130 (2005) MATHEMATICA BOHEMICA No. 2, 161–166

THE WEIERSTRASS THEOREM ON POLYNOMIAL APPROXIMATION

Rudolf Výborný, Queensland

(Received June 20, 2004)

Abstract. In the paper a simple proof of the Weierstrass approximation theorem on a function continuous on a compact interval of the real line is given. The proof is elementary in the sense that it does not use uniform continuity.

Keywords: approximation by polynomials

MSC 2000: 41A10

1. Introduction

Theorems on polynomial approximation belong to the most important theorems not only in analysis but in applications as well. The Weierstrass theorem in its formulation requires only the concept of continuity but advanced means are used in the proof very often. For Weierstrass own original formulation see the inserted box.

> Damit ist bewiesen:
>
> B. »Ist $f(x)$ eine Function von der angegebenen Beschaffenheit, und wird die Veränderliche x auf irgend ein endliches Intervall beschränkt, so lässt sich, nach Annahme einer beliebig kleinen positiven Grösse g, auf mannigfaltige Weise eine ganze rationale Function $G(x)$ bestimmen, welche in dem festgesetzten Intervalle sich der Function $f(x)$ so genau anschliesst, dass die Differenz $f(x)-G(x)$ ihrem absoluten Betrage nach beständig kleiner als g ist.«

Weierstrass [8] used the singular integral

$$\frac{1}{k\sqrt{\pi}}\int_{-\infty}^{\infty} f(u)\mathrm{e}^{-\left(\frac{u-x}{k}\right)^2}\,\mathrm{d}u$$

for the proof. This integral is for $k = \sqrt{t}$ the solution of the Cauchy problem for the heat equation with the initial condition f. The solution is known to be immediately analytic for positive time. It is natural to expect that the solution is close to the initial condition and then to further expect that the analytic function could be approximated by a polynomial. Weierstrass did not mention this, it is of no relevance for the proof. However, Weierstrass did prove first that a function continuous on an interval can be uniformly approximated by an entire analytic function. Other singular integrals have been used for the proof, it is generally believed that one of the simplest proofs is based on Landau's singular integral

$$\frac{1}{K_n} \int_0^1 f(t)[1-(t-x)^2]^n \, \mathrm{d}t,$$

where $K_n = \int_{-1}^{1} \left(1-u^2\right)^n$. See [4] and [2]. Bernstein's proof [5] employs the so called Bernsteins polynomials defined on the interval $[0,1]$ by

$$B_n = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}.$$

The theorem is proved first for the interval $[0,1]$ and then a linear transformation is used to obtain the general version. Bernsteins proof can be generalized to functions of several variables. A far reaching generalization is the Stone-Weierstrass theorem, it works not only in the more dimensional case but covers a lot of other approximation theorems as special cases, e.g. approximation of continuous functions by piecewise linear functions and by trigonometric functions. A treatment of the Stone-Weierstrass theorem accessible to undergraduates can be found in [7] and an account by Stone himself in [6].

It would be convenient to teach Weierstrass theorem on polynomial approximation early in the curriculum, the obstacle is the proof. An elementary proof was given by Kuhn [3]. The main ingredient in the proof is the Bernoulli inequality. We use a very similar approach but arrange the proof differently and in doing so we do not use uniform continuity. This in itself is perhaps not of great importance, because sooner or later students have to encounter uniform continuity anyhow, but by avoiding it in our proof, we can then legitimately use the Weierstrass theorem in proving other theorems and the uniform continuity itself. Thus reinforcing the point that having Weierstrass early is convenient.

Kuhn's idea was used in the proof of Stone-Weierstrass theorem [1] and although this proof can be viewed as elementary it also uses some topological concepts which perhaps prevent the teacher using it at early.


Weierstraß

2. The discussion

The theorem we wish to prove is

Theorem 1. *If $f\colon [a,b] \mapsto \mathbb{C}$ is continuous and $\varepsilon > 0$ then there exists a polynomial P such that*

$$|f(x) - P(x)| < \varepsilon \tag{1}$$

for all $x \in [a,b]$.

Obviously it suffices to prove the theorem for a real valued f. The main idea of the proof consists in showing that if f can be approximated by a polynomial on some interval that it can also be approximated by a polynomial on a slightly larger interval. For the proof we also need a polynomial approximation U of a function which is 1 on a part of the interval and 0 on another part of $[a,b]$. We state this as a

Lemma 1. *If $a < c-k < c \leqslant b$ then for every $\eta > 0$ there exists a polynomial U such that*

$$1-\eta < U(x) \leqslant 1 \quad \text{for} \quad a \leqslant x \leqslant c-k \tag{2}$$

$$0 \leqslant U(x) \leqslant 1 \quad \text{for} \quad c-k < x < c \tag{3}$$

$$0 \leqslant U(x) \leqslant \eta \quad \text{for} \quad c \leqslant x \leqslant b. \tag{4}$$

