I AND I^*-CONVERGENCE IN TOPOLOGICAL SPACES

B. K. Lahiri, Pratulananda Das, West Bengal

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Abstract. We extend the idea of I-convergence and I^*-convergence of sequences to a topological space and derive several basic properties of these concepts in the topological space.

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1. Introduction

The concept of convergence of a sequence of real numbers has been extended to statistical convergence independently by Fast [4] and Schoenberg [14]. Any convergent sequence is statistically convergent but the converse is not true [12]. Moreover a statistically convergent sequence need not even be bounded [12]. Here and throughout $\mathbb{N}$ denotes the set of natural numbers. If $K \subset \mathbb{N}$, then K_n will denote the set $\{k \in K;\ k \leqslant n\}$ and $|K_n|$ stands for the cardinality of K_n. The natural density of K is defined by

$$d(K) = \lim_n \frac{|K_n|}{n},$$

if the limit exists ([5], [11]). A real sequence $x = \{x_n\}$ is statistically convergent to l if for every $\varepsilon > 0$ the set

$$K(\varepsilon) = \{k \in \mathbb{N};\ |x_k - l| \geqslant \varepsilon\}$$

has natural density zero ([4], [14]).

The concept of I-convergence of real sequences ([6], [7]) is a generalization of statistical convergence which is based on the structure of the ideal I of subsets of the set of natural numbers. In the recent literature several works on I-convergence including remarkable contributions by Šalát et al have occured ([1], [3], [6], [7], [9], [10]).

I-convergence of real sequences coincides with the ordinary convergence if I is the ideal of all finite subsets of $\mathbb{N}$ and with the statistical convergence if I is the ideal of subsets of $\mathbb{N}$ of natural density zero.

The concept of I^*-convergence of real sequences arises from the following result on statistical convergence [12]: a real sequence $x = \{x_n\}$ is statistically convergent to ξ if and only if there exists a set

$$M = \{m_1 < m_2 < m_3 < \ldots < m_k < \ldots\} \subset \mathbb{N}$$

such that $d(M) = 1$ and $\lim_k x_{m_k} = \xi$, and extensive work has been done by Šalát et al [6] on this concept also.

The idea of I-convergence has been extended from real number space to metric space [6] and to a normed linear space [13] in recent works. It seems therefore reasonable to think if the concept of I-convergence can be extended to an arbitrary topological space and in that case enquire how the basic properties are affected. In this paper our object is in this line where we extend the concepts of I-convergence and I^*-convergence to a topological space and observe that the basic properties are preserved also in a topological space.

2. I-convergence in a topological space

We recall the following definitions ([8], p. 34).

Definition 1. If X is a nonvoid set then a family of sets $I \subset 2^X$ is an *ideal* if
(i) $\emptyset \in I$,
(ii) $A, B \in I$ implies $A \cup B \in I$ and
(iii) $A \in I, B \subset A$ implies $B \in I$.

The ideal is called *nontrivial* if $I \neq \{\emptyset\}$ and $X \notin I$.

Definition 2. A nonempty family F of subsets of a nonvoid set X is called a *filter* if
(i) $\emptyset \notin F$,
(ii) $A, B \in F$ implies $A \cap B \in F$ and
(iii) $A \in F, A \subset B$ implies $B \in F$.

If I is a nontrivial ideal on X then $F = F(I) = \{A \subset X\,;\ X \setminus A \in I\}$ is clearly a filter on X and conversely.

A nontrivial ideal I is called *admissible* if it contains all the singleton sets. Several examples of nontrivial admissible ideals may be seen in [6].

Throughout (X, τ) will stand for a topological space and I for a nontrivial ideal of $\mathbb{N}$, the set of all positive integers.

We now introduce the following definition.

Definition 3. A sequence $\{x_n\}$ in X is said to be *I-convergent* to $x_0 \in X$ if for any nonvoid open set U containing x_0, $\{n \in \mathbb{N};\ x_n \notin U\} \in I$.

In this case we write $I\text{-}\lim x_n = x_0$ and x_0 is called the I-limit of $\{x_n\}$.

Note 1. If I is admissible then ordinary convergence implies I-convergence and in addition if I does not contain any infinite set then both concepts coincide.

We examine below which usual properties of convergence in a topological space are preserved in I-convergence.

