MULTIPLIERS OF SPACES OF DERIVATIVES

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(Received August 29, 2003)

Abstract. For subspaces, X and Y, of the space, D, of all derivatives $M(X,Y)$ denotes the set of all $g \in D$ such that $fg \in Y$ for all $f \in X$. Subspaces of D are defined depending on a parameter $p \in [0,\infty]$. In Section 6, $M(X,D)$ is determined for each of these subspaces and in Section 7, $M(X,Y)$ is found for X and Y any of these subspaces. In Section 3, $M(X,D)$ is determined for other spaces of functions on $[0,1]$ related to continuity and higher order differentiation.

Keywords: spaces of derivatives, Peano derivatives, Lipschitz function, multiplication operator

MSC 2000: 26A21, 47B37

1. Introduction

A derivative is a function, f, that is everywhere the derivative of another function, F. At the close of the 19th century it was observed that the product of two derivatives need not be a derivative (see [7]). In fact if f is a derivative, f^2 need not be a derivative. (For a treatment of this topic see [1].) Yet it is easy to see that the product of a derivative with a continuously differentiable function *is* a derivative. However, we must not drop the word, "continuously". For example if $\varphi(x) = x^2 \cos x^{-3}$ and $\psi(x) = x^2 \sin x^{-3}$ for $x \neq 0$ and $\varphi(0) = \psi(0) = 0$, then φ and ψ are both everywhere differentiable. Setting $\omega = \varphi'\psi - \varphi\psi'$, a simple calculation shows that $\omega(0) = 0$ while $\omega(x) = 3$ for all $x \neq 0$. Thus ω is not a derivative because derivatives have the Darboux property. Since $\varphi'\psi + \varphi\psi'$ is a derivative (of $\varphi\psi$) and $\varphi'\psi - \varphi\psi'$ is not, neither $\varphi'\psi$ nor $\varphi\psi'$ can be derivatives. These observations lead naturally to the problem of describing the system, W, of all functions, g, such that fg is a derivative for every derivative, f. As was mentioned above, not every derivative, nor even every

Professor Jan Mařík died on January 6, 1994.

differentiable function belongs to W. On the other hand it can be shown that W contains some discontinuous functions as can be seen from the characterization of the class W given by Fleissner in [2] (also see [3]). In Theorem 6.4 a simpler description of W is given.

In [3] Fleissner posed the similar problem of finding the system of all functions g such that fg is a summable derivative for each summable derivative f. (This question was answered in [4].) It seems natural to investigate the following more general problem. Let X and Y be classes of derivatives. Describe the class of all functions g such that $fg \in Y$ for each $f \in X$. This task is accomplished here for several subspaces of the space of all derivatives; some of which are familiar while others are introduced for the first time in this paper.

In the next section we introduce some of the notation and terminology to be used and prove several auxiliary results which will be employed throughout the remainder of the work. Section 3 deals with the spaces of derivatives, continuous functions and Peano differentiable functions. In Section 4 a continuum of new spaces of derivatives is introduced and several preliminary results are established. Section 5 contains additional auxiliary results which will be used in Section 6 to obtain the first set of main results of the article; namely, characterizing the multipliers of the spaces introduced in Section 4 into the space of all derivatives. In the final section the multipliers between the spaces of Section 4 are found. The results of Section 6 are used there.

2. Notation and auxiliary results

Throughout, $\mathbb{N}$ will denote the natural number, $\mathbb{R}$ will denote the real line, and $\mathbb{R}^+ = \{x \in \mathbb{R}\colon\ x > 0\}$. The interval $[0,1]$ is denoted by I. The major space of functions dealt with, the derivatives, is denoted by D and defined by

$$D = \{f\colon\ I \to \mathbb{R};\quad \text{there is an } F\colon\ I \to \mathbb{R} \text{ such that } F'(x) = f(x) \text{ for each } x \in I\}$$

where differentiation at the endpoints of I is in the unilateral sense. Clearly D is a vector space. The symbols Δ, C, C_{ap} denote respectively the space of all differentiable, continuous and approximately continuous functions on I. Thus $D = \{F'\,;\ F \in \Delta\}$. The space C_{ap} plays a major role in Section 7. For any class S of functions, bS and S^+ denote respectively the bounded and nonnegative function in S. It is easy to verify that $bC_{\mathrm{ap}} \subset D$. For an open interval $J \subset \mathbb{R}$, $C(J)$ and $C_\infty(J)$ will denote respectively the continuous functions and the infinitely differentiable functions on J with the convention that $C_\infty = C_\infty(\mathbb{R})$.

Measure and measurable refer to the Lebesgue concepts. The measure of a measurable set S will be denoted by $|S|$. On the other hand, integrable means Denjoy-Perron integrable and summable means absolutely (i.e., Lebesgue) integrable. The symbols $\int_J f$ and $\int_a^b f$ will denote the Denjoy-Perron integral of f (or the Lebesgue integral in case f is summable). As usual $\int_a^b f = -\int_b^a$ if $b < a$ and if $\int_b^a f$ exists. The reader is reminded that $f \in D$ need not be summable (that is, Lebesgue integrable) on I but is (Denjoy-Perron) integrable on I. Indeed if $F(x) = \int_0^x f$, then $F'(x) = f(x)$ for each $x \in I$.

Let J be a compact subinterval of $\mathbb{R}$ and $f\colon\ J \to \mathbb{R}$. Then $\operatorname{osc}(J, f)$ and $\operatorname{Var}(J, f)$ denote respectively the oscillation and variation of f on J. If a and b are the endpoints of J, then we write $\operatorname{osc}(a, b, f)$ and $\operatorname{Var}(a, b, f)$ even if $b < a$.

Now the second major concept of this article, multiplier, is defined and elementary properties stated. Let $X, Y \subset D$. Then

$$M(X, Y) = \{g \in D;\ fg \in Y \text{ for each } f \in X\}.$$

In case $Y = D$ we write $M(X)$; that is, $M(X) = M(X, D)$. In Section 6, $M(X)$ is characterized for the continuum of subspaces of D that will be introduced in Section 4, and in Section 7 $M(X, Y)$ is found where X and Y are any of these same subspaces. The proofs of the first six assertions about $M(X, Y)$ are easy and left to the reader.

Proposition 2.1. *Let $X, Y \subset D$ with Y a vector space. Then $M(X, Y)$ is a vector space.*

Proposition 2.2. *Let $X, Y \subset D$ with $1 \in X$. (That is, the function $f(x) = 1$ for all $x \in I$ belongs to X.) Then $M(X, Y) \subset Y$.*

Proposition 2.3. *Let $X_1 \subset X \subset D$ and $Y \subset Y_1 \subset D$. Then $M(X, Y) \subset M(X_1, Y_1)$.*

Proposition 2.4. *Let $X \subset D$ be a vector space. Then $M(X, X)$ is an algebra.*

Proposition 2.5. *Let $1 \in X \subset D$ with X closed under multiplication. Then $M(X, X) = X$.*

Proposition 2.6. *Let $X, Y \subset D$ and for each $\alpha \in \Omega$ let $X_\alpha, Y_\alpha \subset D$. Then $M\Big(\bigcup_{\alpha\in\Omega} X_\alpha, Y\Big) = \bigcap_{\alpha\in\Omega} M(X_\alpha, Y)$ and $M\Big(X, \bigcap_{\alpha\in\Omega} Y_\alpha\Big) = \bigcap_{\alpha\in\Omega} M(X, Y_\alpha)$.*

Proposition 2.7. *Let $X \subset D$. Then $M(M(M(X))) = M(X)$.*

Proof. Obviously $X \subset M(M(X))$ and consequently $M(X) \subset M(M(M(X)))$. By Proposition 2.3 the first containment implies $M(X) \supset M(M(M(X)))$. $\square$

Fundamental to many of the remaining results of this and later sections is the Second Mean Value Theorem for the Denjoy-Perron integral the proof of which can be found on page 246 of Saks' book [8]. For the remainder of the section let $a, b \in \mathbb{R}$ with $a < b$ and let $J = [a, b]$.

Theorem 2.8. *Let $f\colon J \to \mathbb{R}$ be monotone and $g\colon J \to \mathbb{R}$ be integrable. Then there is a $\xi \in J$ such that $\int_J fg = f(a) \int_a^\xi g + f(b) \int_\xi^b g$.*

Lemma 2.9. *Let $\varepsilon, \tau \in \mathbb{R}^+$, let $p \in (0,1)$ and let $g\colon (0,\tau) \to \mathbb{R}$ be integrable. Suppose $\lim\limits_{x\to 0^+} \frac{1}{x}\int_0^x g = 0$. Then there is an $f \in C_\infty^+$ such that $f = 0$ on $\mathbb{R} \setminus (0,\tau)$, $\int_{\mathbb{R}} f = 1$, $\int_{\mathbb{R}} f^p < \varepsilon$ and $|\int_0^\tau fg| < \varepsilon$.*

Proof. There is a $\delta \in (0,\tau)$ such that $3^p\delta^{1-p} < \varepsilon$ and $|\int_0^x g| < \frac{1}{4}\varepsilon x$ for each $x \in (0,\delta]$. Let $\gamma = \frac{1}{3}\delta$. There is an $h \in C_\infty$ such that $h = 0$ on $\mathbb{R} \setminus (0,\delta)$, $h = 1$ on $(\gamma, 2\gamma)$ and h is monotone on $(0,\gamma)$ as well as on (γ,δ). Clearly $\int_0^\delta h > \gamma$. By Theorem 2.8 there are $\alpha \in [0,\gamma]$ and $\beta \in [\gamma,\delta]$ such that $\int_0^\gamma hg = \int_\alpha^\gamma g$ and $\int_\gamma^\delta hg = \int_\gamma^\beta g$. Hence $|\int_0^\delta hg| = |\int_\alpha^\beta g| \leqslant |\int_0^\alpha g| + |\int_0^\beta g| < \frac{1}{4}\varepsilon(\gamma+\delta) = \varepsilon\gamma$. Set $f = h/\int_0^\delta h$. Since $f \leqslant 1/\gamma$, we have $\int_0^\tau f^p \leqslant \delta(3/\delta)^p = 3^p\delta^{1-p} < \varepsilon$ and $|\int_0^\tau fg| = |\int_0^\delta hg|/\int_0^\delta h < \varepsilon$. $\square$

Proposition 2.10. *Let $\varepsilon > 0$, let $p \in (0,1)$ and let $G\colon J \to \mathbb{R}$ be integrable. Suppose $\lim\limits_{x\to a^+} \frac{1}{x-a}\int_a^x G = G(a)$ and $\lim\limits_{x\to b^-} \frac{1}{b-x}\int_x^b G = G(b)$. Then there is an $f \in C_\infty$ such that $f = 0$ on $\mathbb{R} \setminus J$, $\int_J f = 0$, $\int_J |f| = 2$, $-1 \leqslant \int_a^x f \leqslant 0$ for each $x \in J$, $\int_J |f|^p < \varepsilon$ and*

$$\left|G(b) - G(a) - \int_J fG\right| < \varepsilon. \tag{1}$$

Proof. Let $c \in (a,b)$, let $J_1 = [a,c]$ and $J_2 = [c,b]$. By Lemma 2.9 for $i = 1,2$ there is $f_i \in C_\infty^+$ such that $f_i = 0$ on $\mathbb{R} \setminus J_i$, $\int_J f_i = 1$, $\int_J f_i^p < \frac{1}{2}\varepsilon$ and $|G(a) - \int_J f_1 G| = |\int_J (G(a) - G) f_1| < \frac{1}{2}\varepsilon$, $|G(b) - \int_J f_2 G| < \frac{3}{2}$. Take $f = f_2 - f_1$. $\square$

The proof of the next lemma is complicated due to the lack of absolute integrability for the Denjoy-Perron integral.

Lemma 2.11. *Let $g\colon J \to \mathbb{R}$ be integrable and let $\varepsilon > 0$. Then there is an $F \in C_\infty$ such that*

$$F = 0 \text{ on } \mathbb{R} \setminus J, \quad 0 \leqslant F \leqslant 1 \text{ on } J \tag{2}$$

and

$$\left| \int_J g - \int_J Fg \right| < \varepsilon. \tag{3}$$

Proof. For each $x \in J$ let $G(x) = \int_a^x g$. Then G is continuous on J; so G satisfies the hypotheses of Proposition 2.10. Let f be as in the conclusion of Proposition 2.10 and put $F(x) = -\int_a^x f$. Then (2) is obvious. From integration by parts $\int_J Fg = \int_J fG$ which combined with (1) yields (3). □

The lack of absolute integrability means that in the next assertion its possible for $\int_J |g| = +\infty$.

Proposition 2.12. *Let $g\colon J \to \mathbb{R}$ be integrable and let $Q \in \mathbb{R}$ with $Q < \int_J |g|$. Then there are $f_1, f_2 \in C_\infty$ such that for $i = 1, 2$, $|f_i| \in C_\infty$, $f_i = 0$ on $\mathbb{R} \setminus J$, $|f_i| \leqslant 1$ on J, $\int_J f_1 g > Q$, $\int_J f_2 = 0$ and $\int_J f_2 g > \frac{1}{2}(Q - |\int_J g|)$.*

Proof. As is well known $\int_J |g|$ is the variation of any indefinite integral of g. Consequently there is a partition $a = x_0 < x_1 < \ldots < x_n = b$ of $[a, b]$ such that $S = \sum\limits_{k=1}^{n} |\int_{x_{k-1}}^{x_k} g| > Q$. For $k = 1, \ldots, n$ let $J_k = [x_{k-1}, x_k]$ and let $\alpha_k = \int_{J_k} g$. Let $\varepsilon = \frac{S-Q}{2n}$. By Lemma 2.11, for each $k = 1, \ldots, n$ there is $\varphi_k \in C_\infty^+$ such that $\varphi_k = 0$ on $\mathbb{R} \setminus J_k$, $0 \leqslant \varphi_k \leqslant 1$ on J and if $\beta_k = \int_{J_k} \varphi_k g$, then $|\alpha_k - \beta_k| < \varepsilon$. Let $K = \{1, \ldots, n\}$, $K_0 = \{k \in K;\ \alpha_k > 0\}$, $K_1 = K \setminus K_0$. For $i = 0, 1$ let $S_i = \sum\limits_{k \in K_i} |\alpha_k|$, $h_i = \sum\limits_{k \in K_i} \varphi_k$ and $B_i = \int_J h_i$. Clearly $S_0 + S_1 = S$ and $S_0 - S_1 = \int_J g$. Replacing g by $-g$ if necessary it may be assumed that $B_0 \leqslant B_1$. There is an $r \in [0, 1]$ such that $B_0 = rB_1$. Set $f_1 = h_0 - h_1$ and $f_2 = h_0 - rh_1$. Then

$$\int_J f_1 g = \sum_{k \in K_0} \beta_k - \sum_{k \in K_1} \beta_k > \sum_{k \in K_0} \alpha_k - \sum_{k \in K_1} \alpha_k - n\varepsilon = S - n\varepsilon = \frac{S+Q}{2} > Q$$

and

$$\begin{aligned} \int_J f_2 g &= \sum_{k \in K_0} \beta_k - \sum_{k \in K_1} r\beta_k > \sum_{k \in K_0} \alpha_k - r \sum_{k \in K_1} \alpha_k - n\varepsilon \geqslant S_0 - n\varepsilon \\ &= \frac{S + \int_J g}{2} - n\varepsilon = \frac{Q + \int_J g}{2} > \frac{Q - |\int_J g|}{2}. \end{aligned}$$

Clearly $\int_J f_2 = B_0 - rB_1 = 0$ and the rest is obvious. □

Corollary 2.13. *Let $g\colon J \to \mathbb{R}$ be integrable, let $\varphi\colon (a,b) \to \mathbb{R}^+$ be continuous and let $Q \in \mathbb{R}$ with $Q < \int_J \varphi|g|$. Then there is an $f \in C_\infty$ such that $|f| \in C_\infty$, $f = 0$ on $\mathbb{R} \setminus J$, $|f| \leqslant \varphi$ on (a,b) and $\int_J fg > Q$.*

Proof. Since φ is continuous on (a,b), there is an $n \in \mathbb{N}$ and $x_k \in [a,b]$ for $k = 0,1,\dots,n$ with $a = x_0 < x_1 < \dots < x_n = b$ such that for $k = 1,\dots,n$ there is a $c_k \in [0,\infty)$ with $c_k \leqslant \varphi$ on $J_k = [x_{k-1}, x_k]$ such that $\sum\limits_{k=1}^{n} \int_{J_k} c_k|g| > Q$. For each $k = 1,\dots,n$ there is a $Q_k < \int_{J_k} c_k|g|$ such that $\sum\limits_{k=1}^{n} Q_k \geqslant Q$. If $c_k = 0$, let $f_k = 0$ on $\mathbb{R}$. If $c_k > 0$, by Proposition 2.12 there is an $f_k \in C_\infty$ such that $|f_k| \in C_\infty$, $f_k = 0$ on $\mathbb{R} \setminus J_k$, $|f_k| \leqslant 1$ on J_k and $\int_{J_k} f_k g > \frac{Q_k}{c_k}$. Set $f = \sum\limits_{k=1}^{n} c_k f_k$. Then $\int_{J_k} fg = \int_{J_k} c_k f_k g > Q_k$ for each $k = 1,\dots,n$ (even if $c_k = 0$ since then $Q_k < 0$). Thus $\int_J fg > Q$. Obviously $|f| \in C_\infty$ and $|f| < \varphi$ on (a,b). $\square$

3. Multipliers of continuous functions and Peano differentiable functions

Let $\Delta_2 = \{f\colon I \to \mathbb{R};\ f$ is twice differentiable on $I\}$ (Recall $I = [0,1]$.) and let $f \in \Delta_2$. Then $f' \in C$ and using integration by parts it follows that $fg \in D$ for each $g \in D$. Thus $M(\Delta_2) = D$. Now set $P_0 = C$ and for $n \in \mathbb{N}$ let $P_n = \{f\colon I \to \mathbb{R};\ f$ is n-times Peano differentiable on $I\}$. (A function, f, is n-times Peano differentiable at $y \in I$ means there is a polynomial, F, (of degree $\leqslant n$) with $F(y) = f(y)$ such that $f(x) - F(x) = o(|y-x|^n)$.) It is well known that $P_1 = \Delta$ and that $\Delta \subsetneqq P_n$ if $n \in \mathbb{N} \setminus \{1\}$. In fact if $n \in \mathbb{N}$, $n \neq 1$, there are functions $f \in P_n$ with f' discontinuous. Consequently finding $M(P_2)$ is more difficult than finding $M(\Delta_2)$. Corollary 3.6 characterizes $M(P_n)$ for $n \in \mathbb{N} \cup \{0\}$. Theorem 3.9 is a modification of Corollary 3.6 for $n = 1$. In the assertions to follow the reader will notice a duality which is a recurring theme in this article. It is between a certain limit being zero and an associated limit superior being finite. One of these conditions will appear in the assumption and the other in the conclusion. For the first occurrence of this duality compare Lemmas 3.1 and 3.3.

