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Abstract

We consider the motion of a viscous compressible fluid confined to a physical space with a time dependent kinematic boundary. We suppose that the characteristic speed of the fluid is dominated by the speed of sound and perform the low Mach number limit in the framework of weak solutions. The standard incompressible Navier-Stokes system is identified as the target problem.

Key words: Compressible Navier-Stokes system, low Mach number limit, time dependent domain

Contents

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1 Introduction

Since the seminal works of Ebin [5], Klainerman and Majda [10], there have been numerous studies of singular limits of systems of equations arising in mathematical fluid dynamics, see the surveys by Gallagher [9], Masmoudi [13], or Schochet [15]. While many of the results are either formal or restricted to (possibly) very short life span of classical solutions of the underlying systems of nonlinear equations, the mathematical theory of weak solutions to the barotropic Navier-Stokes system opened a new possibility to attack the problem rigorously in the framework of weak solutions, see Desjardins and Grenier [3], Desjardins et al. [4], Lions and Masmoudi [11], [12], Masmoudi [14], among others.

A rigorous analysis of singular limits is seriously hampered by the presence of physical boundaries, in particular for viscous fluids. As a matter of fact, many of the results mentioned above deal with the problems posed on the whole space R^N, $N = 1, 2, 3$ or in the idealized space-periodic setting, where the effect of the physical boundary is eliminated. On the other hand, Desjardins et al. [4] showed that the boundary layer created in the incompressible limit of viscous fluids may eliminate the effect of acoustic waves, at least for certain shapes of the kinematic boundaries.

In the present paper, we consider the incompressible (low Mach number) limit of a barotropic compressible and viscous fluid in a domain with a time dependent boundary. To the best of our knowledge, all the available results concerning such a situation are based on formal computations under the principal hypothesis that the underlying system of equations admits smooth solutions, see Alì [1]. Note that the fluid motion is driven by the variable boundary rather than by external volume forces in many real world applications.

The motion of a compressible viscous fluid is described by the *mass density* $\varrho = \varrho(t, \mathbf{x})$ and the *velocity field* $\mathbf{u} = \mathbf{u}(t, \mathbf{x})$. In the low Mach number regime, where the characteristic fluid velocity is dominated by the speed of sound, the time evolution of the dimensionless state variables $\varrho \approx \varrho/\varrho_{\rm char}$, $\mathbf{u} \approx \mathbf{u}/\mathbf{u}_{\rm char}$ is described by the (scaled) *Navier-Stokes system*:

$$\partial_t \varrho + {\rm div}_x (\varrho \mathbf{u}) = 0, \tag{1.1}$$

$$\partial_t (\varrho \mathbf{u}) + {\rm div}_x (\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\varepsilon^2} \nabla_x p(\varrho) = {\rm div}_x \mathbb{S}(\nabla_x \mathbf{u}) + \varrho \nabla_x g, \tag{1.2}$$

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} {\rm div}_x \mathbf{u} \mathbb{I} \right) + \eta {\rm div}_x \mathbf{u} \mathbb{I},\ \mu > 0,\ \eta \geq 0, \tag{1.3}$$

where p is the pressure, $\nabla_x g$ the external force, and $\mathbb{S}$ is the viscous stress tensor determined by Newton's law (1.3), with the shear viscosity coefficient μ and the bulk viscosity coefficient η. The parameter ε is the *Mach number*,

$$\varepsilon = \frac{\mathbf{u}_{\rm char}}{\sqrt{p_{\rm char}/\varrho_{\rm char}}},$$

where $X_{\rm char}$ denotes the characteristic value of a quantity X.

At each instant $\tau \geq 0$, the fluid occupies a bounded domain $\Omega_\tau \subset R^3$, with

$$\Omega_\tau = \mathbf{X}(\tau, \Omega_0),\ \Omega_0 \subset R^3,\ \frac{{\rm d}\mathbf{X}(t)}{{\rm d}t} = \mathbf{V}(t, \mathbf{X}(t)), \tag{1.4}$$

where $\mathbf{V}(t, \cdot) : R^3 \to R^3$ is a given regular velocity field. In addition, we assume that

$${\rm div}_x \mathbf{V}(\tau, \cdot) = 0 \text{ yielding } |\Omega_\tau| = |\Omega_0| \text{ for any } \tau \geq 0. \tag{1.5}$$

Finally, we suppose that the boundary $\partial\Omega_\tau$ is impermeable and the fluid satisfies the complete slip boundary conditions

$$(\mathbf{u} - \mathbf{V}) \cdot \mathbf{n}|_{\partial\Omega_\tau} = 0,\ (\mathbb{S}(\nabla_x \mathbf{u}) \cdot \mathbf{n}) \times \mathbf{n}|_{\partial\Omega_\tau} = 0, \tag{1.6}$$

where $\mathbf{n}$ denotes the outer normal vector to $\partial\Omega_\tau$.

We consider a family of *weak solutions* $[\varrho_\varepsilon, \mathbf{u}_\varepsilon]$ of the problem (1.1 - 1.3), (1.5), (1.6) emanating from the initial data

$$\varrho_\varepsilon(0, \cdot) = \varrho_{0,\varepsilon} = \overline{\varrho} + \varepsilon \varrho_{0,\varepsilon}^{(1)},\ \mathbf{u}_\varepsilon(0, \cdot) = \mathbf{u}_{0,\varepsilon},\ \overline{\varrho} > 0 \text{ a positive constant}, \tag{1.7}$$

where

$$\|\varrho_{0,\varepsilon}^{(1)}\|_{L^\infty(\Omega_0)} \leq c, \ \mathbf{u}_{0,\varepsilon} \to \mathbf{u}_0 \text{ weakly in } L^2(\Omega_0; R^3). \tag{1.8}$$

Our main goal is to show that

$$\varrho_\varepsilon \to \overline{\varrho}, \ \mathbf{u}_\varepsilon \to \mathbf{U} \text{ as } \varepsilon \to 0 \text{ in a certain sense,}$$

where $\mathbf{U}$ satisfies the standard incompressible Navier-Stokes system

$$\mathrm{div}_x \mathbf{U} = 0 \tag{1.9}$$

$$\overline{\varrho} \left(\partial_t \mathbf{U} + \mathbf{U} \cdot \nabla_x \mathbf{U} \right) + \nabla_x \Pi = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{U}) \tag{1.10}$$

$$\mathbb{S}(\nabla_x \mathbf{U}) = \mu \left(\nabla_x \mathbf{U} + \nabla_x^t \mathbf{U} \right), \tag{1.11}$$

supplemented with the boundary conditions

$$(\mathbf{U}(\tau, \cdot) - \mathbf{V}(\tau, \cdot)) \cdot \mathbf{n}|_{\partial \Omega_\tau} = 0, \ (\mathbb{S}(\nabla_x \mathbf{U}) \cdot \mathbf{n}) \times \mathbf{n}|_{\partial \Omega_\tau} = 0, \tag{1.12}$$

and the initial condition

$$\mathbf{U}(0, \cdot) = \mathbf{U}_0 = \mathbf{H}_0[\mathbf{u}_0] \text{ in } \Omega_0, \tag{1.13}$$

where $\mathbf{H}_0$ denotes the Helmholtz projection onto the space of solenoidal functions in Ω_0.

Our method is based on careful analysis of propagation of acoustic waves represented by the gradient component of the velocity, which is supposed to "disappear" in the limit $\varepsilon \to 0$. In particular, we study the time evolution of the Helmholtz projection operator $\mathbf{H}_\tau$ associated to the domain Ω_τ and its gradient counterpart. Moreover, the propagation of acoustic waves is governed by the Neumann Laplacean $\Delta_{N,\tau}$, whose spectral properties and their dependence on τ are examined in detail. Note that an alternative approach would be to adapt the local method proposed by Lions and Masmoudi [12]. However, besides some unnecessary restrictions that would have to be imposed on the state equation $p = p(\varrho)$, a direct use of the local method is also hampered by the fact that the Helmholtz projection depends on time in our setting.

The plan of the paper is as follows. In Section 2, we summarize the known facts concerning the weak solutions of the compressible Navier-Stokes system on time dependent domains and formulate our main result. Sections 3 and 4 are the heart of the paper. Here, we discuss the spectral properties of the Neumann Laplacean and their dependence on the domain as well as similar problems for the Helmholtz projection. The convergence towards the target system is shown in Section 5.

