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MEROMORPHIC FUNCTIONS THAT SHARE A NONZERO POLYNOMIAL IM

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Abstract. We study the uniqueness theorems of meromorphic functions concerning differential polynomials sharing a nonzero polynomial IM, and obtain two theorems which will supplement two recent results due to X. M. Li and L. Gao.

Keywords: uniqueness, meromorphic function, differential polynomials

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1. Introduction, definitions and results

In this paper, by meromorphic functions we will always mean meromorphic functions in the complex plane. We adopt the standard notation in the Nevanlinna theory of meromorphic functions as explained in [7], [14] and [15]. For a nonconstant meromorphic function h, we denote by $T(r,h)$ the Nevanlinna characteristic of h and by $S(r,h)$ any quantity satisfying $S(r,h)=o\{T(r,h)\}$ as $r\to\infty$ possibly outside a set of finite linear measure. A meromorphic function $a(z)$ ($\not\equiv\infty$) is called a small function with respect to f provided that $T(r,a)=S(r,f)$.

Let f and g be two nonconstant meromorphic functions, and let a be a finite value. We say that f and g share the value a CM provided that $f-a$ and $g-a$ have the same zeros with the same multiplicities. Similarly, we say that f and g share a IM provided that $f-a$ and $g-a$ have the same zeros ignoring multiplicities. In addition, we say that f and g share ∞ CM if $1/f$ and $1/g$ share 0 CM, and we say that f and g share ∞ IM if $1/f$ and $1/g$ share 0 IM (see [15]). Throughout this paper, we need the following definition:

$$\Theta(a,f)=1-\limsup_{r\to\infty}\frac{\overline{N}(r,a;f)}{T(r,f)},$$

where a is a value in the extended complex plane.

In 1959, W. K. Hayman proved the following theorem:

Theorem A (see [6, Corollary of Theorem 9]). *Let* f *be a transcendental meromorphic function, and let* $n \geqslant 3$ *be an integer. Then* $f^n f' = 1$ *has infinitely many solutions.*

In 1997, C. C. Yang and X. H. Hua proved the following result, which corresponded to Theorem A.

Theorem B (see [13, Theorem 1]). *Let* f *and* g *be two nonconstant meromorphic functions, and let* $n \geqslant 11$ *be a positive integer. If* $f^n f'$ *and* $g^n g'$ *share* 1 *CM, then either* $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, *where* c_1, c_2 *and* c *are three finite nonzero complex numbers satisfying* $(c_1 c_2)^{n+1} c^2 = -1$, *or* $f = tg$ *for a finite complex number* t *such that* $t^{n+1} = 1$.

In 2000, M. L. Fang proved the following result:

Theorem C (see [4, Theorem 2]). *Let* f *be a transcendental meromorphic function, and let* $n \geqslant 1$ *be a positive integer. Then* $f^n f' - z = 0$ *has infinitely many solutions.*

In 2000, M. L. Fang and H. L. Qiu proved the following result, which corresponded to Theorem C.

Theorem D (see [5, Theorem 1]). *Let* f *and* g *be two nonconstant meromorphic functions, and let* $n \geqslant 11$ *be a positive integer. If* $f^n f' - z$ *and* $g^n g' - z$ *share* 0 *CM, then either* $f(z) = c_1 e^{cz^2}$ *and* $g(z) = c_2 e^{-cz^2}$, *where* c_1, c_2 *and* c *are three finite nonzero complex numbers satisfying* $4(c_1 c_2)^{n+1} c^2 = -1$, *or* $f = tg$ *for a finite complex number* t *such that* $t^{n+1} = 1$.

In 2003, W. Bergweiler and X. C. Pang proved the following result:

Theorem E (see [3, Theorem 1.1]). *Let* f *be a transcendental meromorphic function, and let* $R \not\equiv 0$ *be a rational function. If all zeros and poles of* f *are multiple, except possibly finitely many, then* $f' - R = 0$ *has infinitely many solutions.*

Now the following question arises:

Question 1. Similarly to Theorem B and Theorem D, does there exist a unicity theorem corresponding to Theorem E?

Recently X. M. Li and L. Gao proved the following uniqueness theorems dealing with Question 1.

Theorem F (see [11, Theorem 1.1]). *Let f and g be two transcendental meromorphic functions, let $n \geqslant 11$ be a positive integer, and let $P \not\equiv 0$ be a polynomial with its degree $\gamma_P \leqslant 11$. If $f^n f' - P$ and $g^n g' - P$ share 0 CM, then either $f = tg$ for a complex number t satisfying $t^{n+1} = 1$, or $f = c_1 \mathrm{e}^{cQ}$ and $g = c_2 \mathrm{e}^{-cQ}$, where c_1, c_2 and c are three nonzero complex numbers satisfying $(c_1 c_2)^{n+1} c^2 = -1$, and Q is a polynomial satisfying $Q = \int_0^z P(\eta)\,\mathrm{d}\eta$.*

Theorem G (see [11, Theorem 1.2]). *Let f and g be two transcendental meromorphic functions, let $n \geqslant 15$ be a positive integer, and let $P \not\equiv 0$ be a polynomial. If $(f^n(f-1))' - P$ and $(g^n(g-1))' - P$ share 0 CM and $\Theta(\infty, f) > 2/n$, then $f = g$.*

Naturally one may ask the following question which is the motivation of the present paper.

