A GENERALIZED NOTION OF n-WEAK AMENABILITY

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Abstract. In the current work, a new notion of n-weak amenability of Banach algebras using homomorphisms, namely (φ,ψ)-n-weak amenability is introduced. Among many other things, some relations between (φ,ψ)-n-weak amenability of a Banach algebra $\mathcal{A}$ and $M_m(\mathcal{A})$, the Banach algebra of $m\times m$ matrices with entries from $\mathcal{A}$, are studied. Also, the relation of this new concept of amenability of a Banach algebra and its unitization is investigated. As an example, it is shown that the group algebra $L^1(G)$ is (φ,ψ)-n-weakly amenable for any bounded homomorphisms φ and ψ on $L^1(G)$.

Keywords: Banach algebra; continuous homomorphism; (φ,ψ)-derivation; n-weak amenability

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1. Introduction

The notion of amenability for Banach algebras was introduced by Johnson in [7]. A Banach algebra $\mathcal{A}$ is *amenable* if $H^1(\mathcal{A},X^*)=\{0\}$ for every Banach $\mathcal{A}$-module X, where $H^1(\mathcal{A},X^*)$ is the *first Hochschild cohomology group* of $\mathcal{A}$ with coefficients in X^*. One of the fundamental results of Johnson [7] was that the group algebra $L^1(G)$ is an amenable Banach algebra if and only if G is an amenable locally compact group. Dales et al. introduced the notion of n-weak amenability of Banach algebras in [4]. A Banach algebra $\mathcal{A}$ is *n-weakly amenable* if $H^1(\mathcal{A},\mathcal{A}^{(n)})=\{0\}$, where $\mathcal{A}^{(n)}$ is the n-th dual space of $\mathcal{A}$ (1-weak amenability is called weak amenability). A Banach algebra is called *permanently weakly amenable* if it is n-weakly amenable for each positive integer n. It is well known that for any locally compact group G, $L^1(G)$ is permanently weakly amenable (see [3], [4] and [8]). Then n-weak amenability of some Banach algebras is investigated in [6].

In [1], Bodaghi et al. generalized the concept of weak amenability of a Banach algebra $\mathcal{A}$ to that of (φ,ψ)-weak amenability, where φ and ψ are continuous homomorphisms on $\mathcal{A}$ (the case of amenability has been earlier developed by Moslehian and Motlagh in [12]). They determined the relations between weak amenability and (φ, ψ)-weak amenability of a Banach algebra $\mathcal{A}$. In [5], Eshaghi and Jabbari showed that for a locally compact group G, $L^1(G)$ is (φ, ψ)-weakly amenable for all continuous homomorphisms φ and ψ from $L^1(G)$ into $L^1(G)$.

In this paper, we shall extend the concept of n-weak amenability to that of (φ, ψ)-n-weak amenability of Banach algebras which is somewhat different from the notion (φ)-n-weak amenability introduced in [11]. We investigate some relations between (φ, ψ)-n-weak amenability of a Banach algebra $\mathcal{A}$ and $M_m(\mathcal{A})$, the Banach algebra of $m \times m$ matrices with entries from $\mathcal{A}$. Among other examples, we show that $L^1(G)$ is (φ, ψ)-n-weakly amenable for all bounded homomorphisms φ and ψ on $L^1(G)$.

2. (φ, ψ)-n-weak amenability

Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras. We denote by $\mathrm{Hom}(\mathcal{A}, \mathcal{B})$ the space of all bounded homomorphisms from $\mathcal{A}$ into $\mathcal{B}$, with the operator norm, and denote $\mathrm{Hom}(\mathcal{A}, \mathcal{A})$ by $\mathrm{Hom}(\mathcal{A})$. Throughout the paper, by continuity we mean that a homomorphism or derivation is continuous in norm topology.

Let $\mathcal{A}$ be a Banach algebra and let φ and ψ be in $\mathrm{Hom}(\mathcal{A})$. We consider the following module actions on $\mathcal{A}$:

$$a \cdot x = \varphi(a) \cdot x, \quad x \cdot a = x \cdot \psi(a) \quad \forall a, x \in \mathcal{A}.$$

We denote the above $\mathcal{A}$-module by $\mathcal{A}_{(\varphi,\psi)}$. Let $n \in \mathbb{N}$. The natural $\mathcal{A}$-module actions on $(\mathcal{A}_{(\varphi,\psi)})^{(n)}$(the n-th dual of $\mathcal{A}$) are as follows:

$$a \cdot a^{(2n)} = \varphi(a) \cdot a^{(2n)}, \quad a^{(2n)} \cdot a = a^{(2n)} \cdot \psi(a) \quad \forall a \in \mathcal{A}, a^{(2n)} \in (\mathcal{A}_{(\varphi,\psi)})^{(2n)}.$$
$$a \cdot a^{(2n-1)} = \psi(a) \cdot a^{(2n-1)}, \quad a^{(2n-1)} \cdot a = a^{(2n-1)} \cdot \varphi(a) \quad \forall a \in \mathcal{A}, a^{(2n-1)} \in (\mathcal{A}_{(\varphi,\psi)})^{(2n-1)}.$$

A bounded linear map $D\colon \mathcal{A} \to (\mathcal{A}_{(\varphi,\psi)})^{(2n)}$ is called a (φ, ψ)-derivation if $D(ab) = D(a) \cdot \psi(b) + \varphi(a) \cdot D(b)$, for all $a, b \in \mathcal{A}$. For the odd case, a bounded linear map $D\colon \mathcal{A} \to (\mathcal{A}_{(\varphi,\psi)})^{(2n-1)}$ is a (φ, ψ)-derivation if $D(ab) = D(a) \cdot \varphi(b) + \psi(a) \cdot D(b)$ for all $a, b \in \mathcal{A}$. A bounded linear map $D\colon \mathcal{A} \to (\mathcal{A}_{(\varphi,\psi)})^{(2n)}$ is called (φ, ψ)-inner if there exists $x \in (\mathcal{A}_{(\varphi,\psi)})^{(2n)}$ such that $D(a) := \delta_a(a) = \varphi(a) \cdot x - x \cdot \psi(a)$, for all $a \in \mathcal{A}$. Also $D\colon \mathcal{A} \to (\mathcal{A}_{(\varphi,\psi)})^{(2n-1)}$ is (φ, ψ)-inner if there exists $x \in (\mathcal{A}_{(\varphi,\psi)})^{(2n-1)}$ such that $D(a) = x \cdot \varphi(a) - \psi(a) \cdot x$ for all $a \in \mathcal{A}$. The Banach algebra $\mathcal{A}$ is called (φ, ψ)-n-weakly amenable if every (φ, ψ)-derivation $D\colon \mathcal{A} \to (\mathcal{A}_{(\varphi,\psi)})^{(n)}$ is (φ, ψ)-inner.