Lemma 3.1. *Let $\varphi\colon (0,1] \to \mathbb{R}^+$ be continuous and let $g\colon I \to \mathbb{R}$. Suppose $\lim\limits_{x\to 0^+} \frac{1}{x}\int_0^x fg = 0$ for each $f \in (C_\infty(\mathbb{R}^+))^+$ such that $\limsup\limits_{x\to 0^+} f(x)/\varphi(x) < \infty$. Then $\lim\limits_{x\to 0^+} \frac{1}{x}\int_0^x \varphi|g| = 0$.*

Proof. It is easy to construct a strictly positive $h \in C_\infty(\mathbb{R}^+)$ such that $h \leqslant \varphi$ on $(0,1]$. By assumption $\lim\limits_{x\to 0^+} \frac{1}{x}\int_0^x hg = 0$. It follows that $\int_0^b hg$ exists for some $b \in (0,1)$. Because h is strictly positive, $\int_a^b g$ exists for each $a \in (0,b)$.

For each $n \in \mathbb{N}\cup\{0\}$, let $x_n = \frac{b}{2^n}$ and $J_n = [x_n, x_{n-1}]$. Let $n \in \mathbb{N}$. If $\int_{J_n} \varphi|g| = \infty$, set $A_n = 1$. Otherwise set $A_n = \int_{J_n} \varphi|g| - \frac{x_n}{n}$. By applying Corollary 2.13 to J_n for each $n \in \mathbb{N}$, it follows that there is an $f \in C_\infty(\mathbb{R}^+)$ such that $|f| \in C_\infty(\mathbb{R}^+)$, $|f| \leqslant \varphi$ on $(0,1]$ and $\int_{J_n} fg > A_n$ for each $n \in \mathbb{N}$. Let $f_1 = \frac{1}{2}(2|f| + f)$ and $f_2 = \frac{1}{2}(2|f| - f)$. Clearly for $j = 1,2$, $f_j \in (C_\infty(\mathbb{R}^+))^+$ and $\frac{1}{2} \leqslant f_j \leqslant \frac{3}{2}\varphi$ on $(0,1]$. By assumption $\lim\limits_{x\to 0^+} \frac{1}{x}\int_0^x f_j g = 0$ for $j = 1,2$ and consequently $\lim\limits_{x\to 0^+} \frac{1}{x}\int_0^x fg = 0$. It follows that the set of numbers $n \in \mathbb{N}$ for which $A_n = 1$, that is, for which $\int_{J_n} \varphi|g| = \infty$, is finite. Hence there is an $N \in \mathbb{N}$ such that $n \geqslant N$ implies $\int_{J_n} \varphi|g| < \int_{J_n} fg + \frac{x_n}{n}$. Let $x \in (0, x_N)$. Then there is an $n \geqslant N$ such that $x \in J_n$, and using the previous inequality

$$\frac{1}{x}\int_0^x \varphi|g| \leqslant \frac{1}{x_n}\int_0^{x_{n-1}} fg + \frac{1}{nx_n}\sum_{i=n}^\infty \frac{b}{2^i} = 2\biggl(\frac{1}{x_{n-1}}\int_0^{x_{n-1}} fg\biggr) + \frac{2}{n}.$$

Since $\lim\limits_{x\to 0^+} \frac{1}{x}\int_0^x fg = 0$, it follows easily that $\lim\limits_{x\to 0^+} \frac{1}{x}\int_0^x \varphi|g| = 0$. $\square$

Lemma 3.2. *Let $G\colon (0,1) \to \mathbb{R}$ be nonnegative and measurable. Suppose $\limsup\limits_{x\to 0^+} \frac{1}{x}\int_0^x G\beta < \infty$ for each strictly increasing function $\beta \in C_\infty(\mathbb{R}^+)$ such that $\lim\limits_{x\to 0^+} \beta(x) = 0$. Then $\limsup\limits_{x\to 0^+} \frac{1}{x}\int_0^x G < \infty$.*

Proof. Suppose to the contrary that $\limsup\limits_{x\to 0^+} \frac{1}{x}\int_0^x G = \infty$. Define two sequences $\{x_n\}$ and $\{y_n\}$ as follows. Let $y_1 = 1$ and $x_1 = \frac{1}{2}$. Given x_{n-1} by assumption there is a $y_n \in (0, x_{n-1})$ with $\int_0^{y_n} G > n^2 y_n$. Let $x_n \in (0, \frac{y_n}{2})$ with $\int_{x_n}^{y_n} G > n^2 y_n$. It is easy to construct a strictly increasing $\beta \in C_\infty(\mathbb{R}^+)$ such that $\beta \geqslant \frac{1}{n}$ on (x_n, y_n) with $\lim\limits_{x\to 0^+} \beta(x) = 0$. Then $n \in \mathbb{N}$ implies $\int_0^{y_n} G\beta \geqslant \frac{1}{n}\int_{x_n}^{y_n} G > ny_n$; or $\frac{1}{y_n}\int_0^{y_n} G\beta > n$. Thus $\limsup\limits_{x\to 0^+} \frac{1}{x}\int_0^x G\beta = \infty$ contrary to hypothesis. $\square$

Lemma 3.3. *Let $\varphi\colon (0,1] \to \mathbb{R}^+$ be continuous and let $g\colon I \to \mathbb{R}$. Suppose $\lim\limits_{x\to 0^+} \frac{1}{x}\int_0^x fg = 0$ for each $f \in (C_\infty(\mathbb{R}^+))^+$ such that $\lim\limits_{x\to 0^+} \frac{f(x)}{\varphi(x)} = 0$. Then*

$$\limsup_{x\to 0^+} \frac{1}{x}\int_0^x \varphi|g| < \infty. \tag{4}$$

Proof. Let $\beta \in C_0(\mathbb{R}^+)^+$ be strictly increasing with $\lim\limits_{x\to 0^+} \beta(x) = 0$. Let $\psi = \varphi\beta$. Then ψ is continuous on $(0,1]$. Let $f \in (C_\infty(\mathbb{R}^+))^+$ with $\limsup\limits_{x\to 0^+} \frac{f(x)}{\psi(x)} < \infty$.

Since $\lim\limits_{x\to 0^+}\beta(x) = 0$, $\lim\limits_{x\to 0^+}\frac{f(x)}{\varphi(x)} = 0$. By assumption $\lim\limits_{x\to 0^+}\frac{1}{x}\int_0^x fg = 0$. By Lemma 3.1, $\lim\limits_{x\to 0}\frac{1}{x}\int_0^x \varphi\beta|g| = 0$. By Lemma 3.2 with $G = \varphi|g|$, (4) follows. $\square$

The duality alluded to earlier is connected to multipliers as is exhibited in the following two assertions.

Theorem 3.4. *Let $\varphi\colon (0,1] \to \mathbb{R}^+$ be continuous with $\lim\limits_{x\to 0^+}\varphi(x) = 0$ and let P be a class of functions with $C \subset P \subset W$. Let*

$$S = \Big\{ f \in D;\ \text{for each } y \in I \text{ there is a } F \in P \text{ with } F(y) = f(y) \\ \text{such that } \limsup_{x\to y}\frac{|f(x)-F(x)|}{\varphi(|x-y|)} < \infty \Big\}.$$

Let

$$T = \Big\{ g \in D;\ y \in I \text{ implies } \lim_{x\to y}\frac{1}{x-y}\int_y^x \varphi(|t-y|)|g(t)|\,\mathrm{d}t = 0 \Big\}.$$

Then $M(S) = T$.

P r o o f. Let $g \in T$. To show that $g \in M(S)$ let $f \in S$ and let $y \in I$. Let F be the function in P from the definition of S and set $f_1 = f - F$. Since $g \in T$, $\lim\limits_{x\to y}\frac{1}{x-y}\int_y^x f_1 g = \lim\limits_{x\to y}\int_y^x \frac{f(t)-F(t)}{\varphi(|t-y|)}\varphi(|t-y|)g(t)\,\mathrm{d}t = 0$. Since $g \in D$ and since $F \in P \subset W$, $Fg \in D$ and consequently $\lim\limits_{x\to y}\frac{1}{x-y}\int_y^x Fg = F(y)g(y)$. Hence $\lim\limits_{x\to y}\frac{1}{x-y}\int_y^x fg = F(y)g(y) = f(y)g(y)$. Therefore $fg \in D$. Thus $g \in M(S)$.

Now let $g \in M(S)$. By definition $g \in D$. Let $y \in [0,1)$ and let $f_0 \in (C_\infty(\mathbb{R}^+))^+$ with $\limsup\limits_{x\to 0^+}\frac{f_0(x)}{\varphi(x)} < \infty$. Set $f = 0$ on $[0,y]$ and $f(t) = f_0(t-y)$ for $t \in (y,1]$. Because $\lim\limits_{x\to 0^+}\varphi(x) = 0$, $f \in C$. To show that $f \in S$, let $z \in I$. If $z \leqslant y$, set $F = 0$. If $z > y$, let $F = f$. Because $f_0 \in C_\infty$, $F \in C \subset P$. Hence $f \in S$. Since $g \in M(S)$, $fg \in D$ so that

$$\lim_{x\to 0^+}\frac{1}{x}\int_0^x f_0(t)g(y+t)\,\mathrm{d}t = \lim_{x\to 0^+}\frac{1}{x}\int_y^{y+x} fg = f(y)g(y) = 0.$$

By Lemma 3.1, $\lim\limits_{x\to 0^+}\frac{1}{x}\int_0^x \varphi(t)|g(y+t)|\,\mathrm{d}t = 0$. Similarly if $y \in (0,1]$, it can be shown that $\lim\limits_{x\to 0^-}\frac{1}{x}\int_0^x \varphi(|t|)|g(y+t))|\,\mathrm{d}t = 0$. It follows that $g \in T$. $\square$

Theorem 3.5. *Let $\varphi\colon (0,1] \to \mathbb{R}^+$ be continuous with $\lim_{x\to 0^+} \varphi(x) = 0$ and let $C \subset P \subset W$. Let*

$$S = \left\{ f\colon \in D;\ \text{ for each } y \in I \text{ there is a } F \in P \text{ with } F(y) = f(y) \right.$$
$$\left. \text{such that } \lim_{x\to y} \frac{|f(x) - F(x)|}{\varphi(|x-y|)} = 0 \right\}.$$

Let

$$T = \left\{ g \in D;\ y \in I \text{ implies } \limsup_{x\to y} \frac{1}{x-y} \int_y^x \varphi(|t-y|)|g(t)|\,\mathrm{d}t < \infty \right\}.$$

Then $T = M(S)$.

P r o o f. The proof parallels that of the previous one except that Lemma 3.3 is used in place of Lemma 3.1. □

Corollary 3.6. *Let $n \in \mathbb{N} \cup \{0\}$. Then*

$$M(P_n) = \left\{ g \in D;\ y \in I \text{ implies } \limsup_{x\to y} \frac{1}{y-x} \int_y^x |t-y|^n |g(t)|\,\mathrm{d}t < \infty \right\}.$$

P r o o f. In Theorem 3.5 let $\varphi(x) = x^n$ and choose P to be the set of all polynomials of degree no more than n. The reader can easily verify that $S = P_n$ and the assertion follows immediately from Theorem 3.5. □

The next lemma is used in the proofs of remaining two theorems of this section. The first of these theorems characterizes the multipliers of the locally Lipschitz functions while the second characterizes $M(\Delta)$.

Lemma 3.7. *Let $\varphi\colon (0,1) \to \mathbb{R}$ be nonnegative and measurable. Suppose there is a $K > 0$ such that for all $z \in (0, \frac{1}{2})$ and for all $t \in [z, 2z]$ $\frac{1}{K}\varphi(z) \leqslant \varphi(t) \leqslant K\varphi(z)$. Let $f\colon (0,1) \to \mathbb{R}$ be nonnegative and measurable. Then*

(a) $\lim_{x\to 0^+} \frac{\varphi(x)}{x} \int_x^{2x} f = 0$ *if and only if* $\lim_{x\to 0^+} \frac{1}{x} \int_0^x \varphi f = 0$

and

(b) $\limsup_{x\to 0^+} \frac{\varphi(x)}{x} \int_x^{2x} f < \infty$ *if and only if* $\limsup_{x\to 0^+} \frac{1}{x} \int_0^x \varphi f < \infty$.

P r o o f. $\Rightarrow$ (for both (a) and (b)). Let $e > 0$ and $\delta > 0$ such that $\frac{\varphi(x)}{x} \int_x^{2x} f < e$ for all $x \in (0, \delta)$. Let $x \in (0, \delta)$. For each $n \in \mathbb{N}$ let $z_n = \frac{x}{2^n}$. Then

$$\int_0^x \varphi f = \sum_{n\in\mathbb{N}} \int_{z_n}^{2z_n} \varphi f \leqslant \sum_{n\in\mathbb{N}} K\varphi(z_n) \int_{z_n}^{2z_n} f \leqslant K \sum_{n\in\mathbb{N}} e z_n = Kex.$$

$\Leftarrow$ (for both (a) and (b)). Let $e > 0$ and $\delta > 0$ such that $\frac{1}{x}\int_0^x \varphi f < e$ for all $x \in (0,\delta)$. Let $x \in (0,\frac{\delta}{2})$. Then for $t \in [x,2x]$,

$$\varphi(x)\int_x^{2x} f \leqslant K\int_x^{2x} \varphi f \leqslant K\int_0^{2x} \varphi f < Ke2x.$$

$\square$

Theorem 3.8. *Let*

$$\mathrm{Lip}_{\mathrm{loc}} = \{f\colon\ I \to \mathbb{R};\ \ \text{for each } y \in I \text{ there is a } K \in (0,\infty) \text{ such that}\\ |f(x) - f(y)| \leqslant K|x-y| \text{ for all } x \in I\}.$$

Then

$$M(\mathrm{Lip}_{\mathrm{loc}}) = \left\{g \in D;\ \ \text{for each } y \in I\ \lim_{h\to 0}\int_{y+h}^{y+2h} |g| = 0\right\}.$$

P r o o f. For each $x \in (0,1]$ let $\varphi(x) = x$. Then $\mathrm{Lip}_{\mathrm{loc}}$ is the class S for the function φ where for $f \in \mathrm{Lip}_{\mathrm{loc}}$ and for $y \in I$ let F be the constant function $f(y)$. Then Theorem 3.4 gives one form for $M(\mathrm{Lip}_{\mathrm{loc}})$. By part (a) of Lemma 3.7 this form of $M(\mathrm{Lip}_{\mathrm{loc}})$ is equivalent to that in the conclusion of Theorem 3.8. $\square$

Theorem 3.9. $M(\Delta) = \{g \in D$*; for each* $y \in I\ \limsup_{h\to 0} |\int_{y+h}^{y+2h} |g|| < \infty\}$.

The proof is the same as that of Theorem 3.8 except that Theorem 3.5 and part (b) of Lemma 3.7 are used in place of Theorem 3.4 and part (a) of Lemma 3.7.

4. Norms and products of derivatives

In this section the spaces that are the main focus of this article are introduced and some elementary properties are established. The main results of this section are contained in the last two assertions which establish a connection between these spaces and powers of derivatives.

N o t a t i o n 4.1. Throughout this section $J \subset \mathbb{R}$ will denote a compact interval with $|J| > 0$. Let $f\colon\ J \to \mathbb{R}$ be measurable and let $p \in (0,\infty)$. Put

$$\|f\|_{J,p} = \left(\frac{1}{|J|}\int_J |f|^p\right)^{1/p}.$$

(If $\int_J |f|^p = \infty$, we set $\|f\|_{J,p} = \infty$.) We set $\|f\|_{J,\infty} = \operatorname{ess\,sup}\{|f(x)|\colon\ x \in J\}$. Moreover if a and b are the endpoints of J, then we also write $\|f\|_{a,b,p}$ for $\|f\|_{J,p}$

even if $b < a$. If the meaning of J is clear from the context, we will write $\|f\|_p$ for $\|f\|_{J,p}$. The essential fact to remember is that the function identically 1 has norm 1 for any p and any J. Of course the triangle inequality holds if $p \in [1, \infty]$. For $p \in (0,1)$ we will use the following substitute.

Lemma 4.2. *Let $f, g\colon J \to \mathbb{R}$ be measurable and let $p \in (0,1)$. Then*

$$\|f+g\|_{J,p} \leqslant 2^{1/p} \max\{\|f\|_{J,p}, \|g\|_{J,p}\}.$$

P r o o f. Let $Q = \max\{\|f\|_{J,p}, \|g\|_{J,p}\}$. Since $p \in (0,1)$,

$$|J|\,\|f+g\|_p^p = \int_J |f+g|^p \leqslant \int_J |f|^p + \int_J |g|^p \leqslant 2|J|Q^p$$

from which the assertion follows easily. □

N o t a t i o n 4.3. In Section 7 of the paper we will often have three exponents, p, q and $r \in (0, \infty]$ with $q \leqslant p$ satisfying $\frac{1}{p} + \frac{1}{r} = \frac{1}{q}$. We adopt the standard conventions that $\frac{1}{\infty} = 0$. In case $q = 1$, then $p \geqslant 1$ and we denote the corresponding exponent r as usual by p' so that $\frac{1}{p} + \frac{1}{p'} = 1$.

The following useful fact is a consequence of Hölders inequality.

Lemma 4.4. *Let f, g and J be as in Lemma 4.2 and let $p, q, r \in (0, \infty]$ with $q \leqslant p$ and $\frac{1}{p} + \frac{1}{r} = \frac{1}{q}$. Then $\|fg\|_{J,q} \leqslant \|f\|_{J,p}\|g\|_{J,r}$.*

P r o o f. Suppose $q \in (0, \infty)$. By Hölders inequality

$$\|\,|f|^q|g|^q\|_{J,1} \leqslant \|\,|f|^q\|_{J,p/q}\|\,|g|^q\|_{J,r/q}.$$

The assertion then follows easily. If $q = \infty$, then $p, r = \infty$ and the assertion is clear. □

Proposition 4.5. *Let f and J be as in Lemma 4.2 and let $p, q \in (0, \infty]$ with $q < p$. Then $\|f\|_{J,q} \leqslant \|f\|_{J,p}$.*

P r o o f. Let $r \in (0, \infty]$ satisfy $\frac{1}{p} + \frac{1}{r} = \frac{1}{q}$. Then by Lemma 4.4 $\|f\|_{J,q} = \|f \cdot 1\|_{J,q} \leqslant \|f\|_{J,p}\|1\|_{J,r} = \|f\|_{J,p}$. □

The following theorem can be obtained using standard techniques of functional analysis.

Theorem 4.6. *Let $p \in [1, \infty)$ and let $g\colon J \to \mathbb{R}$ be measurable. Then*

$$\|g\|_p = \sup\left\{\frac{1}{|J|}\int_J fg;\ f \in C(\mathbb{R}),\ f = 0 \text{ on } \mathbb{R}\setminus J \text{ and } \|f\|_{p'} \leqslant 1\right\}.$$

In the following definition and in the remainder of the paper the variable x will always be assumed to lie in the domain of function in question.