Question 2. Can one obtain IM-analogues of Theorem F and Theorem G?

We will prove the following results, which deal with Question 2.

Theorem 1. *Let f and g be two transcendental meromorphic functions, let n ($\geqslant 23$) be a positive integer, and let $P \not\equiv 0$ be a polynomial with its degree $\gamma_P \leqslant 23$. If $f^n f' - P$ and $g^n g' - P$ share 0 IM, then either $f = tg$ for a complex number t satisfying $t^{n+1} = 1$, or $f = c_1 \mathrm{e}^{cQ}$ and $g = c_2 \mathrm{e}^{-cQ}$, where c_1, c_2 and c are three nonzero complex numbers satisfying $(c_1 c_2)^{n+1} c^2 = -1$, and Q is a polynomial satisfying $Q = \int_0^z P(\eta)\,\mathrm{d}\eta$.*

Theorem 2. *Let f and g be two transcendental meromorphic functions, let n, m be two positive integers, and let $P \not\equiv 0$ be a polynomial. If $(f^n(f-1)^m)' - P$ and $(g^n(g-1)^m)' - P$ share 0 IM, then each of the following assertions hold:*

- (i) *when $m = 1$, $n \geqslant 30$ and $\Theta(\infty, f) + \Theta(\infty, g) > 4/n$, then $f = g$;*
- (ii) *when $m \geqslant 2$ and $n \geqslant 4m + 26$, then either $f = g$ or f and g satisfy the algebraic equation $R(f, g) = 0$, where*

$$R(w_1, w_2) = w_1^n (w_1 - 1)^m - w_2^n (w_2 - 1)^m.$$

We now explain some definitions and notations which are used in the paper.

Definition 1 [9]. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $N(r, a; f \mid = 1)$ the counting functions of simple a-points of f. For a positive integer p we denote by $N(r, a; f \mid \leqslant p)$ the counting function of those a-points of f (counted with proper multiplicities) whose multiplicities are not greater than p. By $\overline{N}(r, a; f \mid \leqslant p)$ we denote the corresponding reduced counting function. In an analogous manner we define $N(r, a; f \mid \geqslant p)$ and $\overline{N}(r, a; f \mid \geqslant p)$.

Definition 2 [8]**.** Let k be a positive integer or infinity. We denote by $N_k(r,a;f)$ the counting function of a-points of f, where an a-point of multiplicity m is counted m times if $m \leqslant k$ and k times if $m > k$. Then

$$N_k(r,a;f) = \overline{N}(r,a;f) + \overline{N}(r,a;f \mid\geqslant 2) + \ldots + \overline{N}(r,a;f \mid\geqslant k).$$

Clearly $N_1(r,a;f) = \overline{N}(r,a;f)$.

Definition 3. Let a be any value in the extended complex plane, and let k be an arbitrary nonnegative integer. We define

$$\delta_k(a,f) = 1 - \limsup_{r\to\infty} \frac{N_k(r,a;f)}{T(r,f)}.$$

Remark 1. From the definitions of $\delta_k(a,f)$ and $\Theta(a,f)$, it is clear that

$$0 \leqslant \delta_k(a,f) \leqslant \delta_{k-1}(a,f) \leqslant \delta_1(a,f) \leqslant \Theta(a,f) \leqslant 1.$$

Definition 4 [1], [2]**.** Let f and g be two nonconstant meromorphic functions such that f and g share the value 1 IM. Let z_0 be a 1-point of f with multiplicity p and also a 1-point of g with multiplicity q. We denote by $\overline{N}_L(r,1;f)$ the reduced counting function of the 1-points of f and g with $p > q$, by $N_E^{1)}(r,1;f)$ the counting function of the 1-points of f and g with $p = q = 1$, by $\overline{N}_E^{(2}(r,1;f)$ the reduced counting function of the 1-points of f and g with $p = q \geqslant 2$. In the same manner we can define $\overline{N}_L(r,1;g)$, $N_E^{1)}(r,1;g)$ and $\overline{N}_E^{(2}(r,1;g)$.