The following proposition is analogous to Proposition 1.2 from [4] in a more general setting. Since the proof is similar, it is omitted.

Proposition 2.1. *Let $\mathcal{A}$ be a Banach algebra and let $n \in \mathbb{N}$. If $\mathcal{A}$ is (φ,ψ)-$(n+2)$-weakly amenable, then $\mathcal{A}$ is (φ,ψ)-n-weakly amenable.*

For a Banach algebra $\mathcal{A}$, we put $\mathcal{A}^2 = \operatorname{span}\{ab\colon a,b \in \mathcal{A}\}$. The next proposition is proved in [1, Proposition 2.1].

Proposition 2.2. *Let $\mathcal{A}$ be Banach algebra and let $\varphi,\psi \in \operatorname{Hom}(\mathcal{A})$ such that $\varphi(a)b = a\psi(b)$ for all $a,b \in \mathcal{A}$. If $\mathcal{A}$ is (φ,ψ)-weakly amenable, then $\overline{\mathcal{A}^2} = \mathcal{A}$, where $\overline{\mathcal{A}^2}$ is the closure of $\mathcal{A}^2$ in $\mathcal{A}$.*

Let $\mathcal{A}$ be a non-unital Banach algebra. Then $\mathcal{A}^{\#} = \mathcal{A} \oplus \mathbb{C}e$, the unitization of $\mathcal{A}$, is a unital Banach algebra with the following product:

$$(a,\alpha)(b,\beta) = (ab + \alpha b + \beta a, \alpha\beta) \quad \forall a,b \in \mathcal{A},\ \alpha,\beta \in \mathbb{C}.$$

Define $e^* \in \mathcal{A}^{\#*}$ by requiring $\langle e^*, e\rangle = 1$ and $\langle e^*, a\rangle = 0$ for all $a \in \mathcal{A}$. Then we have the following identification:

$$\begin{aligned} \mathcal{A}^{\#(2n)} &= \mathcal{A}^{(2n)} \oplus \mathbb{C}e \quad \forall n \in \mathbb{N},\\ \mathcal{A}^{\#(2n+1)} &= \mathcal{A}^{(2n+1)} \oplus \mathbb{C}e^* \quad \forall n \in \mathbb{Z}^+. \end{aligned}$$

Let $\varphi,\psi \in \operatorname{Hom}(\mathcal{A})$. Define the map $\widehat{\varphi}\colon \mathcal{A}^{\#} \to \mathcal{A}^{\#}$ via $\widehat{\varphi}(a,\alpha) = (\varphi(a),\alpha)$. It is easy to see that $\widehat{\varphi} \in \operatorname{Hom}(\mathcal{A}^{\#})$. The $\mathcal{A}^{\#}$-module actions on $(\mathcal{A}_{(\varphi,\psi)})^{\#(2n+1)}$ are given by

$$\begin{aligned} (a+\alpha e)\cdot(a^{(2n+1)} + \beta e^*) &= \psi(a)\cdot a^{(2n+1)} + \alpha a^{(2n+1)} + (\alpha\beta + \langle a^{(2n+1)}, a\rangle)e^*,\\ (a^{(2n+1)} + \beta e^*)\cdot(a+\alpha e) &= a^{(2n+1)}\cdot\varphi(a) + \alpha a^{(2n+1)} + (\alpha\beta + \langle a^{(2n+1)}, a\rangle)e^*. \end{aligned}$$

The following result is analogous to [4, Proposition 1.4], but we include the proof for the sake of completeness.

Theorem 2.1. *Let $\mathcal{A}$ be a non-unital Banach algebra and let $\varphi,\psi \in \operatorname{Hom}(\mathcal{A})$, $n \in \mathbb{N}$.*

(i) *Suppose that $\mathcal{A}^{\#}$ is $(\widehat{\varphi},\widehat{\psi})$-$2n$-weakly amenable. Then $\mathcal{A}$ is (φ,ψ)-$2n$-weakly amenable;*

(ii) *Suppose $\varphi(a)b = a\psi(b)$ for all $a,b \in \mathcal{A}$. If $\mathcal{A}$ is (φ,ψ)-$(2n-1)$-weakly amenable, then $\mathcal{A}^{\#}$ is $(\widehat{\varphi},\widehat{\psi})$-$(2n-1)$-weakly amenable.*

Proof. (i) Assume that $D\colon \mathcal{A} \to (\mathcal{A}_{(\varphi,\psi)})^{(n)}$ is a bounded (φ,ψ)-derivation. Consider $\mathcal{A}^{\#}$-module actions on $(\mathcal{A}_{(\varphi,\psi)})^{(2n)}$ as follows:

$$(a,\alpha)\cdot a^{(2n)} = \varphi(a)a^{(2n)} + \alpha a^{(2n)}, \quad a^{(2n)}\cdot(a,\alpha) = a^{(2n)}\cdot\psi(a) + \alpha a^{(2n)},$$

for all $a \in \mathcal{A}$, $a^{(2n)} \in \mathcal{A}^{(2n)}$ and $\alpha \in \mathbb{C}$. Define the map $\widehat{D}\colon \mathcal{A}^{\#} \to \mathcal{A}^{\#(2n)}$ by $\widehat{D}((a,\alpha)) = D(a)$, for all $a \in \mathcal{A}$. One can check that $\widehat{D}$ is a $(\widehat{\varphi},\widehat{\psi})$-derivation. This shows that $\mathcal{A}$ is (φ,ψ)-$2n$-weakly amenable.

(ii) Since $\mathcal{A}^{\#}$ is unital, without loss of generality, we may assume that

$$D\colon \mathcal{A}^{\#} \to (\mathcal{A}_{(\widehat{\varphi},\widehat{\psi})})^{\#(2n-1)}; \quad a \mapsto \langle a^*, a\rangle\, e^* + \widehat{D}(a)$$

is a continuous $(\widehat{\varphi},\widehat{\psi})$-derivation in which $a^* \in \mathcal{A}^*$. It is easy to see that $\widehat{D}\colon \mathcal{A} \to (\mathcal{A}_{(\varphi,\psi)})^{(2n-1)}$ is a continuous (φ,ψ)-derivation. Thus there exists $a_0^{(2n-1)} \in \mathcal{A}^{(2n-1)}$ such that $\widehat{D}(a) = a_0^{(2n-1)}\cdot\varphi(a) - \psi(a)\cdot a_0^{(2n-1)}$ for all $a \in \mathcal{A}$. Given $a, b \in \mathcal{A}$ we have

$$\begin{aligned}\langle a^*, ab\rangle &= \langle \widehat{D}(b), a\rangle + \langle \widehat{D}(a), b\rangle \\ &= \langle a_0^{(2n-1)}\cdot\varphi(b) - \psi(b)\cdot a_0^{(2n-1)}, a\rangle + \langle a_0^{(2n-1)}\cdot\varphi(a) - \psi(a)\cdot a_0^{(2n-1)}, b\rangle \\ &= \langle a_0^{(2n-1)}, \varphi(b)a - a\psi(b)\rangle + \langle a_0^{(2n-1)}, \varphi(a)b - b\psi(a)\rangle = 0.\end{aligned}$$