2 Preliminaries, weak solutions, main result

We suppose that the external force is given by a Lipschitz potential g, specifically,

$$g \in W^{1,\infty}(R^3). \tag{2.1}$$

Moreover, we assume that the pressure $p \in C[0,\infty) \cap C^2(0,\infty)$ is a strictly monotone function of the density satisfying

$$p(0) = 0,\ p'(\varrho) > 0 \text{ for all } \varrho > 0,\ \lim_{\varrho\to\infty} \frac{p'(\varrho)}{\varrho^{\gamma-1}} = p_\infty > 0 \text{ for a certain } \gamma > \frac{3}{2}. \tag{2.2}$$

2.1 Weak solutions of the primitive system

Following [8], we say the $[\varrho, \mathbf{u}]$ is a weak solution of the compressible Navier-Stokes system (1.1 - 1.3), (1.6), (1.7) in the time interval $(0,T)$ if the following holds:

- **Integrability.**

$$\varrho \geq 0,\ \text{the mapping } t \mapsto \|\varrho(t,\cdot)\|_{L^\gamma(\Omega_t)} \text{ belongs to } L^\infty(0,T),$$

$$\text{the mapping } t \mapsto \|\mathbf{u}(t,\cdot)\|_{W^{1,2}(\Omega_t,R^3)} \text{ belongs to } L^2(0,T),$$

$$(\mathbf{u} - \mathbf{V})(\tau,\cdot)\cdot \mathbf{n}|_{\partial\Omega_\tau} = 0 \text{ for a.a. } \tau \in (0,T). \tag{2.3}$$

- **Equation of continuity.** The equation of continuity (1.1) is replaced by a family of integral identities

$$\int_0^T \int_{\Omega_t} \left(\varrho\partial_t\phi + \varrho\mathbf{u}\cdot\nabla_x\phi\right)\ \mathrm{d}\mathbf{x}\mathrm{d}t = -\int_{\Omega_0} \varrho_{0,\varepsilon}\phi(0,\cdot)\ \mathrm{d}\mathbf{x} \tag{2.4}$$

  for any test function $\phi \in C_c^\infty([0,T)\times R^3)$.

- **Momentum equation.** The momentum equation (1.2) is satisfied as

$$\int_0^T \int_{\Omega_t} \left(\varrho\mathbf{u}\cdot\partial_t\boldsymbol{\varphi} + \varrho\mathbf{u}\otimes\mathbf{u} : \nabla_x\boldsymbol{\varphi} + \frac{1}{\varepsilon^2}p(\varrho)\mathrm{div}_x\boldsymbol{\varphi}\right)\mathrm{d}\mathbf{x}\mathrm{d}t \tag{2.5}$$

$$= \int_0^T \int_{\Omega_t} \left(\mathbb{S}(\nabla_x\mathbf{u}) : \nabla_x\boldsymbol{\varphi} - \varrho\nabla_x g\cdot\boldsymbol{\varphi}\right)\mathrm{d}\mathbf{x}\mathrm{d}t - \int_{\Omega_0} \varrho_{0,\varepsilon}\mathbf{u}_{0,\varepsilon}\cdot\boldsymbol{\varphi}\ \mathrm{d}\mathbf{x}$$

  for any test function $\boldsymbol{\varphi} \in C_c^\infty([0,T)\times R^3; R^3)$ such that

$$\boldsymbol{\varphi}(\tau,\cdot)\cdot\mathbf{n}|_{\partial\Omega_\tau} = 0 \text{ for any } \tau \in [0,T].$$

- **Energy inequality.** The energy inequality

$$\int_{\Omega_\tau} \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \frac{1}{\varepsilon^2} P(\varrho) \right) (\tau, \cdot) \ \mathrm{d}\mathbf{x} + \int_0^\tau \int_{\Omega_t} \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \ \mathrm{d}\mathbf{x} \mathrm{d}t \tag{2.6}$$

$$\leq \int_{\Omega_0} \left(\frac{1}{2} \varrho_{0,\varepsilon} |\mathbf{u}_{0,\varepsilon}|^2 + \frac{1}{\varepsilon^2} P(\varrho_{0,\varepsilon}) \right) \ \mathrm{d}\mathbf{x}$$

$$+ \int_{\Omega_\tau} \varrho \mathbf{u} \cdot \mathbf{V}(\tau, \cdot) \ \mathrm{d}\mathbf{x} - \int_{\Omega_0} \varrho_{0,\varepsilon} \mathbf{u}_{0,\varepsilon} \cdot \mathbf{V}(0, \cdot) \ \mathrm{d}\mathbf{x}$$

$$+ \int_0^\tau \int_{\Omega_t} \left[\mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) : \nabla_x \mathbf{V} - \varrho \mathbf{u} \cdot \partial_t \mathbf{V} - \varrho \mathbf{u} \otimes \mathbf{u} : \nabla_x \mathbf{V} \right] \ \mathrm{d}\mathbf{x} \mathrm{d}t$$

$$+ \int_0^\tau \int_{\Omega_t} \varrho \nabla_x g \cdot (\mathbf{u} - \mathbf{V}) \ \mathrm{d}\mathbf{x} \mathrm{d}t$$

  holds for a.a. $\tau \in [0, T]$, where

$$P(\varrho) = \varrho \int_{\overline{\varrho}}^{\varrho} \frac{p(z)}{z^2} \ \mathrm{d}z,$$

  and $\mathbf{V}$ is the vector field determining the motion of the spatial domain introduced in (1.4), (1.5).

The *existence* of global-in-time weak solutions under the hypotheses (2.1), (2.2) was shown in [8, Theorem 2.1]. Note that the specific form of the energy inequality (2.6) results from the fact that, in accordance with the hypothesis (1.5), the vector field $\mathbf{V}$ is solenoidal.

2.2 Weak solutions of the target system

We say that $\mathbf{U}$ is a weak solution of the incompressible Navier-Stokes system (1.9 - 1.13) if:

- **Integrability.**

$$\text{the mapping } t \mapsto \|\mathbf{U}(t, \cdot)\|_{L^2(\Omega_t; R^3)} \text{ belongs to } L^\infty(0, T),$$

$$\text{the mapping } t \mapsto \|\mathbf{U}(t, \cdot)\|_{W^{1,2}(\Omega_t; R^3)} \text{ belongs to } L^2(0, T),$$

$$\mathrm{div}_x \mathbf{U}(\tau, \cdot) = 0 \text{ a.a. in } \Omega_\tau, \ (\mathbf{U}(\tau, \cdot) - \mathbf{V}(\tau, \cdot)) \cdot \mathbf{n}|_{\partial \Omega_\tau} = 0 \text{ for a.a. } \tau \in (0, T). \tag{2.7}$$

- **Momentum equation.** The momentum equation (1.10) is replaced by the family of integral identities

$$\int_0^T \int_{\Omega_t} \overline{\varrho} \left(\mathbf{U} \cdot \partial_t \boldsymbol{\varphi} + \mathbf{U} \otimes \mathbf{U} : \nabla_x \boldsymbol{\varphi} \right) \ \mathrm{d}\mathbf{x}\mathrm{d}t = \int_0^T \int_{\Omega_t} \mathbb{S}(\nabla_x \mathbf{U}) : \nabla_x \boldsymbol{\varphi} \ \mathrm{d}\mathbf{x}\mathrm{d}t - \int_{\Omega_0} \overline{\varrho} \mathbf{U}_0 \cdot \boldsymbol{\varphi}(0, \cdot) \ \mathrm{d}\mathbf{x} \tag{2.8}$$

for any test function $\boldsymbol{\varphi} \in C_c^\infty([0,T) \times R^3; R^3)$ satisfying

$$\mathrm{div}_x \boldsymbol{\varphi} = 0, \ \ \boldsymbol{\varphi}(t, \cdot) \cdot \mathbf{n}|_{\partial \Omega_t} = 0.$$

2.3 The main result

Let

$$Q_\tau = \left\{ (t, x) \ \middle| \ t \in (0, \tau), \ \ x \in \Omega_t \right\}$$

be the space-time cylinder characterizing the fluid domain. Our main result reads:

Theorem 2.1 *Let Ω_0 be a bounded domain of class $C^{2+\nu}$. Suppose that the driving force $\nabla_x g$ and the pressure p satisfy the hypotheses (2.1), (2.2). Assume that the vector field $\mathbf{V}$ belongs to that class*

$$\mathbf{V} \in C^1([0,T]; C^3_c(R^3; R^3)), \ \mathrm{div}_x \mathbf{V}(\tau, \cdot) = 0 \ \textit{for all} \ \tau \in [0,T].$$

