ON A MODIFICATION OF AXIOMS OF GENERAL RELATIONS

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Abstract. Basic concepts concerning binary and ternary relations are extended to relations of arbitrary arities and then investigated.

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0. Introduction

The relations dealt with in the paper are considered in the general sense as systems of maps. More precisely, by a relation we understand a subset $R \subseteq G^H$, where G, H are sets and G^H denotes the set of all maps of H into G. G and H are called the carrier and the index set of R, respectively. Relations with well-ordered index sets, the so-called relations of type α, are studied in [8], while relations with general index sets are studied in [9], [10], [5], [6] and [11]. In this paper, the fundamental concepts concerning binary and ternary relations are extended to general relations and discussed.

We denote by $\mathbb{N}$ the set of all positive integers, for any $n \in \mathbb{N}$ we denote $(n] = \{m \in \mathbb{N};\ m \leqslant n\}$. In the case of a finite set H of cardinality k we will not distinguish between maps of the set H into the set G and k-tuples of elements of the set G. For any $n \in \mathbb{N}$ we denote by S_n the set of all permutations of the set $(n]$; id denotes the identical permutation of the set $(n]$.

For any map $f\colon H \to G$ and any subset $K \subseteq H$, we denote by $f|_K$ the restriction of f to K. The abbreviation w.r.t. will be written instead of the phrase "with respect to".

1.1. Definition. Let $n \in \mathbb{N}$, let H be a set. Then the pair $\mathcal{K} = (\{K_i\}_{i=1}^{n+1}, \{\varphi_i\}_{i=1}^{n-1})$ is called an n-decomposition of the set H if $\{K_i\}_{i=1}^{n+1}$ is a sequence of $n+1$ sets satisfying

(1) $\bigcup\limits_{i=1}^{n+1} K_i = H$,

(2) $K_i \cap K_j = \emptyset$ for all $i, j \in (n+1], i \neq j$,

(3) card $K_i =$ card K_j for all $i, j \in (n]$, and $\{\varphi_i\}_{i=1}^{n-1}$ is a sequence of $n-1$ bijections such that $\varphi_i \colon K_i \to K_{i+1}$ for all $i \in (n-1]$.

1.2. Remark. The concept of an n-decomposition is used here and in [5] in different meanings.

1.3. Definition. Let G, H be sets, let $\mathcal{K} = (\{K_i\}_{i=1}^{n+1}, \{\varphi_i\}_{i=1}^{n-1})$ be an n-decomposition of the set H. Then the relation

$$E_{\mathcal{K}} = \{f \in G^H;\ f|_{K_i} = f|_{K_{i+1}} \circ \varphi_i \quad \text{for all } i \in (n-1]\}$$

is called the diagonal w.r.t. $\mathcal{K}$.

1.4. Remark. Let $\mathcal{K} = (\{K_i\}_{i=1}^{n+1}, \{\varphi_i\}_{i=1}^{n-1})$ be an n-decomposition of the set H. If $K_{n+1} = H$ or $n = 1$, then, obviously, $E_{\mathcal{K}} = G^H$.

1.5. Definition. Let $R \subseteq G^H$ be a relation, let $\mathcal{K} = (\{K_i\}_{i=1}^{n+1}, \{\varphi_i\}_{i=1}^{n-1})$ be an n-decomposition of the set H, $\psi \in S_n$. Then we define the relation $R_{\mathcal{K},\psi} \subseteq G^H$ by $R_{\mathcal{K},\psi} = \{f \in G^H;\ \exists\, g \in R$:

$$\begin{aligned}
f|_{K_i} &= g|_{K_i} \text{ if } i \in (n],\ i = \psi(i) \text{ or } i = n+1,\\
f|_{K_i} &= g|_{K_{\psi(i)}} \circ \varphi_{\psi(i)-1} \circ \ldots \circ \varphi_i,\\
g|_{K_i} &= f|_{K_{\psi(i)}} \circ \varphi_{\psi(i)-1} \circ \ldots \circ \varphi_i \quad \text{if} \quad i \in (n], i < \psi(i),\\
f|_{K_{\psi(i)}} &= g|_{K_i} \circ \varphi_{i-1} \circ \ldots \circ \varphi_{\psi(i)},\\
g|_{K_{\psi(i)}} &= f|_{K_i} \circ \varphi_{i-1} \circ \ldots \circ \varphi_{\psi(i)} \quad \text{if} \quad i \in (n], i > \psi(i)\}.
\end{aligned}$$

Then $R_{\mathcal{K},\psi}$ is called the $(\mathcal{K}, \psi)$-modification of the relation R.

1.6. Remark. Let $R \subseteq G^H$ be a relation, let $\mathcal{K} = (\{K_i\}_{i=1}^{n+1}, \{\varphi_i\}_{i=1}^{n-1})$ be an n-decomposition of the set H, $\psi \in S_n$. Clearly, then

(1) $R_{\mathcal{K},\mathrm{id}} = R$,

(2) $\emptyset_{\mathcal{K},\psi} = \emptyset$.

1.7. E x a m p l e. Let $R \subseteq G^H$ be a relation, $H = \{1,2\}$ (i.e. R is binary), $\mathcal{K} = (\{K_i\}_{i=1}^3, \{\varphi_1\})$, $K_1 = \{1\}, K_2 = \{2\}$, let ψ be the permutation of the set (2] defined by $\psi(1) = 2, \psi(2) = 1$. Then $R_{\mathcal{K},\psi} = R^{-1}$. Hence, in this case, the $(\mathcal{K}, \psi)$-modification of a binary relation coincides with its standard inverse.

1.8. Definition. Let $R_1, \ldots, R_n \subseteq G^H$ be relations, $\mathcal{K} = (\{K_i\}_{i=1}^{n+1}, \{\varphi_i\}_{i=1}^{n-1})$ be an n-decomposition of the set H. Then we define the relation $(R_1 \ldots R_n)_{\mathcal{K}} \subseteq G^H$ by $(R_1 \ldots R_n)_{\mathcal{K}} = \{f \in G^H ;\ \exists\ f_i \in R_i$ for all $i \in (n]$ such that

$$\begin{aligned}
&f|_{K_i} = f_i|_{K_i} \quad \text{for all} \quad i \in (n], \\
&f|_{K_{n+1}} = f_i|_{K_{n+1}} \quad \text{for all} \quad i \in (n], \\
&f_i|_{K_j} \circ \varphi_{j-1} \circ \ldots \circ \varphi_i = f_j|_{K_i} \quad \text{for all} \quad i, j \in (n], i < j\}.
\end{aligned}$$

$(R_1 \ldots R_n)_{\mathcal{K}}$ is called the composition of $R_1, \ldots, R_n$ w.r.t. $\mathcal{K}$.