Theorem 1. *If X is Hausdorff then an I-convergent sequence has a unique I-limit.*

Proof. If possible suppose that an I-convergent sequence $\{x_n\}$ has two distinct I-limits x_0 and y_0, say. There exist $U, V \in \tau$ such that $x_0 \in U$ and $y_0 \in V$, $U \cap V = \emptyset$. Since $\{k;\ x_k \notin U\} \in I$ and $\{k;\ x_k \notin V\} \in I$, we have $\{k;\ x_k \in (U \cap V)^c\} \subset \{k;\ x_k \in U^c\} \cup \{k;\ x_k \in V^c\} \in I$ where c stands for the complement. Since I is nontrivial, there exists $k_0 \in \mathbb{N}$ such that $k_0 \notin \{k;\ x_k \in (U \cap V)^c\}$. But then $x_{k_0} \in U \cap V$, a contradiction and the theorem is proved. $\square$

We have stated earlier that if I is admissible then (ordinary) convergence of a sequence in X implies its I-convergence. The following theorem is a kind of converse.

Theorem 2. *If I is an admissible ideal and if there exists a sequence $\{x_n\}$ of distinct elements in a set $E \subset X$ which is I-convergent to $x_0 \in X$ then x_0 is a limit point of E.*

Proof. Let U be an arbitrary open set containing x_0. Since $I\text{-}\lim x_n = x_0$, $\{n;\ x_n \notin U\} \in I$ and so $\{n;\ x_n \in U\} \notin I$ (since I is nontrivial). Also this set should be infinite because I is admissible. Choose $k_0 \in \{n;\ x_n \in U\}$ such that $x_{k_0} \neq x_0$. Then $x_{k_0} \in U \cap (E - \{x_0\})$. Thus x_0 is a limit point of E. This proves the theorem. $\square$

Theorem 3. *A continuous function $g\colon X \to X$ preserves I-convergence.*

Proof. Let $I\text{-}\lim x_n = x$. Let V be an open set containing $g(x)$. There exists then an open set U containing x such that $g(U) \subset V$. Clearly

$$\{n;\ g(x_n) \notin V\} \subset \{n;\ x_n \notin U\}$$

and since $\{n;\ x_n \notin U\} \in I$ we have $\{n;\ g(x_n) \notin V\} \in I$ which shows that $I\text{-}\lim g(x_n) = g(x)$ and this proves the theorem. $\square$

N o t e 2. If I is admissible and X is a first axiom T_1 space, then the continuity of $g\colon X \to X$ is necessary to preserve I-convergence. Because suppose that g is not continuous at $x \in X$. Then there is a sequence $\{x_n\}$ of distinct points in X such that $x_n \to x$ but $g(x_n)$ does not tend to $g(x)$. So there is an open set V containing $g(x)$ and a subsequence $\{x_{k_n}\}$ such that $g(x_{k_n}) \notin V$ for all n. Put $y_n = x_{k_n}$. Then $y_n \to x$ and so $\{y_n\}$ is I-convergent to x but as $\{n;\ g(y_n) \notin V\} = \mathbb{N} \notin I$, $\{g(y_n)\}$ is not I-convergent to $g(x)$.

3. I^*-convergence in X

We now see that the notion of I^*-convergence of a sequence in X which is closely related to ordinary convergence and is defined below has certain connection with that of I-convergence of the sequence.

Definition 4. A sequence $\{x_n\}$ in X is I^*-*convergent* to $x \in X$ if and only if there exists a set $M \in F(I)$ (i.e. $\mathbb{N} \setminus M \in I$), $M = \{m_1 < m_2 < \ldots < m_k < \ldots\}$ such that $\lim\limits_{k\to\infty} x_{m_k} = x$.

In this case we write $I^*\text{-}\lim x_n = x$ and x is called the I^*-limit of $\{x_n\}$.

Theorem 4. *If I is admissible then $I^*\text{-}\lim x_n = x$ implies $I\text{-}\lim x_n = x$ and so in addition if X is Hausdorff then $I^*\text{-}\lim x_n$ is unique.*

P r o o f. There exists a set $K \in I$ such that for $M = \mathbb{N} \setminus K = \{m_1 < m_2 < \ldots < m_k < \ldots\}$ we have $\lim x_{m_k} = x$. Then for any open set U containing x, $x_{m_k} \in U$ for $k > k_0$ (say). Clearly

$$\{n;\ x_n \notin U\} \subset K \cup \{m_1, m_2, \ldots, m_{k_0}\} \in I$$

and so $I\text{-}\lim x_n = x$. This proves the theorem. □

First part of Theorem 4 may be restated as follows.