Therefore $a^*|_{\mathcal{A}^2} = 0$. By Proposition 2.1, $\mathcal{A}$ is (φ,ψ)-weakly amenable. Now, Proposition 2.2 shows that $\mathcal{A}^2$ is dense in $\mathcal{A}$. Hence $a^* = 0$ and thus $D = \widehat{D}$ is a (φ,ψ)-inner derivation. □

Theorem 2.2. *Let $\mathcal{A}$ be a Banach algebra and $\psi,\varphi,\lambda \in \mathrm{Hom}(\mathcal{A})$. If φ is an epimorphism and $\mathcal{A}$ is $(\psi\circ\varphi, \lambda\circ\varphi)$-$n$-weakly amenable, then $\mathcal{A}$ is (ψ,λ)-n-weakly amenable. The converse is true if φ^2 is an identity map.*

Proof. We show the proof for the even case. The odd case is similar. Let $D\colon \mathcal{A} \to (\mathcal{A}_{(\psi,\lambda)})^{(2n)}$ be a continuous (ψ,λ)-derivation and $\widetilde{D} = D\circ\varphi$. For each $a, b, c \in \mathcal{A}$, we have

$$\begin{aligned}\widetilde{D}(ab) &= (D\circ\varphi)(ab) = D(\varphi(a)\varphi(b)) \\ &= D(\varphi(a))\cdot\lambda(\varphi(b)) + \psi(\varphi(a))\cdot D(\varphi(b)) \\ &= \widetilde{D}(a)\cdot(\lambda\circ\varphi)(b) + (\psi\circ\varphi)(a)\cdot\widetilde{D}(b).\end{aligned}$$

Thus $\widetilde{D}$ is an $(\psi\circ\varphi, \lambda\circ\varphi)$-derivation. So there exists $\Phi \in (\mathcal{A}_{(\psi\circ\varphi,\lambda\circ\varphi)})^{(2n)}$ such that for each $a \in \mathcal{A}$, $\widetilde{D}(a) = (\psi\circ\varphi)(a)\cdot\Phi - \Phi\cdot(\lambda\circ\varphi)(a)$. Let $b \in \mathcal{A}$. Then there exists $a \in \mathcal{A}$ such that $\varphi(a) = b$ and so

$$D(b) = D(\varphi(a)) = \widetilde{D}(a) = \psi(\varphi(a))\cdot\Phi - \Phi\cdot\lambda(\varphi(a))\cdot\Phi = \psi(b)\cdot\Phi - \Phi\cdot\lambda(b).$$

Therefore D is an (ψ,λ)-inner derivation.

Conversely, suppose that $D\colon \mathcal{A} \to (\mathcal{A}_{(\psi\circ\varphi,\lambda\circ\varphi)})^{(2n)}$ is a $(\psi \circ \varphi, \lambda \circ \varphi)$-derivation and let $\widetilde{D} = D \circ \varphi^{-1}$. For every $a, b \in \mathcal{A}$, we get

$$\begin{aligned}\widetilde{D}(ab) = D \circ \varphi^{-1}(ab) &= D(\varphi^{-1}(a)\varphi^{-1}(b)) \\ &= D(\varphi^{-1}(a)) \cdot \lambda \circ \varphi(\varphi^{-1}(b)) + \psi \circ \varphi(\varphi^{-1}(a)) \cdot D(\varphi^{-1}(b)) \\ &= D(\varphi^{-1}(a)) \cdot \lambda(b) + \psi(a) \cdot D(\varphi^{-1}(b)) \\ &= \widetilde{D}(a) \cdot \lambda(b) + \psi(a) \cdot \widetilde{D}(b).\end{aligned}$$

Due to (ψ, λ)-n-weak amenability of $\mathcal{A}$, there exists an $\Psi \in (\mathcal{A}_{(\psi,\lambda)})^{(2n)}$ such that for all $a \in \mathcal{A}$, $\widetilde{D}(a) = \psi(a) \cdot \Psi - \Psi \cdot \lambda(a)$ and thus we have $D(a) = D(\varphi^{-1}(\varphi(a)) = \widetilde{D}(\varphi(a)) = \psi(\varphi(a)) \cdot \Psi - \Psi \cdot \lambda(\varphi(a))$. Therefore D is an $(\psi\circ\varphi, \lambda\circ\varphi)$-inner derivation. $\square$

Corollary 2.1. *Let $\mathcal{A}$ be a Banach algebra and let $\varphi \in \mathrm{Hom}(\mathcal{A})$. Then the following statements hold:*

(i) *If φ is an epimorphism and $\mathcal{A}$ is (φ^m, φ^m)-n-weakly amenable for some $m \in \mathbb{N}$, then $\mathcal{A}$ is n-weakly amenable;*

(ii) *If $\mathcal{A}$ is n-weakly amenable such that $\varphi^2 = 1_{\mathcal{A}}$, then $\mathcal{A}$ is (φ^2, φ^2)-n-weakly amenable.*

Let $\Box$ and $\Diamond$ be the first and second Arens products on the second dual space $\mathcal{A}^{**}$, then $\mathcal{A}^{**}$ is a Banach algebra with respect to both of these products. Similar to [11, Proposition 4.4], we have the following result:

Proposition 2.3. *Let $\mathcal{A}$ be a Banach algebra, $\varphi, \psi \in \mathrm{Hom}(\mathcal{A})$ and let X be a Banach $\mathcal{A}$-bimodule. Suppose $D\colon \mathcal{A} \to X$ is a continuous (φ, ψ)-derivation. Then $D''\colon (\mathcal{A}^{**}, \Box) \to X^{**}$ is a continuous (φ'', ψ'')-derivation.*

Proposition 2.4. *Let $\mathcal{A}$ be a Banach algebra with a bounded approximate identity and let X be a Banach $\mathcal{A}$-bimodule. If $\varphi, \psi \in \mathrm{Hom}(\mathcal{A})$, $D\colon \mathcal{A} \to X^*$ is a continuous (φ, ψ)-derivation and there exists $\sigma \in X^*$ such that*

$$\langle D(a), \varphi(b) \cdot x \cdot \psi(c)\rangle = \langle \psi(a) \cdot \sigma - \sigma \cdot \varphi(a), \varphi(b) \cdot x \cdot \psi(c)\rangle,$$

for all $a, b, c \in \mathcal{A}$ and $x \in X$, then D is (φ, ψ)-inner.