1.9. Definition. Let $R \subseteq G^H$ be a relation, let $\mathcal{K}$ be an n-decomposition of the set H. Then we put ${R_{\mathcal{K}}}^1 = R$, ${R_{\mathcal{K}}}^2 = (R \ldots R)_{\mathcal{K}}$, ${R_{\mathcal{K}}}^m = ({R_{\mathcal{K}}}^{m-1} R \ldots R)_{\mathcal{K}} \cup (R\ {R_{\mathcal{K}}}^{m-1} R \ldots R)_{\mathcal{K}} \cup \ldots \cup (R \ldots R\ {R_{\mathcal{K}}}^{m-1})_{\mathcal{K}}$ for any $m \in \mathbb{N}, m \geqslant 3$. $R_{\mathcal{K}}^m$ is called the m-th power of R w.r.t. $\mathcal{K}$.

1.10. E x a m p l e. Let $R_1, R_2 \subseteq G^H$ be relations, $H = \{1,2\}$ (i.e. R_1, R_2 are binary), $\mathcal{K} = (\{K_i\}_{i=1}^3, \{\varphi_1\})$, $K_1 = \{1\}$, $K_2 = \{2\}$. Then $(R_1R_2)_{\mathcal{K}} = R_1R_2$. Hence, in this case, the composition w.r.t. $\mathcal{K}$ coincides with the standard composition of binary relations.

1.11. R e m a r k. Let $R_1, \ldots, R_n \subseteq G^H$ be relations, let $\mathcal{K} = (\{K_i\}_{i=1}^{n+1}, \{\varphi\}_{i=1}^{n-1})$ be an n-decomposition of the set H. If $K_{n+1} = H$, $(R_1 \ldots R_n)_{\mathcal{K}} \neq \emptyset$, then, evidently, there exists an $f \in \bigcap\limits_{i=1}^{n} R_i$.

1.12. Notation. Let H be a set, let $\mathcal{K} = (\{K_i\}_{i=1}^{n+1}, \{\varphi_i\}_{i=1}^{n-1})$ be an n-decomposition of the set H. Then $\mathcal{K}^* = (\{K_i^*\}_{i=1}^{n+1}, \{\varphi_i^*\}_{i=1}^{n-1})$ is the n-decomposition of the set H defined by

$$K_i^* = \begin{cases} K_{i+1} & \text{for all} \quad i \in (n-1] \\ K_1 & \text{for} \quad i = n, \\ K_{n+1} & \text{for} \quad i = n+1, \end{cases}$$

$$\varphi_i^* = \begin{cases} \varphi_{i+1} & \text{for all} \quad i \in (n-2], \\ \varphi_1^{-1} \circ \ldots \circ \varphi_{n-1}^{-1} & \text{for} \quad i = n-1. \end{cases}$$

Further, for any $\psi \in S_n$, ψ^* denotes the permutation of $(n]$ defined by

$$\psi^*(i) = \begin{cases} \psi(i+1)-1 & \text{if} \quad i \in (n-1], \psi(i+1) \neq 1, \\ \psi(1)-1 & \text{if} \quad i = n, \psi(1) \neq 1 \\ n & \text{otherwise.} \end{cases}$$

1.13. Proposition. *Let $R, R_1, \ldots, R_n \subseteq G^H$ be relations, $\mathcal{K}$ an n-decomposition of H, let $\psi \in S_n, m \in \mathbb{N}$. Then*

(1) $\mathcal{K}\underbrace{^{*\ldots*}}_{n \text{ times}} = \mathcal{K}$.

(2) $E_\mathcal{K} = E_{\mathcal{K}^*}$.

(3) $R_{\mathcal{K},\psi} = R_{\mathcal{K}^*,\psi^*}$.

(4) $(R_1 \ldots R_n)_\mathcal{K} = (R_2 \ldots R_n R_1)_{\mathcal{K}^*}$.

(5) $R^m_\mathcal{K} = R^m_{\mathcal{K}^*}$.

Proof is obvious.

1.14. Definition. Let $R \subseteq G^H$ be a relation, let $\mathcal{K}$ be an n-decomposition of the set H, $\psi \in S_n$. Then we put $R^1_{\mathcal{K},\psi} = R_{\mathcal{K},\psi}$, $R^m_{\mathcal{K},\psi} = (R^{m-1}_{\mathcal{K},\psi})_{\mathcal{K},\psi}$ for any $m \in \mathbb{N}, m \geqslant 2$.

1.15. Remark. If $R \subseteq G^H$ is a relation, $\mathcal{K} = (\{\mathcal{K}_i\}_{i=1}^{n+1}, \{\varphi_i\}_{i=1}^{n-1})$ an n-decomposition of the set H, $\psi, \chi \in S_n$, then $(R_{\mathcal{K},\psi})_{\mathcal{K},\chi} = R_{\mathcal{K},\psi\circ\chi}$ need not hold in general.

If, for example, $n = 3, K_1 = \{1,2\}, K_2 = \{3,4\}, K_3 = \{5,6\}, K_4 = \emptyset$, $G = \{x,y,z\}$, $\varphi_1(1) = 3$, $\varphi_1(2) = 4$, $\varphi_2(3) = 5$, $\varphi_2(4) = 6$, $\psi(1) = 1$, $\psi(2) = 3$, $\psi(3) = 2$, $\chi(1) = 2$, $\chi(2) = 3$, $\chi(3) = 1$, $R = \{(x,y,z,x,y,z)\}$, then $R_{\mathcal{K},\psi} = \{(x,y,y,z,z,x)\}$, $(R_{\mathcal{K},\psi})_{\mathcal{K},\chi} = \emptyset$, while $R_{\mathcal{K},\psi\circ\chi} = \{(y,z,z,x,x,y)\}$.

1.16. Proposition. *Let J be a nonempty set, let $R, R_1, \ldots, R_1, R'_n, \ldots, R'_n, T, T_j$ for all $j \in J$ be relations with the carrier G and the index set H. Let $\mathcal{K}$ be an n-decomposition of the set H, $\psi \in S_n$. Let $k \in (n], m \in \mathbb{N}$. Then*

(1) $E_\mathcal{K} = (E_\mathcal{K})_{\mathcal{K},\psi} = (E_\mathcal{K})^2_\mathcal{K}$.

(2) $(E_\mathcal{K} \ldots E_\mathcal{K} \underset{\uparrow\, k\text{-th place}}{R} E_\mathcal{K} \ldots E_\mathcal{K})_\mathcal{K} \subseteq R$.