Theorem 5. *Suppose that I is admissible and $x = \{x_n\}$. If there is a set $K = \{n_1, n_2, \ldots\} \in F(I)$ such that $\lim x_{n_k} = \xi$ then $I\text{-}\lim x_n = \xi$.*

The converse holds under a certain assumption.

Theorem 6. *If X has no limit point then I and I^*-convergence coincide for every admissible ideal I.*

Proof. Let $I\text{-}\lim x_n = x_0$. Because of Theorem 4 we have only to show that $I^*\text{-}\lim x_n = x_0$. Since X has no limit point, $U = \{x_0\}$ is open. Since $I\text{-}\lim x_n = x_0$, we have $\{n;\ x_n \notin U\} \in I$. Hence $\{n;\ x_n \in U\} = \{n;\ x_n = x_0\} \in F(I)$ and thus $I^*\text{-}\lim x_n = x_0$. □

Equivalence of I and I^*-convergence is further studied in Section 4.

Theorem 7. *If a first axiom T_2 space X has a limit point x then there exists an admissible nontrivial ideal I and a sequence $\{y_n\}$ of X such that $I\text{-}\lim y_n = x$ but $I^*\text{-}\lim y_n$ does not exist.*

Proof. The proof of the theorem is patterned after Theorem 3.1 [6] with necessary modifications. Let $\{B_n(x)\}$ be a monotonically decreasing open base at x. We can find a sequence $\{x_n\}$ of distinct elements in X such that $x_n \in B_n(x) \backslash B_{n+1}(x)$ for all n and $x_n \to x$. We now consider the following ideal from ([6], Ex. 3.1g).

Let $\mathbb{N} = \bigcup\limits_{j=1}^{\infty} \Delta_j$ be a decomposition of $\mathbb{N}$ such that each Δ_j is infinite and $\Delta_i \cap \Delta_j = \emptyset$ for $i \neq j$. Let I denote the class of all $A \subset \mathbb{N}$ which intersect at most a finite number of Δ_js. Then I is an admissible nontrivial ideal. Note that any Δ_j is a member of I.

Let $\{y_n\}$ be a sequence defined by $y_n = x_j$ if $n \in \Delta_j$. Let U be an open set containing x. Choose a positive integer m such that $B_n(x) \subset U$ for $n > m$. Then $\{n;\ y_n \notin U\} \subset \Delta_1 \cup \Delta_2 \cup \ldots \cup \Delta_m$ (say) and so $\{n;\ y_n \notin U\} \in I$ because each Δ_j is a member of I and thus $I\text{-}\lim y_n = x$.

Now suppose if possible, that $I^*\text{-}\lim y_n = x$. Then there exists $H \in I$ such that for $M = \mathbb{N} \setminus H = \{m_1 < m_2 < \ldots < m_k < \ldots\}$ we have $\lim\limits_{k\to\infty} y_{m_k} = x$. From the formation of I it follows that there exists $l \in \mathbb{N}$ such that $H \subset \Delta_1 \cup \Delta_2 \cup \ldots \cup \Delta_l$ and then $\Delta_i \subset \mathbb{N} \backslash H = M$ for $i \geqslant l+1$. So for each $i \geqslant l+1$ there exist infinitely many k's (note that each Δ_j contains an infinite number of elements of $\mathbb{N}$) such that $y_{m_k} = x_i$. But then $\lim y_{m_k}$ does not exist because $x_i \neq x_j$ for $i \neq j$, a contradiction. Also the assumption $I^*\text{-}\lim y_n = y \neq x$ leads similarly to a contradiction. This proves the theorem. □

4. Condition (AP) and equivalence of I and I^*-convergence

In this section we consider a condition under which I-convergence and I^*-convergence coincide. This condition is similar to the condition required in [6] which again is similar to the (APO) condition used in [2] and [4].