2. LEMMAS

Lemma 1 [12]**.** *Let f be a transcendental meromorphic function, and let $P_n(f)$ be a differential polynomial in f of the form*

$$P_n(f) = a_n f^n(z) + a_{n-1} f^{n-1}(z) + \ldots + a_1 f(z) + a_0,$$

where a_n $(\neq 0)$, $a_{n-1}, \ldots, a_1, a_0$ are complex numbers. Then

$$T(r, P_n(f)) = nT(r,f) + O(1).$$

Lemma 2 [7]**.** *Let f be a nonconstant meromorphic function, k a positive integer, and let c be a nonzero finite complex number. Then*

$$\begin{aligned}T(r,f) &\leqslant \overline{N}(r,\infty;f)+N(r,0;f)+N(r,c;f^{(k)})-N(r,0;f^{(k+1)})+S(r,f)\\ &\leqslant \overline{N}(r,\infty;f)+N_{k+1}(r,0;f)+\overline{N}(r,c;f^{(k)})-N_0(r,0;f^{(k+1)})+S(r,f),\end{aligned}$$

where $N_0(r,0;f^{(k+1)})$ denotes the counting function which counts only the points such that $f^{(k+1)}=0$ but $f(f^{(k)}-c)\neq 0$.

Lemma 3 [16]**.** *Let f and g be two nonconstant meromorphic functions, and let p, k be two positive integers. Then*

$$N_p(r,0;f^{(k)})\leqslant N_{p+k}(r,0;f)+k\overline{N}(r,\infty;f)+S(r,f).$$

Lemma 4 [7], [14]**.** *Let f be a transcendental meromorphic function, and let $a_1(z)$, $a_2(z)$ be two distinct meromorphic functions such that $T(r,a_i(z))=S(r,f)$, $i=1,2$. Then*

$$T(r,f)\leqslant \overline{N}(r,\infty;f)+\overline{N}(r,a_1;f)+\overline{N}(r,a_2;f)+S(r,f).$$

Lemma 5. *Let f and g be two transcendental meromorphic functions such that $f^{(k)}-P$ and $g^{(k)}-P$ share 0 IM, where k is a positive integer, $P\not\equiv 0$ is a polynomial. If*

$$\begin{aligned}\Delta_1 &= (2k+4)\Theta(\infty,f)+(2k+3)\Theta(\infty,g)+\Theta(0,f)+\Theta(0,g)\\ &\quad+3\delta_{k+1}(0,f)+2\delta_{k+1}(0,g)>4k+13\end{aligned}\tag{2.1}$$

and

$$\begin{aligned}\Delta_2 &= (2k+4)\Theta(\infty,g)+(2k+3)\Theta(\infty,f)+\Theta(0,g)+\Theta(0,f)\\ &\quad+3\delta_{k+1}(0,g)+2\delta_{k+1}(0,f)>4k+13,\end{aligned}\tag{2.2}$$

then either $f^{(k)}g^{(k)}=P^2$ or $f=g$.

Proof. Since f and g are two transcendental meromorphic functions, $f^{(k)}$ and $g^{(k)}$ are also two transcendental meromorphic functions. Let

$$F=\frac{f^{(k)}}{P},\quad G=\frac{g^{(k)}}{P},$$

and let

$$H=\Big(\frac{F''}{F'}-\frac{2F'}{F-1}\Big)-\Big(\frac{G''}{G'}-\frac{2G'}{G-1}\Big). \tag{2.3}$$

Let $z_0\notin\{z\colon\ P(z)=0\}$ be a common simple zero of $f^{(k)}-P$ and $g^{(k)}-P$. Then z_0 is a common simple zero of $F-1$ and $G-1$. Substituting their Taylor series at z_0 into (2.3), we see that z_0 is a zero of H. Thus we have

$$N_E^{1)}(r,1;F)\leqslant N(r,0;H)\leqslant T(r,H)+O(1)\leqslant N(r,\infty;H)+S(r,F)+S(r,G). \tag{2.4}$$

Let $z_1\notin\{z\colon\ P(z)=0\}$ be a pole of H. Then z_1 possibly is a zero of f or of g, possibly a pole of f or of g, possibly a common 1-point of F and G which has different multiplicities related to F and G, or possibly a zero of F' or of G', which is neither a zero of $f(F-1)$ nor a zero of $g(G-1)$. Hence we have

$$\begin{aligned}N(r,\infty;H)\leqslant{}&\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}(r,0;f)+\overline{N}(r,0;g)+\overline{N}_L(r,1;F)\\&+\overline{N}_L(r,1;G)+N_0(r,0;F')+N_0(r,0;G')+O(\log r),\end{aligned} \tag{2.5}$$

where $N_0(r,0;F')$ denotes the counting function of those zeros of F' which are not the zeros of $f(F-1)$, $N_0(r,0;G')$ is similarly defined. Since f is a transcendental meromorphic functions we have

$$T(r,P)=o\{T(r,f)\}. \tag{2.6}$$

By Lemma 2, we have

$$T(r,f)\leqslant\overline{N}(r,\infty;f)+N_{k+1}(r,0;f)+\overline{N}(r,1;F)-N_0(r,0;F')+S(r,f). \tag{2.7}$$