Let $[\varrho_\varepsilon, \mathbf{u}_\varepsilon]$ be a family of weak solutions to the compressible Navier-Stokes system (1.1 - 1.3), (1.6), (1.7) emanating from the initial data satisfying (1.7), (1.8),

$$\int_{\Omega_0} \varrho^{(1)}_{0,\varepsilon} \ \mathrm{d}x = 0.$$

Then

$$\operatorname*{ess\,sup}_{t \in (0,T)} \| \varrho_\varepsilon(t, \cdot) - \overline{\varrho} \|_{L^q(\Omega_t)} \to 0 \ \textit{as} \ \varepsilon \to 0, \ q = \min\{\gamma, 2\}, \tag{2.9}$$

and, for a suitable subsequence,

$$\left\{ \begin{array}{c} \mathbf{u}_\varepsilon \to \mathbf{U} \ \textit{weakly in} \ L^2(Q_T; R^3), \\ \\ \nabla_x \mathbf{u}_\varepsilon \to \nabla_x \mathbf{U} \ \textit{weakly in} \ L^2(Q_T; R^{3\times 3}), \end{array} \right\} \ \textit{as} \ \varepsilon \to 0, \tag{2.10}$$

where $\mathbf{U}$ is a weak solution of the incompressible Navier-Stokes system (1.9 - 1.13), with

$$\mathbf{U}_0 = \mathbf{H}_0[\mathbf{u}_0].$$

The rest of the paper is devoted to the proof of Theorem 2.1.

2.4 Uniform estimates

In this section, we derive the necessary uniform bounds on the family of solutions $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon\}_{\varepsilon > 0}$ *independent* of the small parameter ε.

Lemma 2.1 *Under the assumptions of Theorem 2.1 the following version of the energy inequality holds for a.a. $\tau \in [0,T]$*

$$\int_{\Omega_\tau}\left(\frac{1}{2}\varrho_\varepsilon|\mathbf{u}_\varepsilon|^2+\frac{1}{\varepsilon^2}\Big[P(\varrho_\varepsilon)-P'(\bar\varrho)(\varrho_\varepsilon-\bar\varrho)-P(\bar\varrho)\Big]\right)(\tau,\cdot)\ \mathrm{d}\mathbf{x}+\int_0^\tau\int_{\Omega_t}\mathbb{S}(\nabla_x\mathbf{u}_\varepsilon):\nabla_x\mathbf{u}_\varepsilon\ \mathrm{d}\mathbf{x}\mathrm{d}t \tag{2.11}$$

$$\leq\int_{\Omega_0}\left(\frac{1}{2}\varrho_{0,\varepsilon}|\mathbf{u}_{0,\varepsilon}|^2+\frac{1}{\varepsilon^2}\Big[P(\varrho_{0,\varepsilon})-P'(\bar\varrho)(\varrho_\varepsilon-\bar\varrho)-P(\bar\varrho)\Big]\right)\ \mathrm{d}\mathbf{x}$$

$$+\int_{\Omega_\tau}\varrho_\varepsilon\mathbf{u}_\varepsilon\cdot\mathbf{V}(\tau,\cdot)\ \mathrm{d}\mathbf{x}-\int_{\Omega_0}\varrho_{0,\varepsilon}\mathbf{u}_{0,\varepsilon}\cdot\mathbf{V}(0,\cdot)\ \mathrm{d}\mathbf{x}$$

$$+\int_0^\tau\int_{\Omega_t}\left[\mu\left(\nabla_x\mathbf{u}_\varepsilon+\nabla_x^t\mathbf{u}_\varepsilon-\frac{2}{3}\mathrm{div}_x\mathbf{u}_\varepsilon\mathbb{I}\right):\nabla_x\mathbf{V}-\varrho_\varepsilon\mathbf{u}_\varepsilon\cdot\partial_t\mathbf{V}-\varrho_\varepsilon\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon:\nabla_x\mathbf{V}\right]\ \mathrm{d}\mathbf{x}\mathrm{d}t$$

$$+\int_0^\tau\int_{\Omega_t}\varrho_\varepsilon\nabla_x g\cdot(\mathbf{u}_\varepsilon-\mathbf{V})\ \mathrm{d}\mathbf{x}\mathrm{d}t.$$

Next, it is convenient to introduce the *essential part*

$$[f_\varepsilon]_{ess}:=f_\varepsilon 1_{\{\frac{\bar\varrho}{2}\leq\varrho_\varepsilon\leq 2\bar\rho\}},$$

and the *residual part*

$$[f_\varepsilon]_{res}:=f_\varepsilon-[f_\varepsilon]_{ess}$$

for any measurable function f in Q_T. From the energy inequality (2.11) we obtain the following uniform estimates:

Lemma 2.2 *Under assumptions of Theorem 2.1 it holds*

$$\mathrm{ess}\ \sup_{t\in(0,T)}\int_{\Omega_t}\varrho_\varepsilon|\mathbf{u}_\varepsilon|^2\mathrm{d}\mathbf{x}\leq c,$$

$$\mathrm{ess}\ \sup_{t\in(0,T)}\int_{\Omega_t}\left[\frac{\varrho_\varepsilon-\bar\varrho}{\varepsilon}\right]_{ess}^2\mathrm{d}\mathbf{x}\leq c,$$

$$\mathrm{ess}\ \sup_{t\in(0,T)}\int_{\Omega_t}[\varrho_\varepsilon]_{res}^\gamma\,\mathrm{d}\mathbf{x}\leq\varepsilon^2c,$$

$$\mathrm{ess}\ \sup_{t\in(0,T)}\int_{\Omega_t}1_{res}\mathrm{d}\mathbf{x}\leq\varepsilon^2c,$$

$$\mathrm{ess}\ \sup_{t\in(0,T)}\int_{\Omega_t}\left[\frac{\varrho_\varepsilon-\bar\varrho}{\varepsilon}\right]_{res}^q\mathrm{d}\mathbf{x}\leq\varepsilon^{2-q}c,\ \ q\in[1,\min\{\gamma,2\}],$$

$$\int_0^T\int_{\Omega_t}\left|\nabla_x\mathbf{u}_\varepsilon+\nabla_x\mathbf{u}_\varepsilon^T-\frac{2}{3}\mathrm{div}_x\mathbf{u}_\varepsilon\mathcal{I}\right|^2\mathrm{d}\mathbf{x}\mathrm{d}t\leq c,$$

where the generic constant c is independent of ε.

These estimates, including the energy inequality in Lemma 2.1, can be obtained in a similar way as those on fixed domain and we refer to [7, Chapter 5] for details. We also note that the previous estimates, combined with a version of Korn's inequality (see [7, Theorem 10.17]), give rise to

$$\int_0^T \|\mathbf{u}_\varepsilon\|^2_{W^{1,2}(\Omega_t)} \,\mathrm{d}t \le c. \tag{2.12}$$

2.5 Weak convergence

We start our investigation concerning the convergence of density, velocity and momentum. To this end, it is convenient to extend the density by the constant value $\bar{\varrho}$ outside the fluid domain Q_T. Similarly, we extend the velocity to the whole space R^3, where we use the standard extension $E_t : W^{1,2}(\Omega_t) \to W^{1,2}(R^3)$, uniformly bounded with respect to $t \in [0,T]$. Note that the fluid domain is regular at each time instant.

Keeping the previously introduced convention in mind, we deduce from Lemma 2.2 and the continuity equation that

$$\begin{aligned} &\varrho_\varepsilon - \overline{\varrho} \to 0 && \text{in } L^\infty(0,T;L^\gamma(R^3)) \\ &\varrho_\varepsilon - \overline{\varrho} \to 0 && \text{in } C_{weak}([0,T];L^r(R^3)),\ r \in [1,\gamma). \end{aligned}$$

Furthermore, we get

$$E_t\mathbf{u}_\varepsilon \to \mathbf{U} \text{ weakly in } L^2(0,T;W^{1,2}(R^3;R^3)),$$

passing to a suitable subsequence as the case may be.

Consequently, for any $[T_1,T_2] \times K \subset Q_T$, where K denotes a compact set, it follows that

$$\varrho_\varepsilon\mathbf{u}_\varepsilon \to \overline{\varrho}\mathbf{U} \text{ weakly-* in } L^\infty(T_1,T_2;L^{\frac{2\gamma}{\gamma+1}}(K;R^3)).$$

Similarly, using (2.12), together with the standard Sobolev embedding $W^{1,2}(\Omega_t) \subset L^6(\Omega_t)$, we deduce that

$$\varrho_\varepsilon\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon \to \overline{\varrho\mathbf{u}\otimes\mathbf{u}} \text{ weakly in } L^q((T_1,T_2)L^q(K;R^3)) \text{ for a certain } q > 1$$

if $\gamma > \frac{3}{2}$. Since the compact set K can taken arbitrarily close to the boundary of Q_T, the above results yield weak convergence on the whole "fluid" cylinder Q_T.