P r o o f. Replacing D with $D - \delta_\sigma$, we may suppose that

$$\langle D(a), \varphi(b) \cdot x \cdot \psi(c)\rangle = 0 \qquad \forall a, b, c \in \mathcal{A},\ x \in X.$$

The above equality shows that

$$\psi(c)\cdot D(a)\cdot\varphi(b)=0 \tag{2.1}$$

for all $a,b,c\in\mathcal{A}$. Assume that $(e_j)\subseteq\mathcal{A}$ is a bounded approximate identity for which the iterated weak*-limit $\sigma_0=\lim_j\lim_k(\psi(e_j)\cdot D(e_k)-D(e_k)\cdot\varphi(e_j))$ exists. For each $b\in\mathcal{A}$ and $x\in X$, by applying (2.1) we get

$$\begin{aligned}
\langle D(b),x\rangle &= \lim_j\lim_k\langle D(e_jbe_k),x\rangle\\
&= \lim_j\lim_k\langle D(e_j)\cdot\varphi(b)\varphi(e_k)+\psi(e_j)\cdot D(be_k),x\rangle\\
&= \lim_j\lim_k\langle D(e_j)\cdot\varphi(b)\varphi(e_k)+\psi(e_j)\cdot D(b)\cdot\varphi(e_k)+\psi(e_j)\psi(b)\cdot D(e_k),x\rangle\\
&= \lim_k\langle D(e_k)\cdot\varphi(b)+\psi(b)\cdot D(e_k),x\rangle\\
&= \lim_j\lim_k\langle\psi(b)\cdot[\psi(e_j)\cdot D(e_k)-D(e_k)\cdot\varphi(e_j)],x\rangle\\
&\quad-\lim_j\lim_k\langle[\psi(e_j)\cdot D(e_k)-D(e_k)\cdot\varphi(e_j)]\cdot\varphi(b),x\rangle\\
&= \langle\psi(b)\cdot\sigma_0-\sigma_0\cdot\varphi(b),x\rangle.
\end{aligned}$$

Consequently, D is a (φ,ψ)-inner derivation. □

Let $m,n\in\mathbb{N}$ and let $\mathcal{A}$ be a Banach algebra. The set of $m\times m$ matrices with entries from $\mathcal{A}$, denoted by $M_m(\mathcal{A})$, is a Banach algebra with product in the obvious way and ℓ^1-norm. Supposing that $\varphi,\psi\in\mathrm{Hom}(\mathcal{A})$, we consider $M_m(\mathcal{A}_{(\varphi,\psi)})$ as a Banach $M_m(\mathcal{A})$-module as follows:

$$(a\cdot x)_{ij}=\sum_{k=1}^{m}\varphi(a_{ik})\cdot x_{kj},\quad (x\cdot a)_{ij}=\sum_{k=1}^{m}x_{ik}\psi(a_{kj}),$$

where $a=(a_{ij})\in M_m(\mathcal{A})$, $x=(x_{ij})\in M_m(\mathcal{A}_{(\varphi,\psi)})$. We identify $M_m(\mathcal{A}_{(\varphi,\psi)})^{(n)}$ with $M_m((\mathcal{A}_{(\varphi,\psi)})^{(n)})$ as Banach $\mathcal{A}$-modules and thus

$$(a\cdot x^{(2n)})_{ij}=\sum_{k=1}^{m}\varphi(a_{jk})\cdot x_{ik}^{(2n)},\ (x^{(2n)}\cdot a)_{ij}=\sum_{k=1}^{m}x_{kj}^{(2n)}\cdot\psi(a_{ki}) \tag{2.2}$$

$$(a\cdot x^{(2n-1)})_{ij}=\sum_{k=1}^{m}\psi(a_{jk})\cdot x_{ik}^{(2n-1)},\ (x^{(2n-1)}\cdot a)_{ij}=\sum_{k=1}^{m}x_{kj}^{(2n-1)}\cdot\psi(a_{ki}) \tag{2.3}$$

where $a=(a_{ij})\in M_m(\mathcal{A})$ and $x^{(n)}=(x_{ij}^{(n)})\in M_m((\mathcal{A}_{(\varphi,\psi)})^{(n)})$. For $a\in\mathcal{A}$ and $i,j\in\mathbb{N}$, we put $(a)_{ij}=a\otimes\varepsilon_{ij}\in M_m(\mathcal{A})$, where ε_{ij} is the matrix whose entries is 1 if $i=j$, and zero otherwise.

Theorem 2.3. *Let $\mathcal{A}$ be a Banach algebra with identity, $\varphi, \psi \in \operatorname{Hom}(\mathcal{A})$ and let $I\colon M_m(\mathcal{A}) \to M_m(\mathcal{A})$ be the identity map. Then*

(i) *$\mathcal{A}$ is (φ,ψ)-$2n$-weakly amenable if and only if $M_m(\mathcal{A})$ is $(\varphi\otimes I, \psi\otimes I)$-$2n$-weakly amenable;*

(ii) *$\mathcal{A}$ is (φ,ψ)-$(2n-1)$-weakly amenable if and only if $M_m(\mathcal{A})$ is $(\varphi\otimes I, \psi\otimes I)$-$(2n-1)$-weakly amenable.*

Proof. (i) Suppose $\mathcal{A}$ is (φ,ψ)-$2n$-weakly amenable and $D\colon M_m(\mathcal{A}) \to M_m((\mathcal{A}_{(\varphi,\psi)})^{2n})$ is a $(\varphi\otimes I, \psi\otimes I)$-derivation. We are regarding M_m, the Banach algebra of $m\times m$ matrices with entries from $\mathbb{C}$, as a subalgebra $M_m(\mathcal{A})$. Since M_m is amenable, there exists $x^{(2n)} = (x_{ij}^{(2n)}) \in M_m((\mathcal{A}_{(\varphi,\psi)})^{2n})$ such that $D|_{M_m} = \delta_{x^{(2n)}}|_{M_m}$. Replacing $D - \delta_{x^{(2n)}}$ by D, we may suppose $D|_{M_m} = 0$. For $a \in \mathcal{A}$ and $r,s\in\mathbb{N}$, set $D((a)_{rs}) = (d_{ij}^{r,s}\colon i,j\in\mathbb{N}_m) \in M_m((\mathcal{A}_{(\varphi,\psi)})^{2n})$ and $d_{11}^{(1,1)} = d(a)$. We have $D((a)_{rs}) = D(\varepsilon_{r1}(a)_{11}\varepsilon_{1s}) = \varepsilon_{r1}\cdot D((a)_{11})\cdot\varepsilon_{1s}$, since $D(\varepsilon_{r1}) = D(\varepsilon_{1s}) = 0$. According to (2.2), we have $d_{ij}^{(r,s)} = 0$ unless $(i,j) = (r,s)$, and in this case set $d_{rs}^{(r,s)} = d(a)$. It is easy to see that $d\colon \mathcal{A}\to\mathcal{A}^{(2n)}$ is a (φ,ψ)-derivation. By assumption there exists $a^{(2n)}\in\mathcal{A}^{(2n)}$ such that

$$d(a) = \varphi(a)\cdot a^{(2n)} - a^{(2n)}\cdot\psi(a) \quad \forall a\in\mathcal{A}. \tag{2.4}$$