Similarly,

$$T(r,g)\leqslant\overline{N}(r,\infty;g)+N_{k+1}(r,0;g)+\overline{N}(r,1;G)-N_0(r,0;G')+S(r,g). \tag{2.8}$$

Since $f^{(k)}-P$ and $g^{(k)}-P$ share 0 IM, using (2.4) and (2.5) we obtain

$$\begin{aligned}\overline{N}(r,1;F)+\overline{N}(r,1;G)={}&2N_E^{1)}(r,1;F)+2\overline{N}_L(r,1;F)\\&+2\overline{N}_L(r,1;G)+2\overline{N}_E^{(2}(r,1;F)\\\leqslant{}&N_E^{1)}(r,1;F)+\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\\&+\overline{N}(r,0;f)+\overline{N}(r,0;g)+3\overline{N}_L(r,1;F)\\&+3\overline{N}_L(r,1;G)+N_0(r,0;F')+N_0(r,0;G')\\&+2\overline{N}_E^{(2}(r,1;F)+S(r,f)+S(r,g).\end{aligned} \tag{2.9}$$

Obviously

$$
\begin{aligned}
(2.10)\qquad N_E^{1)}(r,1;F)+2\overline{N}_E^{(2}(r,1;F)+\overline{N}_L(r,1;F)+2\overline{N}_L(r,1;G)&\\
\leqslant N(r,1;G)+S(r,f)+S(r,g)&\\
\leqslant T(r,G)+S(r,f)+S(r,g)&\\
\leqslant T(r,g)+k\overline{N}(r,\infty;g)+S(r,f)+S(r,g).&
\end{aligned}
$$

Also, by Lemma 3 we have

$$
\begin{aligned}
(2.11)\qquad \overline{N}_L(r,1;F)&\leqslant N(r,1;F)-\overline{N}(r,1;F)\\
&\leqslant N\Big(r,\infty;\frac{F}{F'}\Big)\\
&\leqslant N\Big(r,\infty;\frac{F'}{F}\Big)+S(r,f)\\
&\leqslant \overline{N}(r,0;F)+\overline{N}(r,\infty;f)+S(r,f)\\
&\leqslant N_{k+1}(r,0;f)+(k+1)\overline{N}(r,\infty;f)+S(r,f).
\end{aligned}
$$

Similarly,

$$
(2.12)\qquad \overline{N}_L(r,1;G)\leqslant N_{k+1}(r,0;g)+(k+1)\overline{N}(r,\infty;g)+S(r,g).
$$

From (2.7)–(2.12), we obtain

$$
\begin{aligned}
(2.13)\qquad T(r,f)\leqslant{}&(2k+4)\overline{N}(r,\infty;f)+(2k+3)\overline{N}(r,\infty;g)+\overline{N}(r,0;f)+\overline{N}(r,0;g)\\
&+3N_{k+1}(r,0;f)+2N_{k+1}(r,0;g)+S(r,f)+S(r,g).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
(2.14)\qquad T(r,g)\leqslant{}&(2k+4)\overline{N}(r,\infty;g)+(2k+3)\overline{N}(r,\infty;f)+\overline{N}(r,0;g)+\overline{N}(r,0;f)\\
&+3N_{k+1}(r,0;g)+2N_{k+1}(r,0;f)+S(r,f)+S(r,g).
\end{aligned}
$$

Suppose that there exists a subset $I\subseteq\mathbb{R}^+$ satisfying $\operatorname{mes} I=\infty$ such that $T(r,g)\leqslant T(r,f)$, $r\in I$. Hence from (2.13) we have

$$
\begin{aligned}
\Delta_1={}&(2k+4)\Theta(\infty,f)+(2k+3)\Theta(\infty,g)+\Theta(0,f)+\Theta(0,g)\\
&+3\delta_{k+1}(0,f)+2\delta_{k+1}(0,g)\leqslant 4k+13,
\end{aligned}
$$

contradicting (2.1). Similarly, if there exists a subset $I\subseteq\mathbb{R}^+$ satisfying $\operatorname{mes} I=\infty$ such that $T(r,f)\leqslant T(r,g)$, $r\in I$, from (2.14) we obtain

$$
\begin{aligned}
\Delta_2={}&(2k+4)\Theta(\infty,g)+(2k+3)\Theta(\infty,f)+\Theta(0,g)+\Theta(0,f)\\
&+3\delta_{k+1}(0,g)+2\delta_{k+1}(0,f)\leqslant 4k+13,
\end{aligned}
$$

contradicting (2.2). We now assume that $H = 0$. That is,

$$\Big(\frac{F''}{F'} - \frac{2F'}{F-1}\Big) - \Big(\frac{G''}{G'} - \frac{2G'}{G-1}\Big) = 0.$$

Integrating both sides of the above equality twice we get

$$\frac{1}{F-1} = \frac{A}{G-1} + B, \tag{2.15}$$

where A ($\neq 0$) and B are finite complex constants. We now discuss the following three cases.

