)$ and $r \in R$ we have $rL \in L_n(R)$. Indeed,

$$(rL)H_n = r(LH_n) = r(H_nL) = H_n(rL).$$

□

If I is an ideal of R, the set $L_n(I)$ of all generalized Latin squares of order n over I is an ideal of the R-algebra $L_n(R)$.

E x a m p l e 2. Let M be the Heavenly turtle magic square from Example 1. Then $M \in L^3(\mathbb{Z})$ and

$$M^{-1} = \frac{1}{360}\begin{bmatrix} 23 & -52 & 53 \\ 38 & 8 & -22 \\ -37 & 68 & -7 \end{bmatrix},$$

We observe that M^{-1} is $\mathbb{R}$-magic square, however, $M^{-1} \notin L^3(\mathbb{Z})$.

Nevertheless, the following statement is true.

Proposition 3. *Let L be an invertible element of $L_n(R)$. Then $L^{-1} \in L_n(R)$ if and only if the inverse element of $w(L)$ exists and $[w(L)]^{-1} \in R$.*

P r o o f. Suppose that $L^{-1} \in L_n(R)$. Then

$$L^{-1}H_n = H_nL^{-1} = w(L^{-1})H_n.$$

This implies

$$L(L^{-1}H_n) = (LH_n)L^{-1} = (w(L)H_n)L^{-1} = w(L)(w(L^{-1})H_n.$$

But by (2) we have $L(H_nL^{-1}) = L(L^{-1}H_n) = H_n$ and then $w(L)(w(L^{-1}))H_n = H_n$. Therefore, $w(L)(w(L^{-1})) = e$, and analogously $w(L^{-1})w(L) = e$. This implies that $w(L^{-1}) = [w(L)]^{-1}$.

Conversely, assume that $[w(L)]^{-1}$ exists. Since $L \in L_n(R)$, we obtain $LH_n = w(L)H_n$. Consequently, L^{-1} exists provided $L^{-1}(LH_n) = w(L)(L^{-1}H_n)$. This yields that $w(L)(L^{-1}H_n) = H_n$ and thus we have $L^{-1}H_n = [w(L)]^{-1}H_n$. Analogously, we can prove that $H_nL^{-1} = [w(L)]^{-1}H_n$. This implies that $L^{-1} \in L_n(R)$. □

As an immediate consequence of Proposition 1 we have

Theorem 2. *The quotient R-algebra $L_n(R)/\operatorname{Ker} w$ is isomorphic to R.*

The proof is a routine analogous to rings.

Definition 7. Let A be an R-algebra. If there exists a non-trivial homomorpism $h\colon A \to R$, then A is called a *baric* R-algebra.

From Proposition 1 it follows that $L_n(R)$ is a baric R-algebra. If F is a commutative field and if w is a weight of $L_n(F)$, then $\operatorname{Ker} w$ is a maximal two-sided ideal of $L_n(F)$.

R e m a r k. Baric $\mathcal{R}$-algebras, where $\mathcal{R}$ is the field of all real numbers, play important role in genetics (see [26]). It is interesting that the $\mathbb{R}$-algebras $L_n(\mathbb{R})$ have algebraic properties analogous to some algebras which appear in genetics (see [26, pp. 12–14]). This suggests that the algebras $L_n(\mathbb{R})$ could have some importance also in genetics.

