URYSOHN'S LEMMA, GLUING LEMMA AND CONTRACTION* MAPPING THEOREM FOR FUZZY METRIC SPACES

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Abstract. In this paper the concept of a fuzzy contraction* mapping on a fuzzy metric space is introduced and it is proved that every fuzzy contraction* mapping on a complete fuzzy metric space has a unique fixed point.

Keywords: fuzzy contraction mapping, fuzzy continuous mapping

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1. Introduction

The theory of fuzzy sets was introduced by Zadeh in 1965 [7]. Since then many authors (Zi-ke 1982 [8], Erceg 1979 [1], George and Veeramani 1994 [2], Kaleva and Seikkala 1984 [5]) have introduced the concept of a fuzzy metric space in different ways. In this paper we follow the definition of a metric space given by George and Veeramani [2] since the topology induced by the fuzzy metric according to the definition of George and Veeramani [2] is Hausdorff. Motivated by the concept of a metric space, Urysohn's lemma and gluing lemma are studied. Based on the concept of a fuzzy contraction mapping [6], the fuzzy contraction* mapping theorem is established.

2. Preliminaries

Definition 1 [4]. A binary operation $*\colon [0,1]\times[0,1]\to[0,1]$ is a continuous t-norm if $*$ satisfies the following conditions:

1. $*$ is associative and commutative,
2. $*$ is continuous,

3. $a * 1 = a$ for all $a \in [0, 1]$,
4. $a * b \leqslant c * d$ whenever $a \leqslant c$ and $b \leqslant d$, $(a, b, c, d \in [0, 1])$.

Definition 2 [2]. The triple $(X, M, *)$ is said to be a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions:

1. $M(x, y, t) > 0$,
2. $M(x, y, t) = 1$ if and only if $x = y$,
3. $M(x, y, t) = M(y, x, t)$,
4. $M(x, y, t) * M(y, z, s) \leqslant M(x, z, t + s)$, $x, y, z \in X$ and $t, s > 0$,
5. $M(x, y, \cdot)\colon X^2 \times (0, \infty) \to [0, 1]$ is continuous, $x, y, z \in X$ and $t, s > 0$.

R e m a r k 1 [2]. $M(x, y, t)$ can be thought of as the degree of nearness between x and y with respect to t. We identify $x = y$ with $M(x, y, t) = 1$, for $t > 0$ and $M(x, y, t) = 0$ with $x = \infty$ or $y = \infty$.

R e m a r k 2 [2]. In a fuzzy metric space $(X, M, *)$, whenever $M(x, y, t) > 1 - r$ for x, y in X, $t > 0$, $0 < r < 1$, we can find a t_0, $0 < t_0 < 1$ such that $M(x, y, t_0) > 1 - r$.

Definition 3 [4]. A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is said to be a Cauchy sequence if for each ε, $0 < \varepsilon < 1$ and $t > 0$ there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for all $n, m \geqslant n_0$.

Definition 4 [2]. Let $X(X, M, *)$ be a fuzzy metric space. We define the open ball $B(x, r, t)$ with centre $x \in X$ and radius r, $0 < r < 1$, $t > 0$ as

$$B(x, r, t) = \{y \in X\colon\ M(x, y, t) > 1 - r\}.$$

Definition 5 [4]. Let $(X, M, *)$ be a fuzzy metric space. Define $T = \{A \subset X\colon\ x \in A$ if and only if there exist r, $t > 0$, $0 < r < 1$ such that $B(x, r, t) \subset A\}$. Then T is topology on X. This topology is called the topology induced by the fuzzy metric.

Then by Theorem 3.11 of (George and Veeramani 1994 [2]) we know that a sequence $x_n \to x$ (x_n converges to x) if and only if $M(x_n, x, t) \to 1$ as $n \to \infty$.

Definition 6 [2]. A fuzzy metric space is said to be complete if every Cauchy sequence is convergent.

N o t a t i o n. $M_A(x, y, t)$ denotes the degree of nearness between x and y with respect to t when $x, y \in A$.