(3) *If $R \subseteq E_\mathcal{K}$, then (2) becomes the equality.*

(4) *$R \subseteq T$ implies $R_{\mathcal{K},\psi} \subseteq T_{\mathcal{K},\psi}$.*

(5) $(\bigcup_{j\in J} T_j)_{\mathcal{K},\psi} = \bigcup_{j\in J} (T_j)_{\mathcal{K},\psi}$.

(6) $(\bigcap_{j\in J} T_j)_{\mathcal{K},\psi} = \bigcap_{j\in J} (T_j)_{\mathcal{K},\psi}$.

(7) *$R_i \subseteq R'_i$ for all $i \in (n]$ imply $(R_1 \ldots R_n)_\mathcal{K} \subseteq (R'_1 \ldots R'_n)_\mathcal{K}$.*

(8) *$R \subseteq T$ implies $R^m_\mathcal{K} \subseteq T^m_\mathcal{K}$.*

P r o o f. The assertions follow directly from the definitions of the operations. For example, let us prove (2) and (3). Suppose that $\mathcal{K} = (\{\mathcal{K}_i\}_{i=1}^{n+1}, \{\varphi_i\}_{i=1}^{n-1})$.

(2) Let $f \in (\underset{\uparrow\ k\text{-th place}}{E_{\mathcal{K}} \dots E_{\mathcal{K}} R E_{\mathcal{K}} \dots E_{\mathcal{K}}})_{\mathcal{K}}$. Then there exist $f_i \in E_{\mathcal{K}}$ for all $i \in (n], i \neq k$, and an $f_k \in R$ such that

$$\begin{aligned} &f|_{K_i} = f_i|_{K_i} \quad \text{for all} \quad i \in (n], \\ &f|_{K_{n+1}} = f_i|_{K_{n+1}} \quad \text{for all} \quad i \in (n], \\ &f_i|_{K_j} \circ \varphi_{j-1} \circ \dots \circ \varphi_i = f_j|_{K_i} \quad \text{for all} \quad i, j \in (n], i < j. \end{aligned}$$

We have $f|_{K_k} = f_k|_{K_k}, f|_{K_{n+1}} = f_k|_{K_{n+1}}$. Let $i \in (n], i < k$. Then $f|_{K_i} = f_i|_{K_i} = f_i|_{K_k} \circ \varphi_{k-1} \circ \dots \circ \varphi_i = f_k|_{K_i}$. Let $i \in (n], i > k$. Then $f|_{K_i} = f_i|_{K_i}$, hence $f|_{K_i} \circ \varphi_{i-1} \circ \dots \circ \varphi_k = f_i|_{K_i} \circ \varphi_{i-1} \circ \dots \circ \varphi_k = f_i|_{K_k} = f_k|_{K_i} \circ \varphi_{i-1} \circ \dots \circ \varphi_k$. Thus, again, $f|_{K_i} = f_k|_{K_i}$. We obtain $f = f_k \in R$.

(3) Let $f \in R \subseteq E_{\mathcal{K}}$. Put $f_k = f, f_i|_{K_i} = f|_{K_i}, f_i|_{K_{n+1}} = f|_{K_{n+1}}$ for all $i \in (n]$. Further, put

$$f_i|_{K_j} = \begin{cases} f|_{K_i} \circ \varphi_{i-1} \circ \dots \circ \varphi_j & \text{for all} \quad i, j \in (n], i > j, \\ f|_{K_i} \circ \varphi_i^{-1} \circ \dots \circ \varphi_{j-1}^{-1} & \text{for all} \quad i, j \in (n], i < j. \end{cases}$$

Then $f_i \in E_{\mathcal{K}}$ for all $i \in (n]$ and $f_k \in R$. For any $i, j \in (n], i < j$, we have

$$f_i|_{K_j} \circ \varphi_{j-1} \circ \dots \circ \varphi_i = f|_{K_i} = f|_{K_j} \circ \varphi_{j-1} \circ \dots \circ \varphi_i = f_j|_{K_i},$$

so that

$$f \in (\underset{\uparrow\ k\text{-th place}}{E_{\mathcal{K}} \dots E_{\mathcal{K}} R E_{\mathcal{K}} \dots E_{\mathcal{K}}})_{\mathcal{K}}.$$

1.17. R e m a r k. In 1.16, part (2), the inclusion cannot be replaced by the equality unless $R \subseteq E_{\mathcal{K}}$. If, for example, $n = 3$, $K_1 = \{1, 2\}$, $K_2 = \{3, 4\}$, $K_3 = \{5, 6\}$, $K_4 = \emptyset$, $G = \{x, y\}$, $\varphi_1(1) = 3$, $\varphi_1(2) = 4$, $\varphi_2(3) = 5$, $\varphi_2(4) = 6$, $R = \{(x, x, x, x, y, x)\}$, then $(x, x, x, x, y, x) \notin (E_{\mathcal{K}} R\ E_{\mathcal{K}})_{\mathcal{K}}$.

1.18. Definition. Let $R \subseteq G^H$ be a relation, $\mathcal{K}$ an n-decomposition of the set H, let $\psi \in S_n$ be the permutation defined by

$$\pi(i) = \begin{cases} i + 1 & \text{for all} \quad i \in (n-1], \\ 1 & \text{for} \quad i = n. \end{cases}$$

Then we define ${}^1R_{\mathcal{K}} = R_{\mathcal{K},\pi}$, ${}^mR_{\mathcal{K}} = {}^1({}^{m-1}R_{\mathcal{K}})_{\mathcal{K}}$ for any $m \in \mathbb{N}$, $m \geqslant 2$. ${}^mR_{\mathcal{K}}$ is called the m-th cyclic transposition of R w.r.t. $\mathcal{K}$.

1.19. Proposition. *Let $R \subseteq G^H$ be a relation, $\mathcal{K}$ an n-decomposition of the set H. Then*
(1) ${}^1R_{\mathcal{K}} = {}^1R_{\mathcal{K}^*}$.
(2) $E_{\mathcal{K}} = {}^1(E_{\mathcal{K}})_{\mathcal{K}}$.

P r o o f. (1) follows from the fact that $\pi^* = \pi$. (2) follows from 1.16 (1). $\square$

1.20. Proposition. *Let J be a nonempty set, R, T, T_j for all $j \in J$ relations with the carrier G and the index set H. Let $\mathcal{K}$ be an n-decomposition of the set H. Then*
(1) $R \subseteq T$ *implies* ${}^1R_{\mathcal{K}} \subseteq {}^1T_{\mathcal{K}}$.
(2) ${}^1(\bigcup\limits_{j \in J} T_j)_{\mathcal{K}} = \bigcup\limits_{j \in J} {}^1(T_j)_{\mathcal{K}}$.
(3) ${}^1(\bigcap\limits_{j \in J} T_j)_{\mathcal{K}} = \bigcap\limits_{j \in J} {}^1(T_j)_{\mathcal{K}}$.