Passing to the limit in the continuity equation we get

$$\mathrm{div}_x\mathbf{U} = 0 \text{ a.e. in } Q_T,$$

while the momentum equation (2.5) gives rise to

$$\int_0^T \int_{\Omega_t} \left(\overline{\varrho} \mathbf{U} \cdot \partial_t \boldsymbol{\varphi} + \overline{\varrho \mathbf{u} \otimes \mathbf{u}} : \nabla_x \boldsymbol{\varphi} \right) \mathrm{d}\mathbf{x}\mathrm{d}t \tag{2.13}$$
$$= \int_0^T \int_{\Omega_t} \mathbb{S}(\nabla_x \mathbf{U}) : \nabla_x \boldsymbol{\varphi} \, \mathrm{d}\mathbf{x}\mathrm{d}t - \int_{\Omega_0} \overline{\varrho} \mathbf{u}_0 \cdot \boldsymbol{\varphi}(0, \cdot) \, \mathrm{d}\mathbf{x},$$

for all $\boldsymbol{\varphi} \in C_c^\infty([0,T) \times R^3; R^3)$, $\mathrm{div}_x \boldsymbol{\varphi} = 0$, $\boldsymbol{\varphi}(t, \cdot) \cdot \mathbf{n}|_{\partial\Omega_t} = 0$ for all $t \in [0,T]$.

Consequently, it remains to clarify in which sense we may assert the relation

$$\overline{\varrho \mathbf{u} \otimes \mathbf{u}} \approx \overline{\varrho} \mathbf{U} \otimes \mathbf{U}$$

that is necessary for completing the proof of Theorem 2.1.

To discuss this issue we investigate the spectral properties of the Neumann Laplacean and the Helmholtz projection on moving domains.

3 Spectral properties of the Neumann Laplacean and the Helmholtz projection

3.1 Domain dependence of the Helmholtz projection

The Helmholtz decomposition $\mathbf{u} = \mathbf{H}_t[\mathbf{u}] + \mathbf{H}_t^\perp[\mathbf{u}]$ in $L^2(\Omega_t; R^3)$ is defined in terms of the projection $\mathbf{H}_t^\perp[\mathbf{u}] = \nabla_x \Psi$ which is given as the unique solution to the Neumann problem

$$\Delta \Psi = \mathrm{div}_x \mathbf{u} \text{ in } \Omega_t, \quad \nabla_x \Psi \cdot \mathbf{n} = \mathbf{u} \cdot \mathbf{n} \text{ on } \partial\Omega_t, \quad \int_{\Omega_t} \Psi \, \mathrm{d}\mathbf{x} = 0. \tag{3.1}$$

It is easy to see that $\mathrm{div}_x \mathbf{H}_t[\mathbf{u}] = 0$ in Ω_t and $\mathbf{H}_t[\mathbf{u}] \cdot \mathbf{n} = 0$ on $\partial\Omega_t$.

For a function $\mathbf{u} \in L^2(Q_T; R^3)$ we set:

$$\mathbf{H}[\mathbf{u}](t, \mathbf{x}) := \mathbf{H}_t[\mathbf{u}(t, \mathbf{x})] \quad \text{and} \quad \mathbf{H}^\perp[\mathbf{u}](t, \mathbf{x}) := \mathbf{H}_t^\perp[\mathbf{u}(t, \mathbf{x})].$$

We are going to show the differentiability of the mapping $t \mapsto \mathbf{H}_t[\mathbf{u}]$. For this reason we adopt some results on shape sensitivity analysis of the Neumann problem, see e.g. Sokolowski-Zolesio [16]. In this context, the symbol $\partial_t \mathbf{u}$ plays the role of the shape derivative of $\mathbf{u}$. By the shape differential calculus (see [16, Section 3.3]) one identifies the shape derivative $\partial_t \Psi$ of solutions to (3.1) with the unique solution of the problem

$$\begin{aligned} &\Delta \partial_t \Psi = \partial_t \mathrm{div}_x \mathbf{u}, && \text{in } \Omega_t, \\ &\nabla_x \partial_t \Psi \cdot \mathbf{n} = \mathrm{div}_{\partial\Omega_t}(\mathbf{V} \cdot \mathbf{n} \, \nabla_{\partial\Omega_t} \Psi) + [\mathrm{div}_x \mathbf{u} + \kappa \mathbf{u} \cdot \mathbf{n}] \, \mathbf{V} \cdot \mathbf{n} + \partial_t \mathbf{u} \cdot \mathbf{n} - \mathbf{u} \cdot \nabla_x \mathbf{V} \mathbf{n}, && \text{on } \partial\Omega_t \end{aligned} \tag{3.2}$$

where $\mathrm{div}_{\partial\Omega_t}$, $\nabla_{\partial\Omega_t}$, κ is the tangential divergence, the tangential gradient and the mean curvature of $\partial\Omega_t$. In particular, if $\partial_t \mathrm{div}_x \mathbf{u} \in L^2(\Omega_t; R^3)$ and $\partial_t \mathbf{u} \in W^{1/2,2}(\partial\Omega_t)$ then from [16, Proposition 3.3] it follows $\partial_t \Psi \in W^{1,2}(\Omega_t)$.

Assuming more regularity, specifically, $\partial_t \mathrm{div}_x \mathbf{u} \in W^{2,2}(\Omega_t; R^3)$ and $\partial_t \mathbf{u} \in W^{3/2,2}(\partial\Omega_t)$, we obtain $\partial_t \mathbf{H}_t^{\perp}[\mathbf{u}] = \nabla_x \partial_t \Psi \in W^{1,2}(\Omega_t)$ and $\partial_t \mathbf{H}_t[\mathbf{u}] = \partial_t \mathbf{u} - \partial_t \mathbf{H}_t^{\perp}[\mathbf{u}] \in W^{1,2}(\Omega_t)$.

3.2 Compactness of the solenoidal part of velocity

Since $\mathbf{U} \cdot \mathbf{n} = \mathbf{V} \cdot \mathbf{n}$ on $\partial\Omega_t$ and $\mathrm{div}_x \mathbf{U} = \mathrm{div}_x \mathbf{V} = 0$, we can write

$$\mathbf{U} = \mathbf{H}[\mathbf{U}] + \nabla_x W, \text{ where } \nabla_x W := \mathbf{H}^{\perp}[\mathbf{V}].$$

In accordance with the standard elliptic regularity, cf. for instance [6],

$$\|\mathbf{H}_\tau[\mathbf{u}]\|_{L^q(\Omega_\tau)} \le c(q) \|\mathbf{u}\|_{L^q(\Omega_\tau)} \text{ for any } 2 \le q < \infty \text{ and } \tau \in [0,T], \tag{3.3}$$

where, since the domains Ω_τ are regular, the constant $c(q)$ can be chosen to be independent of $\tau \in [0,T]$. Moreover, as

$$\int_0^T \int_{\Omega_t} \mathbf{H}[\mathbf{u}_\varepsilon - \mathbf{U}] \cdot \boldsymbol{\varphi} \, \mathrm{d}\mathbf{x}\mathrm{d}t = \int_0^T \int_{\Omega_t} (\mathbf{u}_\varepsilon - \mathbf{U}) \cdot \mathbf{H}[\boldsymbol{\varphi}] \, \mathrm{d}\mathbf{x}\mathrm{d}t \text{ for any } \boldsymbol{\varphi} \in L^2(Q_T; R^3),$$

the weak convergence of $\mathbf{u}_\varepsilon$ and (3.3) implies that

$$\mathbf{H}[\mathbf{u}_\varepsilon] \to \mathbf{H}[\mathbf{U}] \text{ weakly in } L^2(Q_T; R^3).$$

Using a similar argument we obtain that

$$\nabla_x \mathbf{H}[\mathbf{u}_\varepsilon] \to \nabla_x \mathbf{H}[\mathbf{U}] \text{ weakly in } L^2(Q_T; R^{3\times 3})$$

and consequently

$$\mathbf{H}^{\perp}[\mathbf{u}_\varepsilon] \to \nabla_x W \text{ weakly in } L^2(Q_T; R^3), \quad \nabla_x \mathbf{H}^{\perp}[\mathbf{u}_\varepsilon] \to \nabla_x^2 W \text{ weakly in } L^2(Q_T; R^{3\times 3}).$$