In the next definition we introduce the subspaces of D that will be central to the rest of the paper.

Definition 4.7. For each $p \in (0, \infty)$ let

$$S_p = \{f \in D;\ \lim_{x\to y} \|f - f(y)\|_{x,y,p} = 0 \text{ for each } y \in I\}$$

and

$$T_p = \{f \in D;\ \limsup_{x\to y} \|f\|_{x,y,p} < \infty \text{ for each } y \in I\}.$$

For each $p \in [0, \infty)$ let

$$\underline{S}_p = \{f \in D;\ \text{for each } y \in I \text{ there is a } q \in (p, \infty) \text{ with } \lim_{x\to y} \|f - f(y)\|_{x,y,q} = 0\}$$

and let

$$\underline{T}_p = \{f \in D;\ \text{for each } y \in I \text{ there is a } q \in (p, \infty) \text{ with } \limsup_{x\to y} \|f\|_{x,y,q} < \infty\}.$$

For each $p \in (0, \infty]$ let $\overline{S}_p = \bigcap_{q\in(0,p)} S_q$ and let $\overline{T}_p = \bigcap_{q\in(0,p)} T_p$. Finally, let $S_0 = D \cap C_{\mathrm{ap}}$, $T_0 = D$, $S_\infty = M(T_1)$ and $T_\infty = bD$.

The reader might think that a more logical choice for S_∞ would be, C, the continuous functions on I and indeed from the interpretation given to $\|\cdot\|_{J,p}$, such a choice would seem to correspond to the case $p = \infty$. The definition of T_∞ certainly corresponds to the definition of T_p when $p = \infty$. However according to Corollary 3.6 $M(C) = T_1$, but $M(T_1)$ contains discontinuous functions. The selection of $M(T_1)$ for S_∞ will be justified in Theorem 6.5.

Note that S_1 is the class of all Lebesgue function. Moreover if $f\colon I \to \mathbb{R}$ is such that for each $y \in I$ there is a $q \in [1, \infty)$ with $\lim_{x\to y} \|f - f(y)\|_{y,x,q} = 0$, then by Proposition 4.5, q may be replaced by 1; that is, f is a Lebesgue function and consequently $f \in D$. Thus the condition $f \in D$ in the definition of S_p for $p \geqslant 1$ is redundant. Also note that all of the classes introduced in Definition 4.7 are vector spaces.

The proof of the next assertion uses Proposition 4.5 and standard arguments.

Proposition 4.8. *Let $p_1, p_2 \in (0,\infty)$ with $p_1 < p_2$. Then the following containments hold.*

$$\begin{array}{ccccccccccccccccccc}
T_\infty & \subset & \overline{T}_\infty & \subset & \ldots & \subset & \underline{T}_{p_2} & \subset & T_{p_2} & \subset & \overline{T}_{p_2} & \subset & \ldots & \subset & \underline{T}_{p_1} & \subset & T_{p_1} & \subset & \overline{T}_{p_1} & \subset & \ldots & \subset & \underline{T}_0 & \subset & T_0 \\
 & & \cup & & & & \cup & & \cup & & \cup & & & & \cup & & \cup & & \cup & & & & \cup & & \cup \\
 & & \overline{S}_\infty & \subset & \ldots & \subset & \underline{S}_{p_2} & \subset & S_{p_2} & \subset & \overline{S}_{p_2} & \subset & \ldots & \subset & \underline{S}_{p_1} & \subset & S_{p_1} & \subset & \overline{S}_{p_1} & \subset & \ldots & \subset & \underline{S}_0 & \subset & S_0.
\end{array}$$

The missing containments; namely, $S_\infty \subset \overline{S}_\infty$ and $S_\infty \subset T_\infty$ are established in Section 6. The first is part of Theorem 6.7 while the second can be found early in the proof of Proposition 6.10.

The next lemma is used here and again in the proof of Theorem 6.12.

Lemma 4.9. *Let $h\colon (0,1) \to \mathbb{R}$ be measurable with $h(x) \geqslant 0$ for each $x \in (0,1)$ and let $p \in (1,\infty)$. Suppose $\lim\mathrm{ap}_{x\to 0^+} h(x) = 0$ and $\limsup\limits_{x\to 0^+} \frac{1}{x}\int_0^x h^p < \infty$. Then $\lim\limits_{x\to 0^+} \frac{1}{x}\int_0^x h = 0$.*

Proof. Let $\varepsilon > 0$. Set $\varphi = \min\{h, \frac{1}{\varepsilon}\}$. If $h(x) > \frac{1}{\varepsilon}$, then $\varepsilon^{p-1}h^p(x) > h(x)$. So $\frac{1}{x}\int_0^x h \leqslant \frac{1}{x}\int_0^x \varphi \leqslant \varepsilon^{p-1}\frac{1}{x}\int_0^x h^p$. Since φ is bounded and since $\lim\mathrm{ap}_{x\to 0^+}\varphi(x) = 0$, we have $\lim\limits_{x\to 0^+} \frac{1}{x}\int_0^x \varphi = 0$. Consequently $\limsup\limits_{x\to 0^+} \frac{1}{x}\int_0^x h \leqslant \varepsilon^{p-1} \limsup\limits_{x\to 0^+} \frac{1}{x}\int_0^x h^p$ from which the desired conclusion follows at once. □

The following two assertions are used frequently in Section 7.

Proposition 4.10. *For $p \in [0,\infty)$, $\underline{S}_p = \underline{T}_p \cap C_{\mathrm{ap}}$. For $p \in (0,\infty]$, $\overline{S}_p = \overline{T}_p \cap C_{\mathrm{ap}}$.*

Proof. By Proposition 4.8 for $p \in [0,\infty)$, $\underline{S}_p \subset \underline{T}_p \cap C_{\mathrm{ap}}$ and for $p \in (0,\infty]$, $\overline{S}_p \subset \overline{T}_p \cap C_{\mathrm{ap}}$. Now let $f \in \underline{T}_p \cap C_{\mathrm{ap}}$ and let $y \in I$. By definition there is an $r \in (p,\infty)$ such that $\limsup\limits_{x\to y} \|f\|_{y,x,r} < \infty$. By Lemma 4.2 if $p < 1$ or by the triangle inequality if $1 \leqslant p$, $\limsup\limits_{x\to y} \|f - f(y)\|_{y,x,r} < \infty$. Let $r_1 \in (p,r)$. Then by Lemma 4.9, $\lim\limits_{x\to y} \|f - f(y)\|_{y,x,r_1} = 0$. Thus by definition $f \in \underline{S}_p$. Then $\underline{T}_p \cap C_{\mathrm{ap}} \subset \underline{S}_p$. The remaining containment is proved similarly. □

The assertion obtained from Proposition 4.10 by omitting the underlines (or overlines) is false. It is standard to construct a function $f\colon [0,1] \to \mathbb{R}$, continuous on $(0,1]$, with $f(0) = 0$ which is approximately continuous at 0 such that $\lim\limits_{x\to 0^+} \frac{1}{x}\int_0^x f = 0$ but $\lim\limits_{x\to 0^+} \frac{1}{x}\int_0^x |f| = 1$. So $f \in (T_1 \cap C_{\mathrm{ap}}) \setminus S_1$.

Theorem 4.11. *Let $p, q \in [0,\infty]$ with $q \leqslant p$ and define r by $\frac{1}{p}+\frac{1}{r}=\frac{1}{q}$.*

(i) *If $p<\infty$, if $f \in S_p$ and if $g \in T_r \cap C_{\mathrm{ap}}$, then $fg \in S_q$.*

(ii) *If $p<\infty$, if $f \in \underline{S}_p$ and if $g \in \overline{S}_r$, then $fg \in \underline{S}_q$.*

(iii) *If $f \in \overline{S}_p$ and if $g \in \overline{S}_r$, then $fg \in \overline{S}_q$.*

Proof of (i). Let $y \in I$ and write

$$fg - f(y)g(y) = (f - f(y))g + f(y)(g - g(y)).$$

By Lemma 4.4

$$\lim_{x\to y} \|(f - f(y))g\|_{x,y,q} \leqslant \lim_{x\to y} \|f - f(y)\|_{x,y,p} \limsup_{x\to y} \|g\|_{x,y,r} = 0.$$

The second term is dealt with in two cases. First assume $p = q$. Then $r = \infty$ and hence $g \in bC_{\mathrm{ap}}$. Clearly $\lim_{x\to y} \|f(y)(g - g(y))\|_{x,y,q} = 0$. Now assume $q < p$. Apply Lemma 4.9 with $h(x) = |g(y+x) - g(y)|^q$ and with exponent $\frac{r}{q}$ to obtain $\lim_{x\to y} \|(g - g(y))\|_{x,y,r} = 0$. Thus $\lim_{x\to y} \|f(y)(g - g(y))\|_{x,y,q} = 0$.

Proof of (ii). As above write

$$fg - f(y)g(y) = (f - f(y))g + f(y)(g - g(y)).$$

Since $f \in \underline{S}_p$, there is a $t \in (p,\infty)$ such that $\lim_{x\to y} \|f - f(y)\|_{x,y,t} = 0$. Because $\frac{1}{t}+\frac{1}{r} < \frac{1}{p}+\frac{1}{r} = \frac{1}{q}$, there is a $v < r$ such that $\frac{1}{t}+\frac{1}{v} = \frac{1}{u} < \frac{1}{p}+\frac{1}{r} = \frac{1}{q}$. Since $g \in \overline{S}_r$, $\lim_{x\to y} \|g - g(y)\|_{x,y,v} = 0$. As in the proof of (i), with p, r, q replaced by t, v, u respectively, $\lim_{x\to y} \|fg - f(y)g(y)\|_{x,y,u} = 0$. Because $u > q$, $fg \in \underline{S}_q$.

Proof of (iii). It is shown that (ii) implies (iii). Let $f \in \overline{S}_p$ and let $g \in \overline{S}_r$. Choose $u < q$. Then $\frac{1}{u} > \frac{1}{q} = \frac{1}{p}+\frac{1}{r}$. There are $t < p$ and $v < r$ such that $\frac{1}{t}+\frac{1}{v} = \frac{1}{u}$. Since $f \in \overline{S}_p$, $f \in \underline{S}_t$ and because $g \in \overline{S}_r$, $g \in \underline{S}_v$. By (ii) with p, q, r replaced by t, u, v respectively, $fg \in \underline{S}_u \subset S_u$. By definition $fg \in \overline{S}_q$. □

The restriction $p < \infty$ is essential for (i) while (ii) makes no sense for $p = \infty$.

Corollary 4.12. *Let p, q and r be as in Theorem 4.11 with $p < \infty$. Suppose $f \in S_p$ and $g \in S_r$. Then $fg \in S_q$.*

Proof. Because $g \in S_r$, by Proposition 4.8, $g \in T_r \cap C_{\mathrm{ap}}$. Now apply Theorem 4.11 (i). □

Corollary 4.13. *The space $\overline{S}_\infty$ is an algebra.*

Proof. As was already noted, $\overline{S}_\infty$ is a vector space. Let $f, g \in \overline{S}_\infty$ and let $q \in (1, \infty)$. Then $f, g \in S_{2q}$ and by Corollary 4.12, $fg \in S_q$. By definition of $\overline{S}_\infty$, $fg \in \overline{S}_\infty$. $\square$

The remainder of this section is devoted to characterizing the algebra $\overline{S}_\infty$ from which it is concluded that $\overline{S}_\infty$ is the largest algebra contained in D. We begin by stating two results that can be found elsewhere.

Lemma 4.14. *Let $f \in D$. Suppose there is a strictly convex $\varphi\colon \mathbb{R} \to \mathbb{R}$ such that the composition $\varphi \circ f \in D$. Then $f \in S_1$.*

For the proof see [6] Lemma 4.4, page 811.

Lemma 4.15. *Let $f, g \in C_{\mathrm{ap}}$ with $|g| \leqslant f \in D$. Then $g \in S_1$.*

For the proof see [5], 1.8, page 121.

Lemma 4.16. *Let $x, y \in [0, \infty)$ and let $p \in (1, \infty)$. Then*
(i) $(x+y)^p \leqslant 2^p \max\{x^p, y^p\}$
(ii) $|x^p - y^p| \leqslant 2^p \max\{|x-y|y^{p-1}, |x-y|^p\}$.

Proof. Assertion (i) is obvious. In fact it holds for $p \in [0, \infty)$. To prove (ii) let

$$\varphi(t) = \begin{cases} \frac{t^p - y^p}{t - y} & \text{if } t \in [0, \infty) \setminus \{y\} \\ pt^{p-1} & \text{if } t = y. \end{cases}$$

Since the function t^p is strictly convex, φ is an increasing function. Thus if $t \leqslant 2y$, then $|t^p - y^p| \leqslant |t-y|\varphi(2y) \leqslant |t-y|2^p y^{p-1}$. If $t > 2y$, then $t < 2(t-y)$ and hence $t^p - y^p \leqslant t^p \leqslant 2^p(t-y)^p$. $\square$

Proposition 4.17. *Let $p \in (1, \infty)$ and let $f \in S_p$. Then $|f|^p \in S_1$.*

Proof. Let $y \in I$ and set $g = |f|^p$. By Lemma 4.16 (ii),

$$|g - g(y)| \leqslant 2^p \max\{|f - f(y)||f(y)|^{p-1}, |f - f(y)|^p\}.$$

Since $p > 1$, Proposition 4.5, implies $\lim\limits_{x \to y} \|f - f(y)\|_{x,y,1} = 0$. It then follows easily that $\lim\limits_{x \to y} \|g - g(y)\|_{y,x,1} = 0$. $\square$

Proposition 4.18. *Let $p \in (1, \infty)$, let $f \in C_{\mathrm{ap}}$ and let $|f|^p \in D$. Then $f \in S_p$.*

Proof. Let $y \in I$ and set $g = |f - f(y)|^p$. By Lemma 4.16 (i),

$$g \leqslant 2^p(|f|^p + |f(y)|^p).$$

By Lemma 4.15, $g \in S_1 \subset D$, so that $\lim\limits_{x\to y} \|f - f(y)\|_{y,x,p} = g(y) = 0$. □

Proposition 4.19. *Let $p \in (1, \infty)$. Then $f \in S_p$ if and only if $f, |f|^p \in D$.*

Proof. Let $f \in S_p$. By Proposition 4.17, $|f|^p \in S_1 \subset D$. By Proposition 4.8, $f \in S_1 \subset D$.

Suppose $f, |f|^p \in D$. By Lemma 4.14, $f \in S_1 \subset C_{\mathrm{ap}}$ and by Proposition 4.18, $f \in S_p$. □

The above assertion for $p = 1$ is false. For example take

$$f(x) = \begin{cases} 1 + \sin \frac{1}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0. \end{cases}$$

Then $|f| = f \in D$ but $f \notin S_1$.

Proposition 4.20. *Let $n \in \mathbb{N}$ with $n > 1$, let $p \in [n, \infty)$ and let $f \in S_p$. Then $f^n \in S_1$.*

Proof. Since $S_p \subset C_{\mathrm{ap}}$, $f \in C_{\mathrm{ap}}$ and hence $f^n \in C_{\mathrm{ap}}$. Moreover $|f^n| \leqslant 1 + |f|^p$. By Proposition 4.17, $|f|^p \in S_1 \subset D$ and hence Lemma 4.15 implies $f^n \in S_1$. □

Theorem 4.21. $\overline{S}_\infty = \{f;\ f^n \in D$ *for each* $n \in \mathbb{N}\}$.

Proof. By Corollary 4.13, $\overline{S}_\infty$ is an algebra and hence $\overline{S}_\infty \subset \{f\colon f^n \in D$ for each $n \in \mathbb{N}\}$. Suppose $f^n \in D$ for each $n \in \mathbb{N}$ and let $p \in (1, \infty)$. Choose $n \in \mathbb{N}$ so that $2n > p$. Then $f \in D$ and $|f|^{2n} = f^{2n} \in D$. By Proposition 4.19, $f \in S_{2n}$. Since $2n > p$, $S_{2n} \subset S_p$. Thus $f \in S_p$ completing the proof. □

The final result of this section is an immediate consequence of the preceding theorem.

Theorem 4.22. *Let $A \subset D$ be an algebra. Then $A \subset \overline{S}_\infty$.*

5. Preliminary results

In this section we present the assertions that are used in Section 6 to find $M(X)$ where X is any of the spaces introduced in the previous section. In addition, other resutls are proved that are employed in Section 7 to find $M(X,Y)$ where X and Y are any of the same subpaces of D. We begin with some technical lemmas whose proofs depend on the propositions of Section 2.