P r o o f. The assertions follow from 1.16 (4), (5), and (6). $\square$

2. Properties of relations

2.1. Definition. Let $R \subseteq G^H$ be a relation, $\mathcal{K} = (\{K_i\}_{i=1}^{n+1}, \{\varphi_i\}_{i=1}^{n-1})$ an n-decomposition of the set H, $\psi \in S_n$. Then R is called

(1) reflexive (irreflexive) w.r.t. $\mathcal{K}$ if $E_{\mathcal{K}} \subseteq R$ ($R \cap E_{\mathcal{K}} = \emptyset$),
(2) symmetric (assymmetric, antisymmetric) w.r.t. $\mathcal{K}$ and ψ if $R_{\mathcal{K},\psi} \subseteq R$ ($R \cap R_{\mathcal{K},\psi} = \emptyset$, $R \cap R_{\mathcal{K},\psi} \subseteq E_{\mathcal{K}}$),
(3) transitive (atransitive) w.r.t. $\mathcal{K}$ if $R^2_{\mathcal{K}} \subseteq R$ ($R \cap R^m_{\mathcal{K}} = \emptyset$ for any $m \in \mathbb{N}, m \geqslant 2$),
(4) complete w.r.t. $\mathcal{K}$ if $f \in G^H$, $f|_{K_i} \neq f|_{K_j} \circ \varphi_{j-1} \circ \ldots \circ \varphi_i$ for all $i, j \in (n]$, $i < j$ imply the existence of a $\chi \in S_n$ such that $f \in R_{\mathcal{K},\chi}$.

2.2. Proposition. *Let J be a nonempty set, $j_0 \in J$. Let $R, R_1, \ldots, R_n, T_j$ for all $j \in J$ be relations with the carrier G and the index set H. Let $\mathcal{K}$ be an n-decomposition of the set H, $\psi \in S_n$. Then*
(1) *If T_{j_0} is reflexive w.r.t. $\mathcal{K}$, then $\bigcup\limits_{j \in J} T_j$ is reflexive w.r.t. $\mathcal{K}$.*
(2) *If $R, R_1, \ldots, R_n$ and T_j for all $j \in J$ are reflexive w.r.t. $\mathcal{K}$, then $\bigcap\limits_{j \in J} T_j, R_{\mathcal{K},\psi}$ and $(R_1 \ldots R_n)_{\mathcal{K}}$ are reflexive w.r.t. $\mathcal{K}$.*
(3) *If R and T_j for all $j \in J$ are irreflexive (symmetric) w.r.t. $\mathcal{K}$ (and ψ), then $\bigcup\limits_{j \in J} T_j$, $\bigcap\limits_{j \in J} T_j$ and $R_{\mathcal{K},\psi}$ have the same property.*
(4) *If T_j for all $j \in J$ are transitive w.r.t. $\mathcal{K}$, then $\bigcap\limits_{j \in J} T_j$ is transitive w.r.t. $\mathcal{K}$.*

(5) *If T_{j_0} is atransitive (assymmetric, antisymmetric) w.r.t. $\mathcal{K}$ (and ψ), then $\bigcap_{j\in J} T_j$ has the same property.*

(6) *If R is asymmetric (antisymmetric) w.r.t. $\mathcal{K}$ and ψ, then $R_{\mathcal{K},\psi}$ has the same property.*

(7) *If T_{j_0} is complete w.r.t. $\mathcal{K}$, then $\bigcup_{j\in J} T_j$ is complete w.r.t. $\mathcal{K}$.*

Proof. The assertion (1) is evident, the others follow from 1.6 (2), 1.16 (1), (4)–(6), and (8). □

2.3. Remark. Let $R \subseteq G^H$ be a relation, $\mathcal{K}$ an n-decomposition of H, let $\psi \in S_n$. It can be easily obtained from 2.2 (3) by induction that if R is symmetric w.r.t. $\mathcal{K}$ and ψ, then $R^{m+1}_{\mathcal{K},\psi} \subseteq R^{m}_{\mathcal{K},\psi}$ for any $m \in \mathbb{N}$.

2.4. Proposition. *Let $R \subseteq G^H$ be a relation, $\mathcal{K}$ an n-decomposition of the set H, let $\psi \in S_n$. Then:*

(1) *If R is reflexive (irreflexive, transitive, atransitive, complete) w.r.t. $\mathcal{K}$, then it has the same property w.r.t. $\mathcal{K}^*$.*

(2) *If R is symmetric (asymmetric, antisymmetric) w.r.t. $\mathcal{K}$ and ψ, then it has the same property w.r.t. $\mathcal{K}^*$ and ψ^*.*

Proof. The assertions follow from 1.13 (2), (3), and (5). □

2.5. Definition. Let $R \subseteq G^H$ be a relation, $\mathcal{K}$ an n-decomposition of the set H. Then R is called

(1) cyclic (acyclic, anticyclic) w.r.t. $\mathcal{K}$ if it is symmetric (asymmetric, antisymmetric) w.r.t. $\mathcal{K}$ and π,

(2) symmetric (asymmetric, antisymmetric) w.r.t. $\mathcal{K}$ if it is symmetric w.r.t. $\mathcal{K}$ and ψ for any $\psi \in S_n$ (asymmetric, antisymmetric w.r.t. $\mathcal{K}$ and ψ for any odd permutation $\psi \in S_n$).

2.6. Proposition. *Let J be a nonempty set, $j_0 \in J$. Let R, T_j for all $j \in J$ be relations with the carrier G and the index set H. Let $\mathcal{K}$ be an n-decomposition of the set H, $\psi \in S_n$. Then:*

(1) *If R and T_j for all $j \in J$ are cyclic w.r.t. $\mathcal{K}$, then $\bigcup_{j\in J} T_j$, $\bigcap_{j\in J} T_j$ and ${}^1R_{\mathcal{K}}$ are cyclic w.r.t. $\mathcal{K}$.*

(2) *If T_j for all $j \in J$ are symmetric w.r.t. $\mathcal{K}$, then $\bigcup_{j\in J} T_j$ and $\bigcap_{j\in J} T_j$ are symmetric w.r.t. $\mathcal{K}$.*

(3) *If R and T_{j_0} are acyclic (anticyclic) w.r.t. $\mathcal{K}$, then $\bigcap_{j\in J} T_j$ and ${}^1R_{\mathcal{K}}$ have the same property.*

(4) *If T_{j_0} is asymmetric (antisymmetric) w.r.t. $\mathcal{K}$, then $\bigcap_{j\in J} T_j$ has the same property.*

(5) *If R is complete w.r.t. $\mathcal{K}$, then ${}^1R_{\mathcal{K}}$ is complete w.r.t. $\mathcal{K}$.*

Proof. The assertions follow from 2.2 (3), (5), and (6). $\square$

2.7. Remark. Let $R \subseteq G^H$ be a relation, $\mathcal{K}$ an n-decomposition of the set H. Putting $\psi = \pi$ in 2.3, we obtain that if R is cyclic w.r.t. $\mathcal{K}$, then ${}^{m+1}R_{\mathcal{K}} \subseteq {}^{m}R_{\mathcal{K}}$ for any $m \in \mathbb{N}$.