Definition 5. An admissible ideal I is said to satisfy the *condition* (AP) if for every countable family of mutually disjoint sets $\{A_1, A_2, \ldots\}$ belonging to I there exists a countable family of sets $\{B_1, B_2, \ldots\}$ such that $A_j \Delta B_j$ is finite for all $j \in \mathbb{N}$ and $B = \bigcup B_j \in I$.

Note that $B_j \in I$ for all $j \in \mathbb{N}$.

Theorem 8. *Let I be an admissible ideal.*

(i) *If I has the property* (AP) *and (X, τ) is a first axiom space then for arbitrary sequence $\{x_n\}$ in X, I-$\lim x_n = x$ implies I^*-$\lim x_n = x$.*

(ii) *If (X, τ) is a first axiom T_1 space containing at least one limit point and for each $x \in X$, I-$\lim x_n = x$ implies I^*-$\lim x_n = x$ then I has the property* (AP).

Proof. (i) Let I-$\lim x_n = x$. Then for an arbitrary open set U containing x, $\{n;\ x_n \notin U\} \in I$. Let $B_n(x)$ be a monotonically decreasing local base at x. Let $A_1 = \{n;\ x_n \notin B_1(x)\}$ and for $m \geqslant 2$, $A_m = \{n;\ x_n \notin B_m(x) \text{ but } x_n \in B_{m-1}(x)\}$. Then it follows that $\{A_1, A_2, \ldots\}$ is a sequence of sets in I with $A_i \cap A_j = \emptyset$ for $i \neq j$. By the condition (AP) there exists a countable family of sets $\{B_1, B_2, \ldots\}$ in I such that $A_j \Delta B_j$ are finite for all j and $B = \bigcup B_j \in I$. Let $M = \mathbb{N} \backslash B = \{m_1 < m_2 < \ldots\}$ (say). We will show that $\lim\limits_{k \to \infty} x_{m_k} = x$.

For this let U be any open set containing x. Then there exists $k_1 \in \mathbb{N}$ such that $B_n(x) \subset U$ for all $n \geqslant k_1$. Now $\{n;\ x_n \notin U\} \subset \bigcup\limits_{j=1}^{k_1} A_j$. Since $A_j \Delta B_j$, $j = 1, 2, \ldots, k_1$ are finite, there exists $n_0 \in \mathbb{N}$ such that

$$\bigcup_{j=1}^{k_1} B_j \cap \{n;\ n > n_0\} = \bigcup_{j=1}^{k_1} A_j \cap \{n;\ n > n_0\}.$$

Choose $m_l \in \mathbb{N}$ such that $m_l > n_0$. Then for all $p > l$, $m_p \notin B$ and this implies from the above that $m_p \notin \bigcup\limits_{j=1}^{k_1} A_j$ and so $x_{m_p} \in B_{k_1}(x) \subset U$. This shows that $\lim x_{m_k} = x$ and so I^*-$\lim x_n = x$.

(ii) Suppose that $x \in X$ is a limit point of X. We can as before find a sequence $\{x_n\}$ of distinct points in X such that $\lim x_n = x$ and $x_n \in B_n(x)$ for all n, $x_n \neq x$ for $n \in \mathbb{N}$, where $\{B_n(x)\}$ is a monotone decreasing local base at x. Let $\{A_n\}$ be a mutually disjoint countable family of nonvoid sets from I. Define a sequence $\{y_n\}$ (as before) by $y_n = x_j$ if $n \in A_j$ and $y_n = x$ if $n \notin A_j$ for any j. Let U be any open set containing x. Then there exists $m \in \mathbb{N}$ such that $B_n(x) \subset U$ for all $n \geqslant m$. Now

$$\{n;\ y_n \notin U\} \subset A_1 \cup A_2 \cup \ldots \cup A_{m-1}$$

and so belongs to I which implies that I-$\lim y_n = x$. By our assumption I^*-$\lim y_n = x$. Hence there exists a set $H \in I$ such that for $M = \mathbb{N} \setminus H = \{m_1 < m_2 < \ldots\}$, say, we have