Case 1. Let $B \neq 0$ and $A = B$. If $B = -1$, we obtain from (2.15) $FG = 1$, i.e., $f^{(k)}g^{(k)} = P^2$.

If $B \neq -1$, from (2.15) we get

$$\frac{1}{F} = \frac{BG}{(1+B)G-1} \quad \text{and} \quad G = \frac{-1}{b(F-(1+B)/B)}.$$

So by Lemma 3 we obtain

$$\begin{aligned} \overline{N}\Big(r, \frac{1}{1+B}; G\Big) \leqslant \overline{N}(r, 0; F) &\leqslant N_{k+1}(r, 0; f) + k\overline{N}(r, \infty; f) \\ &\quad + O(\log r) + S(r, f) \end{aligned} \tag{2.16}$$

and

$$\overline{N}\Big(r, \frac{1+B}{B}; F\Big) \leqslant \overline{N}(r, \infty; g) + O(\log r). \tag{2.17}$$

Using Lemma 2, (2.16) and (2.17) we obtain

$$\begin{aligned} T(r, g) &\leqslant N_{k+1}(r, 0; g) + \overline{N}\Big(r, \frac{1}{1+B}; G\Big) + \overline{N}(r, \infty; g) \\ &\quad - N_0(r, 0; G') + S(r, g) \\ &\leqslant N_{k+1}(r, 0; g) + N_{k+1}(r, 0; f) + k\overline{N}(r, \infty; f) \\ &\quad + \overline{N}(r, \infty; g) + S(r, f) + S(r, g) \end{aligned} \tag{2.18}$$

and

$$\begin{aligned} T(r, f) &\leqslant N_{k+1}(r, 0; f) + \overline{N}\Big(r, \frac{1+B}{B}; F\Big) + \overline{N}(r, \infty; f) \\ &\quad - N_0(r, 0; F') + S(r, f) \\ &\leqslant N_{k+1}(r, 0; f) + \overline{N}(r, \infty; f) + \overline{N}(r, \infty; g) + S(r, f). \end{aligned} \tag{2.19}$$

Suppose that there exists a subset $I \subseteq \mathbb{R}^+$ satisfying $\operatorname{mes} I = \infty$ such that $T(r,f) \leqslant T(r,g)$, $r \in I$. So from (2.18) we obtain

$$k\Theta(\infty,f) + \Theta(\infty,g) + \delta_{k+1}(0,f) + \delta_{k+1}(0,g) \leqslant k+2,$$

which by (2.1) gives

$$(k+4)\Theta(\infty,f)+(2k+2)\Theta(\infty,g)+\Theta(0,f)+\Theta(0,g)+2\delta_{k+1}(0,f)+\delta_{k+1}(0,g) > 3k+11,$$

a contradiction with Remark 1. If there exists a subset $I \subseteq \mathbb{R}^+$ satisfying $\operatorname{mes} I = \infty$ such that $T(r,g) \leqslant T(r,f)$, $r \in I$, by the same argument we obtain a contradiction from (2.1) and (2.19).

Case 2. Let $B \neq 0$ and $A \neq B$. If $B = -1$, from (2.15) we obtain $F = -A/(G-(a+1))$.

If $B \neq -1$, from (2.15) we obtain $F - (1+B)/B = -A/B^2(G+(A-B)/B)$. Using the same argument as in case 1 we obtain a contradiction in both the cases.

Case 3. Let $B = 0$. Then from (2.15) we get

$$g = Af + (1-A)P_1, \tag{2.20}$$

where P_1 is a polynomial of degree $\gamma_{P_1} \geqslant k$. If $A \neq 1$, by Lemma 4 and (2.20) we get

$$\begin{aligned} T(r,g) &\leqslant \overline{N}(r,0;g) + \overline{N}(r,\infty;g) + \overline{N}(r,(1-A)P_1;g) + S(r,g) \\ &\leqslant \overline{N}(r,0;g) + \overline{N}(r,\infty;g) + \overline{N}(r,0;f) + S(r,g). \end{aligned} \tag{2.21}$$

Since f and g are transcendental meromorphic functions, from (2.20) we have

$$T(r,f) = T(r,g) + O(\log r).$$

So from (2.21) we obtain

$$\Theta(0,f) + \Theta(0,g) + \Theta(\infty,g) \leqslant 2,$$

which by (2.1) gives

$$(2k+4)\Theta(\infty,f) + (2k+2)\Theta(\infty,g) + 3\delta_{k+1}(0,f) + 2\delta_{k+1}(0,g) > 4k+11,$$

a contradiction with Remark 1. Thus $A = 1$ and so $f = g$. This proves the lemma.