2.8. Proposition. *Let $R \subseteq G^H$ be a relation, $\mathcal{K}$ an n-decomposition of the set H. If R has any of the properties defined in 2.5 w.r.t. $\mathcal{K}$, then it has the same property w.r.t. $\mathcal{K}^*$.*

Proof. The proposition follows from 2.4 (2) and from the facts that $\pi^* = \pi$ and $\{\psi^*;\ \psi \in S_n\} = S_n$. $\square$

3. Hulls of relations

3.1. Definition. Let $R \subseteq G^H$ be a relation, $\mathcal{K}$ an n-decomposition of the set H, $\psi \in S_n$. Let (p) be any of the properties defined in 2.1 or 2.5. A relation $Q \subseteq G^H$ is called the (p)-hull of R w.r.t. $\mathcal{K}$ (and ψ) if

(1) $R \subseteq Q$,
(2) Q has the property (p),
(3) if $T \subseteq G^H$ is any relation having the property (p) and such that $R \subseteq T$, then $Q \subseteq T$.

3.2. Remark. Let $R \subseteq G^H$ be a relation, $\mathcal{K}$ an n-decomposition of the set H, $\psi \in S_n$. Let (p) be any of the properties defined in 2.1 or 2.5. Obviously, then R has the property (p) w.r.t. $\mathcal{K}$ (and ψ) if and only if the (p)-hull Q of R w.r.t. $\mathcal{K}$ (and ψ) exists and $R = Q$.

3.3. Proposition. *Let $R, T \subseteq G^H$ be relations, $\mathcal{K}$ an n-decomposition of the set H, $\psi \in S_n$. Let (p) be any of the properties defined in 2.1 or 2.5, $R^{(p)}_{\mathcal{K}(,\psi)}$ ($T^{(p)}_{\mathcal{K}(,\psi)}$) the (p)-hull of $R(T)$ w.r.t. $\mathcal{K}$ (and ψ). Then $R \subseteq T$ implies $R^{(p)}_{\mathcal{K}(,\psi)} \subseteq T^{(p)}_{\mathcal{K}(,\psi)}$.*

Proof. Let $R \subseteq T$. We have $T \subseteq T^{(p)}_{\mathcal{K}(,\psi)}$. Thus $R \subseteq T^{(p)}_{\mathcal{K}(,\psi)}$. As $T^{(p)}_{\mathcal{K}(,\psi)}$ has the property (p), we obtain $R^{(p)}_{\mathcal{K}(,\psi)} \subseteq T^{(p)}_{\mathcal{K}(,\psi)}$. $\square$

3.4. Definition. Let $R \subseteq G^H$ be a relation, $\mathcal{K}$ an n-decomposition of the set H. Then we define ${}_1R_{\mathcal{K}} = R$, ${}_mR_{\mathcal{K}} =_{m-1} R_{\mathcal{K}} \cup ({}_{m-1}R_{\mathcal{K}})^2_{\mathcal{K}}$ for any $m \in \mathbb{N}, m \geqslant 2$.

3.5. Remark. Let $R \subseteq G^H$ be a relation, $\mathcal{K}$ an n-decomposition of the set H. Clearly, then ${}_mR_{\mathcal{K}} \subseteq {}_{m+1}R_{\mathcal{K}}$ for any $m \in \mathbb{N}$.

3.6. Proposition. *Let $R \subseteq G^H$ be a relation, $\mathcal{K}$ an n-decomposition of the set H. Let $\psi \in S_n$. Then the following relations exist:*
(1) *the reflexive hull $R^{(r)}_{\mathcal{K}}$ of R w.r.t. $\mathcal{K}$ and we have $R^{(r)}_{\mathcal{K}} = R \cup E_{\mathcal{K}}$,*
(2) *the symmetric hull $R^{(s)}_{\mathcal{K},\psi}$ of R w.r.t. $\mathcal{K}$ and ψ and we have $R^{(s)}_{\mathcal{K},\psi} = R \cup \bigcup\limits_{i=1}^{\infty} R^i_{\mathcal{K},\psi}$,*
(3) *the transitive hull $R^{(t)}$ of R w.r.t. $\mathcal{K}$ and we have $R^{(t)}_{\mathcal{K}} = \bigcup\limits_{i=1}^{\infty} {}_iR_{\mathcal{K}}$.*

Proof. (1) is evident.

(2) Put $Q = R \cup \bigcup\limits_{i=1}^{\infty} R^i_{\mathcal{K},\psi}$. Clearly, then $R \subseteq Q$. We have $Q_{\mathcal{K},\psi} = (R \cup \bigcup\limits_{i=1}^{\infty} R^i_{\mathcal{K},\psi})_{\mathcal{K},\psi} = R_{\mathcal{K},\psi} \cup \bigcup\limits_{i=1}^{\infty} R^{i+1}_{\mathcal{K},\psi} = \bigcup\limits_{i=1}^{\infty} R^i_{\mathcal{K},\psi} \subseteq Q$ by 1.16 (5) and Q is symmetric w.r.t. $\mathcal{K}$ and ψ. Further, let $T \subseteq G^H$ be symmetric w.r.t. $\mathcal{K}$ and ψ and let $R \subseteq T$. By virtue of 1.16 (4) and using induction we obtain $Q = R \cup \bigcup\limits_{i=1}^{\infty} R^i_{\mathcal{K},\psi} \subseteq T \cup \bigcup\limits_{i=1}^{\infty} T^i_{\mathcal{K},\psi} \subseteq T$ due to 2.3.