Take $X\in M_m((\mathcal{A}_{(\varphi,\psi)})^{2n})$ to be the matrix that has $a^{(2n)}$ in each diagonal position and zero elsewhere. By (2.2) and (2.4), we have

$$D((a)_{ij}) = (\varphi(a))_{ij}\cdot X - X\cdot(\psi(a))_{ij} = (\varphi\otimes I)(a\otimes\varepsilon_{ij})\cdot X - (\psi\otimes I)(a\otimes\varepsilon_{ij}).$$

On the other hand,

$$D((a_{ij})) = (\varphi\otimes I)((a_{ij}))\cdot X - X\cdot(\psi\otimes I)((a_{ij})).$$

The above equalities show that $M_m((\mathcal{A}_{(\varphi,\psi)})^{(2n)})$ is $(\varphi\otimes I,\psi\otimes I)$-$2n$-weakly amenable.

Conversely, assume that $M_m((\mathcal{A}_{(\varphi,\psi)})^{(2n)})$ is $(\varphi\otimes I,\psi\otimes I)$-$2n$-weakly amenable and $D\colon\mathcal{A}\to(\mathcal{A}_{(\varphi,\psi)})^{(2n)}$ is a (φ,ψ)-derivation. It is easy to check that $D\otimes I\colon \mathcal{A}\otimes M_m \to (\mathcal{A}_{(\varphi,\psi)})^{(2n)}\otimes M_m$ is a $(\varphi\otimes I,\psi\otimes I)$-derivation. We identify $M_m((\mathcal{A}_{(\varphi,\psi)})^{(2n)})$ with $(\mathcal{A}_{(\varphi,\psi)})^{(2n)}\otimes M_m$. By assumption there exists $x\in(\mathcal{A}_{(\varphi,\psi)})^{(2n)}\otimes M_m$ such that $x = \sum\limits_{i,j=1}^{m} x_{ij}^{(2n)}\otimes\varepsilon_{ij}$ and $D\otimes I = \delta_x$. For each $a\in\mathcal{A}$, we have

$$\begin{aligned} D\otimes I(a\otimes\varepsilon_{11}) &= D(a)\otimes\varepsilon_{11} = (a\otimes\varepsilon_{11})\cdot x - x\cdot(a\otimes\varepsilon_{11}) \\ &= (\varphi(a)\otimes\varepsilon_{11})\cdot x - x\cdot(\psi(a)\otimes\varepsilon_{11}) \\ &= \sum_{i=1}^{m}(\varphi(a)\cdot x_{i1}^{(2n)})\otimes\varepsilon_{i1} - \sum_{j=1}^{m}(x_{1j}^{(2n)}\cdot\psi(a))\otimes\varepsilon_{1j}. \end{aligned}$$

The above equalities imply that $D(a) = \varphi(a) \cdot x_{11}^{(2n)} - x_{11}^{(2n)} \cdot \psi(a)$, so $\mathcal{A}$ is (φ, ψ)-$2n$-weakly amenable.

(ii) The proof is similar to (i). $\square$

Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras, $n \in \mathbb{N}$ and $\theta\colon \mathcal{A} \to \mathcal{B}$ is a continuous homomorphism. Then $\mathcal{B}^{(n)}$ (the n-th dual of $\mathcal{B}$) can be regarded as $\mathcal{A}$-module under the module actions

$$a \cdot b^{(n)} = \theta(a) \cdot b^{(n)}, \quad b^{(n)} \cdot a = b^{(n)} \cdot \theta(a) \quad \forall a \in \mathcal{A},\ b^{(n)} \in \mathcal{B}^{(n)}.$$

Let $n \in \mathbb{N}$. Then, the n-th adjoint map of θ is $\mathcal{A}$-module homomorphism.

Theorem 2.4. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras and let $\varphi, \psi \in \operatorname{Hom}(\mathcal{B})$. Let $\theta_1 \in \operatorname{Hom}(\mathcal{A}, \mathcal{B})$ and $\theta_2 \in \operatorname{Hom}(\mathcal{B}, \mathcal{A})$ such that $\theta_1 \circ \theta_2 = I_{\mathcal{B}}$, $n \in \mathbb{N}$. Then the maps $\widetilde{\varphi} = \theta_2 \circ \varphi \circ \theta_1$ and $\widetilde{\psi} = \theta_2 \circ \psi \circ \theta_1$ are in $\operatorname{Hom}(\mathcal{A})$ and*

(i) *If $\mathcal{A}$ is $(\widetilde{\varphi}, \widetilde{\psi})$-$2n$-weakly amenable, then $\mathcal{B}$ is (φ, ψ)-$2n$-weakly amenable;*

(ii) *If $\mathcal{A}$ is $(\widetilde{\varphi}, \widetilde{\psi})$-$(2n-1)$-weakly amenable, then $\mathcal{B}$ is (φ, ψ)-$(2n-1)$-weakly amenable.*

Proof. Obviously $\widetilde{\varphi}, \widetilde{\psi} \in \operatorname{Hom}(\mathcal{A})$.

(i) Suppose $\mathcal{A}$ is $(\widetilde{\varphi}, \widetilde{\psi})$-$2n$-weakly amenable and $D\colon \mathcal{B} \to (\mathcal{B}_{(\varphi,\psi)})^{(2n)}$ is a (φ, ψ)-derivation. The map $\widetilde{D} = \theta_2^{(2n)} \circ D \circ \theta_1\colon \mathcal{A} \to (\mathcal{A}_{(\varphi,\psi)})^{(2n)}$ is a bounded linear map. For each $a_1, a_2 \in \mathcal{A}$, we have

$$\begin{aligned}
\widetilde{D}(a_1 a_2) &= \theta_2^{(2n)} \circ D \circ \theta_1(a_1 a_2) = \theta_2^{(2n)}(D(\theta_1(a_1)\theta_1(a_2))) \\
&= \theta_2^{(2n)}(D(\theta_1(a_1)) \cdot \psi(\theta_1(a_2)) + \varphi(\theta_1(a_1)) \cdot D(\theta_1(a_2))) \\
&= \theta_2^{(2n)} \circ D \circ \theta_1(a_1) \cdot \theta_2(\psi(\theta_1(a_2))) + \theta_2(\varphi(\varphi(a_1))) \cdot \theta_2^{(2n)} \circ D \circ \theta_1(a_2) \\
&= \theta_2^{(2n)} \circ D \circ \theta_1(a_1) \cdot \widetilde{\psi}(a_2) + \widetilde{\varphi}(a_1) \cdot \theta_2^{(2n)} \circ D \circ \theta_1(a_2).
\end{aligned}$$