□

Lemma 6 [11]**.** *Let f and g be two transcendental meromorphic functions, let $n \geqslant 2$ be a positive integer, and let P be a nonconstant polynomial with its degree $\gamma_P \leqslant n$. If $f^n f' g^n g' = P^2$, then f and g are expressed as $f = c_1 e^{cQ}$ and $g = c_2 e^{-cQ}$ respectively, where c_1, c_2 and c are three nonzero complex numbers satisfying $(c_1c_2)^{n+1}c^2 = -1$, and Q is a polynomial satisfying $Q = \int_0^z P(\eta)\,\mathrm{d}\eta$.*

Lemma 7. *Let f and g be two transcendental meromorphic functions, let n, m be two positive integers and let P be a nonconstant polynomial. If $m = 1$, $n \geqslant 6$ or if $m \geqslant 2$, $n \geqslant m+3$, then*

$$(f^n(f-1)^m)'(g^n(g-1)^m)' \neq P^2.$$

Proof. On the contrary, assume

$$(f^n(f-1)^m)'(g^n(g-1)^m)' = P^2. \tag{2.22}$$

We discuss the following two cases.

Case 1. Let $m \geqslant 2$. Then from (2.22) we obtain

$$f^{n-1}(f-1)^{m-1}(cf-d)f'g^{n-1}(g-1)^{m-1}(cg-d)g' = P^2, \tag{2.23}$$

where $c = n+m$ and $d = n$.

Let $z_0 \notin \{z\colon P(z) = 0\}$ be a 1-point of f with multiplicity p_0 ($\geqslant 1$). Then from (2.23) it follows that z_0 is a pole of g. Suppose that z_0 is a pole of g of order q_0 ($\geqslant 1$). Then we have $mp_0 - 1 = (n+m)q_0 + 1$, i.e., $mp_0 = (n+m)q_0 + 2 \geqslant n+m+2$, and so

$$p_0 \geqslant \frac{n+m+2}{m}.$$

Let $z_1 \notin \{z\colon P(z) = 0\}$ be a zero of $cf - d$ with multiplicity p_1 ($\geqslant 1$). Then from (2.23) it follows that z_1 is a pole of g. Suppose that z_1 is a pole of g of order q_1 ($\geqslant 1$). Then we have $2p_1 - 1 = (n+m)q_1 + 1$, and so

$$p_1 \geqslant \frac{n+m+2}{2}.$$

Let $z_2 \notin \{z\colon P(z) = 0\}$ be a zero of f with multiplicity p_2 ($\geqslant 1$). Then it follows from (2.23) that z_2 is a pole of g. Suppose that z_2 is a pole of g of order q_2 ($\geqslant 1$). Then we have

$$np_2 - 1 = (n+m)q_2 + 1. \tag{2.24}$$

From (2.24) we get $mq_2+2=n(p_2-q_2)\geqslant n$, i.e., $q_2\geqslant(n-2)/m$. Thus from (2.24) we obtain $np_2=(n+m)q_2+2\geqslant(n+m)(n-2)/m+2$, and so

$$p_2\geqslant\frac{n+m-2}{m}.$$

Let $z_3\notin\{z\colon\ P(z)=0\}$ be a pole of f. Then it follows from (2.23) that z_3 is a zero of $g(g-1)(cg-d)$ or a zero of g'. So we have

$$\begin{aligned}\overline{N}(r,\infty;f)&\leqslant\overline{N}(r,0;g)+\overline{N}(r,1;g)+\overline{N}\Big(r,\frac{d}{c};g\Big)+\overline{N}_0(r,0;g')\\&\quad+S(r,f)+S(r,g)\\&\leqslant\Big(\frac{m+2}{n+m+2}+\frac{m}{n+m-2}\Big)T(r,g)+\overline{N}_0(r,0;g')\\&\quad+S(r,f)+S(r,g),\end{aligned}$$

where $\overline{N}_0(r,0;g')$ denotes the reduced counting function of those zeros of g' which are not zeros of $g(g-1)(cg-d)$.