The key property we need to show in this section is the strong convergence

$$\mathbf{H}[\mathbf{u}_\varepsilon] \to \mathbf{H}[\mathbf{U}] \text{ in } L^2(Q_T; R^3). \tag{3.4}$$

To this end we show that

$$\mathbf{H}[\varrho_\varepsilon \mathbf{u}_\varepsilon] \to \mathbf{H}[\overline{\varrho}\mathbf{U}] = \overline{\varrho}\mathbf{H}[\mathbf{U}] \quad \text{strongly in } C_{weak}([T_1,T_2], L^{\frac{2\gamma}{\gamma+1}}(K)) \text{ provided } [T_1,T_2] \times \overline{K} \subset Q_T. \tag{3.5}$$

Remark: *In the proof below we systematically use of the continuity of the Helmhotz decomposition on moving domains.*

For any $\boldsymbol{\varphi} \in C^\infty(\overline{Q}_T)$ with $\boldsymbol{\varphi}(T,\cdot) = 0$ and $\boldsymbol{\varphi} \cdot \mathbf{n} = 0$ on $\partial\Omega_t$ we define

$$I^\varepsilon_{\boldsymbol{\varphi}}(t) := \left(\int_{\Omega_t} \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \mathbf{H}[\boldsymbol{\varphi}] \, \mathrm{d}\mathbf{x} \right)(t) = \left(\int_{\Omega_t} \mathbf{H}[\varrho_\varepsilon \mathbf{u}_\varepsilon] \cdot \boldsymbol{\varphi} \, \mathrm{d}\mathbf{x} \right)(t).$$

To use the Arzelá-Ascoli theorem we want to show that

$$\left| I^\varepsilon_{\boldsymbol{\varphi}}(t) - I^\varepsilon_{\boldsymbol{\varphi}}(t') \right| \leq C \left| t - t' \right|^\alpha$$

for some $\alpha > 0$. Using $\mathbf{H}[\boldsymbol{\varphi}]$ as a test function in the momentum equation (2.5) we obtain:

$$I^\varepsilon_{\boldsymbol{\varphi}}(t) - I^\varepsilon_{\boldsymbol{\varphi}}(t') = \int_t^{t'} \int_{\Omega_s} \varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : \nabla_x \mathbf{H}[\boldsymbol{\varphi}] - \mathbb{S}(\mathbf{u}_\varepsilon) : \nabla_x \mathbf{H}[\boldsymbol{\varphi}] + \varrho_\varepsilon \nabla_x g \cdot \mathbf{H}[\boldsymbol{\varphi}] + \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \partial_t \mathbf{H}[\boldsymbol{\varphi}] \, \mathrm{d}\mathbf{x}\mathrm{d}s.$$

The first three terms are easy to estimate using the apriori estimates we have and the smoothness of $\mathbf{H}[\boldsymbol{\varphi}]$ and $\nabla_x \mathbf{H}[\boldsymbol{\varphi}]$, for the last term we use the result of the previous section.

Using Arzelá-Ascoli Theorem we have that the set of functions $I^\varepsilon_{\boldsymbol{\varphi}}(t)$ is precompact in $C(T_1, T_2)$ which together with diagonalization argument yields (3.5).

Moreover

$$(\varrho_\varepsilon - \overline{\varrho})\mathbf{u}_\varepsilon \to 0 \quad \text{strongly in } L^2(T_1, T_2, L^{\frac{6}{5}}(K))$$

and thus

$$\mathbf{H}[(\varrho_\varepsilon - \overline{\varrho})\mathbf{u}_\varepsilon] \to 0 \text{ and } \mathbf{H}^\perp[(\varrho_\varepsilon - \overline{\varrho})\mathbf{u}_\varepsilon] \to 0 \quad \text{strongly in } L^2(T_1, T_2, L^{\frac{6}{5}}(K))$$

for all T_1, T_2, K such that $[T_1, T_2] \times \overline{K} \subset Q_T$. Now writing

$$\overline{\varrho} |\mathbf{H}[\mathbf{u}_\varepsilon]|^2 = \mathbf{H}[(\overline{\varrho} - \varrho_\varepsilon)\mathbf{u}_\varepsilon] \cdot \mathbf{H}[\mathbf{u}_\varepsilon] + \mathbf{H}[\varrho_\varepsilon \mathbf{u}_\varepsilon] \cdot \mathbf{H}[\mathbf{u}_\varepsilon]$$

we see that

$$\mathbf{H}[\mathbf{u}_\varepsilon] \to \mathbf{H}[\mathbf{U}] \quad \text{strongly in } L^2(Q_T; R^3).$$

3.3 Decomposition of the convective term

Our aim is to show that

$$\int_0^T \int_{\Omega_t} \varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : \nabla_x \boldsymbol{\varphi} \; \mathrm{d}\mathbf{x}\mathrm{d}t \to \overline{\varrho} \int_0^T \int_{\Omega_t} \mathbf{U} \otimes \mathbf{U} : \nabla_x \boldsymbol{\varphi} \; \mathrm{d}\mathbf{x}\mathrm{d}t$$

for all $\boldsymbol{\varphi} \in C^\infty(\overline{Q}_T)$ such that $\boldsymbol{\varphi}(T,\cdot) = 0$, $\mathrm{div}_x \boldsymbol{\varphi} = 0$ and $\boldsymbol{\varphi} \cdot \mathbf{n} = 0$ on $\partial\Omega_t$ for all $t \in [0,T)$. To this end, the convective term can be decomposed as follows:

$$\int_0^T \int_{\Omega_t} \varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : \nabla_x \boldsymbol{\varphi} \, \mathrm{d}\mathbf{x}\mathrm{d}t = \int_0^T \int_{\Omega_t} \mathbf{H}^\perp[\varrho_\varepsilon \mathbf{u}_\varepsilon] \otimes \mathbf{H}[\mathbf{u}_\varepsilon] : \nabla_x \boldsymbol{\varphi} \, \mathrm{d}\mathbf{x}\mathrm{d}t +$$
$$+ \int_0^T \int_{\Omega_t} \mathbf{H}[\varrho_\varepsilon \mathbf{u}_\varepsilon] \otimes \mathbf{u}_\varepsilon : \nabla_x \boldsymbol{\varphi} \, \mathrm{d}\mathbf{x}\mathrm{d}t + \int_0^T \int_{\Omega_t} \mathbf{H}^\perp[\varrho_\varepsilon \mathbf{u}_\varepsilon] \otimes \mathbf{H}^\perp[\mathbf{u}_\varepsilon] : \nabla_x \boldsymbol{\varphi} \, \mathrm{d}\mathbf{x}\mathrm{d}t.$$

We can pass to the limit in the first two terms using the convergence stated above to achieve:

$$\int_0^T \int_{\Omega_t} \mathbf{H}[\varrho_\varepsilon \mathbf{u}_\varepsilon] \otimes \mathbf{u}_\varepsilon : \nabla_x \boldsymbol{\varphi} \, \mathrm{d}\mathbf{x}\mathrm{d}t \to \int_0^T \int_{\Omega_t} \overline{\varrho} \mathbf{H}[\mathbf{U}] \otimes \mathbf{U} : \nabla_x \boldsymbol{\varphi} \, \mathrm{d}\mathbf{x}\mathrm{d}t$$

and

$$\int_0^T \int_{\Omega_t} \mathbf{H}^\perp[\varrho_\varepsilon \mathbf{u}_\varepsilon] \otimes \mathbf{H}[\mathbf{u}_\varepsilon] : \nabla_x \boldsymbol{\varphi} \, \mathrm{d}\mathbf{x}\mathrm{d}t =$$
$$= \int_0^T \int_{\Omega_t} \mathbf{H}^\perp[(\varrho_\varepsilon - \overline{\varrho})\mathbf{u}_\varepsilon] \otimes \mathbf{H}[\mathbf{u}_\varepsilon] : \nabla_x \boldsymbol{\varphi} \, \mathrm{d}\mathbf{x}\mathrm{d}t + \overline{\varrho} \int_0^T \int_{\Omega_t} \mathbf{H}^\perp[\mathbf{u}_\varepsilon] \otimes \mathbf{H}[\mathbf{u}_\varepsilon] : \nabla_x \boldsymbol{\varphi} \, \mathrm{d}\mathbf{x}\mathrm{d}t \to$$
$$\to \overline{\varrho} \int_0^T \int_{\Omega_t} \nabla_x W \otimes \mathbf{H}[\mathbf{U}] : \nabla_x \boldsymbol{\varphi} \, \mathrm{d}\mathbf{x}\mathrm{d}t.$$