Lemma 5.1. *Let $J=[a,b]\subset\mathbb{R}$, let $g\colon J\to\mathbb{R}$ be summable, let $\alpha\in(0,\infty)$ with $|\int_L g|<\alpha$ for each subinterval $L\subset J$, and let $m\in\mathbb{N}$. Then there is an $f\in C_\infty$ such that $|f|\in C_\infty$, $f=0$ on $\mathbb{R}\setminus J$, $|f|\leqslant 1$ on J, $\int_J f=0$, $|\int_a^x f|\leqslant\frac{|J|}{m}$ for each $x\in J$ and $\int_J fg>\frac12(\int_J|g|-m\alpha)$.*

P r o o f. For each $k=0,1,2,\dots,m$ let $x_k=a+\frac{k|J|}{m}$ and let $J_k=[x_{k-1},x_k]$ for $k=1,\dots,m$. By Proposition 2.12 applied to each J_k with $Q=\int_{J_k}|g|+|\int_{J_k}g|-\alpha$, there is a function $f_k\in C_\infty$ (the f_2 of Proposition 2.12) such that $|f_k|\in C_\infty$, $f_k=0$ on $\mathbb{R}\setminus J_k$, $|f_k|\leqslant 1$ on $\mathbb{R}$, $\int_{J_k}f_k=0$ and $\int_{J_k}f_kg>\frac12(\int_{J_k}|g|-\alpha)$. It is easy to see that $f=\sum\limits_{k=1}^{m}f_k$ is the desired function. $\square$

Lemma 5.2. *Let $J=[a,b]\subset\mathbb{R}$, let $g\colon J\to\mathbb{R}$ be integrable with $\int_J|g|=\infty$ and let $\varepsilon,T\in(0,\infty)$. Then there is an $f\in C_\infty$ such that $|f|\in C_\infty$, $f=0$ on $\mathbb{R}\setminus J$, $|f|\leqslant 1$ on J, $\int_J f=0$, $|\int_a^x f|<\varepsilon$ for each $x\in J$ and $\int_J fg>P$.*

P r o o f. Let $m\in\mathbb{N}$ with $m>\frac{|J|}{\varepsilon}$. Define x_k and J_k as in the preceding proof. There is an $\ell\in\{1,2,\dots,m\}$ such that $\int_{J_\ell}|g|=\infty$. Applying Proposition 2.12 to J_ℓ with $Q=2T+|\int_{J_\ell}g|$ we obtain an $f\in C_\infty$ (again the f_2 of Proposition 2.12) such that $|f|\in C_\infty$, $f=0$ on $\mathbb{R}\setminus J_\ell$, $|f|\leqslant 1$ on J, $\int_J f=0$ and $\int_J fg>T$. Clearly $|\int_a^x f|\leqslant\frac{|J|}{m}<\varepsilon$ for each $x\in J$. $\square$

Proposition 5.3. *Let $g\colon(0,1)\to\mathbb{R}$ be a derivative with $\limsup\limits_{x\to0^+}\operatorname{Var}(x,2x,g)=\infty$. Then there is an $f\in C_\infty(\mathbb{R}^+)$ such that $\lim\limits_{x\to0^+}\frac1x\int_0^x f=0$, $\lim\limits_{x\to0^+}\frac1x\int_0^x|f|^p=0$ for each $p\in(0,1)$ but $\lim\limits_{x\to0^+}\frac1x\int_0^x fg=0$ doesn't hold.*

P r o o f. For each $k\in\mathbb{N}$ there is an $x_k\in(0,1)$ with $2x_{k+1}<x_k$ and $\operatorname{Var}(J_k,g)>k+1$ where $J_k=[x_k,2x_k]$. For each $k\in\mathbb{N}$ set $p_k=1-\frac{1}{k+1}$. Let $k\in\mathbb{N}$. Then there is a partition $x_k=t_0<t_1<\ldots<t_\ell=2x_k$ of $[x_k,2x_k]$ such that $\sum\limits_{j=1}^{\ell}|g(t_i)-g(t_{i-1})|$

$> k+1$. By Proposition 2.10 there are $\varphi_j \in C_\infty$, such that, setting $L_j = [t_{j-1}, t_j]$, we have $\varphi_j = 0$ on $\mathbb{R} \setminus L_j$, $\int_{L_j} \varphi_j = 0$, $\left|\int_0^x \varphi_j\right| \leqslant 1$ for each $x \in (0,1]$, $\int_{L_j} \left|\varphi_j\right|^{p_k} \leqslant \frac{x_k}{\ell}$ and $\left|\int_{L_j} \varphi_j g\right| > \left|g(t_j) - g(t_{j-1})\right| - \frac{1}{\ell}$. Set $\sigma_j = \operatorname{sgn}\left(\int_{L_j} \varphi_j g\right)$ and $f_k = \sum\limits_{j=1}^{\ell} \sigma_j \varphi_j$. Then $f_k \in C_\infty$, $f_k = 0$ on $\mathbb{R} \setminus J_k$, $\int_{J_k} f_k = 0$, $\left|\int_0^x f_k\right| \leqslant 1$ for each $x \in (0,1]$ and $\int_{J_k} \left|f_k\right|^{p_k} < x_k$. Consequently for each $k \in \mathbb{N}$

$$\|f_k\|_{J_k, p_k} < 1 \qquad \text{and} \qquad \int_{J_k} f_k g > k. \tag{5}$$

Set $f = \sum\limits_{k=1}^{\infty} x_k \frac{f_k}{k}$ and let $p \in (0,1)$. There is an $m \in \mathbb{N}$ such that $p_m > p$. Let $n \in \mathbb{N}$ with $n > m$ and let $x \in (x_n, x_{n-1}]$. Then $\int_0^x |f|^p \leqslant \sum\limits_{k=n}^{\infty} \int_{J_k} |f|^p$. Let $k \geqslant n$. Then $p < p_k$, $\|f_k\|_{J_k,p} \leqslant \|f_k\|_{J_k,p_k} < 1$ and hence $\frac{1}{x_k} \int_{J_k} |f_k|^p < 1$. By definition of $f, \int_{J_k} |f|^p = (\frac{x_k}{k})^p \int_{J_k} |f_k|^p$. Since $k \geqslant n$ and since $x_k < 1$, $\int_{J_k} |f|^p < \frac{1}{n^p} |J_k|$. Therefore $\int_0^x |f|^p < \frac{1}{n^p} \sum\limits_{k=n}^{\infty} |J_k| < \frac{2x}{n^p}$. Thus $\lim\limits_{x \to 0^+} \frac{1}{x} \int_0^x |f|^p = 0$. Using $|\int_0^x f_k| \leqslant 1$ a similar argument proves that $\lim\limits_{x \to 0^+} \frac{1}{x} \int_0^x f = 0$. However by (5) we have $\int_{J_k} fg > x_k$ for each $k \in \mathbb{N}$ so that $\limsup\limits_{x \to 0^+} \frac{1}{x} \int_0^x fg \geqslant 1$ and hence $\lim\limits_{x \to 0^+} \frac{1}{x} \int_0^x fg = 0$ can't hold.

□

Proposition 5.4. *Let $g\colon (0,1) \to \mathbb{R}$ be a derivative with $\limsup\limits_{x \to 0^+} g(x) = \infty$ and let $p \in [1, \infty)$. Then there is an $f \in (C_\infty(\mathbb{R}^+))^+$ such that $\lim\limits_{x \to 0^+} \|f\|_{0,x,p} = 0$, $\limsup\limits_{x \to 0^+} \|fg\|_{0,x,p} = \infty$ but $\lim\limits_{x \to 0^+} \frac{1}{x} \int_0^x fg = 0$ doesn't hold.*

Proof. Let $a_0 = 1$. For each $n \in \mathbb{N}$ there is an $a_n \in (0,1)$ such that $2a_n < a_{n-1}$ and $g(a_n) > n^2$. Because g is the derivative of its indefinite integral, for each $n \in \mathbb{N}$ there is a $b_n \in (a_n, a_{2n})$ such that, setting $J_n = [a_n, b_n]$, yields $\int_{J_n} g > n^2 |J_n|$. Let $v_n = (\frac{a_n}{n|J_n|})^{1/p}$ and set $L_n = [a_n, a_{n-1}]$. It is easy to construct a function $f \in (C_\infty(\mathbb{R}^+))^+$ such that $f = v_n$ on J_n (so that $\int_{J_n} f^p = v_n^p |J_n|$) and $\int_{L_n} f^p < 2\frac{a_n}{n}$. Let $x \in L_n$. Because $a_n < \frac{1}{2^k}$, $\int_0^x f^p \leqslant \sum\limits_{k=n}^{\infty} \int_{L_k} f^p < \frac{2}{n} \sum\limits_{k=n}^{\infty} a_k < 4\frac{a_n}{n} < 4\frac{x}{n}$. Thus $\lim\limits_{x \to 0^+} \|f\|_{0,x,p} = 0$. Let $n \in \mathbb{N}$. Then $\frac{1}{|J_n|} \int_{J_n} |g|^p = \|g\|_{J_n,p}^p \geqslant \|g\|_{J_n,1}^p \geqslant (\frac{1}{|J_n|} \int_{J_n} g)^p \geqslant n^{2p}$. Hence

$$\int_0^{2a_n} |fg|^p \geqslant \int_{J_n} |fg|^p = v_n^p \int_{J_n} |g|^p > v_n^p |J_n| n^{2p} = n^{2p-1} a_n.$$

Consequently $\limsup\limits_{x\to 0^+} \|fg\|_{0,x,p} = \infty$. If $p = 1$, then also $\int_{J_n} fg > v_n n^2 |J_n| = na_n$ which together with $1 < \frac{b_n}{a_n} < 2$ implies that both $\lim\limits_{n\to\infty} \frac{1}{a_n}\int_0^{a_n} fg = 0$ and $\lim\limits_{n\to\infty} \frac{1}{b_n}\int_0^{b_n} fg = 0$ can't hold. Thus $\lim\limits_{x\to 0^+} \frac{1}{x}\int_0^x fg = 0$ can't hold. $\square$

Proposition 5.5. *Let $g\colon (0,1) \to \mathbb{R}$ with $\lim\limits_{x\to 0^+} \frac{1}{x}\int_0^x g = 0$ and suppose that $\lim\limits_{x\to 0^+} \frac{1}{x}\int_0^x fg = 0$ for each $f \in bC_\infty(\mathbb{R}^+)$ satisfying $\lim\limits_{x\to 0^+} \frac{1}{x}\int_0^x f = 0$. Then we have $\lim\limits_{x\to 0^+} \frac{1}{x}\int_0^x |g| = 0$.*

Proof. There is a $\delta \in \mathbb{R}^+$ such that $\int_0^\delta g$ exists. For each $n \in \mathbb{N}$ set $x_n = \frac{\delta}{2^n}$ and $J_n = [x_n, 2x_n] = [x_n, x_{n-1}]$. For each $n \in \mathbb{N}$, there is an $\eta_n > 0$ with $|\int_0^x g| < \eta_n x$ for each $x \in J_n$ such that $\lim\limits_{n\to\infty} \eta_n = 0$. For each $n \in \mathbb{N}$ choose $m_n \in \mathbb{N}$ such that $\lim\limits_{n\to\infty} m_n = \infty$ and $\lim\limits_{n\to\infty} \eta_n m_n = 0$. Let $n \in \mathbb{N}$. If $\int_{J_n} |g| = \infty$, then by Lemma 5.2 there is an $f_n \in C_\infty$ such that $f_n = 0$ on $\mathbb{R} \setminus J_n$, $\int_{\mathbb{R}} f_n = 0$, $|\int_{x_n}^x f_n| < \frac{x_n}{m_n}$ for each $x \in J_n$ and $\int_{J_n} f_n g > 1$. If $\int_{J_n} |g| < \infty$, then by Lemma 5.1 with $\alpha = 4\eta_n x_n$ and $m = m_n$, there is an $f \in C_\infty$ satisfying all of the above properties except $\int_{J_n} f_n g > \frac{1}{2}\int_{J_n} |g| - 2m_n\eta_n x_n$. Put $f = \sum\limits_{n=1}^{\infty} f_n$ on $\mathbb{R}^+$. Clearly $f \in bC_\infty(\mathbb{R}^+)$. Using an argument similar to the one employed in the proof of Proposition 5.3 it can be shown that $\lim\limits_{x\to 0^+} \frac{1}{x}\int_0^x f = 0$. By assumption $\lim\limits_{x\to 0^+} \frac{1}{x}\int_0^x fg = 0$. Hence $\int_{J_n} f_n g > 1$ can hold for only finitely many $n \in \mathbb{N}$; i.e., $\int_{J_n} |g| = \infty$ holds for only finitely many $n \in \mathbb{N}$. Thus there is an $N_0 \in \mathbb{N}$ such that $n > N_0$ implies $\int_{J_n} |g| < \infty$ and, by the choice of f_n in that case, $\int_{J_n} |g| < 2\int_{J_n} fg + 4m_n\eta_n x_n$. For $x \in J_n$,

$$\frac{1}{x}\int_0^x |g| \leqslant \frac{1}{x_n}\sum_{k\geqslant n}\int_{J_k} |g| \leqslant \frac{1}{x_n}\left(2\int_0^{2x_n} fg + 8\sup\{m_k\eta_k;\ k \geqslant n\}x_n\right)$$

from which $\lim\limits_{x\to 0^+} \frac{1}{x}\int_0^x |g| = 0$ follows. $\square$

Proposition 5.6. *Let $g \in D$ with $g(0) \neq 0$ and let $p \in [0,\infty)$. Then there is an $f \in C(\mathbb{R}^+)$ such that $\lim\limits_{x\to 0^+} \frac{1}{x}\int_0^x f = 0$, $\lim\operatorname{ap}_{x\to 0^+} f(0) = 0$, $\limsup\limits_{x\to 0^+} \frac{1}{x}\int_0^x |fg|^q = \infty$ for each $q > p$ and if $p > 0$, then $\lim\limits_{x\to 0^+} \frac{1}{x}\int_0^x |f|^p = 0$.*

Proof. We may suppose that $g(0) > 1$. Let $S = \{x;\ g(x) > 1\}$. Since $g \in D$, $|S \cap (0,\delta)| > 0$ for each $\delta \in (0,1)$ and hence for each $n \in \mathbb{N}$ there is an $x_n \in (0,1)$ such that x_n is a point of density of S and $2x_{n+1} < x_n$. For each $n \in \mathbb{N}$ there is a $y_n \in (x_n, 2x_n)$ such that if $J_n = (x_n, y_n)$, $|J_n \setminus S| < \frac{|J_n|}{3}$ and $|J_n| < \frac{x_n}{n^{np+1}}$. Let v_n satisfy $|J_n| v_n^{p+\frac{1}{n}} = x_n$. Then $n^{np+1} < v_n^{p+\frac{1}{n}}$, or $n^n < v_n$.

For each $n \in \mathbb{N}$ there is an $f_n \in C(\mathbb{R})$ such that $f_n = 0$ on $\mathbb{R} \setminus J_n$, $|f_n| \leqslant v_n$ on J_n, $\int_{J_n} f_n = 0$, $0 \leqslant \int_0^x f_n \leqslant \frac{x_n}{n}$ for each $x \in I$ and $|B_n| < \frac{|J_n|}{3}$ where $B_n = \{x \in J_n;\ |f_n(x)| < v_n\}$. Set $f = \sum\limits_{n=1}^{\infty} f_n$ on $\mathbb{R}^+$. Clearly $f \in C(\mathbb{R}^+)$. For $x \in [x_n, x_{n-1})$, $\int_0^x f = \int_0^x f_n \leqslant \frac{x_n}{n} < \frac{x}{n}$. Because $0 \leqslant \int_0^x f_n$, $\lim\limits_{x\to 0^+} \frac{1}{x}\int_0^x f = 0$. Set $p_n = p + \frac{2}{n}$ and $A_n = (S \cap J_n) \setminus B_n$. Then $|A_n| > \frac{|J_n|}{3}$.

Let $q > p$. There is an $n \in \mathbb{N}$ with $p_n < q$. For any such n

$$\int_0^{y_n} |fg|^{p_n} > \int_{A_n} |fg|^{p_n} \geqslant v_n^{p_n}|A_n| > v_n^{p_n}\frac{|J_n|}{3} = \frac{x_n}{3} v_n^{p_n-(p+\frac{1}{n})} > \frac{y_n v_n^{1/n}}{6} > \frac{n y_n}{6}.$$

Hence $\|fg\|_{0,y_n,q} \geqslant \|fg\|_{0,y_n,p_n} > (\frac{n}{6})^{1/p_n}$. Because $\frac{1}{p_n} \geqslant 2$ for $n \geqslant 2$, we have $\liminf\limits_{x\to 0^+} \|fg\|_{0,x,q} = \infty$.

Let $V = \bigcup\limits_{n=1}^{\infty} J_n$. Since $|J_n| n^{np+1} < x_n$, $p \geqslant 0$ implies $|J_n| < \frac{x_n}{n}$. Let $x \in (x_n, x_{n-1}]$. Then $|V \cap (0,x)| \leqslant \sum\limits_{k=n}^{\infty} |J_k| < \frac{1}{n} \sum\limits_{k=n}^{\infty} x_k < \frac{2x_n}{n} < \frac{2x}{n}$. It follows that $\lim \mathrm{ap}_{x\to 0^+} f(x) = 0$.

Finally assume $p > 0$. If $x \in (x_n, x_{n-1}]$, then

$$\int_0^x |f|^p \leqslant \sum_{k=n}^{\infty} v_k^p |J_k| = \sum_{k=n}^{\infty} x_k \frac{1}{k} \leqslant \frac{1}{n} 2x_n \leqslant \frac{2}{n} x.$$

Thus, $\lim\limits_{x\to 0^+} \frac{1}{x}\int_0^x |f|^p = 0$. $\square$

Proposition 5.7. *Let g be as in Proposition 5.6 and let $p \in (0,\infty)$. There is an $f \in C(\mathbb{R}^+)$ such that $\lim\limits_{x\to 0^+} \frac{1}{x}\int_0^x f = 0$, $\limsup\limits_{x\to 0^+} \frac{1}{x}\int_0^x |fg|^p = \infty$ and $\lim\limits_{x\to 0^+} \frac{1}{x}\int_0^x |f|^q = 0$ for each $q \in (0,p)$.*

Proof. As before assume $g(0) > 1$. Let S, x_n, y_n and J_n be as before. For $n \leqslant \frac{1}{p}$ set $w_n = v_n$ and for $n > \frac{1}{p}$ define w_n by $|J_n| w_n^{p-\frac{1}{n}} = x_n$. In either case $w_n \geqslant v_n > n^n$. For each $n \in \mathbb{N}$ there is an $f_n \in C(\mathbb{R})$ such that $f_n = 0$ on $\mathbb{R} \setminus J_n$, $|f_n| \leqslant w_n$ on J_n, $\int_{J_n} f_n = 0$, $0 \leqslant \int_0^x f_n \leqslant \frac{x_n}{n}$ for each x and $|B_n^*| < \frac{|J_n|}{3}$ where $B_n^* = \{x \in J_n;\ |f_n(x)| < w_n\}$. Set $f = \sum\limits_{n=1}^{\infty} f_n$ on $\mathbb{R}^+$. Clearly $f \in C(\mathbb{R}^+)$ and a now-familiar argument shows that $\lim\limits_{x\to 0^+} \frac{1}{x}\int_0^x f = 0$.

Let $A_n^* = (S \cap J_n) \setminus B_n^*$. Then $|A_n^*| > \frac{1}{3}|J_n|$ and for $n > \frac{1}{p}$,

$$\int_0^{y_n} |fg|^p \geqslant \int_{A_n^*} |fg|^p \geqslant w_n^p |A_n^*| > w_n^{p-\frac{1}{n}} |J_n| \frac{1}{3} w_n^{1/n} > \frac{1}{6} n y_n.$$

Hence $\limsup\limits_{x\to 0^+} \frac{1}{x}\int_0^x |fg|^p = \infty$.

Let $q \in (0,p)$ and set $q_n = p - \frac{2}{n}$. There is an $m \in \mathbb{N}$ with $m > \frac{1}{p}$ such that $q_m > q$. Then $n > m$ implies $\|f_n\|_{J_n,q} \leqslant \|f_n\|_{J_n,q_n} \leqslant w_n$. Hence $\int_{J_n} |f|^q = |J_n| \|f_n\|^q_{J_n,q} \leqslant |J_n| w_n^q < |J_n| w_n^{q_n} = |J_n| \frac{w_n^{p-1/n}}{w_n^{1/n}} < \frac{x_n}{n}$. Once again it follows that $\lim\limits_{x\to 0^+} \frac{1}{x} \int_0^x |f|^q = 0$.