Then $\widetilde{D}$ is a $(\widetilde{\varphi}, \widetilde{\psi})$-derivation, hence there exists $x \in (\mathcal{A}_{\varphi,\psi})^{(2n)}$ such that

$$\widetilde{D}(a) = \widetilde{\varphi}(a) \cdot x - x \cdot \widetilde{\psi}(a) \quad \forall a \in \mathcal{A}. \tag{2.5}$$

It is easy to check that $\theta_1^{(2n)}(\widetilde{\varphi}(\theta_2(b)) \cdot x) = \varphi(b) \cdot \theta_1^{(2n)}(x)$ and $\theta_1^{(2n)}(x \cdot \widetilde{\psi}(\theta_2)) = \theta_1^{(2n)}(x) \cdot \psi(b)$ for $b \in \mathcal{B}$. Also, $\theta_1^{(2n)} \circ \theta_2^{(2n)} = I_{\mathcal{B}^{(2n)}}$. By (2.5), we obtain

$$\begin{aligned}
D(b) &= \theta_1^{(2n)} \circ \theta_2^{(2n)} \circ D \circ \theta_1 \circ \theta_2(b) \\
&= \theta_1^{(2n)}(\theta_2^{(2n)} \circ D \circ \theta_1 \circ \theta_2(b)) \\
&= \theta_1^{(2n)}(\widetilde{\varphi}(\theta_2(b)) \cdot x - x \cdot \widetilde{\psi}(\theta_2(b))) \\
&= \varphi(b) \cdot \theta_1^{(2n)}(x) - \theta_1^{(2n)}(x) \cdot \psi(b),
\end{aligned}$$

for all $b \in \mathcal{B}$ and $\theta_1^{(2n)}(x) \in (\mathcal{B}_{(\varphi,\psi)})^{(2n)}$. Therefore $\mathcal{B}$ is (φ, ψ)-$2n$-weakly amenable.

(ii) The proof is similar to (i). $\square$

In the case when φ and ψ are identity maps, we see that the homomorphic image of an n-weakly amenable Banach algebra is again n-weakly amenable.

Corollary 2.2. *Let $\varphi, \psi \in \mathrm{Hom}(\mathcal{B}), n \in \mathbb{N}$. Let $\mathcal{A}$ be a Banach algebra such that $\mathcal{A} = \mathcal{B} \oplus I$ for a closed subalgebra $\mathcal{B}$ and closed ideal I. If $\mathcal{A}$ is $(\iota \circ \varphi \circ P, \iota \circ \psi \circ P)$-$n$-weakly amenable, then $\mathcal{B}$ is (φ, ψ)-n-weakly amenable, where $P\colon \mathcal{A} \to \mathcal{B}$ is the natural projection and $\iota\colon \mathcal{B} \to \mathcal{A}$ is the inclusion map.*

3. Examples

For any Banach space X, we will say that a net $(m_\alpha) \subseteq X^*$ converges weak$^{\approx}$ to $m \in X^*$ if $m_\alpha \to m$ in weak* topology and $\|m_\alpha\| \to \|m\|$. This notion was introduced by Lau and Loy in [9]. In particular, if $\mu \in M(G)$, assume that $\nu \in L^\infty(G)^*$ is a norm preserving extension of μ. Then there exists a net $(\varphi_j) \subseteq L^1(G)$ with $\|\varphi_j\| \leqslant \|\mu\|$ and $\varphi_j \to \nu$. Passing to a suitable subnet we may assume that $\|\varphi_j\| \to \|\mu\|$. Hence, we have $\varphi_j \to \mu$ in weak$^{\approx}$ topology. If $\varphi \in \mathrm{Hom}(L^1(G))$, the we can extend φ to a homomorphism $\widehat{\varphi}$ on $M(G)$. Now, we need the following result which is analogous to [5, Theorem 2.4]. Since the proof is similar, it is omitted.

Theorem 3.1. *Let G be a locally compact group and let $\varphi, \psi \in \mathrm{Hom}(L^1(G))$. Let $L^1(G)$ be a $M(G)$-bimodule by module actions $\mu \cdot f = \widehat{\varphi}(\mu) * f$ and $f \cdot \mu = f * \widehat{\psi}(\mu)$ for each $f \in L^1(G)$ and $\mu \in M(G)$. Then every (φ, ψ)-derivation $D\colon L^1(G) \to L^1(G)^{(2n)}$ extends to a unique $(\widehat{\varphi}, \widehat{\psi})$-derivation $M(G)$ into $L^1(G)^{(2n)}$.*

Let G be a locally compact group, $\varphi, \psi \in \mathrm{Hom}(L^1(G))$ and let X be a Banach space. Suppose that G acts on X from left (right), i.e., we have a continuous mapping $(g, x) \mapsto g \cdot x$ ($(x, g) \mapsto x \cdot g$) from $G \times X$ into X in which $g \cdot x = \varphi(g) \cdot x$ ($x \cdot g = x \cdot \psi(g)$). A map $d\colon G \to X$ is called a (φ, ψ)-*derivation* if

$$d(gh) = d(g) \cdot \varphi(h) + \psi(g) \cdot d(h) \qquad \forall g, h \in G.$$

The (φ, ψ)-derivation d is called (φ, ψ)-*inner* if there exists $x \in X$ such that $d(g) = x \cdot \varphi(g) - \psi(g) \cdot x$, for every $g \in G$. In this case we write $d = ad_x$. A map $T\colon G \to X$ is called a (φ, ψ)-*crossed homomorphism* if

$$T(gh) = \psi(g) \cdot T(h) \cdot \varphi(g)^{-1} + T(g),$$

for every $g, h \in G$, and T is called (φ, ψ)-*principal* if there exists $x \in X$ such that $T(s) = \psi(g) \cdot x \cdot \varphi(g)^{-1} - x$, for every $g \in G$. Let $d\colon G \to X$ be a (φ, ψ)-derivation and set $T(g) = d(g) \cdot \varphi(g)^{-1}$, for $g \in G$. Then T is a crossed homomorphism and T is principal if d is (φ, ψ)-inner. Conversely, let $T\colon G \to X$ be a (φ, ψ)-crossed homomorphism. Set $d(g) = T(g) \cdot \varphi(g)$ for $g \in G$. Then d is a (φ, ψ)-derivation and d is (φ, ψ)-inner if T is principal. Let $D\colon \ell^1(G) \to X^*$ be continuous (φ, ψ)-derivation. Set $d(g) = D(\delta_g)$ for every $g \in G$. Then d is a (φ, ψ)-derivation and it is clear that if D is an (φ, ψ)-inner derivation then so is d. Similar to [10, Theorem 1.1], we have the following result:

Theorem 3.2. *Let G be a (discrete) group and X a locally compact space on which G has a 2-sided action as above. Then any bounded (φ, ψ)-derivation $D\colon G \to M(X)$ is (φ, ψ)-inner.*

In the following example, we use techniques of the proofs from [3] and [4, Theorem 4.1] to show that $L^1(G)$ is (φ, ψ)-n-weakly amenable for all $n \in \mathbb{N}$ and $\varphi, \psi \in \mathrm{Hom}(L^1(G))$.