Proof. We denote $l = b-a$ and $d = c - \frac{1}{2}k$. First we find a polynomial p with values between 0 and $\frac{1}{2}$ on $[a,d]$ and between $\frac{1}{2}$ and 1 on $[d,b]$. This is easy

$$p(x) = \frac{1}{2} + \frac{x-d}{2l}.$$

Then we define

$$U(x) = (1 - [p(x)]^n)^{2^n}$$

with some $n \in \mathbb{N}$ which we choose suitably later. Obviously

$$0 \leqslant U(x) \leqslant 1 \tag{5}$$

for $a \leqslant x \leqslant b$. Employing the Bernoulli inequality where n is replaced by 2^n, gives

$$U(x) \geqslant 1 - [2p(x)]^n \geqslant 1 - [2p(c-k)]^n \tag{6}$$

for $a \leqslant x \leqslant c-k$. On the other hand we have for $c \leqslant x \leqslant b$ that

$$(7) \qquad U(x) \leqslant \frac{1}{[2p(x)]^n} U(x)(1+[2p(x)]^n) \leqslant \frac{1}{[2p(x)]^n}(1-[p(x)]^{2n})^{2^n} \leqslant \frac{1}{[2p(c)]^n}.$$

Since both $2p(c-k) < 1$ and $2p(c) > 1$ it follows that both $[2p(c-k)]^n \to 0$ and $[2p(c)]^{-n} \to 0$. We can therefore find $n \in \mathbb{N}$ such that

$$[2p(c-k)]^n < \eta \text{ and } \frac{1}{[2p(c)]^n} < \eta.$$

Equations (2), (3) and (4) now follow from equations (5), (6) and (7), respectively.

□

Proof of the Weierstrass Theorem. For a given $\varepsilon > 0$ let S_ε be the set of all $t \leqslant b$ such that there exists a polynomial P_ε with the property that

$$(8) \qquad |f(x) - P_\varepsilon(x)| < \varepsilon$$

for $a \leqslant x \leqslant t$. By continuity of f at a there exists $t_0 > a$ such that

$$|f(x) - f(a)| \leqslant \varepsilon$$

for $a \leqslant x \leqslant t_0$. Consequently f can be approximated by a constant $f(a)$ on $[a, t_0]$ and $S_\varepsilon \neq \emptyset$. Let $s = \sup S_\varepsilon$. Clearly $a < s \leqslant b$. By continuity of f at s there is $\delta > 0$ such that

$$(9) \qquad |f(x) - f(s)| \leqslant \frac{\varepsilon}{3}$$

for $s - \delta \leqslant x \leqslant s + \delta$ and $x \leqslant b$. By the definition of the least upper bound there is a c with $s - \delta < c \leqslant s$ and $s \in S_\varepsilon$. This means there is a polynomial P_ε satisfying equation (8) for $a \leqslant x \leqslant c$. Let

$$m = \max\{|f(x) - P_\varepsilon(x)|;\ a \leqslant x \leqslant c\}$$

and M so large that

$$M > |f(x) - P_\varepsilon(x)| + |f(x) - f(s)|$$

for all $x \in [a, b]$. We apply the lemma for $c - k = s - \delta$ to find the function U with $0 < \eta < 1$ so small that

$$(10) \qquad m + M\eta < \varepsilon,$$

$$(11) \qquad M\eta < \frac{2\varepsilon}{3}.$$

This is possible since $m < \varepsilon$. Now we define

$$P(x) = f(s) + [P_\varepsilon(x) - f(s)]U(x), \tag{12}$$

and show that it satisfies (1) on $[a, b]$. First we have

$$|f(x) - P(x)| \leqslant |f(x) - P_\varepsilon(x)|U(x) + |f(x) - f(s)|\,(1 - U(x))\,.$$

It follows from (2) and (10) that on the interval $[a, s - \delta]$

$$|f(x) - P(x)| \leqslant m + M\eta < \varepsilon.$$

On the interval $[s - \delta, c]$ clearly

$$|f(x) - P(x)| \leqslant \varepsilon U(x) + \frac{\varepsilon}{3}\,(1 - U(x)) < \varepsilon.$$

Finally, using (4), (9) and (11) we have

$$|f(x) - P(x)| \leqslant MU(x) + \frac{\varepsilon}{3}\,(1 - U(x)) < \varepsilon,$$

on $[c, s + \delta] \cap [c, b]$. This proves that $s = b$, otherwise the inequality (1) with P defined by equation (12) would hold on $[a, s + \delta]$, contrary to the definition of s. Hence equation (1) holds on $[a, s] = [a, b]$ and the proof is complete.

References

[1] *B. Brosowski, F. Deutsch*: An elementary proof of the Stone-Weierstrass theorem. Proc. Am. Math. Soc. *81* (1981), 89–92.

[2] *D. Jackson*: A proof of Weierstrass's theorem. Am. Math. Mon. *41* (1934), 309–312.

[3] *H. Kuhn*: Ein elementarer Beweis des Weierstrasschen Approximationsatzes. Arch. Math. *15* (1964), 316–317.

[4] *E. Landau*: Über die Approximationen einer stetigen Funktion durch eine ganze rationale Funktion. Palermo Rend. *25* (1908), 337–345.

[5] *I. P. Natanson*: Theory of Functions of a Real Variable (Engl. transl.) Vol 1. Ungar, New York, 1974.

[6] *M. H. Stone*: A generalized Weierstrass approximation theorem. Stud. Math. *1* (1962), 30–87.

[7] *Béla Sz.-Nagy*: Introduction to Real Functions and Orthogonal Expansions. Akadémiai Kiadó, Budapest, 1964.

[8] *K. Weierstrass*: Mathematische Werke. Preussische Akademie der Wissenschaften, 1903.

Author's address: *Rudolf Výborný*, 15 Rialanna Street, Kenmore, Qld 4069, Australia e-mail: `rudolf.vyborny@mailbox.uq.edu.au`.