□

Lemma 5.8. *Let $\{a_n\}$ be a decreasing sequence with $\lim\limits_{n\to\infty} a_n = 0$ and for each $n \in \mathbb{N}$, let $b_n \in \mathbb{R}^+$. Then there is an $f \in (C(\mathbb{R}^+))^+$ such that $n \in \mathbb{N}$ implies $f(a_n) = b_n$ and $\lim\limits_{x\to 0^+} \frac{1}{x} \int_0^x f^p = 0$ for each $p \in (0,\infty)$.*

P r o o f. For each $n \in \mathbb{N}$, set $\beta_n = \max\{b_n, \frac{2^n}{a_n}\}$ and $\delta_n = \mathrm{e}^{-\beta_n}$. Choose $d_n \in (0, \min\{\delta_n, \frac{a_n}{2}\})$ with $a_{n+1} + d_{n+1} < a_n - d_n$ and set $J_n = (a_n - d_n, a_n + d_n)$. There is an $f \in (C(\mathbb{R}^+))^+$ such that for each $n \in \mathbb{N}, f(a_n) = b_n$, $f \leqslant b_n$ on J_n and $f = 0$ on $\mathbb{R}^+ \setminus \bigcup\limits_{n=1}^{\infty} J_n$. Let $p \in (0,\infty)$ and set $\mu = \max\{x^{p+1}\mathrm{e}^{-x};\ x \in (0,\infty)\}$. Then $x \in (a_n - d_n, a_{n-1} - d_{n-1}]$ implies $\int_0^x f^p \leqslant \sum\limits_{k=n}^{\infty} \int_{J_k} f^p \leqslant \sum\limits_{k=n}^{\infty} 2d_k b_k^p$. Note that $d_k \leqslant \delta_k \leqslant \frac{\mu}{\beta_k^{p+1}} = \mu \frac{1}{\beta_k} \frac{1}{\beta_k^p} < \mu \frac{a_k}{2^k} \frac{1}{b_k^p}$ and $x > \frac{1}{2} a_n$. Thus

$$\int_0^x f^p \leqslant \sum_{k=n}^{\infty} \frac{2\mu a_k}{2^k} < 2\mu a_n \sum_{k=n}^{\infty} \frac{1}{2^k} = \frac{4\mu a_n}{2^n} < \frac{\delta \mu x}{2^n}$$

from which again $\lim\limits_{x\to 0^+} \int_0^x f^p = 0$ follows. □

Proposition 5.9. *Let g be as in Propositions* 5.6 *and* 5.7. *Then there is an $f \in (C(\mathbb{R}^+))^+$ such that $\lim\limits_{x\to 0^+} \frac{1}{x} \int_0^x f^p = 0$ for each $p \in (0,\infty)$ and $\limsup\limits_{x\to 0^+} |(fg)(x)| = \infty$.*

P r o o f. Again assume $g(0) > 1$. There is a decreasing sequence $\{a_n\}$ in $(0,1)$ with $\lim\limits_{n\to\infty} a_n = 0$ such that $n \in \mathbb{N}$ implies $g(a_n) > 1$. Now apply Lemma 5.8 with $b_n = n$. □

The next series of results leads to the two assertions at the end of this section which will provide the basis for the proofs of two of the major theorems in the next section.

Lemma 5.10. *Let $g\colon (0,1) \to \mathbb{R}$ be measurable and nonnegative, let $A \in (0,\infty)$ and let $a \in (0,1]$. Suppose $\int_0^a g > aA$. Then there is $b \in \left(0, \frac{a}{2}\right]$ such that $\int_b^{2b} g > bA$.*

P r o o f. For each $n \in N$ set $a_n = \frac{a}{2^n}$. If $\int_{a_n}^{2a_n} g \leqslant a_n A$ for each $n \in \mathbb{N}$, then $\int_0^a g \leqslant A \sum\limits_{n=1}^{\infty} a_n = Aa$ which is a contradiction. So there is an $n \in \mathbb{N}$ with $\int_{a_n}^{2a_n} g > a_n A$ and we let $b = a_n$. □

Proposition 5.11. *For each $n \in \mathbb{N}$ let $r_n \in (0,\infty)$ and let $g\colon (0,1) \to \mathbb{R}$ be measurable such that $\limsup\limits_{x\to 0^+} \|g\|_{0,x,r_n} = \infty$ for each $n \in \mathbb{N}$. Then for each $n \in \mathbb{N}$ there is an $a_n \in (0,1]$ such that $2a_{n+1} < a_n$ and $\|g\|_{a_n,2a_n,r_n} > n^2$ for each $n \in \mathbb{N}$.*

Proof. Set $a_0 = 1$. Let $n \in \mathbb{N}$ and suppose a_{n-1} has been defined. Since $\limsup\limits_{x\to 0^+} \|g\|_{0,x,r_n} = \infty$, there is a $c \in (0, a_{n-1})$ such that $\frac{1}{c}\int_0^c |g|^{r_n} > n^{2r_n}$. By Lemma 5.10, there is an $a_n \in \left(0, \frac{c}{2}\right]$ such that $\frac{1}{a_n}\int_{a_n}^{2a_n} |g|^{r_n} > n^{2r_n}$ which is the desired result. □

Theorem 5.12. *For each $n \in \mathbb{N}$, let $s_n \in (1,\infty)$ with $s_1 \leqslant s_2 \leqslant \ldots$ and let $r_n = s_n'$. Let $g\colon (0,1) \to \mathbb{R}$ be integrable such that $\limsup\limits_{x\to 0^+} \|g\|_{0,x,r_n} = \infty$ for each $n \in \mathbb{N}$. Then there is an $f \in C(\mathbb{R}^+)$ such that $\lim\limits_{x\to 0^+} \|f\|_{0,x,s_n} = 0$ for each $n \in \mathbb{N}$ and $\limsup\limits_{x\to 0^+} \frac{1}{x}\int_0^x fg = +\infty$.*

Proof. For each n let a_n satisfy the conclusion of Proposition 5.11 and set $J_n = [a_n, 2a_n]$. Let $n \in \mathbb{N}$. Since $\|g\|_{J_n,r_n} > n^2$, by Theorem 4.6 there is an $f_n \in C(\mathbb{R})$ with $f_n = 0$ on $\mathbb{R} \setminus J_n$ such that $\|f_n\|_{J_n,s_n} \leqslant 1$ and $\int_{J_n} f_n g > a_n n^2$. Set $f = \sum\limits_{n=1}^{\infty} \frac{f_n}{n}$. Let $n \in \mathbb{N}$. For $x \in (0, a_n]$ choose $k \in \mathbb{N}$ so that $x \in (a_k, a_{k-1}]$. Then $k > n$. So for $m \geqslant k$ Proposition 4.5 implies $\|f_m\|_{J_m,s_n} \leqslant \|f_m\|_{J_m,s_m} \leqslant 1$. Consequently $\int_{J_m} |f_m|^{s_n} \leqslant a_m$. Thus

$$\int_0^x |f|^{s_n} \leqslant \sum_{m=k}^{\infty} \int_{J_m} \left|\frac{f_m}{m}\right|^{s_n} \leqslant \frac{1}{k^{s_n}} \sum_{m=k}^{\infty} \int_{J_m} |f_m|^{s_n} \leqslant \frac{1}{k^{s_n}} \sum_{m=k}^{\infty} a_m \leqslant \frac{2a_k}{k^{s_n}} < \frac{2x}{k^{s_n}}.$$

Thus $\lim\limits_{x\to 0^+} \frac{1}{x}\int_0^x |f|^{s_n} = 0$. On the other hand for $x = 2a_m$,

$$\frac{1}{x}\int_0^x fg \geqslant \frac{1}{2a_m}\int_{J_m} fg > \frac{1}{2a_m}\frac{a_m m^2}{m} = \frac{m}{2}$$

and hence $\limsup\limits_{x\to 0^+} \frac{1}{x}\int_0^x fg = +\infty$. □

Theorem 5.13. *For each $n \in \mathbb{N}$, let $s_n \in (0,\infty)$ and let $t \in (0,\infty)$ with $t < s_1 \leqslant s_2 \leqslant \ldots$. For each $n \in \mathbb{N}$ define r_n by $\frac{1}{s_n} + \frac{1}{r_n} = \frac{1}{t}$. Suppose $g\colon (0,1) \to \mathbb{R}$ is measurable with $\limsup\limits_{x\to 0^+} \|g\|_{0,x,r_n} = \infty$ for each $n \in \mathbb{N}$. Then there is an $f \in C(\mathbb{R}^+)$ such that $\limsup\limits_{x\to 0+} \|fg\|_{0,x,t} = +\infty$ and $\lim\limits_{x\to 0^+} \|f\|_{0,x,s_n} = 0$ for each $n \in \mathbb{N}$.*

Proof. As before let a_n satisfy the conclusion of Proposition 5.11 and set $J_n = [a_n, 2a_n]$. Let $n \in \mathbb{N}$. Set $p = \frac{r_n}{t}$. Then $p' = \frac{s_n}{t}$. Since $\||g|^t\|_{J_n,p} = \|g\|^t_{J_n,r_n} > n^{2t}$, by Theorem 4.6 there is an $f_n = C(\mathbb{R})$ $(f_n = |f|^{\frac{1}{t}})$ with $f_n = 0$ on $\mathbb{R} \setminus J_n$ such that $\|f_n\|_{J_n,s_n} = \| \, |f_n|^t \|_{J_n,p'}^{\frac{1}{t}} \leqslant 1$ and $\int_{J_n} |f_n g|^t > a_n n^2$. Set $f = \sum\limits_{n=1}^{\infty} \frac{f_n}{n}$. Proceeding as in the previous proof it follows that $\lim\limits_{x\to 0^+} \frac{1}{x}\int_0^x |f|^{s_n} = 0$ for each $n \in \mathbb{N}$. Also as before for $x = 2a_m$ we have $\frac{1}{x}\int_0^x |fg|^t > \frac{m}{2}$ and hence $\limsup\limits_{x\to 0^+} \frac{1}{x}\int_0^x fg = +\infty$. $\square$

Theorem 5.14. *For each $n \in \mathbb{N}$ let $s \in (0,\infty)$ and let $t_n \in (s,\infty)$ with $t_1 \leqslant t_2 \leqslant \ldots < s$. For each $n \in \mathbb{N}$ define r_n by $\frac{1}{s} + \frac{1}{r_n} = \frac{1}{t_n}$. Suppose $g\colon (0,1) \to \mathbb{R}$ is measurable and $\limsup\limits_{x\to 0^+} \|g\|_{0,x,r_n} = +\infty$ for each $n \in \mathbb{N}$. Then there is an $f \in C(\mathbb{R}^+)$ such that $\lim\limits_{x\to 0^+} \|f\|_{0,x,s} = 0$ and $\limsup\limits_{x\to 0^+} \|fg\|_{0,x,t_n} = +\infty$ for each $n \in \mathbb{N}$.*

P r o o f. Proceed as in the proof of Theorem 5.13. $\square$

6. Multipliers of various spaces

The main results of this paper are contained in the next two sections. In this section we find $M(X)$ where X is any of the spaces introduced in Section 4: $\underline{S}_p$, S_p, $\overline{S}_p$, $\underline{T}_p$, T_p, or $\overline{T}_p$ with the appropriate limitations on p. We begin with $M(X)$ for any X with $\overline{S}_1 \subset X$.

Definition 6.1. Let

$$W = \{g \in D\colon \limsup_{h\to 0} \operatorname{Var}(x+h, x+2h, g) < \infty \text{ for each } x \in I\}.$$

The space W is what is referred to in [2] as the space of functions of distant bounded variation. It was shown there that $M(D) = W$. First we present a new proof of that result and somewhat more, beginning with two lemmas.

Lemma 6.2. *Let $\delta, C \in (0,\infty)$ with $\delta < 1$ and let $g\colon (0,1) \to \mathbb{R}$ be integrable such that $\lim\limits_{x\to 0^+} \frac{1}{x}\int_0^x g = C$. For each $n \in \mathbb{N}$ set $z_n = 2^{-n}\delta$ and $J_n = [z_n, 2z_n]$. Let $V = \limsup\limits_{n\to\infty} \operatorname{osc}(J_n, g)$. Then*

$$C - V \leqslant \liminf_{x\to 0^+} g(x) \leqslant \limsup_{x\to 0^+} g(x) \leqslant C + V.$$

P r o o f. Let $x \in (0,1)$. Then there is an $n \in \mathbb{N}$ such that $x \in J_n$. Clearly $g \leqslant g(x) + \operatorname{osc}(J_n, g)$ on J_n. Hence $g(x) \geqslant \frac{1}{z_n}\int_{J_n} g - \operatorname{osc}(J_n, g)$. Since $\lim\limits_{n\to\infty} \frac{1}{z_n}\int_{J_n} g = C$, $C - V \leqslant \liminf\limits_{x\to 0^+} g(x)$. The other inequality has a similar proof. $\square$

Lemma 6.3. *Let $f,g\colon (0,1)\to\mathbb{R}$ be measurable such that* $\lim\limits_{x\to 0^+}\frac{1}{x}\int_0^x f=0$, $\limsup\limits_{x\to 0^+}|g(x)|<\infty$ *and* $\limsup\limits_{x\to 0^+}\operatorname{Var}(x,2x,g)<\infty$. *Then* $\lim\limits_{x\to 0^+}\frac{1}{x}\int_0^x fg=0$.

P r o o f. Let $\varepsilon_0\in(0,\infty)$. By assumption there are $\delta_0,B,C\in(0,\infty)$ such that $|g(x)|<B$ and $\operatorname{Var}(x,2x,g)<C$ for each $x\in(0,\delta_0)$. Put $\varepsilon=\frac{\varepsilon_0}{8(2B+C)}$. There is a $\delta\in(0,\delta_0)$ such that $|\int_0^x f|<\varepsilon x$ for each $x\in(0,\delta]$. Let $x_0\in(0,\delta]$. We first must show that fg is integrable on $[0,x_0]$ a task made more difficult because we are dealing with the Denjoy-Perron integral. For each $n\in\mathbb{N}$ let $x_n=2^{-n}x_0$. For $n\in\mathbb{N}$ and for $x\in[x_n,2x_n]=[x_n,x_{n-1}]$ let $G(x)=\operatorname{Var}(x_n,x,g)$, $g_1(x)=\frac{1}{2}(G(x)+g(x)-g(x_n))$ and $g_2(x)=\frac{1}{2}(G(x)-g(x)+g(x_n))$. Then g_1 and g_2 are nondecreasing on $[x_n,2x_n]$ with $g_1(x_n)=g_2(x_n)=0$. So fg_i is integrable on $[x_n,2x_n]$ for $i=1,2$ and consequently fg is integable on $[x_n,2x_n]$. Moreover by Theorem 2.8

$$\left|\int_{x_n}^{2x_n} fg\right|=\left|\int_{x_n}^{2x_n} f(g_1-g_2)\right|\leqslant\left|\int_{x_n}^{2x_n} fg_1\right|+\left|\int_{x_n}^{2x_n} fg_2\right|<4(C+2B)\varepsilon x_n.$$

Hence for any $n\in\mathbb{N}$

$$\begin{aligned}\left|\int_{x_n}^{x_0} fg\right|&=\left|\sum_{m=1}^{n}\int_{x_m}^{2x_m} fg\right|\leqslant\sum_{m=1}^{n}\left|\int_{x_m}^{2x_m} fg\right|\\&<(C+2B)4\varepsilon\sum_{m=1}^{n}x_m<(C+2B)4\varepsilon x_0 2=\varepsilon_0x_0.\end{aligned}$$

By the theory of the Denjoy-Perron integral, $\int_0^\delta fg$ exists and $|\int_0^\delta fg|\leqslant\varepsilon_0\delta$. □

Theorem 6.4. *Let $\overline{S}_1\subset X\subset D$. Then $M(X)=M(D)=W$.*

P r o o f. By Proposition 2.3, $M(D)\subset M(X)\subset M(\overline{S}_1)$. Consequently it suffices to prove $W\subset M(D)$ and $M(\overline{S}_1)\subset W$. To prove the former, first note that for any interval J and any $h\colon J\to\mathbb{R}$ we have $\operatorname{osc}(J,h)\leqslant\operatorname{Var}(J,h)$. Consequently if $g\in W$, then according to Lemma 6.2 g is bounded. Then Lemma 6.3 easily proves that if $g\in W$ and if $f\in D$, then $fg\in D$. For the second containment suppose $g\in D\setminus W$. We may assume $\limsup\limits_{x\to 0^+}\operatorname{Var}(x,2x,g)=\infty$. By Proposition 5.3 there is an $f\in C_\infty(\mathbb{R}^+)$ such that $\lim\limits_{x\to 0^+}\frac{1}{x}\int_0^x f=0$, $\lim\limits_{x\to 0^+}\frac{1}{x}\int_0^x|f|^p=0$ for each $p\in(0,1)$ but $\lim\limits_{x\to 0^+}\frac{1}{x}\int_0^x fg=0$ doesn't hold. The first two conditions imply $f\in\overline{S}_1$. The third says $fg\notin D$. Thus $g\notin M(\overline{S}_1)$. □

By Proposition 4.8 for each $p\in(0,1)$ we have $\overline{S}_1\subset\underline{S}_p\subset S_p\subset\overline{S}_p$ and $\overline{S}_1\subset\underline{T}_p\subset T_p\subset\overline{T}_p$. Also $\overline{S}_1\subset\underline{S}_0\subset S_0$ and $\overline{S}_1\subset\underline{T}_0\subset T_0$. Thus for each of these spaces, X, we have $M(X)=W$. We now deal with the remaining spaces. The next theorem sets the pattern for the second major theorem of this section, Theorem 6.13.

Theorem 6.5. $M(S_1) = T_\infty$, $M(T_\infty) = S_1$, $M(S_\infty) = T_1$, $M(T_1) = S_\infty$.

Proof. It follows from Proposition 5.4 with $p = 1$ that $M(S_1) \subset T_\infty$. That every bounded derivative (that is, an element of T_∞) is in $M(S_1)$ is easy and is left to the reader.

Proposition 5.5 implies $M(T_\infty) \subset S_1$ and again the opposite containment is easy. By Corollary 3.6 with $n = 0$, $M(C) = T_1$. By Proposition 2.7, $T_1 = M(C) = M(M(M(C))) = M(M(T_1)) = M(S_\infty)$. The last equality is just the definition of S_∞. $\square$

Remark 6.6. The relation $M(S_1) = T_\infty$ is also proved in [5]. The equality $M(T_\infty) = S_1$ was stated without proof in [1].

Theorem 6.7. $M(D) \subset S_\infty \subset bC_{\mathrm{ap}} \subset \overline{S}_\infty$.

Proof. Since $T_1 \subset D$, by Proposition 2.3 $M(D) \subset M(T_1) = S_\infty$. Since $T_\infty \cap S_1 \subset T_1$, Theorem 6.5 implies $S_\infty = M(T_1) \subset M(S_1) \cap M(T_\infty) = T_\infty \cap S_1 = bC_{\mathrm{ap}}$. It is easy to see that $bC_{\mathrm{ap}} \subset S_p$ for each $p \in [0, \infty)$. Thus $bC_{\mathrm{ap}} \subset \overline{S}_\infty$. $\square$

Remark 6.8. Let $f \in M(D)$. Then by the preceding theorem f is approximately continuous. Consequently f is continuous on any interval on which it is of bounded variation. That $M(D) = W$ implies that there are many such intervals. In fact it implies that the union of all open intervals $(a, b) \subset I$ such that f is continuous and of bounded variation on each $[c, d]$ with $a < c < d < b$ is all of I except for a finite set.