Example 3.1. Let G be a locally compact group and $\varphi, \psi \in \mathrm{Hom}(L^1(G))$ be nonzero (for the cases where φ or ψ is zero homomorphism, refer to Example 3.2). It is known that $L^1(G)$ has a bounded approximate identity (e_α) with $\|e_\alpha\| \leqslant 1$ for all α. By [2, Proposition 28.7], there exists $E \in L^1(G)^{**}$ such that $\|E\| = 1$ and E is a right identity for $(L^1(G)^{**}, \Box)$. Since $L^1(G)$ is a closed ideal of measure algebra $M(G)$, the Banach algebra $(L^1(G)^{**}, \Box)$ is a closed ideal in $(M(G)^{**}, \Box)$. Hence, the map $\mathcal{T}\colon M(G) \to (L^1(G)^{**}, \Box)$ defined by $\mathcal{T}(\mu) = E \,\Box\, \mu$ is an isometric embedding. We write E_g for $E \,\Box\, \delta_g$, where $g \in G$. Obviously, $E_{gh} = E_g \,\Box\, E_h$ for all $g, h \in G$.

Let $X = L^1(G)^{(2k+2)}$ and $D\colon L^1(G) \to X$ be a (φ, ψ)-derivation. Then $D''\colon (\mathcal{A}^{**}, \Box) \to X^{**}$ is a bounded (φ'', ψ'')-derivation by Proposition 2.3. For any $g, h \in G$, we have

$$D''(E_{gh}) = D''(E_g) \cdot \varphi''(E_h) + \psi''(E_g) \cdot D''(E_h)$$

and thus

$$\begin{aligned} \psi''(E_{(gh)^{-1}}) \cdot D''(E_{gh}) = \psi''(E_{h^{-1}}) \cdot (\psi''(E_{g^{-1}}) \cdot D''(E_g)) \cdot \varphi''(E_h) \\ + \psi''(E_{h^{-1}}) \cdot D''(E_h). \end{aligned} \tag{3.1}$$

Since X^* is the underling space of a commutative von Neumann algebra, it is an L^∞-space. Thus the real-valued functions in X^* form the space $X^*_{\mathbb{R}}$ which is a complete lattice, that is, every non-empty bounded subset of $X^*_{\mathbb{R}}$ has a supremum. Easily, we can see that

$$\mathrm{Re}(\psi''(E_g) \cdot \Phi) = \psi''(E_g) \cdot (\mathrm{Re}\ \Phi) \qquad \forall g \in G,\ \Phi \in X^*. \tag{3.2}$$

Similar to the proof of [4, Theorem 4.1], we can prove that

$$\psi''(E) \cdot \sup\{\psi''(E_g) \cdot \Phi\colon\ \Phi \in Y\} = \psi''(E_g) \cdot \sup\{\psi''(E) \cdot \Phi\colon\ \Phi \in Y\} \tag{3.3}$$

and

$$\sup\{\psi''(E_g) \cdot \Phi\colon\ \Phi \in Y\} \cdot \psi''(E) = \sup\{\psi''(E) \cdot \Phi\colon\ \Phi \in Y\} \cdot \psi''(E_g) \tag{3.4}$$

for all $g \in G$, where Y is an arbitrary bounded subset of $X^*_{\mathbb{R}}$. Put

$$S = \sup\{\psi''(E_{g^{-1}}) \cdot \operatorname{Re} D''(E_g)\colon\ g \in G\};$$

the supremum being taken in the complete lattice $X^*_{\mathbb{R}}$. It follows from (3.1)–(3.4) that

$$\psi''(E) \cdot S \cdot \psi''(E) = \psi''(E_{h^{-1}}) \cdot S \cdot \psi''(E_h) + \psi''(E_{h^{-1}}) \cdot \operatorname{Re} D''(E_h) \cdot \psi''(E).$$

If $\psi''(E_h)$ acts from the left on the above equality, we get

$$\psi''(E) \cdot \operatorname{Re} D''(E_h) \cdot \varphi''(E) = \psi''(E_h) \cdot S \cdot \varphi''(E) - \psi''(E) \cdot S \cdot \varphi''(E_h). \tag{3.5}$$

Similarly for the imaginary part of $D''(E_h)$, there exists an element T such that

$$\psi''(E) \cdot \operatorname{Im} D''(E_h) \cdot \varphi''(E) = \psi''(E_h) \cdot T \cdot \varphi''(E) - \psi''(E) \cdot T \cdot \varphi''(E_h). \tag{3.6}$$

Taking $\Psi = S + \mathrm{i}T \in X^{***}$ and using (3.5) and (3.6), we deduce that

$$\psi''(E) \cdot D''(E_h) \cdot \varphi''(E) = \psi''(E_h) \cdot \Psi \cdot \varphi''(E) - \psi''(E) \cdot \Psi \cdot \varphi''(E_h).$$

Therefore for each discrete measure $\zeta \in \ell^1(G)$, we have

$$\psi''(E) \cdot D''(E \,\Box\, \zeta) \cdot \varphi''(E) = \psi''(E \,\Box\, \zeta) \cdot \Psi \cdot \varphi''(E) - \psi''(E) \cdot \Psi \cdot \varphi''(\zeta) \cdot \varphi''(E).$$

Now, assume that $f, g \in L^1(G)$, then

$$\psi(f) \cdot D''(E \,\Box\, \zeta) \cdot \varphi(g) = \psi(f * \zeta) \cdot \Psi \cdot \varphi(g) - \psi(f) \cdot \Psi \cdot \varphi(\zeta * g). \tag{3.7}$$

Given $h \in L^1(G)$, there is a net (ζ_j) of discrete measure such that $\zeta_j \to h$ in the strong operator topology on $L^1(G)$. So $\lim_j \varphi(\zeta_j * f) = \varphi(h * f)$ and $\lim_j \varphi(f * \zeta_j) = \varphi(f * h)$ for all $f \in L^1(G)$. Similarly, we have the same for ψ.