Now we make a remark on new algebras of the generalized Latin squares. Denote by F a commutative field whose characteristic is not two. If $L_i \in L_n(F)$, $i = 1, 2$, then we introduce in $L_n(F)$ a new multiplication by setting

$$L_1 \circ L_2 = \tfrac{1}{2}(L_1 L_2 + L_2 L_1).$$

Denote this F-algebra by $(L_n(F))^+$. This algebra is evidently commutative, but it is not associative though its fulfils the Jordan identity

$$(L_1 \circ L_2) \circ L_1^2 = L_1 \circ (L_2 \circ L_1^2), \quad L_1, L_2 \in L_n(F).$$

Thus, the algebra $(L_n(F))^+$ is a *Jordan algebra.* It is clear that $L_1 \circ L_2 \in L_n(F)$ for $L_1, L_2 \in L_n(F)$. We define now a mapping $h\colon (L_n(F))^+ \to F$ by setting $h(L) = s$ if $L \in (L_n(F))^+$ and $H_n \circ L = sH_n$. It can be easily shown that h is an epimorphism of $(L_n(F))^+$ onto F. It follows that $(L_n(F))^+$ is a baric Jordan algebra.

4. Generation of R-algebras $L_n(R)$

One of the main problems which concerns the R-algebras $L_n(R)$ consists in determination of a system of generators of $L_n(R)$. In order to do it we introduce first a proposition which will be a motivation for our further investigations.

Proposition 4. *Let $a_{ij} \in R$, $i, j = 1, \dots, n-1$, be given and let $s \in R$. Then the matrix L given by*

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1,n-1} & s - \sum\limits_{i=1}^{n-1} a_{1i} \\ a_{21} & a_{22} & \dots & a_{2,n-1} & s - \sum\limits_{i=1}^{n-1} a_{2i} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1,1} & a_{n-1,2} & \dots & a_{n-1,n-1} & s - \sum\limits_{i=1}^{n-1} a_{n-1,i} \\ s - \sum\limits_{i=1}^{n-1} a_{i1} & s - \sum\limits_{i=1}^{n-1} a_{i2} & \dots & s - \sum\limits_{i=1}^{n-1} a_{i,n-1} & \sum\limits_{i,j=1}^{n-1} a_{ij} - (n-2)s \end{bmatrix}$$

is a generalized R-Latin square of order n with a weight $w(L) = s$.

The proof is evident.

From this proposition it follows that for a generation of a generalized R-Latin square of order n it suffices to choose $(n-1)^2+1$ elements of R, i.e., a_{ij}, $i, j = 1, \dots, n-1$, and s.

Another inspiration comes from the Birkhoff theorem. Namely, the permutation matrices are generalized R-Latin squares (as was said above) and therefore, it seems that they are suitable entries for a construction of generalized R-Latin squares.

Definition 8. Suppose that R is a commutative ring whose characteristic is different from two. The elements a_i, $i = 1, \dots, n$, of an R-algebra A are *linearly dependent,* if there exist elements $r_1, \dots, r_n$ of R not all zero such that $\sum\limits_{i=1}^{n} r_i a_i = 0$; otherwise they are *linearly independent.*

Suppose that a_i, $i = 1, \dots, n$ is a maximal set of linearly independent elements of an R-algebra A. If the set of all linear combinations of elements a_i, $i = 1, \dots, n$ is equal to the R-algebra A, then the set $G = \{a_i,\ i = 1, 2, \dots, n\}$ is called a system of generators of A.

As a system of generators of the R-algebra $L_n(R)$ we take the set of permutation matrices of order n over R. More precisely, it seems that the following conjecture is correct.

Conjecture. *A system of generators of the R-algebra $L_n(R)$ for $n \geqslant 3$ consists of a set of permutation matrices of order n over R, $G_n = \{P_1, \dots, P_n, \dots, P_{(n-1)^2+1}\}$.*

For the time being this conjecture has not been proved. Nevertheless, we proved its validity for $n = 2, \dots, 6$.

a) First let $n = 2$. A system $G_2 = G_2'$ of generators of $L_2(R)$ is a set given by

$$P_1 = \begin{bmatrix} e & 0 \\ 0 & e \end{bmatrix}, \qquad P_2 = \begin{bmatrix} 0 & e \\ e & 0 \end{bmatrix}.$$