Thus it remains to show that

$$\int_0^T \int_{\Omega_t} \mathbf{H}^\perp[\varrho_\varepsilon \mathbf{u}_\varepsilon] \otimes \mathbf{H}^\perp[\mathbf{u}_\varepsilon] : \nabla_x \boldsymbol{\varphi} \, \mathrm{d}\mathbf{x}\mathrm{d}t \to \overline{\varrho} \int_0^T \int_{\Omega_t} \nabla_x W \otimes \nabla_x W : \nabla_x \boldsymbol{\varphi} \, \mathrm{d}\mathbf{x}\mathrm{d}t.$$

We split this term further in the following way:

$$\int_0^T \int_{\Omega_t} \mathbf{H}^\perp[\varrho_\varepsilon \mathbf{u}_\varepsilon] \otimes \mathbf{H}^\perp[\mathbf{u}_\varepsilon] : \nabla_x \boldsymbol{\varphi} \, \mathrm{d}\mathbf{x}\mathrm{d}t = \int_0^T \int_{\Omega_t} \mathbf{H}^\perp[\varrho_\varepsilon \mathbf{V}] \otimes \mathbf{H}^\perp[\mathbf{u}_\varepsilon] : \nabla_x \boldsymbol{\varphi} \, \mathrm{d}\mathbf{x}\mathrm{d}t +$$
$$+ \int_0^T \int_{\Omega_t} \mathbf{H}^\perp[\varrho_\varepsilon(\mathbf{u}_\varepsilon - \mathbf{V})] \otimes \mathbf{H}^\perp[\mathbf{V}] : \nabla_x \boldsymbol{\varphi} \, \mathrm{d}\mathbf{x}\mathrm{d}t +$$
$$+ \int_0^T \int_{\Omega_t} \mathbf{H}^\perp[\varrho_\varepsilon(\mathbf{u}_\varepsilon - \mathbf{V})] \otimes \mathbf{H}^\perp[(\mathbf{u}_\varepsilon - \mathbf{V})] : \nabla_x \boldsymbol{\varphi} \, \mathrm{d}\mathbf{x}\mathrm{d}t.$$

It is easy to see that the first term converges to $\int_0^T \int_{\Omega_t} \overline{\varrho} \nabla_x W \otimes \nabla_x W : \nabla_x \boldsymbol{\varphi} \, \mathrm{d}\mathbf{x}\mathrm{d}t$, while the second term converges to zero. Finally, we have to show that the last term converges to zero. This will be done in the next two sections.

4 Spectral analysis of the wave equation

First we rewrite our system of equation (1.1), (1.2) in the form of an acoustic analogy. Denoting

$$r_\varepsilon := \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}, \qquad \mathbf{z}_\varepsilon := \varrho_\varepsilon(\mathbf{u}_\varepsilon - \mathbf{V}),$$

we obtain

$$\varepsilon \partial_t r_\varepsilon + \mathrm{div}_x \mathbf{z}_\varepsilon = \mathcal{O}(\varepsilon),\ \ \varepsilon \partial_t \mathbf{z}_\varepsilon + p'(\overline{\varrho}) \nabla_x r_\varepsilon = \mathcal{O}(\varepsilon),\ \ \mathbf{z}_\varepsilon \cdot \mathbf{n}|_{\partial\Omega_t} = 0,$$

or, more specifically in the weak formulation

$$\int_0^T \int_{\Omega_t} \varepsilon r_\varepsilon \partial_t \varphi + \mathbf{z}_\varepsilon \cdot \nabla_x \varphi \,\mathrm{d}\mathbf{x}\mathrm{d}t = -\int_0^T \int_{\Omega_t} \varepsilon r_\varepsilon \mathbf{V} \cdot \nabla_x \varphi \,\mathrm{d}\mathbf{x}\mathrm{d}t \tag{4.1}$$

for all $\varphi \in C_c^\infty((0,T) \times \overline{\Omega_t})$, and

$$\begin{aligned}
\int_0^T \int_{\Omega_t} \varepsilon \mathbf{z}_\varepsilon \cdot \partial_t \boldsymbol{\varphi} + p'(\overline{\varrho}) r_\varepsilon \mathrm{div}_x \boldsymbol{\varphi} \,\mathrm{d}\mathbf{x}\mathrm{d}t &= \varepsilon \int_0^T \int_{\Omega_t} -(\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) : \nabla_x \boldsymbol{\varphi} \,\mathrm{d}\mathbf{x}\mathrm{d}t + \\
+\varepsilon \int_0^T \int_{\Omega_t} \mathbb{S}(\nabla_x \mathbf{u}_\varepsilon) : \nabla_x \boldsymbol{\varphi} &+ \frac{(p(\varrho_\varepsilon) - p'(\overline{\varrho})(\varrho_\varepsilon - \overline{\varrho}) - p(\overline{\varrho}))}{\varepsilon^2} \mathrm{div}_x \boldsymbol{\varphi} \,\mathrm{d}\mathbf{x}\mathrm{d}t + \\
+\varepsilon \int_0^T \int_{\Omega_t} \varrho_\varepsilon \partial_t \mathbf{V} \cdot \boldsymbol{\varphi} &+ \varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{V} : \nabla_x \boldsymbol{\varphi} + \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla_x \mathbf{V} \cdot \boldsymbol{\varphi} \,\mathrm{d}\mathbf{x}\mathrm{d}t
\end{aligned} \tag{4.2}$$

for all $\boldsymbol{\varphi} \in C_c^\infty((0,T) \times \overline{\Omega_t})$ such that $\boldsymbol{\varphi} \cdot \mathbf{n} = 0$ on $(0,T) \times \partial\Omega_t$.

The idea is to use the spectral analysis to show, roughly speaking, that solutions of (4.1), (4.2), and, in particular, $\mathbf{H}^\perp[\mathbf{z}_\varepsilon] \approx \mathbf{H}^\perp[\mathbf{u}_\varepsilon - \mathbf{V}_\varepsilon]$ can be written as a "small" part and a "compact" part, where the latter makes the integral

$$\int_0^T \int_{\Omega_t} \mathbf{H}^\perp[\varrho_\varepsilon(\mathbf{u}_\varepsilon - \mathbf{V})] \otimes \mathbf{H}^\perp[(\mathbf{u}_\varepsilon - \mathbf{V})] : \nabla_x \boldsymbol{\varphi} \,\mathrm{d}\mathbf{x}\mathrm{d}t$$

converge to zero for solenoidal test functions $\boldsymbol{\varphi}$. To this end, we introduce the following eigenvalue problem

$$\nabla_x \omega = -\lambda(t)\mathbf{a}, \quad \mathrm{div}_x \mathbf{a} = -\lambda(t)\omega \quad \text{ in } \Omega_t, \quad \mathbf{a} \cdot \mathbf{n} = 0 \quad \text{ on } \partial\Omega_t,$$

which is equivalent to the eigenvalue problem for the Laplace equation

$$-\Delta\omega = \Lambda(t)\omega \quad \text{ in } \Omega_t, \quad \nabla_x \omega \cdot \mathbf{n} = 0 \quad \text{ on } \partial\Omega_t, \quad \lambda^2 = -\Lambda. \tag{4.3}$$

The latter problem admits for any $t \in [0,T]$ a system of real eigenfunctions $\{\omega_j(t,\cdot)\}_{j=0}^\infty$ which forms an orthonormal basis in $L^2(\Omega_t)$, corresponding to real eigenvalues

$$0 = \Lambda_0(t) < \Lambda_1(t) \leq \Lambda_2(t) \leq ... \tag{4.4}$$

Accordingly the original system admits solutions in the form

$$\mathbf{a}_j(t,\mathbf{x}) = \frac{\mathrm{i}}{\sqrt{\Lambda_j(t)}}\nabla_x\omega_j(t,\mathbf{x}), \quad \lambda_j(t) = \mathrm{i}\sqrt{\Lambda_j(t)}, \quad j = 1,2,...,$$

where

$$\{\mathbf{a}_j(t,\cdot)\}_{j=1}^{\infty} \text{ forms an orthonormal basis in } \mathbf{H}^{\perp}(L^2(\Omega_t)) = \overline{\left\{\mathrm{span}\,\{\mathrm{i}\mathbf{a}_j(t,\cdot)\}_{j=1}^{\infty}\right\}}^{L^2(\Omega_t;R^3)}$$

and the eigenspace of $\lambda_0(t) = 0$ is $\mathbf{H}(L^2(\Omega_t))$. We also observe that

$$\left\{\frac{\omega_j(t,\cdot)}{\sqrt{1+\Lambda_j(t)}}\right\}_{j=0}^{\infty} \text{ is an orthonormal system in } W^{1,2}(\Omega_t),\ t\in[0,T].$$