For each $f, g \in L^1(G)$, we have

$$\begin{aligned}\lim_j \psi(f) \cdot D''(E \,\square\, \zeta_j) \cdot \varphi(g) &= \lim_j (D''(f * \zeta_j) \cdot \varphi(g) - D''(f) \cdot \varphi(\zeta_j * g)) \\ &= D''(f * h) \cdot \varphi(g) - D''(f) \cdot \varphi(h * g) \\ &= D''(f) \cdot \varphi(h * g) + \psi(f) \cdot D''(h) \cdot \varphi(g) - D''(f) \cdot \varphi(h * g) \\ &= \psi(f) \cdot D''(h) \cdot \varphi(g).\end{aligned}$$

On the other hand,

$$\begin{aligned}\psi(f) \cdot D''(h) \cdot \varphi(g) &= \psi(f * h) \cdot \Psi \cdot \varphi(g) - \psi(f) \cdot \Psi \cdot \varphi(h * g) \\ &= \psi(f) \cdot (\psi(h) \cdot \Psi - \Psi \cdot \varphi(h)) \cdot \varphi(g).\end{aligned}$$

Note that in the above equalities we have used the relation (3.7). Let $P\colon X^{***} \to X^*$ be the natural projection such that P is an $L^1(G)$-bimodule morphism. We have $D = P \circ D''$. Put $\Psi_0 = P(\Psi)$. Then

$$\psi(f) \cdot D(h) \cdot \varphi(g) = \psi(f) \cdot (\psi(h) \cdot \Psi_0 - \Psi_0 \cdot \varphi(h)) \cdot \varphi(g) \quad \forall f, g, h \in L^1(G)$$

and thus

$$\langle D(h), \varphi(g) \cdot x \cdot \psi(f) \rangle = \langle \psi(h) \cdot \Psi_0 - \Psi_0 \cdot \varphi(h), \varphi(g) \cdot x \cdot \psi(f) \rangle$$

for all $f, g, h \in L^1(G)$ and $x \in X$. Now, Proposition 2.4 shows that D is a (φ, ψ)-inner derivation and so $L^1(G)$ is (φ, ψ)-$(2k+1)$-weakly amenable.

Let $D\colon L^1(G) \to L^1(G)^{(2k)}$ be a continuous (φ, ψ)-derivation. By similar techniques as those of Theorem 3.1, we can extend D to a derivation $D\colon M(G) \to L^1(G)^{(2k)}$, where the measure algebra $M(G)$ acts on $L^1(G)^{(2k)}$ through dualizations of the actions on $L^1(G)$ defined in Theorem 3.1. Hence $L^1(G)^{(2k)}$ is isomorphic, as an $M(G)$-bimodule, to $M(X)$ for some compact space X. The action of point masses on $M(X)$ is as follows:

$$\delta_g \cdot \Omega = \varphi(\delta_g) \cdot \Omega, \quad \Omega \cdot \delta_g = \Omega \cdot \psi(\delta_g) \qquad \forall g \in G,\ \Omega \in M(X).$$

These actions are equivalent to actions of G on $M(X)$ and $g \mapsto \widehat{D}(\delta_g)$ is a bounded (φ, ψ)-derivation from G into $M(X)$. By Theorem 3.2, this derivation is (φ, ψ)-inner and this suffices to conclude that $D\colon M(G) \to L^1(G)^{(2k)}$ is inner, by weak* continuity of D. Therefore $L^1(G)$ is (φ, ψ)-$(2k)$-weakly amenable.

Example 3.2. Let $\mathcal{A}$ be a Banach algebra with a bounded approximate identity. It is proved in [1, Example 4.2] that $\mathcal{A}$ is $(0,\psi)$-weakly amenable and $(\varphi,0)$-weakly amenable. The same process can be applied to show that $\mathcal{A}$ is $(0,\psi)$-n-weakly amenable and $(\varphi,0)$-n-weakly amenable for all $n\in\mathbb{N}$. Therefore every group algebra and C^*-algebra is $(\varphi,0)$ and $(0,\psi)$-n-weakly amenable for all $n\in\mathbb{N}$.

Example 3.3. Suppose that X is an infinite set and x_0 is a fixed element in X. Define an algebra product in $l^1(X)$ via $ab := a(x_0)b$ for all $a,b\in l^1(X)$. This Banach algebra has been introduced by Yong Zang in [13]. For every $\varphi,\psi\in\mathrm{Hom}(\mathcal{A})$, we wish to show that $\mathcal{A}$ is (φ,ψ)-$(2n-1)$-weakly amenable for all $n\in\mathbb{N}$. This Banach algebra has a left identity e_0 defined by

$$e_0(x)=\begin{cases}1 & \text{if } x=x_0,\\ 0 & \text{if } x\neq x_0.\end{cases}$$

The $l^1(X)$-bimodule actions on the dual module $l^1(X)^*=l^\infty(X)$ are in fact formulated as follows:

$$f\cdot a=a(x_0)f \quad a\cdot f=f(a)e_0^* \qquad \forall a\in l^1(X),\ f\in l^\infty(X).$$

where e_0^* is the element of $l^\infty(X)$ satisfying $e_0^*(x_0)=1$ and $e_0^*(x)=0$ for $x\neq x_0$. Let $\varphi\colon l^\infty(X)\to l^\infty(X)$ be a non-zero homomorphism. Then

$$a(x_0)\varphi(b)=\varphi(a(x_0)b)=\varphi(ab)=\varphi(a)\varphi(b)=\varphi(a)(x_0)\varphi(b).$$

Hence, $\varphi(b)(\varphi(a)(x_0)-a(x_0))$ for all $a,b\in l^1(X)$. Since φ is non-zero,

$$\varphi(a)(x_0)=a(x_0) \qquad \forall a\in l^1(X). \tag{3.8}$$

Now, suppose that $\varphi,\psi\in\mathrm{Hom}(l^1(X))$ and $D\colon l^1(X)\to(l^1(X)_{(\varphi,\psi)})^{(2n-1)}$ is a bounded (φ,ψ)-derivation. For each $a,b\in l^1(X)$, we have

$$\begin{aligned}a(x_0)D(b)&=D(a(x_0)b)=D(ab)\\&=D(a)\cdot\varphi(b)+\psi(a)\cdot D(b)\\&=\varphi(b)(x_0)D(a)+\psi(a)\cdot D(b).\end{aligned}$$

Letting $b=a$ in the above equalities and using (3.8), we get $\psi(a)\cdot D(a)=0$ for all $a\in l^1(X)$. The last equality implies that $\psi(a)\cdot D(b)=-\psi(b)\cdot D(a)$ for all $a,b\in l^1(X)$. Thus

$$\begin{aligned}D(a)&=D(e_0a)=D(e_0)\cdot\varphi(a)+\psi(e_0)\cdot D(a)\\&=D(e_0)\cdot\varphi(a)-\psi(a)\cdot D(e_0)\end{aligned}$$

for all $a\in l^1(X)$. Therefore $\mathcal{A}$ is (φ,ψ)-$(2n-1)$-weakly amenable for all $n\in\mathbb{N}$. Since $\mathcal{A}$ does not have a bounded right approximate identity [13], $\mathcal{A}$ can not be amenable.

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