If $L \in L_2(R)$ then there exist $a, b \in R$ such that

$$L = \begin{bmatrix} a & b \\ b & a \end{bmatrix} = aP_1 + bP_2.$$

b) Let $n = 3$. A system G_3 of generators of $L_3(R)$ is given by

$$P_1 = \begin{bmatrix} e & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & e \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0 & e & 0 \\ 0 & 0 & e \\ e & 0 & 0 \end{bmatrix}, \quad P_3 = \begin{bmatrix} 0 & 0 & e \\ e & 0 & 0 \\ 0 & e & 0 \end{bmatrix},$$
$$P_4 = \begin{bmatrix} 0 & e & 0 \\ e & 0 & 0 \\ 0 & 0 & e \end{bmatrix}, \quad P_5 = \begin{bmatrix} 0 & 0 & e \\ 0 & e & 0 \\ e & 0 & 0 \end{bmatrix}.$$

Suppose that $L \in L_3(R)$ is such that $w(L) = s$ and let L be of the form

$$L = \begin{bmatrix} a_1 & a_2 & s - a_1 - a_2 \\ a_3 & a_4 & s - a_3 - a_4 \\ s - a_1 - a_3 & s - a_2 - a_4 & \sum\limits_{i=1}^{4} a_i - s \end{bmatrix}, \quad a_i \in R,\ i = 1, 2, 3, 4.$$

Then

$$L = a_1 P_1 + (s - a_3 - a_4) P_2 + (s - a_2 - a_4) P_3 + \left(\sum_{i=2}^{4} a_i - s \right) P_4 + (a_4 - a_1) P_5.$$

Example 3. If M is a Heavenly turtle magic square, then

$$M = \begin{bmatrix} 4 & 9 & 2 \\ 3 & 5 & 7 \\ 8 & 1 & 6 \end{bmatrix} = 4P_1 + 7P_2 + P_3 + 2P_4 + P_5.$$

Compare with Example 1.

Example 4. The incidence matrix of the projective plane of order 1

$$I = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

can be expressed in the form $I = P_1 + P_2$.

In general, the incidence matrix of the projective plane of order n is a sum of $n+1$ permutation matrices of order $n^2 + n + 1$.

c) Let $n = 4$. A system G_4 of generators of $L_4(R)$ consists of the following permutation matrices of order 4:

$$P_1 = \begin{bmatrix} e & 0 & 0 & 0 \\ 0 & e & 0 & 0 \\ 0 & 0 & e & 0 \\ 0 & 0 & 0 & e \end{bmatrix} P_2 = \begin{bmatrix} 0 & e & 0 & 0 \\ 0 & 0 & e & 0 \\ 0 & 0 & 0 & e \\ e & 0 & 0 & 0 \end{bmatrix} P_3 = \begin{bmatrix} 0 & 0 & e & 0 \\ 0 & 0 & 0 & e \\ e & 0 & 0 & 0 \\ 0 & e & 0 & 0 \end{bmatrix} P_4 = \begin{bmatrix} 0 & 0 & 0 & e \\ e & 0 & 0 & 0 \\ 0 & e & 0 & 0 \\ 0 & 0 & e & 0 \end{bmatrix}$$
$$P_5 = \begin{bmatrix} e & 0 & 0 & 0 \\ 0 & e & 0 & 0 \\ 0 & 0 & 0 & e \\ 0 & 0 & e & 0 \end{bmatrix} P_6 = \begin{bmatrix} e & 0 & 0 & 0 \\ 0 & 0 & 0 & e \\ 0 & 0 & e & 0 \\ 0 & e & 0 & 0 \end{bmatrix} P_7 = \begin{bmatrix} 0 & 0 & e & 0 \\ e & 0 & 0 & 0 \\ 0 & 0 & 0 & e \\ 0 & e & 0 & 0 \end{bmatrix} P_8 = \begin{bmatrix} e & 0 & 0 & 0 \\ 0 & 0 & 0 & e \\ 0 & e & 0 & 0 \\ 0 & 0 & e & 0 \end{bmatrix}$$
$$P_9 = \begin{bmatrix} 0 & e & 0 & 0 \\ 0 & 0 & 0 & e \\ e & 0 & 0 & 0 \\ 0 & 0 & e & 0 \end{bmatrix} P_{10} = \begin{bmatrix} 0 & e & 0 & 0 \\ 0 & 0 & 0 & e \\ 0 & 0 & e & 0 \\ e & 0 & 0 & 0 \end{bmatrix}.$$