Proposition 1 (Urysohn's Lemma)**.** *Let $(X, M, *)$ be a fuzzy metric space. Let T be a topology on X induced by the fuzzy metric. Let A and B be distinct members of r. Then there exists a fuzzy continuous function $f\colon X \to [0,1]$ such that $f = 0$ on A and $f = 1$ on B.*

P r o o f. Define a function $f\colon X \to [0,1]$ by

$$f(x) = \frac{1 - M_A(x,x,t)}{M_B(x,x,t) - M_A(x,x,t)}.$$

Note that $M_B(x,x,t) - M_A(x,x,t) \neq 0$ for any $x \in X$. If $x \in A$, $M_A(x,x,t) = 1$, then $f(x) = 0$. If $x \in B$, $M_B(x,x,t) = 1$, then $f(x) = 1 - M_A(x,x,t)/1 - M_A(x,x,t) = 1$. Since $M(x,y,t)$ is fuzzy continuous (George and Veeramani 1994 [2]), f is fuzzy continuous. □

Proposition 2 (Gluing Lemma)**.** *Let $(X, M, *)$ and $(Y, M, *)$ be two fuzzy metric spaces. Let U_i, $i \in I$ be members of fuzzy induced topology T on X such that $\bigcup\limits_{i\in I} U_i = X$. Assume that there exists a fuzzy continuous function* [3] *$f_i\colon U_i \to Y$ for each $i \in I$ with the property that $f_i(x) = f_j(x)$ for all $x \in U_i \cap U_j$ and $i, j \in I$. Then the function $f\colon X \to Y$ defined by $f(x) = f_i(x)$ if $x \in U_i$ is well defined and fuzzy continuous on X.*

P r o o f. Let $x, y \in X$. Since f_i is continuous for given $r \in (0,1)$, $t > 0$ we can find $r_0 \in (0,1)$, $t/4 > 0$ such that $M(x,y,t_0) > 1 - r_0$ implies $M(f_i(x), f_i(y), t/2) > 1 - r$. Now $M(x,y,t/4) > 1 - r_0$. Let $x \in U_i$, $y \in U_j$ for some $i \neq j$. Let $x_i \in U_i \cap U_j$. Then

$$\begin{aligned} M(f(x), f(y), t/2) &> M(f(x), f(x_i), t/4) * M(f(x_i), f(y), t/4) \\ &= M(f_i(x), f_i(x_i), t/4) * M(f_j(x_i), f_j(y), t/4) \\ &> (1-r) * (1-r) = 1 - r. \end{aligned}$$

Therefore f is fuzzy continuous. □

Definition 7. Let $(X, M, *)$ be a fuzzy metric space. A function $f\colon X \to X$ is called a fuzzy contraction* mapping if $M(x,y,t) \geqslant 1-(1-r^2)$ for all $0 < 1-r^2 < 1$. Then $M(f(x), f(y), t) \geqslant 1-(1-r_0^2)$ for each $x, y \in X$ for some $1-r_0^2 < 1-r^2, 1$.

Example 1. Consider the fuzzy metric space $(\mathbb{R}, M, *)$, where $\mathbb{R}$ is the set of all real numbers and $M(x,y,t) = t/(t+|x-y|)$. Let $f\colon \mathbb{R} \to \mathbb{R}$ and define $f(x) = x/2$. Then $M(x,y,t) = t/(t+|x-y|) \geqslant 1-(1-r^2)$, $t > 0$, $0 < 1-r^2 < 1$ where $1-r^2 \geqslant |x-y|/(t+|x-y|)$. Then

$$\begin{aligned} M(f(x), f(y), t) &= \frac{t}{t+|(x/2)-(y/2)|} \\ &= \frac{1-(|(x/2)-(y/2)|)}{t+|(x/2)-(y/2)|} \geqslant 1-(1-r_0^2) \end{aligned}$$

where

$$1-r_0^2 \geqslant \frac{|(x/2)-(y/2)|}{t+|(x/2)-(y/2)|}.$$

Further,

$$\begin{aligned} (1-r^2)-(1-r_0^2) &\geqslant \frac{|x-y|}{t+|x-y|} - \frac{(|(x/2)-(y/2)|)}{t+|(x/2)-(y/2)|} \\ &\geqslant \frac{|x-y|}{t+|x-y|} - \frac{\frac{1}{2}|x-y|}{t+\frac{1}{2}|x-y|} \\ &\geqslant \frac{|x-y|(t+\frac{1}{2}|x-y|) - \frac{1}{2}|x-y|(t+|x-y|)}{(t+|x-y|)(t+\frac{1}{2}|x-y|)} \\ &\geqslant \frac{|x-y|t - \frac{1}{2}(|x-y|t)}{(t+|x-y|)(t+\frac{1}{2})|x-y|} \\ &\geqslant \frac{(t/2)|x-y|}{(t+|x-y|)(t+\frac{1}{2})|x-y|} = 0, \end{aligned}$$

which implies that f is a fuzzy contraction* by Definition 7.