(3) Put $Q = \bigcup\limits_{i=1}^{\infty} {}_iR_{\mathcal{K}}$. Clearly $R = {}_1R_{\mathcal{K}} \subseteq Q$. Let $f \in Q^2_{\mathcal{K}}$. Then there exists an $f_i \in Q$ for each $i \in (n]$ such that $f|_{K_i} = f_i|_{K_i}$ for each $i \in (n]$, $f|_{K_{n+1}} = f_i|_{K_{n+1}}$ for each $i \in (n]$, $f_i|_{K_j} \circ \varphi_{j-1} \circ \ldots \circ \varphi_i = f_j|_{K_i}$ for each $i, j \in (n]$, $i < j$. For each $i \in (n]$ there exists a $j_i \in \mathbb{N}$ such that $f_i \in {}_{j_i}R_{\mathcal{K}}$. Hence it follows that $f \in ({}_{j_1}R_{\mathcal{K}} \ldots {}_{j_n}R_{\mathcal{K}})_{\mathcal{K}}$. Denote $j_0 = \max\{j_1, \ldots, j_n\}$. By 3.5, we have ${}_{j_i}R_{\mathcal{K}} \subseteq_{j_0} R_{\mathcal{K}}$ for all $i \in (n]$. By 1.16 (7), $f \in ({}_{j_0}R_{\mathcal{K}} \ldots {}_{j_0}R_{\mathcal{K}})_{\mathcal{K}} =_{j_0} R^2_{\mathcal{K}} \subseteq {}_{j_0+1}R_{\mathcal{K}} \subseteq \bigcup\limits_{i=1}^{\infty} {}_iR_{\mathcal{K}} = Q$. Thus $Q^2_{\mathcal{K}} \subseteq Q$ and Q is transitive w.r.t. $\mathcal{K}$. Let $T \subseteq G^H$ be transitive w.r.t. $\mathcal{K}$ and such that $R \subseteq T$. It is easy to prove by induction that ${}_iR_{\mathcal{K}} \subseteq T$ for any $i \in \mathbb{N}$. Hence $Q = \bigcup\limits_{i=1}^{\infty} {}_iR_{\mathcal{K}} \subseteq \bigcup\limits_{i=1}^{\infty} T = T$ and we have $R^{(t)}_{\mathcal{K}} = Q$. □

3.7. Proposition. *Let $R \subseteq G^H$ be a relation, $\mathcal{K}$ an n-decomposition of the set H, $\psi \in S_n$. Then:*
(1) *If R is complete (symmetric, antisymmetric) w.r.t. $\mathcal{K}$ (and ψ), then $R^{(r)}_{\mathcal{K}}$ has the same property.*
(2) *If $n \leqslant 2$ and R is transitive w.r.t. $\mathcal{K}$, then $R^{(r)}_{\mathcal{K}}$ is transitive w.r.t. $\mathcal{K}$.*
(3) *If R is reflexive (irreflexive, complete) w.r.t. $\mathcal{K}$, then $R^{(s)}_{\mathcal{K},\psi}$ has the same property.*
(4) *If R is reflexive (complete) w.r.t. $\mathcal{K}$, then $R^{(t)}_{\mathcal{K}}$ has the same property.*

Proof. (1) follows from 1.16 (1), (5), 2.2 (3), (7), and 3.6 (1).

(2) Let $n \leqslant 2$ and let R be transitive w.r.t. $\mathcal{K}$. Then $R^2_{\mathcal{K}} \subseteq R$. The case of $n = 1$ is trivial. Let $n = 2$. Let $f \in (R^{(r)}_{\mathcal{K}})^2_{\mathcal{K}} = (R \cup E_{\mathcal{K}})^2_{\mathcal{K}}$ (by 3.6 (1)). Then there exist

$f_1, f_2 \in R \cup E_{\mathcal{K}}$ such that $f|_{K_1} = f_1|_{K_1}, f|_{K_2} = f_2|_{K_2}, f|_{K_3} = f_1|_{K_3} = f_2|_{K_3}, f_1|_{K_2} \circ \varphi_1 = f_2|_{K_1}$. If $f_1, f_2 \in R$, then $f \in (R\,R)_{\mathcal{K}} = R^2_{\mathcal{K}} \subseteq R \subseteq R^{(r)}_{\mathcal{K}}$. If $f_1, f_2 \in E_{\mathcal{K}}$, then, by 1.16 (1), $f \in (E_{\mathcal{K}} E_{\mathcal{K}})_{\mathcal{K}} = (E_{\mathcal{K}})^2_{\mathcal{K}} = E_{\mathcal{K}} \subseteq R^{(r)}_{\mathcal{K}}$. If $f_1 \in R, f_2 \in E_{\mathcal{K}}$, then $f|_{K_1} = f_1|_{K_1}, f|_{K_2} = f_2|_{K_2} = f_2|_{K_1} \circ \varphi_1^{-1} = f_1|_{K_2}, f|_{K_3} = f_1|_{K_3}$. Hence $f = f_1 \in R \subseteq R^{(r)}_{\mathcal{K}}$. The case of $f_1 \in E_{\mathcal{K}}, f_2 \in R$ is analogous. Thus $(R^{(r)}_{\mathcal{K}})^2_{\mathcal{K}} \subseteq R^{(r)}_{\mathcal{K}}$ and $R^{(r)}_{\mathcal{K}}$ is transitive w.r.t. $\mathcal{K}$.

(3) and (4) follow from 1.14, 1.16 (1), (2), (4), (6), 3.1 (1), 3.4, and 3.6 (2), (3). □

3.8. Corollary. *Let $R \subseteq G^H$ be a relation, $\mathcal{K}$ an n-decomposition of the set H, $\psi \in S_n$. Then*

(1) $(R^{(r)}_{\mathcal{K}})^{(s)}_{\mathcal{K},\psi} = (R^{(s)}_{\mathcal{K},\psi})^{(r)}_{\mathcal{K}}$.

(2) $(R^{(t)}_{\mathcal{K}})^{(r)}_{\mathcal{K}} \subseteq (R^{(r)}_{\mathcal{K}})^{(t)}_{\mathcal{K}}$.

(3) *If $n \leqslant 2$, then* $(R^{(t)}_{\mathcal{K}})^{(r)}_{\mathcal{K}} = (R^{(r)}_{\mathcal{K}})^{(t)}_{\mathcal{K}}$.