$$\lim_{k\to\infty} y_{m_k} = x. \tag{1}$$

Put $B_j = A_j \cap H$ for all $j \in \mathbb{N}$. Then $B_j \in I$ for all $j \in \mathbb{N}$. Also $\bigcup B_j \subset H$ and so belongs to I. Let $j \in \mathbb{N}$ be fixed. Clearly the set A_j has at most a finite number of elements common with M, for otherwise $y_{m_k} = x_j$ for infinite number of m_k's and $x_j \neq x$ and this contradicts (1). Thus we can choose a $k_0 \in \mathbb{N}$ such that $A_j \subset (A_j \cap B_j) \cup \{m_1, m_2, \ldots, m_{k_0}\}$. Therefore $A_j \Delta B_j = A_j \setminus B_j \subset \{m_1, m_2, \ldots, m_{k_0}\}$ and so is finite. Since this is true for all $j \in \mathbb{N}$, it follows that I has the property (AP). This proves the theorem. $\square$

5. I-limit points and I-cluster points

Definition 6. Let $x = \{x_n\}$ be a sequence of elements of X.

a) $y \in X$ is called an *I-limit point* of x if there exists a set $M = \{m_1 < m_2 < \ldots\} \subset \mathbb{N}$ such that $M \notin I$ and $\lim_{k\to\infty} x_{m_k} = y$.

b) $y \in X$ is called an *I-cluster point* of x if for every open set U containing y, $\{n;\ x_n \in U\} \notin I$.

We denote respectively by $I(L_x)$ and $I(C_x)$ the collection of all I-limit points and I-cluster points of x.

Theorem 9. *If I is an admissible ideal then $I(L_x) \subset I(C_x)$.*

P r o o f. Let $y \in I(L_x)$. Then there exists $M = \{m_1 < m_2 < \ldots\} \subset \mathbb{N}$, $M \notin I$ such that $\lim x_{m_k} = y$. Let U be any open set containing y. Then there exists $k_0 \in \mathbb{N}$ such that $x_{m_k} \in U$ for all $k > k_0$. Then $\{n;\ x_n \in U\} \supset M/\{m_1, \ldots, m_{k_0}\}$ and so $\{n;\ x_n \in U\} \notin I$. This shows that $y \in I(C_x)$ and the theorem is proved. $\square$

Theorem 10. *Let I be an admissible ideal.*

(i) *Then $I(C_x)$ is closed for each sequence $x = \{x_n\}$ in X.*

(ii) *Suppose that (X, τ) is completely separable and let there exist a disjoint sequence of sets $\{M_n\}$ such that $M_n \subset \mathbb{N}$, $M_n \notin I$ for all n. Then for each nonvoid closed set $F \subset X$ there exists a sequence x in X such that $F = I(C_x)$.*

P r o o f. (i) Let $y \in \overline{I(C_x)}$ where bar denotes the closure. Let U be any open set containing y. Then $U \cap I(C_x) \neq \emptyset$. Let $z \in U \cap I(C_x)$. But $z \in U$ and $z \in I(C_x)$ implies $\{n;\ x_n \in U\} \notin I$. Hence $y \in I(C_x)$.

(ii) Since X is completely separable, F is separable and let $A = \{a_1, a_2, \ldots\} \subset F$ be a countable set with $\overline{A} = F$. For $n \in M_i$, let $x_n = a_i$. We thus obtain a subsequence $\{k_n\}$, say, of the sequence $\{n\}$ of positive integers. Let $x = \{x_{k_n}\}$ and $y \in I(C_x)$. If $y = a_i$ for some i then $y \in F$. So let $y \neq a_i$ for any i.

Let U be any open set containing y. Then from definition $\{n;\ x_{k_n} \in U\} \notin I$ and so $\{n;\ x_{k_n} \in U\}$ is not void which implies that at least one of $a_i \in U$. So $F \cap U \neq \emptyset$. This gives that y is a limit point of F and thus $y \in F$. So $I(C_x) \subset F$.

To prove the reverse inclusion, let $z \in F$ and let U be any open set containing z. Then there exists $a_i \in A$ such that $a_i \in U$. Thus $\{n;\ x_{k_n} \in U\} \supset M_i$ and so $\{n;\ x_{k_n} \in U\} \notin I$ and this implies $z \in I(C_x)$ and the theorem is proved. $\square$

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Authors' addresses: *B. K. Lahiri*, B-1/146 Kalyani, West Bengal-741235, India; *Pratulananda Das*, Department of Mathematics, Jadavpur University, Kolkata-700032, West Bengal, India, e-mail: `pratulananda@yahoo.co.in`.