By the second fundamental theorem of Nevanlinna we get

$$\begin{aligned}2T(r,f)&\leqslant\overline{N}(r,0;f)+\overline{N}(r,1;f)+\overline{N}\Big(r,\frac{d}{c};f\Big)+\overline{N}(r,\infty;f)\\&\quad-\overline{N}_0(r,0;f')+S(r,f)\\&\leqslant\Big(\frac{m+2}{n+m+2}+\frac{m}{n+m-2}\Big)\{T(r,f)+T(r,g)\}\\&\quad-\overline{N}_0(r,0;f')+\overline{N}_0(r,0;g')+S(r,f)+S(r,g).\end{aligned}\tag{2.25}$$

Similarly,

$$\begin{aligned}2T(r,g)&\leqslant\Big(\frac{m+2}{n+m+2}+\frac{m}{n+m-2}\Big)\{T(r,f)+T(r,g)\}\\&\quad+\overline{N}_0(r,0;f')-\overline{N}_0(r,0;g')+S(r,f)+S(r,g).\end{aligned}\tag{2.26}$$

Adding (2.25) and (2.26) we obtain

$$\Big(1-\frac{m+2}{n+m+2}-\frac{m}{n+m-2}\Big)\{T(r,f)+T(r,g)\}\leqslant S(r,f)+S(r,g),$$

contradicting the fact that $n\geqslant m+3$.

Case 2. Let $m=1$. Then from (2.22) we obtain

$$f^{n-1}(af-b)f'g^{n-1}(ag-b)g'=P^2,\tag{2.27}$$

where $a=n+1$ and $b=n$.

Let $z_4 \notin \{z\colon P(z)=0\}$ be a pole of f. Then it follows from (2.27) that z_4 is a zero of $g(ag-b)$ or a zero of g'. Then proceeding in a manner similar to Case 1 we obtain

$$\Big(1-\frac{2}{n-1}-\frac{4}{n+3}\Big)\{T(r,f)+T(r,g)\} \leqslant S(r,f)+S(r,g),$$

which contradicts the fact that $n \geqslant 6$. This proves the lemma. □

Lemma 8. *Let f and g be two nonconstant meromorphic functions such that*

$$\Theta(\infty,f)+\Theta(\infty,g) > \frac{4}{n},$$

where $n(\geqslant 3)$ is an integer. Then

$$f^n(af+b) \equiv g^n(ag+b)$$

implies $f \equiv g$, where a, b are two nonzero constants.

Proof. We omit the proof since it can be carried out following the lines of Lemma 6 [10]. □

3. Proofs of the theorems

Proof of Theorem 1. We consider $F_1(z) = f^{n+1}/(n+1)$ and $G_1(z) = g^{n+1}/(n+1)$. Then we see that $F_1' - P$ and $G_1' - P$ share the value 0 IM. Using Lemma 1, we have

$$\begin{aligned}\Theta(0,F_1) &= 1-\limsup_{r\to\infty}\frac{\overline{N}(r,0;F_1)}{T(r,F_1)}\\ &= 1-\limsup_{r\to\infty}\frac{\overline{N}(r,0;f)}{(n+1)T(r,f)}\\ &\geqslant 1-\limsup_{r\to\infty}\frac{T(r,f)}{(n+1)T(r,f)}\\ &\geqslant \frac{n}{n+1}.\end{aligned}\tag{3.1}$$

Similarly,

$$\Theta(0,G_1) \geqslant \frac{n}{n+1}.\tag{3.2}$$

$$\begin{aligned}\Theta(\infty, F_1) &= 1 - \limsup_{r\to\infty} \frac{\overline{N}(r,\infty;F_1)}{T(r,F_1)} \\ &= 1 - \limsup_{r\to\infty} \frac{\overline{N}(r,\infty;f)}{(n+1)T(r,f)} \\ &\geqslant 1 - \limsup_{r\to\infty} \frac{T(r,f)}{(n+1)T(r,f)} \\ &\geqslant \frac{n}{n+1}.\end{aligned} \tag{3.3}$$

Similarly,

$$\Theta(\infty, G_1) \geqslant \frac{n}{n+1}. \tag{3.4}$$

$$\begin{aligned}\delta_2(0, F_1) &= 1 - \limsup_{r\to\infty} \frac{N_2(r,0;F_1)}{T(r,F_1)} \\ &= 1 - \limsup_{r\to\infty} \frac{N_2(r,0;f^n)}{(n+1)T(r,f)} \\ &\geqslant 1 - \limsup_{r\to\infty} \frac{2T(r,f)}{(n+1)T(r,f)} \\ &\geqslant \frac{n-1}{n+1}.\end{aligned} \tag{3.5}$$

Similarly,

$$\delta_2(0, G_1) \geqslant \frac{n-1}{n+1}. \tag{3.6}$$

Using (2.1), (2.2) and (3.1)–(3.6) we obtain

$$\Delta_1 \geqslant \frac{18n-5}{n+1} \quad \text{and} \quad \Delta_2 \geqslant \frac{18n-5}{n+1}.$$

Since $n \geqslant 23$, we get $\Delta_1 > 17$ and $\Delta_2 > 17$. So by Lemma 5 we obtain either $F_1'G_1' = P^2$ or $F_1 = G_1$. Suppose that $F_1'G_1' = P^2$, i.e., $f^n f' g^n g' = P^2$. Hence by Lemma 6 we obtain $f = c_1 \mathrm{e}^{cQ}$ and $g = c_2 \mathrm{e}^{-cQ}$, where c_1, c_2 and c are three nonzero complex numbers satisfying $(c_1c_2)^{n+1}c^2 = -1$, and Q is a polynomial satisfying $Q = \int_0^z P(\eta)\,\mathrm{d}\eta$.