Proof. (1) As $R \subseteq R^{(s)}_{\mathcal{K},\psi}$, we have, by 3.3, $R^{(r)}_{\mathcal{K}} \subseteq (R^{(s)}_{\mathcal{K},\psi})^{(r)}_{\mathcal{K}}$, and again by 3.3, $(R^{(r)}_{\mathcal{K}})^{(s)}_{\mathcal{K},\psi} \subseteq ((R^{(s)}_{\mathcal{K},\psi})^{(r)}_{\mathcal{K}})^{(s)}_{\mathcal{K},\psi}$. By 3.7 (1), $(R^{(s)}_{\mathcal{K},\psi})^{(r)}_{\mathcal{K}}$ is symmetric w.r.t. $\mathcal{K}$ and ψ, consequently, by 3.2, $((R^{(s)}_{\mathcal{K},\psi})^{(r)}_{\mathcal{K}})^{(s)}_{\mathcal{K},\psi} = (R^{(s)}_{\mathcal{K},\psi})^{(r)}_{\mathcal{K}}$. Thus $(R^{(r)}_{\mathcal{K}})^{(s)}_{\mathcal{K},\psi} \subseteq (R^{(s)}_{\mathcal{K},\psi})^{(r)}_{\mathcal{K}}$. As $R \subseteq R^{(r)}_{\mathcal{K}}$, we have, by 3.3, $R^{(s)}_{\mathcal{K},\psi} \subseteq (R^{(r)}_{\mathcal{K}})^{(s)}_{\mathcal{K},\psi}$, and again by 3.3, $(R^{(s)}_{\mathcal{K},\psi})^{(r)}_{\mathcal{K}} \subseteq ((R^{(r)}_{\mathcal{K}})^{(s)}_{\mathcal{K},\psi})^{(r)}_{\mathcal{K}}$. By 3.7 (3), $(R^{(r)}_{\mathcal{K}})^{(s)}_{\mathcal{K},\psi}$ is reflexive w.r.t. $\mathcal{K}$, consequently, by 3.2, $((R^{(r)}_{\mathcal{K}})^{(s)}_{\mathcal{K},\psi})^{(r)}_{\mathcal{K}} = (R^{(r)}_{\mathcal{K}})^{(s)}_{\mathcal{K},\psi}$. Thus $(R^{(s)}_{\mathcal{K},\psi})^{(r)}_{\mathcal{K}} \subseteq (R^{(r)}_{\mathcal{K}})^{(s)}_{\mathcal{K},\psi}$. Combining the two results, we obtain $(R^{(r)}_{\mathcal{K}})^{(s)}_{\mathcal{K},\psi} = (R^{(s)}_{\mathcal{K},\psi})^{(r)}_{\mathcal{K}}$.

(2) and (3) follow analogously from 3.3, 3.7 (4), (2), and 3.2. □

3.9. Remark. The inclusion in 3.8 (2) cannot, in general, be replaced by equality. If, for example, $n = 3$, $K_1 = \{1,2\}$, $K_2 = \{3,4\}$, $K_3 = \{5,6\}$, $K_4 = \emptyset$, $G = \{x,y\}$, $\varphi_1(1) = 3$, $\varphi_1(2) = 4$, $\varphi_2(3) = 5$, $\varphi_2(4) = 6$, $R = \{(x,y,x,x,x,y), (x,y,x,y,y,x)\}$, then $(x,y,x,y,x,y) \in E_{\mathcal{K}} \subseteq R^{(r)}_{\mathcal{K}}$, $(x,y,x,x,x,y) \in R \subseteq R^{(r)}_{\mathcal{K}}$, $(x,y,x,y,y,x) \in R \subseteq R^{(r)}_{\mathcal{K}}$, hence $(x,y,x,x,y,x) \in (R^{(r)}_{\mathcal{K}})^2_{\mathcal{K}} \subseteq (R^{(r)}_{\mathcal{K}})^{(t)}_{\mathcal{K}}$, but $R^2_{\mathcal{K}} = \emptyset$, consequently $R^{(t)}_{\mathcal{K}} = R$, and $(x,y,x,x,y,x) \notin R \cup E_{\mathcal{K}} = R^{(r)}_{\mathcal{K}} = (R^{(t)}_{\mathcal{K}})^{(r)}_{\mathcal{K}}$.

3.10. Corollary. *Let $R \subseteq G^H$ be a relation, $\mathcal{K}$ an n-decomposition of the set H. Then* $(R^{(r)}_{\mathcal{K}})^{(t)}_{\mathcal{K}} = ((R^{(t)}_{\mathcal{K}})^{(r)}_{\mathcal{K}})^{(t)}_{\mathcal{K}}$.

Proof. Similarly as in the proof of 3.8 (1) we get $(R^{(r)}_{\mathcal{K}})^{(t)}_{\mathcal{K}} \subseteq ((R^{(t)}_{\mathcal{K}})^{(r)}_{\mathcal{K}})^{(t)}_{\mathcal{K}}$. By 3.8 (2), $(R^{(t)}_{\mathcal{K}})^{(r)}_{\mathcal{K}} \subseteq (R^{(r)}_{\mathcal{K}})^{(t)}_{\mathcal{K}}$, consequently, by 3.3 and 3.2, $((R^{(t)}_{\mathcal{K}})^{(r)}_{\mathcal{K}})^{(t)}_{\mathcal{K}} \subseteq ((R^{(r)}_{\mathcal{K}})^{(t)}_{\mathcal{K}})^{(t)}_{\mathcal{K}} = (R^{(r)}_{\mathcal{K}})^{(t)}_{\mathcal{K}}$. Thus, $(R^{(r)}_{\mathcal{K}})^{(t)}_{\mathcal{K}} = ((R^{(t)}_{\mathcal{K}})^{(r)}_{\mathcal{K}})^{(t)}_{\mathcal{K}}$. □

3.11. Proposition. *Let $R \subseteq G^H$ be a relation, $\mathcal{K}$ an n-decomposition of the set H. Then the following relations exist:*

(1) *the cyclic hull* $R^{(c)}_{\mathcal{K}}$ *of* R *w.r.t.* $\mathcal{K}$ *and we have* $R^{(c)}_{\mathcal{K}} = R \cup \bigcup\limits_{i=1}^{\infty} {}^{i}R_{\mathcal{K}}$,

(2) *the symmetric hull* $R^{(d)}_{\mathcal{K}}$ *of* R *w.r.t.* $\mathcal{K}$ *and we have*

$$R^{(d)}_{\mathcal{K}} = \bigcup_{i=1}^{\infty} \bigcup_{\psi_1,\psi_2,\ldots,\psi_i \in S_n} (\ldots(R_{\mathcal{K},\psi_1})_{\mathcal{K},\psi_2}\ldots)_{\mathcal{K},\psi_i}.$$

Proof. (1) As $R^{(c)}_{\mathcal{K}} = R^{(s)}_{\mathcal{K},\pi}$, we have, by 3.6 (2), $R^{(c)}_{\mathcal{K}} = R \cup \bigcup\limits_{i=1}^{\infty} R^{i}_{\mathcal{K},\pi} = R \cup \bigcup\limits_{i=1}^{\infty} {}^{i}R_{\mathcal{K}}$.