Definition 8. A mapping from a fuzzy metric space X to a fuzzy metric space Y is said to be fuzzy continuous* if for given $1-r^2$, $t > 0$, $0 < 1-r^2 < 1$ we can find $1-r_0^2 \in (0,1)$, $t_0 > 0$ such that $M(x,y,t_0) > 1-(1-r_0^2)$ implies $M(f(x), f(y), t/2) > 1-(1-r^2)$.

Proposition 3. *Every fuzzy contraction* mapping on a fuzzy metric space is fuzzy continuous*.*

Proof. Let $f\colon X \to X$ be a fuzzy contraction* mapping. Therefore for $x, y \in X$, given $1-r^2 \in (0,1)$, $t>0$, we can find $1-r_0^2 \in (0,1)$, $t/4>0$ such that $1-r^2 = (1-(1-r_0^2)) * (1-(1-r_0^2))$. Now $M(x,y,t/4) > 1-(1-r_0^2)$ implies $M(f(x), f(y), t/4) > 1-(1-s^2) > 1-(1-r_0^2)$ where $1-s^2 \in (0,(1-r_0^2))$ (since f is a fuzzy contraction* mapping). Let $x_1 \in X$. Then

$$\begin{aligned}M(f(x), f(y), t/2) &> M(f(x), f(x_1), t/4) * M(f(x_1), f(y), t/4)\\ &> (1-(1-r_0^2)) * (1-(1-r_0^2)) > (1-(1-r^2), (1-r^2) \in (0,1)\end{aligned}$$

which implies that f is a fuzzy continuous* mapping. □

Remark 3. The converse need not be true as the following example shows.

Example 2. Consider the fuzzy metric space $(\mathbb{R}, M, *)$ [2] where $\mathbb{R}$ is the set of all real numbers and

$$M(x,y,t) = \frac{t}{t+|x-y|}.$$

Let $f\colon \mathbb{R} \to \mathbb{R}$ and define $f(x) = x^2$. Then

$$M(x,y,t) = \frac{t}{t+|x-y| \geqslant 1-(1-r^2)}$$

where $(1-r^2) \geqslant |x-y|/(t+|x-y|)$. Then

$$\begin{aligned}M(f(x), f(y), t/2) &= \frac{(t/2)}{(t/2)+|x^2-y^2|} = \frac{t}{t+2(|x^2-y^2|)}\\ &\geqslant 1-(1-r_0^2) \text{ where } 1-r_0^2 \geqslant \frac{2|x^2-y^2|}{t+2(|x^2-y^2|)}\end{aligned}$$

which implies that f is a fuzzy continuous* mapping. However, $M(f(x), f(y), t) = t/(t+|x^2-y^2|) \geqslant 1-(1-s^2)$ where $1-s^2 \geqslant |x^2-y^2|/(t+|x^2-y^2|)$ since

$$\begin{aligned}(1-s^2)-(1-r^2) &\geqslant \frac{|x^2-y^2|}{t+|x^2-y^2|} - \frac{|x-y|}{t+|x-y|}\\ &\geqslant \frac{(t+|x-y|)|x^2-y^2| - (t+|x^2-y^2|)|x-y|}{(t+|x^2-y^2|)(t+|x-y|)}\\ &\geqslant \frac{t(|x^2-y^2|-|x-y|)}{(t+|x^2-y^2|)(t+|x-y|)} \begin{cases}\geqslant 0 \text{ if } x, y \text{ are integers}\\ \leqslant 0 \text{ if } x, y \text{ are not integers}\end{cases}\end{aligned}$$

and consequently, f is not a fuzzy contraction* mapping.