Suppose now that $a_i \in R$, $i = 1, \ldots, 9$, and $s \in R$ are given. Then

$$L = \begin{bmatrix} a_1 & a_2 & a_3 & s - \sum_{i=1}^{3} a_i \\ a_4 & a_5 & a_6 & s - \sum_{i=4}^{6} a_i \\ a_7 & a_8 & a_9 & s - \sum_{i=7}^{9} a_i \\ s - a_1 - a_4 - a_7 & s - a_2 - a_5 - a_8 & s - a_3 - a_6 - a_9 & \sum_{i=1}^{9} a_i - 2s \end{bmatrix}$$

is a generalized R-Latin square of order 4 such that $w(L) = s$.

It can be shown that

$$\begin{aligned} L = &\left(\sum_{i=1}^{9} (a_i - 2s) P_1 + a_6 \right) P_2 + (s - a_1 - a_2 - a_4) P_3 + \left(s - \sum_{i=1}^{3} a_i \right) P_4 \\ &+ \left(a_5 - \sum_{i=1}^{9} a_i + 2s \right) P_5 + (s - a_2 - a_3 - a_5 - a_8) P_6 + \left(\sum_{i=1}^{4} a_i - s \right) P_7 \\ &+ \left(\sum_{i=1}^{3} a_i + a_8 - s \right) P_8 + (a_1 + a_2 + a_4 + a_7 - s) P_9 + (s - a_1 - a_4 - a_6 - a_7) P_{10}. \end{aligned}$$

d) The cases $n = 5$ and $n = 6$ were verified by computer.

E x a m p l e 5. A Dürer magic square has the form

$$D = \begin{bmatrix} 16 & 3 & 2 & 13 \\ 5 & 10 & 11 & 8 \\ 9 & 6 & 7 & 12 \\ 4 & 15 & 14 & 1 \end{bmatrix}, \quad D \in L_4(\mathbb{Z}), \quad w(D) = s = 34.$$

It can be expressed in terms of the permutation matrices P_i, $i = 1, 2, \ldots, 10$, as

$$D = P_1 + 11P_2 + 10P_3 + 13P_4 + 9P_5 + 13P_6 - 8P_7 - 7P_8 - P_9 - 7P_{10}.$$

E x a m p l e 6. Let $K = (e, g_1, g_2, g_3)$ be the Klein four group which is given by the multiplication matrix

$$L_k = \begin{bmatrix} e & g_1 & g_2 & g_3 \\ g_1 & e & g_3 & g_2 \\ g_2 & g_3 & e & g_1 \\ g_3 & g_2 & g_1 & e \end{bmatrix}.$$

We observe that its 2×2 corner blocks are Latin squares. Denote by $Z(K)$ the group ring of K over Z. A matrix L_k is then a generalized $Z(K)$-Latin square of order 4, where

$$s = w(L_k) = e + \sum_{i=1}^{3} g_i,$$

and at the same time we have

$$L_k = eP_1 + g_3P_2 + (-g_1 + g_2 + g_3)P_3 + g_3P_4 + (g_1 - g_3)P_7 + (g_1 - g_3)P_9.$$

R e m a r k. The matrix L_k is also a multiplication table of the symmetry group of the molecules H_2O and H_2O_2.

It is an open problem whether for $n > 6$ there exist systems of generators of $L_n(R)$ consisting of $(n-1)^2 + 1$ permutation matrices of order n.

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