We shall need some information on the time evolution of the eigenvalues and eigenfunctions. Using [2, Theorem 4.3] and the properties of $\mathbf{V}$ we find that

$$\left|\frac{1}{\Lambda_j(t_1)} - \frac{1}{\Lambda_j(t_2)}\right| \le C|t_1 - t_2|,\ t_1,t_2\in[0,T],$$

i.e. the functions Λ_j^{-1}, $j = 1,2,\dots$ are equi-Lipschitz in $[0,T]$. For the eigenfunctions such a property cannot be expected in general. Instead, it is more convenient to work with the orthogonal projections P_M and $\mathbf{Q}_M$ on the eigenspaces spanned by $\{\omega_j\}_{j=1}^M$, $\{\mathbf{a}_j\}_{j=1}^M$, respectively, defined by

$$P_M[\varphi](t,\cdot) = \sum_{j=1}^M \omega_j(t,\cdot)\int_{\Omega_t}\varphi(t,\mathbf{y})\omega_j(t,\mathbf{y})\,\mathrm{d}\mathbf{y},\ \varphi\in L^2(Q_T),$$

$$\mathbf{Q}_M[\boldsymbol{\varphi}](t,\cdot) := \sum_{j=1}^M \mathbf{a}_j(t,\cdot)\int_{\Omega_t}\boldsymbol{\varphi}(t,\mathbf{y})\cdot\mathbf{a}_j(t,\mathbf{y})\,\mathrm{d}\mathbf{y},\ \boldsymbol{\varphi}\in L^2(Q_T;R^3).$$

By an easy calculation one can check that P_M is also an orthogonal projection with respect to the scalar product in $W^{1,2}$ and that

$$\mathbf{Q}_M[\boldsymbol{\varphi}](t,\cdot) = -\nabla_x P_M[\psi](t,\cdot),\ \text{where } \mathbf{H}^{\perp}[\boldsymbol{\varphi}] = \nabla_x\psi.$$

The Lipschitz continuity of the eigenprojections can be proved under the condition that the set of corresponding eigenvalues remains isolated from the rest of the spectrum. If M were such that

$$\Lambda_{M+1} \ne \Lambda_M \text{ in } [0,T], \tag{4.5}$$

then [2, Theorem 4.4] and the properties of $\mathbf{V}$ would yield that the projection P_M is in certain sense equi-Lipschitz. In particular, for $t_1, t_2 \in [0,T]$, we would have

$$\forall \varphi \in W^{1,2}(\Omega_0): \quad \left\| P_M[\varphi \circ \mathbf{X}_{t_1}^{-1}](t_1) \circ \mathbf{X}_{t_1} - P_M[\varphi \circ \mathbf{X}_{t_2}^{-1}](t_2) \circ \mathbf{X}_{t_2} \right\|_{W^{1,2}(\Omega_0)} \leq c|t_1 - t_2| \, \|\varphi\|_{W^{1,2}(\Omega_0)}.$$

This would further imply that $\partial_t P_M$ exists a.a. in Q_T and satisfies

$$\sup_{\tau \in (0,T)} \|\partial_t P_M(\tau)\|_{\mathcal{L}(W^{1,2}(\Omega_\tau), W^{1,2}(\Omega_\tau))} \leq c.$$

However, in general the multiplicity of eigenvalues of Neumann Laplacean can be very sensitive to the time evolution of the domain Ω_t, so that such M which satisfies (4.5) need not exist even if Ω_t varies in a smooth way. Instead, we will split the interval $[0,T]$ into a finite number of sub-intervals on which the smoothness of certain eigenprojections can be achieved.

Let M be given. Since the eigenvalues are continuous in time and $\lim_{j\to\infty} \Lambda_j(t) = \infty$ for every $t \in [0,T]$, it follows that for every $t \in [0,T]$ there exists $M_t \geq M$ and a neighborhood I_t of t, such that

$$\Lambda_{M_t+1} \neq \Lambda_{M_t} \text{ in } I_t.$$

Consequently, the projections P_{M_t} and $\mathbf{Q}_{M_t}$ are Lipschitz continuous with respect to time in I_t, with the Lipschitz constant independent of t. From the system $\{I_t\}_{t\in[0,T]}$ we can select a finite cover $\{I_{t_l}\}_{l=1}^n$ of $[0,T]$.

Let us fix $l \in \{1, \ldots, n\}$. For simplicity of notation, we will write $I_l := I_{t_l}$ and $M_l := M_{t_l}$ in what follows. We test the acoustic version of the continuity equation (4.1) by the function $\psi(t) P_{M_l}[\varphi](t, \mathbf{x})$, where $\psi \in C_c^\infty(I_l)$ and $\varphi \in C^\infty(\overline{Q}_T)$, which yields:

$$\begin{aligned} &\int_{I_l} \psi \int_{\Omega_t} \left(\varepsilon \partial_t P_{M_l}[r_\varepsilon] - \operatorname{div}_x \mathbf{Q}_{M_l}[\mathbf{z}_\varepsilon]\right) \varphi \, \mathrm{d}\mathbf{x}\mathrm{d}t \\ &= \varepsilon \int_{I_l} \psi \int_{\Omega_t} \left(r_\varepsilon \mathbf{V} \cdot \nabla_x P_{M_l}[\varphi] + r_\varepsilon \partial_t P_{M_l}[\varphi] - \mathbf{V} \cdot \nabla_x (P_{M_l}[r_\varepsilon]\varphi) - P_{M_l} r_\varepsilon \partial_t \varphi \right) \mathrm{d}\mathbf{x}\mathrm{d}t. \end{aligned} \tag{4.6}$$

Similarly, testing (4.2) by $\psi(t)\mathbf{Q}_{M_l}[\boldsymbol{\varphi}]$, where $\psi \in C_c^\infty(I_l)$ and $\boldsymbol{\varphi} \in C^\infty(\overline{Q}_T; R^3)$, $\boldsymbol{\varphi} \cdot \mathbf{n} = 0$, we get:

$$\begin{aligned} &\int_{I_l} \psi \int_{\Omega_t} \left(\varepsilon \partial_t \mathbf{Q}_{M_l}[\mathbf{z}_\varepsilon] - p'(\overline{\varrho}) \nabla_x P_{M_l}[r_\varepsilon]\right) \cdot \boldsymbol{\varphi} \, \mathrm{d}\mathbf{x}\mathrm{d}t \\ &= -\varepsilon \int_{I_l} \psi \int_{\Omega_t} \left[\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{V} - \varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon + \mathbb{S}(\nabla_x \mathbf{u}_\varepsilon)\right] : \nabla_x \mathbf{Q}_{M_l}[\boldsymbol{\varphi}] \, \mathrm{d}\mathbf{x}\mathrm{d}t \\ &\quad -\varepsilon \int_{I_l} \psi \int_{\Omega_t} \left[\frac{(p(\varrho_\varepsilon) - p'(\overline{\varrho})(\varrho_\varepsilon - \overline{\varrho}) - p(\overline{\varrho}))}{\varepsilon^2} \operatorname{div}_x \mathbf{Q}_{M_l}[\boldsymbol{\varphi}] \right] \mathrm{d}\mathbf{x}\mathrm{d}t \\ &\quad -\varepsilon \int_{I_l} \psi \int_{\Omega_t} \left[\varrho_\varepsilon \partial_t \mathbf{V} + \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla_x \mathbf{V}\right] \cdot \mathbf{Q}_{M_l}[\boldsymbol{\varphi}] \, \mathrm{d}\mathbf{x}\mathrm{d}t \\ &\quad -\varepsilon \int_{I_l} \psi \int_{\Omega_t} \left[\mathbf{Q}_{M_l}[\mathbf{z}_\varepsilon] \cdot \partial_t \boldsymbol{\varphi} - \mathbf{z}_\varepsilon \cdot \partial_t \mathbf{Q}_{M_l}[\boldsymbol{\varphi}] + \mathbf{V} \cdot \nabla_x (\mathbf{Q}_{M_l}[\mathbf{z}_\varepsilon] \cdot \boldsymbol{\varphi})\right] \mathrm{d}\mathbf{x}\mathrm{d}t. \end{aligned} \tag{4.7}$$

Denoting $d_{\varepsilon,l} := P_{M_l}[r_\varepsilon]$ and $\nabla_x \Psi_{\varepsilon,l} := \mathbf{Q}_{M_l}[\mathbf{H}^\perp[\mathbf{z}_\varepsilon]]$, identities (4.6)-(4.7) can be rewritten as the system

$$\varepsilon \partial_t d_{\varepsilon,l} + \Delta \Psi_{\varepsilon,l} = \varepsilon f_{\varepsilon,l}, \tag{4.8}$$

$$\varepsilon \partial_t \nabla_x \Psi_{\varepsilon,l} + p'(\overline{\varrho}) \nabla_x d_{\varepsilon,l} = \varepsilon \mathbf{g}_{\varepsilon,l}, \tag{4.9}$$

satisfied in $\{(t,\mathbf{x});\ t \in I_l,\ \mathbf{x} \in \Omega_t\}$. Having collected all the necessary material, we complete the proof of Theorem 2.1 in the next section.