Proposition 4. *Every fuzzy contraction* mapping on a complete fuzzy metric space* [2] *has a unique fixed point.*

P r o o f. Let f be a fuzzy contraction* mapping on a complete fuzzy metric space $(X, M, *)$. □

U n i q u e n e s s part. If possible let $x_0 \neq y_0$ be two fixed points of f. Then we have

$$x_0 = f^1(x_0) = f^2(x_0) = f^3(x_0) = \ldots = f^n(x_0),$$
$$y_0 = f^1(y_0) = f^2(y_0) = f^3(y_0) = \ldots = f^n(y_0) \quad \text{for each } n \in \mathbb{N}.$$

Now

$$\begin{aligned} M(x_0, y_0, t) = M(f^n(x_0), f^n(y_0)t) &\geqslant 1 - (1 - r^2)/k^n \\ &> M(x_0, y_0, t) \ (= 1 - (1 - r^2)) \end{aligned}$$

where $k > 1$, a contradiction, hence $x_0 = y_0$. Therefore the fixed points are unique.

E x i s t e n c e p a r t. Let $x_1 = f(x_0)$, $x_2 = f(x_1), \ldots, x_n = f(x_{n-1}) = f^{n-1}(x_1)$. Then

$$\begin{aligned} M(x_n, x_{n+1}, t) = M(f^{n-1}(x_1), f^{n-1}(x_2), t) &\geqslant 1 - (1 - r^2)/k^{n-1} \\ &\geqslant 1 - \frac{1}{1 - s^2} \quad \text{for some } \frac{1}{1 - s^2} \in (0, 1). \end{aligned}$$

Therefore,

$$\text{(A)} \qquad M(x_n, x_{n+1}, t) \geqslant 1 - \frac{1}{1 - s^2}.$$

For a given $t' = (m - n)t > 0$, $\varepsilon > 0$, choose n_0 such that $1/n_0 < \varepsilon$. Then for $m \geqslant n \geqslant n_0$,

$$\begin{aligned} M(x_n, x_m, t') &\geqslant M(x_n, x_{n+1}, t) * M(x_{n+1}, x_{n+2}, t) * \ldots * M(x_{m-1}, x_m, t) \\ &\geqslant \bigl(1 - (1 - s^2)^{-1}\bigr) * \bigl(1 - (1 - s^2)^{-1}\bigr) * \ldots * \bigl(1 - (1 - s^2)^{-1}\bigr) \\ &\geqslant 1 - \frac{1}{n} \qquad \text{for some } \frac{1}{n} \in (0, 1) \geqslant 1 - \varepsilon \end{aligned}$$

and hence $\{x_n\}$ is a Cauchy sequence. Since X is a complete metric space, this sequence converges to, say, $z_0 \in X$. Now we assert that z_0 is a fixed point of f. Consider $n \in \mathbb{N}$ for $0 < 1 - r^2 < 1$, $t > 0$. Then we have

$$\begin{aligned} M(f(z_0), z_0, t) \geqslant M(f(z_0), f(x_0), t/n + 1) &* M(f(x_0), f^2(x_0), t/n + 1) * \ldots \\ &* M(f^n(x_0), z_0, t/n + 1), \end{aligned}$$

and since f is a fuzzy contraction* mapping, this is for $k > 1$ and $1/(1 - s_n^2) \in (0,1)$ greater than or equal to

$$\begin{aligned}(1-(1-s_n^2)) * (1-(1-r^2)) * (1-(1-r^2)/k) * \ldots \\ * (1-(1-r^2)/k^{n-1}) * M(f^n(x_0), z_0, t/n+1) \\ \geqslant (1-(1-r^2)/k^{n+p}) * M(f^n(x_0), z_0, t/n+1)\end{aligned}$$

for some $p \in \mathbb{N}$. Taking limit on both sides as $n \to \infty$ we obtain

$$\begin{aligned}\lim_{n\to\infty} M(f(z_0), z_0, t) \geqslant \lim_{n\to\infty}(1-(1-r^2)/k^{n+p}) * \lim_{n\to\infty} M(f^n(x_0), z_0, t/n+1) \\ \Rightarrow M(f(z_0), z_0, t) \geqslant 1 * 1 \Rightarrow f(z_0) = z_0.\end{aligned}$$

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