The next lemma is used here and extensively in Section 7.

Lemma 6.9. *Let $p, q \in (0, \infty]$ with $q \leqslant p$. Define $r \in (0, \infty]$ by $\frac{1}{p} + \frac{1}{r} = \frac{1}{q}$. Suppose $f, g, fg \in D$.*

(i) *If $f \in T_p$ and if $g \in T_r$, then $fg \in T_q$*

(ii) *If $p < \infty$, if $f \in \underline{T}_p$ and if $g \in \overline{T}_r$, then $fg \in \underline{T}_q$*

(iii) *If $f \in \overline{T}_p$ and if $g \in \overline{T}_r$, then $fg \in \overline{T}_q$.*

Proof. (i) follows immediately from Lemma 4.4. For (ii) let $y \in I$. Since $f \in \underline{T}_p$, there is a $s \in (p, \infty)$ such that $\limsup\limits_{x \to y} \|f\|_{x,y,s} < \infty$. Since $\frac{1}{s} + \frac{1}{r} < \frac{1}{p} + \frac{1}{r} = \frac{1}{q}$, there is $q_1 > q$ and $r_1 < r$ such that $\frac{1}{s} + \frac{1}{r_1} = \frac{1}{q_1}$. By definition $g \in T_{r_1}$. Thus Lemma 4.4 implies $\limsup\limits_{x \to y} \|fg\|_{x,y,q} < \infty$. By definition $fg \in \overline{T}_q$. The proof of (iii) is easy and hence is omitted. $\square$

Proposition 6.10. *Let $p \in [1,\infty]$. Suppose one of the following holds.*

(i) *$f \in S_p$ and $g \in T_{p'}$.*

(ii) *$p > 1, f \in \overline{S}_p$ and $g \in \underline{T}_{p'}$.*

(iii) *$p < \infty, f \in \underline{S}_p$ and $g \in \overline{T}_{p'}$.*

Then $fg \in T_1$.

Proof. If $f \in S_\infty$ and if $g \in T_1$, then since $S_\infty = M(T_1)$, $fg \in D$. Since $S_1 \subset T_1$, $S_\infty = M(T_1) \subset M(S_1) = T_\infty$ by Theorem 6.5. (This inclusion is one of those missing from Proposition 4.8.) Consequently f is bounded and since $g \in T_1$, it follows that $fg \in T_1$. In each of the remaining cases it is easy to prove, using Lemma 4.4, that for each $y \in I$, $\lim\limits_{x\to y} \|(f - f(y))g\|_{x,y,1} = 0$. Since $fg = (f - f(y))g + f(y)g$, it follows that $fg \in D$. Now apply Lemma 6.9 with $r = 1$. □

Lemma 6.11. *Let $p \in [1,\infty)$ and let $f\colon \mathbb{R}^+ \to \mathbb{R}$ be measurable. Suppose $\limsup\limits_{x\to 0^+} \frac{1}{x}\int_0^x |f|^p < \infty$. Then there is an $h \in C_\infty(\mathbb{R}^+)$ with $\lim\limits_{x\to 0^+} \frac{1}{x}\int_0^x |f-h|^p = 0$.*

Proof. For each $n \in \mathbb{N}$, let $J_n = [2^{-n}, 2^{-n+1}]$. For each $n \in \mathbb{N}$ with $\int_{J_n} |f|^p < \infty$ since $p \in [1,\infty)$, there is an $h_n \in C_\infty(\mathbb{R}^+)$ such that $h_n = 0$ on $\mathbb{R} \setminus J_n$ and $\|f - h_n\|_{J_n,p} < \frac{1}{n}$. If $\int_{J_n} |f|^p = \infty$, set $h_n = 0$. Let $h = \sum\limits_{n=1}^{\infty} h_n$. Since $\limsup\limits_{x\to 0^+} \frac{1}{x}\int_0^x |f|^p < \infty$, there is an $m \in \mathbb{N}$ such that $\int_{J_n} |f|^p < \infty$ for each $n \geqslant m$. Let $n \geqslant m$ and let $x \in J_n$. Then

$$\int_0^x |f-h|^p \leqslant \sum_{k=n}^{\infty} \int_{J_k} |f-h_k|^p \leqslant \sum_{k=n}^{\infty} \frac{|J_k|}{k^p} \leqslant \sum_{k=n}^{\infty} \frac{|J_k|}{n^p} \leqslant \frac{2x}{n^p}$$

from which the desired result follows immediately. □

Theorem 6.12. *Let $p \in [1,\infty]$. Then $M(S_p) = T_{p'}$ and $M(T_p) = S_{p'}$.*

Proof. By Theorem 6.5 we may assume $p \in (1,\infty)$. Let $g \in M(S_p)$. By Theorem 5.12 with $s_n = p$ for each $n \in \mathbb{N}$, $g \in T_{p'}$. Hence $M(S_p) \subset T_{p'}$. The opposite containment follows from Proposition 6.10 (i).

Let $g \in M(T_p)$. Since $T_p \supset T_\infty \cup S_p$, by Propositions 2.3 and 2.6 $g \in M(T_\infty) \cap M(S_p) = S_1 \cap M(S_p) \subset C_{\mathrm{ap}} \cap T_{p'}$. Let $y \in I$, set $g_1 = g - g(y)$ and set $f = |g_1|^{p'-1} \operatorname{sgn} g_1$. Since $p(p'-1) = p'$, $|f|^p = |g_1|^{p'}$. Thus

$$\limsup_{x\to y} \frac{1}{|x-y|}\int_y^x |f|^p = \limsup_{x\to y} \frac{1}{|x-y|}\int_y^x |g_1|^{p'} < \infty. \tag{6}$$

By Lemma 6.11 there is an $h \in C_\infty(\mathbb{R} \setminus \{y\})$ such that

$$\lim_{x\to y} \frac{1}{x-y} \int_y^x |f-h|^p = 0. \tag{7}$$

Let $h(y) = 0$. Then for each $x \in I\setminus\{y\}$ we have $\|h\|_{y,x,p} \leqslant \|h-f\|_{y,x,p} + \|f\|_{y,x,p}$. By (6) and (7) $\limsup\limits_{x\to y} \frac{1}{|x-y|} \int_y^x |h|^p < \infty$. It follows from (7) that $\lim \mathrm{ap}_{x\to y}(f-h)(x) = 0$. Since $g_1 \in C_{\mathrm{ap}}$ and since $g_1(y) = 0$, $\lim \mathrm{ap}_{x\to y} f(x) = 0$. Thus $\lim \mathrm{ap}_{x\to y} h(x) = 0$. By Lemma 4.9, $\lim\limits_{x\to y} \frac{1}{x-y} \int_y^x |h| = 0$ and consequently $h \in D$. Therefore $h \in T_p$. Since $g \in M(T_p)$, $hg_1 \in D$. Hence $\lim\limits_{x\to y} \frac{1}{x-y} \int_y^x hg_1 = 0$. Furthermore by (6) and (7) we have

$$\lim_{x\to y} \frac{1}{|x-y|} \int_y^x |(f-h)g_1| \leqslant \lim_{x\to y} \|f-h\|_{x,y,p} \|g_1\|_{x,y,p'} = 0.$$

Thus

$$\lim_{x\to y} \frac{1}{x-y} \int_y^x |g-g(y)|^{p'} = \lim_{x\to y} \frac{1}{x-y} \int_y^x fg_1 = 0.$$

Therefore $g \in S_{p'}$. Hence $M(T_p) \subset S_{p'}$. Again the opposite containment follows from Proposition 6.10 (i). $\square$

Theorem 6.13. *For $p \in [1,\infty)$, $M(\underline{S}_p) = \overline{T}_{p'}$ and $M(\underline{T}_p) = \overline{S}_{p'}$. For $p \in (1,\infty]$, $M(\overline{S}_p) = \underline{T}_{p'}$ and $M(\overline{T}_p) = \underline{S}_{p'}$.*

Proof. Assume $p \in [1,\infty)$. Let $q' \in (1,p')$. Then $q \in (p,\infty)$. By Proposition 4.8 $S_q \subset \underline{S}_p$. By Proposition 2.6 and Theorem 6.12, $M(\underline{S}_p) \subset M(S_q) = T_{q'}$. Thus $M(\underline{S}_p) \subset \bigcap\limits_{q'\in(1,p')} T_{q'} = \overline{T}_{p'}$. That $\overline{T}_{p'} \subset M(\underline{S}_p)$ follows from Proposition 6.10 (iii). Since $\underline{T}_p \supset \underline{S}_p \cup T_\infty$, by Proposition 2.6, Theorem 6.5, the above and Proposition 4.10, $M(\underline{T}_p) \subset M(\underline{S}_p) \cap M(T_\infty) = \overline{T}_{p'} \cap S_1 \subset \overline{T}_{p'} \cap C_{\mathrm{ap}} = \overline{S}_{p'}$. By Proposition 6.10 (ii) $\overline{S}_{p'} \subset M(\underline{T}_p)$.

Now assume $p \in (1,\infty]$ and let $g \in M(\overline{S}_p)$. For each $n \in \mathbb{N}$ let $p_n \in (1,p)$ such that $\lim\limits_{n\to\infty} p_n = p$. Then for each $n \in \mathbb{N}$, $p_n' > p'$. Let $y \in I$. By Theorem 5.12 it follows that $\limsup\limits_{x\to y} \|g\|_{x,y,p_n'} < \infty$ for some $n \in \mathbb{N}$. Hence $g \in \underline{T}_{p'}$. So $M(\overline{S}_p) \subset \underline{T}_{p'}$. The opposite containment follows from Proposition 6.10 (ii). The proof that $M(\overline{T}_p) = \underline{S}_{p'}$ is similar to that of $M(\underline{T}_p) = \overline{S}_{p'}$ and is omitted. $\square$

It is finally possible to fill in the final missing containment from Proposition 4.8. Because $\underline{T}_1 \subset T_1$, by Proposition 2.3, $S_\infty = M(T_1) \subset M(\underline{T}_1) = \overline{S}_\infty$.

This section is concluded with a theorem whose significance is explained in the subsequent remark.

Theorem 6.14. *Let $X \subset D$. Then $X \subset S_2$ if and only if $X \subset M(X)$.*

P r o o f. If $X \subset S_2$, then $X \subset T_2 = M(S_2) \subset M(X)$. Let $X \subset M(X)$ and let $f \in X$. Then $f^2 \in D$; so by Proposition 4.19 $f \in S_2$. $\square$

R e m a r k 6.15. Let $X \subset D$. Using Proposition 4.18 as in the preceding proof it is easy to see that $M(X) \subset X$ implies $M(X) \subset S_2$. However $M(X) \subset S_2$ can hold even if X does not contain the zero function in which case $M(X) \subset X$ is impossible. At the same time the equality $M(X) = X$ can never hold. For if there were such an X, then by the previous theorem $X \subset S_2$ and hence $T_2 = M(S_2) \subset M(X) = X \subset S_2$ which is false. (There are bounded derivatives; that is, elements of $T_\infty \subset T_2$ that are not approximately continuous; that is, not in $S_0 \supset S_2$.)

7. Multipliers from one space to another

In this section we find the spaces of multipliers $M(X, Y)$ where X and Y are any of the spaces of derivatives investigated in the previous three sections. To carry out the campaign the following notation will be useful.

N o t a t i o n 7.1. For $p \in (0, \infty)$ let

$$\mathcal{S}_p = \{\underline{S}_p, S_p, \overline{S}_p\} \quad \text{and} \quad \mathcal{T}_p = \{\underline{T}_p, T_p, \overline{T}_p\}.$$

Also let

$$\mathcal{S}_0 = \{\underline{S}_0, S_0\},\ \mathcal{T}_0 = \{\underline{T}_0, T_0\},\ \mathcal{S}_\infty = \{S_0, \overline{S}_\infty\} \text{ and } \mathcal{T}_\infty = \{T_\infty, \overline{T}_\infty\}.$$

Finally let $\mathcal{S} = \bigcup_{p \in [0,\infty]} \mathcal{S}_p$ and $\mathcal{T} = \bigcup_{p \in [0,\infty]} \mathcal{T}_p$. Generic elements of $\mathcal{S}$ will be denoted by S and $\widetilde{S}$ while T and $\widetilde{T}$ will denote generic elements of $\mathcal{T}$. Also X and Y will denote elements of $\mathcal{S} \cup \mathcal{T}$.

The problem of determining $M(X, Y)$ is decomposed into four parts: $M(T, S)$, $M(S, T)$, $M(T, \widetilde{T})$ and $M(S, \widetilde{S})$. We take them up in that order.

Theorem 7.2. *Let $X, Y \in \mathcal{S} \cup \mathcal{T}$ with $T_\infty \subset X$ and $Y \subset S_0$. Then $M(X, Y) = \{0\}$.*

P r o o f. Let $g \in M(X, Y)$ and let $y \in I$. Then there is an $f \in T_\infty = bD$ such that f is not approximately continuous at y. By assumption $fg \in Y \subset S_0 = C_{\mathrm{ap}}$. It is easy to see that if $g(y) \neq 0$, then fg is not approximately continuous. Thus $g(y) = 0$. $\square$

As a consequence of Theorem 7.2, $M(T, S) = \{0\}$ for each $T \in \mathcal{T}$ and $S \in \mathcal{S}$. Next the multipliers of the second type, $M(S, T)$, are computed from which the spaces $M(T, \widetilde{T})$ and $M(S, \widetilde{S})$ will be deduced. The results for $M(S, T)$ can best be displayed by a matrix-type chart with the S-spaces corresponding to the rows and the T-spaces, the columns. The intersection of row S and column T being the space $M(S, T)$, see Figure 1. The next two theorems combine to show that below the main diagonal each entry is $\{0\}$.

$X \setminus Y$	T_∞	$\overline{T}_\infty$	$\dots$	$\underline{T}_p$	T_p	$\overline{T}_p$	$\dots$	$\underline{T}_q$	T_q	$\overline{T}_q$	$\dots$	$\underline{T}_1$	T_1	$\overline{T}_1$	$\dots$	$\underline{T}_u$	T_u	$\overline{T}_u$	$\dots$	$\underline{T}_0$	T_0
S_∞	T_∞	$\overline{T}_\infty$	$\dots$	$\underline{T}_p$	T_p	$\overline{T}_p$	$\dots$	$\underline{T}_q$	T_q	$\overline{T}_q$	$\dots$	$\underline{T}_1$	T_1	T_1	$\dots$	T_1	T_1	T_1	$\dots$	T_1	T_1
$\overline{S}_\infty$		$\overline{T}_\infty$	$\dots$	$\underline{T}_p$	$\underline{T}_p$	$\overline{T}_p$	$\dots$	$\underline{T}_q$	$\underline{T}_q$	$\overline{T}_q$	$\dots$	$\underline{T}_1$	$\underline{T}_1$	$\underline{T}_1$	$\dots$	$\underline{T}_1$	$\underline{T}_1$	$\underline{T}_1$	$\dots$	$\underline{T}_1$	$\underline{T}_1$
$\vdots$			$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$
$\underline{S}_p$				$\overline{T}_\infty$	$\overline{T}_\infty$	$\overline{T}_\infty$	$\dots$	$\overline{T}_r$	$\overline{T}_r$	$\overline{T}_r$	$\dots$	$\overline{T}_{p'}$	$\overline{T}_{p'}$	$\overline{T}_{p'}$	$\dots$	$\overline{T}_{p'}$	$\overline{T}_{p'}$	$\overline{T}_{p'}$	$\dots$	$\overline{T}_{p'}$	$\overline{T}_{p'}$
S_p					T_∞	$\overline{T}_\infty$		$\underline{T}_r$	T_r	$\overline{T}_r$	$\dots$	$\underline{T}_{p'}$	$T_{p'}$	$T_{p'}$	$\dots$	$T_{p'}$	$T_{p'}$	$T_{p'}$	$\dots$	$T_{p'}$	$T_{p'}$
$\overline{S}_p$						$\overline{T}_\infty$	$\dots$	$\underline{T}_r$	$\underline{T}_r$	$\overline{T}_r$	$\dots$	$\underline{T}_{p'}$	$\underline{T}_{p'}$	$\underline{T}_{p'}$	$\dots$	$\underline{T}_{p'}$	$\underline{T}_{p'}$	$\underline{T}_{p'}$	$\dots$	$\underline{T}_{p'}$	$\underline{T}_{p'}$
$\vdots$							$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$
$\underline{S}_q$								$\overline{T}_\infty$	$\overline{T}_\infty$	$\overline{T}_\infty$	$\dots$	$\overline{T}_{q'}$	$\overline{T}_{q'}$	$\overline{T}_{q'}$	$\dots$	$\overline{T}_{q'}$	$\overline{T}_{q'}$	$\overline{T}_{q'}$	$\dots$	$\overline{T}_{q'}$	$\overline{T}_{q'}$
S_q									T_∞	$\overline{T}_\infty$	$\dots$	$\underline{T}_{q'}$	$T_{q'}$	$T_{q'}$	$\dots$	$T_{q'}$	$T_{q'}$	$T_{q'}$	$\dots$	$T_{q'}$	$T_{q'}$
$\overline{S}_q$										$\overline{T}_\infty$	$\dots$	$\underline{T}_{q'}$	$\underline{T}_{q'}$	$\underline{T}_{q'}$	$\dots$	$\underline{T}_{q'}$	$\underline{T}_{q'}$	$\underline{T}_{q'}$	$\dots$	$\underline{T}_{q'}$	$\underline{T}_{q'}$
$\vdots$											$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$
$\underline{S}_1$												$\overline{T}_\infty$	$\overline{T}_\infty$	$\overline{T}_\infty$	$\dots$	$\overline{T}_\infty$	$\overline{T}_\infty$	$\overline{T}_\infty$	$\dots$	$\overline{T}_\infty$	$\overline{T}_\infty$
S_1													T_∞	T_∞	$\dots$	T_∞	T_∞	T_∞	$\dots$	T_∞	T_∞
$\overline{S}_1$														W	$\dots$	W	W	W	$\dots$	W	W
$\vdots$															$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$
$\underline{S}_u$																W	W	W	$\dots$	W	W
S_u																	W	W	$\dots$	W	W
$\overline{S}_u$																		W	$\dots$	W	W
$\vdots$																			$\ddots$	$\vdots$	$\vdots$
$\underline{S}_0$																				W	W
S_0																					W

Figure 1. The $M(S, T)$ chart.