(2) Put $Q = \bigcup\limits_{i=1}^{\infty} \bigcup\limits_{\psi_1,\psi_2,\ldots,\psi_i \in S_n} (\ldots(R_{\mathcal{K},\psi_1})_{\mathcal{K},\psi_2}\ldots)_{\mathcal{K},\psi_i}$. By 1.6 (1), we have $R = R_{\mathcal{K},\mathrm{id}} \subseteq Q$. Let $\xi \in S_n$.

By Proposition 1.16 (5), $Q_{\mathcal{K},\xi} = \big(\bigcup\limits_{i=1}^{\infty} \bigcup\limits_{\psi_1,\psi_2,\ldots,\psi_i \in S_n} (\ldots(R_{\mathcal{K},\psi_1})_{\mathcal{K},\psi_2}\ldots)_{\mathcal{K},\psi_i}\big)_{\mathcal{K},\xi} = \bigcup\limits_{i=1}^{\infty} \bigcup\limits_{\psi_1,\psi_2,\ldots,\psi_i \in S_n} ((\ldots(R_{\mathcal{K},\psi_1})_{\mathcal{K},\psi_2}\ldots)_{\mathcal{K},\psi_i})_{\mathcal{K},\xi} \subseteq Q$, and Q is symmetric w.r.t. $\mathcal{K}$. Now, let $R \subseteq T$ where T is symmetric w.r.t. $\mathcal{K}$. Then, by 1.16 (4),

$$\begin{aligned} Q &= \bigcup_{i=1}^{\infty} \bigcup_{\psi_1,\psi_2,\ldots,\psi_i \in S_n} (\ldots(R_{\mathcal{K},\psi_1})_{\mathcal{K},\psi_2}\ldots)_{\mathcal{K},\psi_i} \\ &\subseteq \bigcup_{i=1}^{\infty} \bigcup_{\psi_1,\psi_2,\ldots,\psi_i \in S_n} (\ldots(T_{\mathcal{K},\psi_1})_{\mathcal{K},\psi_2}\ldots)_{\mathcal{K},\psi_i} \subseteq T. \end{aligned}$$

Hence Q is the symmetric hull of R w.r.t. $\mathcal{K}$. □

3.12. Proposition. *Let* $R \subseteq G^H$ *be a relation, let* $\mathcal{K}$ *be an* n*-decomposition of the set* H*.*

(1) *If* R *is reflexive (irreflexive, complete) w.r.t.* $\mathcal{K}$*, then* $R^{(c)}_{\mathcal{K}}$ *and* $R^{(d)}_{\mathcal{K}}$ *have the same property.*

(2) *If* R *is symmetric (antisymmetric) w.r.t.* $\mathcal{K}$*, then* $R^{(r)}_{\mathcal{K}}$ *has the same property.*

Proof. Let R be reflexive w.r.t. $\mathcal{K}$. Then $E_{\mathcal{K}} \subseteq R$. But $R \subseteq R^{(c)}_{\mathcal{K}}, R \subseteq R^{(d)}_{\mathcal{K}}$, hence $E_{\mathcal{K}} \subseteq R^{(c)}, E_{\mathcal{K}} \subseteq R^{(d)}_{\mathcal{K}}$, and both $R^{(c)}_{\mathcal{K}}$ and $R^{(d)}_{\mathcal{K}}$ are reflexive w.r.t. $\mathcal{K}$. Let R be irreflexive w.r.t. $\mathcal{K}$. By 2.2 (3), ${}^{1}R_{\mathcal{K}} = R_{\mathcal{K},\pi}$ is irreflexive w.r.t. $\mathcal{K}$. It follows by induction that ${}^{i}R_{\mathcal{K}}$ is irreflexive w.r.t. $\mathcal{K}$ for all $i \in \mathbb{N}$. By 3.11 (1), $R^{(c)}_{\mathcal{K}} = \bigcup\limits_{i=1}^{\infty} {}^{i}R_{\mathcal{K}}$. Hence, again by 2.2 (3), $R^{(c)}_{\mathcal{K}}$ is irreflexive w.r.t. $\mathcal{K}$. The other properties can be easily verified with the aid of 2.2 (3), 3.11 (2),3.3 (1), and 3.7 (1). □

3.13. Corollary. *Let $R \subseteq G^H$ be a relation, $\mathcal{K}$ an n-decomposition of the set H, $\psi \in S_n$. Then*
(1) $(R^{(r)}_{\mathcal{K}})^{(c)}_{\mathcal{K}} = (R^{(c)}_{\mathcal{K}})^{(r)}_{\mathcal{K}}$.
(2) $(R^{(d)}_{\mathcal{K}})^{(r)}_{\mathcal{K}} = (R^{(r)}_{\mathcal{K}})^{(d)}_{\mathcal{K}}$.
(3) $(R^{(d)}_{\mathcal{K}})^{(s)}_{\mathcal{K},\psi} = (R^{(s)}_{\mathcal{K},\psi})^{(d)}_{\mathcal{K}} = R^{(d)}_{\mathcal{K}}$.
(4) $(R^{(d)}_{\mathcal{K}})^{(c)}_{\mathcal{K}} = (R^{(c)}_{\mathcal{K}})^{(d)}_{\mathcal{K}} = R^{(d)}_{\mathcal{K}}$.

Proof. (1) follows from 3.8 (1) for $\psi = \pi$.

(2) As $R \subseteq R^{(r)}_{\mathcal{K}}$, we have, by 3.3, $R^{(d)}_{\mathcal{K}} \subseteq (R^{(r)}_{\mathcal{K}})^{(d)}_{\mathcal{K}}$, and again by 3.3, $(R^{(d)}_{\mathcal{K}})^{(r)}_{\mathcal{K}} \subseteq ((R^{(r)}_{\mathcal{K}})^{(d)}_{\mathcal{K}})^{(r)}_{\mathcal{K}}$. By 3.12 (1), $R^{(r)}_{\mathcal{K}})^{(d)}_{\mathcal{K}}$ is reflexive w.r.t. $\mathcal{K}$, consequently, by 3.2, $((R^{(r)}_{\mathcal{K}})^{(d)}_{\mathcal{K}})^{(r)}_{\mathcal{K}} = (R^{(r)}_{\mathcal{K}})^{(d)}_{\mathcal{K}}$. Thus $(R^{(d)}_{\mathcal{K}})^{(r)}_{\mathcal{K}} \subseteq (R^{(r)}_{\mathcal{K}})^{(d)}_{\mathcal{K}}$. Similarly, using 3.3, 3.12 (2) and 3.2, we obtain $(R^{(r)}_{\mathcal{K}})^{(d)}_{\mathcal{K}} \subseteq (R^{(d)}_{\mathcal{K}})^{(r)}_{\mathcal{K}}$, which proves the assertion.

(3) follows from 3.3 and 3.2.

(4) is a special case of (3). □

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