If $F_1 = G_1$, then $f = tg$ for a complex number t such that $t^{n+1} = 1$. This completes the proof of Theorem 1. □

Proof of Theorem 2. Let $F_2(z) = f^n(f-1)^m$ and $G_2(z) = g^n(g-1)^m$. Then $F_2' - P$ and $G_2' - P$ share the value 0 IM. Using Lemma 1, we obtain

$$\begin{aligned}\Theta(0, F_2) &= 1 - \limsup_{r\to\infty} \frac{\overline{N}(r, 0; F_2)}{T(r, F_2)} \\ &= 1 - \limsup_{r\to\infty} \frac{\overline{N}(r, 0; f^n(f-1)^m)}{(n+m)T(r, f)} \\ &\geqslant 1 - \limsup_{r\to\infty} \frac{2T(r, f)}{(n+m)T(r, f)} \\ &\geqslant \frac{n+m-2}{n+m}.\end{aligned} \tag{3.7}$$

Similarly,

$$\Theta(0, G_2) \geqslant \frac{n+m-2}{n+m}. \tag{3.8}$$

$$\begin{aligned}\Theta(\infty, F_2) &= 1 - \limsup_{r\to\infty} \frac{\overline{N}(r, \infty; F_2)}{T(r, F_2)} \\ &= 1 - \limsup_{r\to\infty} \frac{\overline{N}(r, \infty; f)}{(n+m)T(r, f)} \\ &\geqslant 1 - \limsup_{r\to\infty} \frac{T(r, f)}{(n+m)T(r, f)} \\ &\geqslant \frac{n+m-1}{n+m}.\end{aligned} \tag{3.9}$$

Similarly,

$$\Theta(\infty, G_2) \geqslant \frac{n+m-1}{n+m}. \tag{3.10}$$

$$\begin{aligned}\delta_2(0, F_2) &= 1 - \limsup_{r\to\infty} \frac{N_2(r, 0; F_2)}{T(r, F_2)} \\ &= 1 - \limsup_{r\to\infty} \frac{N_2(r, 0; f^n(f-1)^m)}{(n+m)T(r, f)} \\ &\geqslant 1 - \limsup_{r\to\infty} \frac{(m+2)T(r, f)}{(n+m)T(r, f)} \\ &\geqslant \frac{n-2}{n+m}.\end{aligned} \tag{3.11}$$

Similarly,

$$\delta_2(0, G_2) \geqslant \frac{n-2}{n+m}. \tag{3.12}$$

Using (2.1), (2.2) and (3.7)–(3.12) we obtain

$$\Delta_1 \geqslant \frac{18n+13m-25}{n+m} \quad \text{and} \quad \Delta_2 \geqslant \frac{18n+13m-25}{n+m}.$$

Since $n \geqslant 4m+26$, we get $\Delta_1 > 17$ and $\Delta_2 > 17$. In view of Lemma 5 and Lemma 7 we conclude that $F_2 = G_2$, i.e.,

$$f^n(f-1)^m = g^n(g-1)^m. \tag{3.13}$$

Let $m = 1$. Then from (3.13) we get

$$f^n(f-1) = g^n(g-1),$$

which gives $f = g$, together with Lemma 8.

Let $m \geqslant 2$. Then from (3.13) we obtain

$$\begin{aligned} f^n[f^m + \ldots + (-1)^i\ {}^mC_i f^{m-i} + \ldots + (-1)^m] \\ = g^n[g^m + \ldots + (-1)^i\ {}^mC_i g^{m-i} + \ldots + (-1)^m]. \end{aligned} \tag{3.14}$$

Let $h = f/g$. If h is a constant, then substituting $f = gh$ in (3.14) we obtain

$$\begin{aligned} g^{n+m}(h^{n+m}-1) + \ldots + (-1)^i\ {}^mC_i g^{n+m-i}(h^{n+m-i}-1) \\ + \ldots + (-1)^m g^n(h^n-1) = 0, \end{aligned}$$

which implies $h = 1$. Hence $f = g$.

If h is not a constant, then from (3.14) we see that f and g satisfy the algebraic equation $R(f,g) = 0$, where

$$R(w_1,w_2) = w_1^n(w_1-1)^m - w_2^n(w_2-1)^m.$$

This completes the proof of Theorem 2. □

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