Theorem 7.3. *Let $p \in (0, \infty]$. Then $M(\overline{S}_p, T_p) = \{0\}$.*

Proof. Let $p \in (0, \infty)$, let $g \in M(\overline{S}_p, T_p)$ and let $y \in I$. Show $g(y) = 0$. Suppose to the contrary that $g(y) \neq 0$. Then by Proposition 5.7 there is an $f \in \overline{S}_p$ with $fg \notin T_p$ contrary to $g \in M(\overline{S}_p, T_p)$. Thus $g(y) = 0$. The case $p = \infty$ follow the same procedure except using Proposition 5.9 in place of Proposition 5.7. $\square$

Theorem 7.4. *Let $p \in [0, \infty)$. Then $M(S_p, \underline{T}_p) = \{0\}$.*

Proof. Proceed as in the proof of Theorem 7.3 using Proposition 5.6 in place of Proposition 5.7. $\square$

From Theorems 7.3 and 7.4 and Proposition 2.3 for $X, Y \in \mathcal{S} \cup \mathcal{T}$ if $\overline{S}_p \subset X$ and $Y \subset T_p$ for $p \in (0, \infty]$ or if $S_p \subset X$ and $Y \subset \underline{T}_p$ for $p \in [0, \infty)$, then $M(X, Y) = \{0\}$. It follows that all entries in the chart for $M(S, T)$ below the main diagonal are $\{0\}$. In addition the corresponding conclusion holds for the charts for $M(T, \widetilde{T})$ and $M(S, \widetilde{S})$. These entries are also denoted by leaving the corresponding space blank.

The next two assertions combine to complete the lower right hand corner of the $M(S, T)$ chart.

Theorem 7.5. *Let $X \in \mathcal{S} \cup \mathcal{T}$. Then $W = M(D) \subset M(X, X)$.*

Proof. Let $g \in W$. By Theorem 6.6, $g \in bC_{\text{ap}}$. Let $f \in X$. Then $fg \in D$. If $X = T_0$, then $M(D) = M(D, D) = M(X, X)$ by choice of T_0. If $X = S_0$, then $fg \in D \cap C_{\text{ap}} = S_0$. Thus $g \in M(S_0, S_0)$. Next assume $p \in (0, \infty]$ and $X = T_p$. Let $y \in I$. By definition $\limsup_{x \to y} \|f\|_{x,y,p} < \infty$. Because $\|g\|_\infty < \infty$, $\limsup_{x \to y} \|fg\|_{x,y,p} < \infty$. So $g \in M(T_p, T_p)$. Similarly if $X = \underline{T}_p$ and if $y \in I$, then by definition there is $q \in (0, p)$ with $\limsup_{x \to y} \|f\|_{x,y,q} < \infty$. Thus $\limsup_{x \to y} \|fg\|_{x,y,q} < \infty$. So $fg \in \underline{T}_p$. It is just as easy to prove that $M(D) \subset M(\overline{T}_p, \overline{T}_p)$ for $p \in [0, \infty)$.

Again assume $p \in (0, \infty]$ but now assume $X = S_p$. Let $f \in S_p$ and let $y \in I$. By definition $\lim_{x \to y} \|f - f(y)\|_{x,y,p} = 0$. If $p < \infty$, then

$$\|fg - f(y)g(y)\|_{x,y,p} \leqslant \|f - f(y)\|_{x,y,p} \|g\|_\infty + |f(y)| \|g - g(y)\|_{x,y,p}$$

and $\lim_{x \to y} \|g - g(y)\|_{x,y,p} = 0$ because $g \in bC_{\text{ap}}$. For $p = \infty$, let $f \in S_\infty$. To show that $fg \in S_\infty = M(T_1)$, let $h \in T_1$. By the previous case for $X = T_1$, $gh \in T_1$. So $f \in S_\infty = M(T_1)$ implies $(fg)h \in T_1$. Thus $fg \in M(T_1) = S_\infty$.

Continuing with $p \in (0, \infty]$ let $X = \underline{S}_p$, let $f \in \underline{S}_p$ and let $y \in I$. By definition there is a $q \in (0, p)$ with $\lim_{x \to y} \|f - f(y)\|_{x,y,q} = 0$. By the first argument of the preceding paragraph, $\lim_{x \to y} \|fg - f(y)g(y)\|_{x,y,q} = 0$. By definition $fg \in \underline{S}_p$. Finally for $p \in [0, \infty)$ the proof that $M(D) \subset M(\overline{S}_p, \overline{S}_p)$ is similar. $\square$

Theorem 7.6. *Let $X, Y \in \mathcal{S} \cup \mathcal{T}$ with $\overline{S}_1 \subset X \subset Y$. Then $M(X,Y) = M(D) = W$.*

P r o o f. By Theorem 6.4, $M(\overline{S}_1) = M(D) = W$. So by Theorem 7.5 and by Proposition 2.3

$$M(D) \subset M(X,X) \subset M(X,Y) \subset M(\overline{S}_1, D) = M(\overline{S}_1) = M(D).$$

$\square$

As a consequence of Theorem 7.6. in the $M(S,T)$ chart all entries on and below the $\overline{S}_1$ row, on and to the right of the $\overline{T}_1$ column and on or above the main diagonal are W. It says the same about the $M(T,\widetilde{T})$ and $M(S,\widetilde{S})$ charts.

The next theorem spells out the part of the $M(S,T)$ chart on and to the right of the T_1 column.

Theorem 7.7. *Let $X, Y \in \mathcal{S} \cup \mathcal{T}$ with $X \subset T_1 \subset Y$. Then $M(X,Y) = M(X,D) = M(X)$.*

P r o o f. Because $Y \subset D$, Proposition 2.3 implies $M(X,Y) \subset M(X)$. So let $g \in M(X)$ and let $f \in X$. By definition $fg \in D$. The possibilities for X are $X = \underline{T}_1$, $X = \underline{S}_1$ or $X \in \mathcal{S}_p \cup \mathcal{T}_p$ for $p \in (1,\infty]$. In either of the first two cases, $f \in \underline{T}_1$ and $g \in M(X) \subset \overline{T}_\infty$. So by Lemma 6.9 (ii), with $q = p = 1$, $fg \in T_1 \subset Y$. In the remaining cases Lemma 6.9 (i) or (ii) implies $fg \in T_1$. So in any case $g \in M(X,Y)$. $\square$

Theorem 7.7 shows that all columns from T_1 to its right and on or above the S_1 row agree with that of the T_0 column. But because $T_0 = D$, this column is known by the results of Section 6. Note that the corresponding assertion is valid for the $M(T,\widetilde{T})$ chart, but not to the $M(S,\widetilde{S})$ chart.

The next four assertions combine to determine the remainder of the $M(S,T)$ chart.

Theorem 7.8. *Let $p, q \in [1,\infty]$ with $q \leqslant p$ and define $r \in [1,\infty]$ by $\frac{1}{p} + \frac{1}{r} = \frac{1}{q}$. Then $M(S_p, T_q) = T_r$.*

P r o o f. First it is shown that $M(S_p, T_q) \subset T_r$. Begin by assuming $q < p < \infty$. Let $g \in D \setminus T_r$. Then $g \in D$ and there is $y \in I$ such that $\limsup\limits_{x \to y} \|g\|_{x,y,r} = \infty$. By Theorem 5.13 with $s_n = p$ for each $n \in \mathbb{N}$, there is an $f \in S_p$ such that $fg \notin T_q$. Thus $g \notin M(S_p, T_q)$. If $q = p < \infty$, proceed as above except using Proposition 5.4 instead of Theorem 5.13. Lastly, assume $p = \infty$. Then $r = q$ and $M(S_\infty, T_q) \subset T_q$ by Proposition 2.2.

Now $T_r \subset M(S_p, T_q)$ is proved. First note that because $q \geqslant 1$, $\frac{1}{p} + \frac{1}{r} = \frac{1}{q} \leqslant 1 = \frac{1}{p} + \frac{1}{p'}$. Thus $\frac{1}{r} \leqslant \frac{1}{p'}$, or $p' \leqslant r$. (This observation is used here and in the proofs of the next three theorems as well.) Let $f \in S_p$ and $g \in T_r \subset T_{p'} = M(S_p)$. Thus $fg \in D$. Because $S_p \subset T_p$, Lemma 6.9 (i) implies $T_r \subset M(S_p, T_q)$. □

Theorem 7.9. *Let p, q and r be as in Theorem 7.8, except that $p < \infty$. Then $M(\underline{S}_p, \overline{T}_q) = M(\underline{S}_p, T_q) = M(\underline{S}_p, \underline{T}_q) = \overline{T}_r$ and if $1 < q$, then $M(\overline{S}_p, \overline{T}_q) = M(S_p, \overline{T}_q) = \overline{T}_r$.*

P r o o f. First $M(\underline{S}_p, \overline{T}_q) \subset \overline{T}_r$ is proved. Let $t \in (p, \infty)$. Then $S_t \subset \underline{S}_p$. By Propositions 2.3 and 2.6, $M(\underline{S}_p, \overline{T}_q) \subset M\big(S_t, \bigcap\limits_{u \in (0,q)} T_u\big) = \bigcap\limits_{u \in (0,q)} M(S_t, T_u)$. By Theorem 7.8, $M(S_t, T_u) = T_{r_1}$ where $\frac{1}{t} + \frac{1}{u} = \frac{1}{r_1}$. Because $u \in (0, q)$, $\frac{1}{r_1} = \frac{1}{u} - \frac{1}{t} > \frac{1}{q} - \frac{1}{t} = \frac{1}{r_2}$. Hence $r_1 < r_2$. It follows that $M(\underline{S}_p, \overline{T}_q) \subset \bigcap\limits_{r_1 \in (0,r_2)} T_{r_1} = \overline{T}_{r_2}$. Because $t > p$, $r_2 < r$. It follows that $M(\underline{S}_p, \overline{T}_q) \subset \bigcap\limits_{u \in (0,r)} T_u = \overline{T}_r$.

Let $f \in \underline{S}_p$ and $g \in \overline{T}_r \subset \overline{T}_{p'} = M(\underline{S}_p)$. Thus $fg \in D$. Because $\underline{S}_p \subset \underline{T}_p$, Lemma 6.9 (ii) implies $\overline{T}_r \subset M(\underline{S}_p, \underline{T}_q)$. By Proposition 2.3, $M(\underline{S}_p, \underline{T}_q) \subset M(\underline{S}_p, T_q) \subset M(\underline{S}_p, \overline{T}_q)$. So by the previous paragraph $M(\underline{S}_p, \underline{T}_q) = M(\underline{S}_p, T_q) = M(\underline{S}_p, \overline{T}_q) = \overline{T}_r$. Now let $1 < q$. Then $p' < r$. Let $f \in \overline{S}_r$ and $g \in \overline{T}_r \subset \underline{T}_{p'} = M(\overline{S}_p)$. Thus $fg \in D$. Since $\overline{S}_p \subset \overline{T}_p$, Lemma 6.9 (iii) implies $\overline{T}_r \subset M(\overline{S}_p, \overline{T}_q)$. By Proposition 2.3, $M(\overline{S}_p, \overline{T}_q) \subset M(S_p, \overline{T}_q) \subset M(\underline{S}_p, \overline{T}_q)$. □

Note that the case $q = 1$ of the preceding theorem was dealt with in Theorem 7.7. Also the inclusion $\overline{T}_r \subset M(\overline{S}_p, \overline{T}_q)$ is valid if $p = \infty$. Recall in that case $r = q$. Thus $\overline{T}_q \subset M(\overline{S}_\infty, \overline{T}_q) \subset M(S_\infty, \overline{T}_q) \subset \overline{T}_q$ again by Propositions 2.3 and 2.2. Thus $M(\overline{S}_\infty, \overline{T}_q) = M(S_\infty, \overline{T}_q) = \overline{T}_q$.

Theorem 7.10. *Let $p, q \in [1, \infty]$ with $q < p$ and define $r \in [1, \infty)$ by $\frac{1}{p} + \frac{1}{r} = \frac{1}{q}$. Then $M(\overline{S}_p, \underline{T}_q) = M(\overline{S}_p, T_q) = \underline{T}_r$.*

P r o o f. First it is shown that $M(\overline{S}_p, T_q) \subset \underline{T}_r$. To that end let $g \in D \setminus \underline{T}_r$. Then $g \in D$ and there is a $y \in I$ such that for each $u > r$, $\limsup\limits_{x \to y} \|g\|_{x,y,u} = \infty$. For each $n \in \mathbb{N}$, let $s_n \in (q, \infty)$ with $s_1 \leqslant s_2 \leqslant \ldots$ and $\lim\limits_{n \to \infty} s_n = p$. For each $n \in \mathbb{N}$, define r_n by $\frac{1}{s_n} + \frac{1}{r_n} = \frac{1}{q}$. Then the sequence $\{r_n\}$ decreases to r. Thus for each $n \in \mathbb{N}$, $\limsup\limits_{x \to y} \|g\|_{x,y,r_n} = \infty$. By Theorem 5.13 there is a function f such that $f \in S_{s_n}$ for all $n \in \mathbb{N}$ and hence $f \in \overline{S}_p$ because $\{s_n\}$ increases to p, while $fg \notin T_q$. Thus $g \notin M(\overline{S}_p, T_q)$.

Now let $f \in \overline{S}_p$ and $g \in \underline{T}_r$. Because $q < p$, $p' < r$ and hence $\underline{T}_r \subset \underline{T}_{p'} = M(\overline{S}_p)$. Thus $fg \in D$. Since $\overline{S}_p \subset \overline{T}_p$, Lemma 6.9 (ii) (with the roles of p and r reversed) implies $\underline{T}_r \subset M(\overline{S}_p, \underline{T}_q)$. (Because $q < p$, $r < \infty$. Thus Lemma 6.9 (ii) applies.) By Proposition 2.3, $M(\overline{S}_p, \underline{T}_q) \subset M(\overline{S}_p, T_q)$. Thus $\underline{T}_r = M(\overline{S}_p, \underline{T}_q) = M(\overline{S}_p, T_q)$. $\square$

Theorem 7.11. *Let p, q and r be as in Theorem 7.10. Then $M(S_p, \underline{T}_q) = \underline{T}_r$.*

Proof. The proof that $M(S_p, \underline{T}_q) \subset \underline{T}_r$ parallels the first part of the proof of Theorem 7.10 except that an increasing sequence $\{t_n\}$ is selected converging to q and Theorem 5.14 is applied instead of Theorem 5.13. By the second part of the proof of Theorem 7.10, $\underline{T}_r \subset M(\overline{S}_p, \underline{T}_q)$. By Proposition 2.3, $M(\overline{S}_p, \underline{T}_q) \subset M(S_p, \underline{T}_q)$. Thus $\underline{T}_r = M(S_p, \underline{T}_q)$. $\square$

With Theorem 7.11 the $M(S,T)$ chart is complete. The chart appears on Figure 1.

The remaining three theorems indicate how the $M(T, \widetilde{T})$ and $M(S, \widetilde{S})$ charts can be obtained from the $M(S,T)$ chart. The following notation is useful in the statements of the remaining two theorems.

Notation 7.12. Let $S \in \mathcal{S}$. Then $\tau(S)$ denotes the corresponding member of $\mathcal{T}$. For example $\tau(\underline{S}_p) = \underline{T}_p$. Similarly for $T \in \mathcal{T}$, $\sigma(T)$ denotes the corresponding member of $\mathcal{S}$.

Theorem 7.13. *Let $S \in \{\underline{S}_1\} \cup \bigcup\limits_{p \in (1,\infty]} \mathcal{S}_p$ and let $T \in \{\underline{T}_1\} \cup \bigcup\limits_{q \in (1,\infty]} \mathcal{T}_q$ with $q \leqslant p$. Then $M(\tau(S), T) = M(S,T) \cap C_{\mathrm{ap}}$.*

Proof. Because $S \subset \tau(S)$ and because $T_\infty \subset \tau(S)$, Proposition 2.3 implies $M(\tau(S), T) \subset M(S,T)$ and $M(\tau(S), T) \subset M(T_\infty, D) = S_1 \subset S_0 \subset C_{\mathrm{ap}}$. Thus $M(\tau(S), T) \subset M(S,T) \cap C_{\mathrm{ap}}$.

To prove the opposite containment, first assume either ($S = \underline{S}_p$ and $T \in \mathcal{T}_q$) or ($S \in \{S_p, \overline{S}_p\}$ and $T = \overline{T}_q$). In each of these cases, $M(S,T) = \overline{T}_r$ where, as before $\frac{1}{p} + \frac{1}{r} = \frac{1}{q}$. By Proposition 4.10, $M(S,T) \cap C_{\mathrm{ap}} = \overline{T}_r \cap C_{\mathrm{ap}} = \overline{S}_r$. Let $g \in \overline{S}_r$. If $q = 1$, then $r = p'$ and $T = \underline{T}_1$. Thus $S = \underline{S}_p$. Let $f \in \tau(S) = \underline{T}_p$. Then $g \in \overline{S}_r = \overline{S}_{p'} = M(\underline{T}_p)$ implies $fg \in D$. If $q > 1$ and if $p = q$, then $r = \infty$ and hence $g \in M(\underline{T}_1)$. Thus for $f \in \tau(S) \subset \underline{T}_1$, $fg \in D$. If $q > 1$ and if $q < p$, then $p' < r$. Hence $g \in \overline{S}_r \subset \underline{S}_{p'} = M(\overline{T}_p)$. Thus for $f \in \tau(S) \subset \overline{T}_p$, $fg \in D$. Thus in any case Lemma 6.9 can be applied. If $S = \underline{S}_p$, then $p < \infty$ and by Lemma 6.9 (ii), $fg \in \underline{T}_q \subset T$. Hence $g \in M(\tau(S), T)$. If $S \in \{S_p, \overline{S}_p\}$, then by Lemma 6.9 (iii), $fg \in \underline{T}_q$.

Now consider all cases resulting in $M(S,T) = \underline{T}_r$. Note that in all such cases, $r < \infty$. That is, assume either ($S = S_p$ and $T = \underline{T}_q$) or ($S = \underline{S}_p$ and $T \in \{\underline{T}_q, T_q\}$). By choice, in all cases $M(S,T) = \underline{T}_r$ and hence again by Proposition 4.10, $M(S,T) \cap C_{\mathrm{ap}} = \underline{T}_r \cap C_{\mathrm{ap}} = \underline{S}_r$. Let $g \in \underline{S}_r$. Because $p' < r$, $\underline{S}_r \subset \underline{S}_{p'} = M(\overline{T}_p)$.

Thus $fg \in D$. So again Lemma 6.9 can be employed. By Lemma 6.9 (ii) with the roles of f and g reversed, $fg \in \underline{T}_q$. Thus $g \in M(\tau(S), T)$.

Finally assume $S = S_p$ and $T = T_q$. Then $M(S_p, T_q) = T_r$. Because $T = T_q$, $q > 1$ and hence $p' < r$. Thus $T_r \subset T_{p'}$. So $M(S_p, T_q) \cap C_{\rm ap} \subset \underline{T}_{p'} \cap C_{\rm ap} = \underline{S}_{p'} \subset S_p = M(T_p)$. Thus $g \in M(S_p, T_q) \cap C_{\rm ap}$ and $f \in T_p$ implies $fg \in D$. So by Lemma 6.9 (i), $fg \in T_q$. Hence $g \in M(T_p, T_q)$. $\square$

The results of the preceding theorem are displayed in Figure 2, $X \cap C_{\rm ap}$ is denoted by $\widehat{X}$.

The final two theorems will complete the $M(S, \widetilde{S})$ chart. Recall that Theorem 7.3, 7.4 and 7.6 fill in part of that chart. But in this case, Theorem 7.7 doesn't apply. The next theorem deals with the remaining part of the chart except for the S_∞ row.

Theorem 7.14. *Let* $S \in \{S_1, \underline{S}_1, \underline{S}_\infty\} \cup \bigcup\limits_{p\in(1,\infty)} \mathcal{S}_p$ *and let* $T \in \bigcup\limits_{q\in[0,\infty)} \mathcal{T}_q$ *with* $q \leqslant p$. *Then* $M(S, \sigma(T)) = M(S, T) \cap C_{\rm ap}$.

P r o o f. Because $\sigma(T) \subset T$, $M(S, \sigma(T)) \subset M(S, T)$. Moreover $M(S, \sigma(T)) \subset \sigma(T) \subset C_{\rm ap}$. Thus $M(S, \sigma(T)) \subset M(S, T) \cap C_{\rm ap}$.

First that part of the chart including and to the right of the column headed S_1 and above and including the row labeled S_1, but excluding S_∞ is handled. So assume $T_1 \subset T$. Each row is dealt with separately. First let $S = \underline{S}_p$. Then $p \in [1, \infty]$ and $M(\underline{S}_p, T) \cap C_{\rm ap} = \overline{T}_{p'} \cap C_{\rm ap} = \overline{S}_{p'}$ by Proposition 4.10. Let $g \in \overline{S}_{p'}$ and let $f \in \underline{S}_p$. By Theorem 4.11 (ii) (with the "q" of that theorem equal 1), $fg \in \underline{S}_1 \subset \sigma(T)$ because $T_1 \subset T$. Thus $M(\underline{S}_p, \sigma(T)) = \overline{S}_{p'}$. Next let $S = S_p$. Then $p \in [1, \infty)$ and $M(S_p, T) \cap C_{\rm ap} = T_{p'} \cap C_{\rm ap}$. Let $g \in T_{p'} \cap C_{\rm ap}$ and $f \in S_p$. By Theorem 4.11 (i), $fg \in S_1 \subset \sigma(T)$. Thus $M(S_p, \sigma(T)) = T_{p'} \cap C_{\rm ap}$. The last case for this part of the chart is $S = \overline{S}_p$. In this case $p \in (1, \infty)$ so that $p' < \infty$ and $M(\overline{S}_p, T) \cap C_{\rm ap} = \underline{T}_{p'} \cap C_{\rm ap} = \underline{S}_{p'}$. Let $g \in \underline{S}_{p'}$ and let $f \in \overline{S}_p$. By Theorem 4.11 (ii) (with the roles of p and $r = p'$ reversed), $fg \in \underline{S}_1 \subset \sigma(T)$. Thus $M(\overline{S}_p, \sigma(T)) = \underline{S}_{p'}$.

Now for the remainder of the chart except for the S_∞ row, let $T \in \{\underline{T}_1\} \cup \bigcup\limits_{q\in(1,\infty]} \mathcal{T}_q$ and let $S \in \{\underline{S}_1\} \cup \bigcup\limits_{p\in(1,\infty]} \mathcal{S}_p$. First consider all cases resulting in $M(S, T) = \overline{T}_r$. Specifically assume either ($S = \underline{S}_p$ and $T \in \mathcal{T}_q$) or ($S \in \{S_p, \overline{S}_p\}$ and $T = \overline{T}_q$). Then in all of these cases $M(S, T) = \underline{T}_r$ where as always $\frac{1}{p} + \frac{1}{r} = \frac{1}{q}$. By Proposition 4.10, $M(S, T) \cap C_{\rm ap} = \overline{T}_r \cap C_{\rm ap} = \overline{S}_r$. Let $g \in \overline{S}_r$ and first suppose $f \in \underline{S}_p$. By Theorem 4.11 (ii), $fg \in \underline{S}_q \subset \sigma(T)$. Hence $M(\underline{S}_p, \sigma(T)) = \overline{S}_r$. Next suppose $S = \overline{S}_p$. Then $T = \overline{T}_q$. Let $g \in \overline{S}_r$ and let $f \in \overline{S}_p$. By Theorem 4.11 (iii), $fg \in \overline{S}_q = \sigma(\overline{T}_q)$. Thus $M(\overline{S}_p, \sigma(\overline{T}_q)) = \overline{S}_r$. The remaining case is $S = S_p$. Because $\underline{S}_p \subset S_p \subset \overline{S}_p$, by Proposition 2.3, $\overline{S}_r = M(\overline{S}_p, \overline{S}_q) \subset M(S_p, \overline{S}_q) \subset M(\underline{S}_p, \overline{S}_q) = \overline{S}_r$. Hence $M(S_p, \overline{S}_q) = \overline{S}_r$.

$X \setminus Y$	T_∞	$\overline{T}_\infty$	$\dots$	$\underline{T}_p$	T_p	$\overline{T}_p$	$\dots$	$\underline{T}_q$	T_q	$\overline{T}_q$	$\dots$	$\underline{T}_1$	T_1	$\overline{T}_1$	$\dots$	$\underline{T}_u$	T_u	$\overline{T}_u$	$\dots$	$\underline{T}_0$	T_0
T_∞	$\hat{T}_\infty$	$\overline{S}_\infty$	$\dots$	$\underline{S}_p$	$\hat{T}_p$	$\overline{S}_p$	$\dots$	$\underline{S}_q$	$\hat{T}_q$	$\overline{S}_q$	$\dots$	$\underline{S}_1$	S_1	S_1	$\dots$	S_1	S_1	S_1	$\dots$	S_1	S_1
$\overline{T}_\infty$		$\overline{S}_\infty$	$\dots$	$\underline{S}_p$	$\underline{S}_p$	$\overline{S}_p$	$\dots$	$\underline{S}_q$	$\underline{S}_q$	$\overline{S}_q$	$\dots$	$\underline{S}_1$	$\underline{S}_1$	$\underline{S}_1$	$\dots$	$\underline{S}_1$	$\underline{S}_1$	$\underline{S}_1$	$\dots$	$\underline{S}_1$	$\underline{S}_1$
$\vdots$			$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$
$\underline{T}_p$				$\overline{S}_\infty$	$\overline{S}_\infty$	$\overline{S}_\infty$	$\dots$	$\overline{S}_r$	$\overline{S}_r$	$\overline{S}_r$	$\dots$	$\overline{S}_{p'}$	$\overline{S}_{p'}$	$\overline{S}_{p'}$	$\dots$	$\overline{S}_{p'}$	$\overline{S}_{p'}$	$\overline{S}_{p'}$	$\dots$	$\overline{S}_{p'}$	$\overline{S}_{p'}$
T_p					$\hat{T}_\infty$	$\overline{S}_\infty$	$\dots$	$\underline{S}_r$	$\hat{T}_r$	$\overline{S}_r$	$\dots$	$\underline{S}_{p'}$	$S_{p'}$	$S_{p'}$	$\dots$	$S_{p'}$	$S_{p'}$	$S_{p'}$	$\dots$	$S_{p'}$	$S_{p'}$
$\overline{T}_p$						$\overline{S}_\infty$	$\dots$	$\underline{S}_r$	$\underline{S}_r$	$\overline{S}_r$	$\dots$	$\underline{S}_{p'}$	$\underline{S}_{p'}$	$\underline{S}_{p'}$	$\dots$	$\underline{S}_{p'}$	$\underline{S}_{p'}$	$\underline{S}_{p'}$	$\dots$	$\underline{S}_{p'}$	$\underline{S}_{p'}$
$\vdots$							$\ddots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$
$\underline{T}_q$								$\overline{S}_\infty$	$\overline{S}_\infty$	$\overline{S}_\infty$	$\dots$	$\overline{S}_{q'}$	$\overline{S}_{q'}$	$\overline{S}_{q'}$	$\dots$	$\overline{S}_{q'}$	$\overline{S}_{q'}$	$\overline{S}_{q'}$	$\dots$	$\overline{S}_{q'}$	$\overline{S}_{q'}$
T_q									$\hat{T}_\infty$	$\overline{S}_\infty$	$\dots$	$\underline{S}_{q'}$	$S_{q'}$	$S_{q'}$	$\dots$	$S_{q'}$	$S_{q'}$	$S_{q'}$	$\dots$	$S_{q'}$	$S_{q'}$
$\overline{T}_q$										$\overline{S}_\infty$	$\dots$	$\underline{S}_{q'}$	$\underline{S}_{q'}$	$\underline{S}_{q'}$	$\dots$	$\underline{S}_{q'}$	$\underline{S}_{q'}$	$\underline{S}_{q'}$	$\dots$	$\underline{S}_{q'}$	$\underline{S}_{q'}$
$\vdots$											$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$
$\underline{T}_1$												$\overline{S}_\infty$	$\overline{S}_\infty$	$\overline{S}_\infty$	$\dots$	$\overline{S}_\infty$	$\overline{S}_\infty$	$\overline{S}_\infty$	$\dots$	$\overline{S}_\infty$	$\overline{S}_\infty$
T_1													S_∞	S_∞	$\dots$	S_∞	S_∞	S_∞	$\dots$	S_∞	S_∞
$\overline{T}_1$														W	$\dots$	W	W	W	$\dots$	W	W
$\vdots$															$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$
$\underline{T}_u$																W	W	W	$\dots$	W	W
T_u																	W	W	$\dots$	W	W
$\overline{T}_u$																		W	$\dots$	W	W
$\vdots$																			$\ddots$	$\vdots$	$\vdots$
$\underline{T}_0$																				W	W
T_0																					W

Figure 2. The $M(T,\widetilde{T})$ Chart.

Next consider all cases resulting in $M(S,T) = \underline{T}_r$. Specifically assume either ($S = S_p$ and $T = \underline{T}_q$) or ($S = \overline{S}_p$ and $T \in \{\underline{T}_q, T_q\}$). Note that in all of these cases, $r < \infty$. Let $g \in \underline{S}_r$. Let $f \in S \in \{S_p, \overline{S}_p\}$. Then $f \in \overline{S}_p$. By Theorem 4.11 (ii) (with the roles of p and r reversed), $fg \in \underline{S}_q \subset \sigma(T)$. Thus in all three cases, $M(S, \sigma(T)) = \underline{S}_r$.

The final case is $S = S_p$ and $T = T_q$. Then $M(S_p, T_q) \cap C_{\text{ap}} = T_r \cap C_{\text{ap}}$. Let $g \in T_r \cap C_{\text{ap}}$ and let $f \in S_p$. If $p < \infty$, then by Theorem 4.11 (i), $fg \in S_q = \sigma(T_q)$. So assume $p = \infty$. By Proposition 2.2 $M(S_\infty, S_q) \subset S_q$. Now assume $f \in S_\infty$ and

$g \in S_q$. Show that $fg \in S_q = M(T_{q'})$. Let $h \in T_{q'}$. Then $gh \in D$. By Lemma 6.9 (i), $gh \in T_1$. Because $f \in S_\infty = M(T_1)$, $fgh \in D$ completing the proof. $\square$

$X \setminus Y$	S_∞	$\overline{S}_\infty$	$\dots$	$\underline{S}_p$	S_p	$\overline{S}_p$	$\dots$	$\underline{S}_q$	S_q	$\overline{S}_q$	$\dots$	$\underline{S}_1$	S_1	$\overline{S}_1$	$\dots$	$\underline{S}_u$	S_u	$\overline{S}_u$	$\dots$	$\underline{S}_0$	S_0
S_∞	S_∞	$\overline{S}_\infty$	$\dots$	$\underline{S}_p$	S_p	$\overline{S}_p$	$\dots$	$\underline{S}_q$	S_q	$\overline{S}_q$	$\dots$	$\underline{S}_1$	S_1	$\hat{T}_1$	$\dots$	$\hat{T}_1$	$\hat{T}_1$	$\hat{T}_1$	$\dots$	$\hat{T}_1$	$\hat{T}_1$
$\overline{S}_\infty$		$\overline{S}_\infty$	$\dots$	$\underline{S}_p$	$\underline{S}_p$	$\overline{S}_p$	$\dots$	$\underline{S}_q$	$\underline{S}_q$	$\overline{S}_q$	$\dots$	$\underline{S}_1$	$\underline{S}_1$	$\underline{S}_1$	$\dots$	$\underline{S}_1$	$\underline{S}_1$	$\underline{S}_1$	$\dots$	$\underline{S}_1$	$\underline{S}_1$
$\vdots$			$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$
$\underline{S}_p$				$\overline{S}_\infty$	$\overline{S}_\infty$	$\overline{S}_\infty$	$\dots$	$\overline{S}_r$	$\overline{S}_r$	$\overline{S}_r$	$\dots$	$\overline{S}_{p'}$	$\overline{S}_{p'}$	$\overline{S}_{p'}$	$\dots$	$\overline{S}_{p'}$	$\overline{S}_{p'}$	$\overline{S}_{p'}$	$\dots$	$\overline{S}_{p'}$	$\overline{S}_{p'}$
S_p					$\hat{T}_\infty$	$\overline{S}_\infty$	$\dots$	$\underline{S}_r$	$\hat{T}_r$	$\overline{S}_r$	$\dots$	$\underline{S}_{p'}$	$\hat{T}_{p'}$	$\hat{T}_{p'}$	$\dots$	$\hat{T}_{p'}$	$\hat{T}_{p'}$	$\hat{T}_{p'}$	$\dots$	$\hat{T}_{p'}$	$\hat{T}_{p'}$
$\overline{S}_p$						$\overline{S}_\infty$	$\dots$	$\underline{S}_r$	$\underline{S}_r$	$\overline{S}_r$	$\dots$	$\underline{S}_{p'}$	$\underline{S}_{p'}$	$\underline{S}_{p'}$	$\dots$	$\underline{S}_{p'}$	$\underline{S}_{p'}$	$\underline{S}_{p'}$	$\dots$	$\underline{S}_{p'}$	$\underline{S}_{p'}$
$\vdots$							$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$
$\underline{S}_q$								$\overline{S}_\infty$	$\overline{S}_\infty$	$\overline{S}_\infty$	$\dots$	$\overline{S}_{q'}$	$\overline{S}_{q'}$	$\overline{S}_{q'}$	$\dots$	$\overline{S}_{q'}$	$\overline{S}_{q'}$	$\overline{S}_{q'}$	$\dots$	$\overline{S}_{q'}$	$\overline{S}_{q'}$
S_q									$\hat{T}_\infty$	$\overline{S}_\infty$	$\dots$	$\underline{S}_{q'}$	$\hat{T}_{q'}$	$\hat{T}_{q'}$	$\dots$	$\hat{T}_{q'}$	$\hat{T}_{q'}$	$\hat{T}_{q'}$	$\dots$	$\hat{T}_{q'}$	$\hat{T}_{q'}$
$\overline{S}_q$										$\overline{S}_\infty$	$\dots$	$\underline{S}_{q'}$	$\underline{S}_{q'}$	$\underline{S}_{q'}$	$\dots$	$\underline{S}_{q'}$	$\underline{S}_{q'}$	$\underline{S}_{q'}$	$\dots$	$\underline{S}_{q'}$	$\underline{S}_{q'}$
$\vdots$											$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$
$\underline{S}_1$												$\overline{S}_\infty$	$\overline{S}_\infty$	$\overline{S}_\infty$	$\dots$	$\overline{S}_\infty$	$\overline{S}_\infty$	$\overline{S}_\infty$	$\dots$	$\overline{S}_\infty$	$\overline{S}_\infty$
S_1													$\hat{T}_\infty$	$\hat{T}_\infty$	$\dots$	$\hat{T}_\infty$	$\hat{T}_\infty$	$\hat{T}_\infty$	$\dots$	$\hat{T}_\infty$	$\hat{T}_\infty$
$\overline{S}_1$														W	$\dots$	W	W	W	$\dots$	W	W
$\vdots$															$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$
$\underline{S}_u$																W	W	W	$\dots$	W	W
S_u																	W	W	$\dots$	W	W
$\overline{S}_u$																		W	$\dots$	W	W
$\vdots$																			$\ddots$	$\vdots$	$\vdots$
$\underline{S}_0$																				W	W
S_0																					W

Figure 3. The $M(S, \widetilde{S})$ chart.

Theorem 7.15. *If $T \in \{\overline{T}_1\} \cup \bigcup_{q \in [0,1)} \mathcal{T}_q$, then $M(S_\infty, \sigma(T)) = T_1 \cap C_{\mathrm{ap}}$. If $T \in \{\underline{T}_1, T_1\} \cup \bigcup_{q \in (1,\infty]} \mathcal{T}_q$, then $M(S_\infty, \sigma(T)) = \sigma(T)$.*

P r o o f. First assume $T \in \{\overline{T}_1\} \cup \bigcup\limits_{q\in[0,1)} \mathcal{T}_q$. By Proposition 2.2, $M(S_\infty, \sigma(T)) \subset \sigma(T) \subset C_{\mathrm{ap}}$ and by Proposition 2.3 $M(S_\infty, \sigma(T)) \subset M(S_\infty) = T_1$. Therefore $M(S_\infty, \sigma(T)) \subset T_1 \cap C_{\mathrm{ap}}$. Let $f \in S_\infty$ and $g \in T_1 \cap C_{\mathrm{ap}}$. Because $S_\infty \subset C_{\mathrm{ap}}$, $fg \in C_{\mathrm{ap}}$. Because $f \in S_\infty$, by Proposition 6.10 (i), $fg \in T_1 \subset \overline{T}_1$. Thus $fg \in \overline{T}_1 \cap C_{\mathrm{ap}} = \overline{S}_1 \subset \sigma(T)$. Therefore $g \in M(S_\infty, \sigma(T))$.

Finally assume $T \in \{\underline{T}_1, T_1\} \cup \bigcup\limits_{q\in(1,\infty]} \mathcal{T}_q$. By Proposition 2.2, $M(S_\infty, \sigma(T)) \subset \sigma(T)$. Let $f \in S_\infty$ and $g \in \sigma(T)$. Show that $fg \in \sigma(T)$. Suppose $T = \underline{T}_q$. Then $q \in [1, \infty)$ and $\sigma(T) = \underline{S}_q = M(\overline{T}_{q'})$. Let $h \in \overline{T}_{q'}$. By Proposition 6.10 (iii), $gh \in T_1$. Because $S_\infty = M(T_1)$, $fgh \in D$. So $fg \in M(\overline{T}_{q'}) = \underline{S}_q$. The remaining two cases, $T = T_q$ and $T = \overline{T}_q$, proceed in an analogous manner. $\square$

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