5 Convergence

In accordance with the previous discussion our ultimate goal is to show that

$$\int_0^T \int_{\Omega_t} \mathbf{H}^\perp[\varrho_\varepsilon(\mathbf{u}_\varepsilon - \mathbf{V})] \otimes \mathbf{H}^\perp[(\mathbf{u}_\varepsilon - \mathbf{V})] : \nabla_x \boldsymbol{\varphi} \, \mathrm{d}\mathbf{x}\mathrm{d}t \to 0 \tag{5.1}$$

for any $\boldsymbol{\varphi} \in C^\infty(\overline{Q}_T; R^3)$, $\mathrm{div}_x \boldsymbol{\varphi}(t,\cdot) = 0$, $\boldsymbol{\varphi} \cdot \mathbf{n}|_{\partial\Omega_t} = 0$. To this end, we can write

$$\mathbf{H}^\perp[\mathbf{z}_\varepsilon] \otimes \mathbf{H}^\perp[(\mathbf{u}_\varepsilon - \mathbf{V})] = \big(\mathbf{Q}_{M_l}[\mathbf{H}^\perp[\mathbf{z}_\varepsilon]] + \big(\mathbf{H}^\perp[\mathbf{z}_\varepsilon] - \mathbf{Q}_{M_l}[\mathbf{H}^\perp[\mathbf{z}_\varepsilon]]\big)\big) \otimes$$
$$\otimes \big(\mathbf{Q}_{M_l}[\mathbf{H}^\perp[(\mathbf{u}_\varepsilon - \mathbf{V})]] + \big(\mathbf{H}^\perp[(\mathbf{u}_\varepsilon - \mathbf{V})] - \mathbf{Q}_{M_l}[\mathbf{H}^\perp[(\mathbf{u}_\varepsilon - \mathbf{V})]]\big)\big)$$

and moreover

$$\mathbf{H}^\perp[\mathbf{z}_\varepsilon] - \mathbf{Q}_{M_l}[\mathbf{H}^\perp[\mathbf{z}_\varepsilon]] = \mathbf{H}^\perp[(\varrho_\varepsilon - \overline{\varrho})(\mathbf{u}_\varepsilon - \mathbf{V})] - \mathbf{Q}_{M_l}[\mathbf{H}^\perp[(\varrho_\varepsilon - \overline{\varrho})(\mathbf{u}_\varepsilon - \mathbf{V})]] +$$
$$+ \overline{\varrho}\left(\mathbf{H}^\perp[\mathbf{u}_\varepsilon - \mathbf{V}] - \mathbf{Q}_{M_l}[\mathbf{H}^\perp[\mathbf{u}_\varepsilon - \mathbf{V}]]\right).$$

It should be noted that we already know that

$$\mathbf{H}^\perp[(\varrho_\varepsilon - \overline{\varrho})(\mathbf{u}_\varepsilon - \mathbf{V})] - \mathbf{Q}_{M_l}[\mathbf{H}^\perp[(\varrho_\varepsilon - \overline{\varrho})(\mathbf{u}_\varepsilon - \mathbf{V})]] \to 0 \quad \text{in } L^2(0,T,L^{\frac{6}{5}}(\Omega_t;R^3)).$$

Next we can use the properties of the functions ω_j and $\mathbf{a}_j$ to write

$$\|\mathrm{div}_x \mathbf{u}_\varepsilon\|^2_{L^2(\Omega_t)} = \|\mathrm{div}_x(\mathbf{u}_\varepsilon - \mathbf{V})\|^2_{L^2(\Omega_t)} = \left\| \sum_{j=1}^\infty \mathrm{div}_x \mathbf{a}_j \int_{\Omega_t} (\mathbf{u}_\varepsilon - \mathbf{V}) \cdot \mathbf{a}_j \, \mathrm{d}\mathbf{x} \right\|^2_{L^2(\Omega_t)} =$$
$$= \left\| \sum_{j=1}^\infty \sqrt{\Lambda_j} \omega_j \int_{\Omega_t} (\mathbf{u}_\varepsilon - \mathbf{V}) \cdot \mathbf{a}_j \, \mathrm{d}\mathbf{x} \right\|^2_{L^2(\Omega_t)} = \sum_{j=1}^\infty \Lambda_j \left(\int_{\Omega_t} (\mathbf{u}_\varepsilon - \mathbf{V}) \cdot \mathbf{a}_j \, \mathrm{d}\mathbf{x} \right)^2$$

and consequently

$$\left\|\mathbf{H}^{\perp}[(\mathbf{u}_\varepsilon-\mathbf{V})]-\mathbf{Q}_{M_l}[\mathbf{H}^{\perp}[(\mathbf{u}_\varepsilon-\mathbf{V})]]\right\|_{L^2(\Omega_t)}=\sum_{j>M_l}\left(\int_{\Omega_t}(\mathbf{u}_\varepsilon-\mathbf{V})\cdot\mathbf{a}_j\,\mathrm{d}\mathbf{x}\right)^2\leq$$
$$\leq\frac{1}{\inf_{j>M_l}\Lambda_j(t)}\|\mathrm{div}_x\mathbf{u}_\varepsilon\|^2_{L^2(\Omega_t)}\leq\frac{1}{\inf_{j>M}\Lambda_j(t)}\|\mathrm{div}_x\mathbf{u}_\varepsilon\|^2_{L^2(\Omega_t)}.$$

Choosing M large enough we can make $1/(\inf_{t\in[0,T],j>M}\Lambda_j(t))$ as small as we want. Therefore it is enough to prove

$$\int_{I_l}\int_{\Omega_t}\mathbf{Q}_{M_l}[\mathbf{H}^{\perp}[\mathbf{z}_\varepsilon]]\otimes\mathbf{Q}_{M_l}[\mathbf{H}^{\perp}[(\mathbf{u}_\varepsilon-\mathbf{V})]]:\nabla_x\boldsymbol{\varphi}\,\mathrm{d}\mathbf{x}\mathrm{d}t\to 0$$

or equivalently

$$\int_{I_l}\int_{\Omega_t}\mathbf{Q}_{M_l}[\mathbf{H}^{\perp}[\mathbf{z}_\varepsilon]]\otimes\mathbf{Q}_{M_l}[\mathbf{H}^{\perp}[\mathbf{z}_\varepsilon]]:\nabla_x\boldsymbol{\varphi}\,\mathrm{d}\mathbf{x}\mathrm{d}t\to 0$$

for any solenoidal test function $\boldsymbol{\varphi}$ and $l=1,\dots,n$.

Now we write (using the summation convention with the indices j,k)

$$\int_{I_l}\int_{\Omega_t}\mathbf{Q}_{M_l}[\mathbf{H}^{\perp}[\mathbf{z}_\varepsilon]]\otimes\mathbf{Q}_{M_l}[\mathbf{H}^{\perp}[\mathbf{z}_\varepsilon]]:\nabla_x\boldsymbol{\varphi}\,\mathrm{d}\mathbf{x}\mathrm{d}t=\int_{I_l}\int_{\Omega_t}\partial_k\Psi_{\varepsilon,l}\partial_j\Psi_{\varepsilon,l}\partial_j\varphi_k\,\mathrm{d}\mathbf{x}\mathrm{d}t=$$
$$=-\int_{I_l}\int_{\Omega_t}\partial_k\Psi_{\varepsilon,l}\Delta\Psi_{\varepsilon,l}\varphi_k\,\mathrm{d}\mathbf{x}\mathrm{d}t-\frac{1}{2}\int_{I_l}\int_{\Omega_t}\partial_k\left|\nabla_x\Psi_{\varepsilon,l}\right|^2\varphi_k\,\mathrm{d}\mathbf{x}\mathrm{d}t,$$

where the second term on the right hand side is equal to zero due to the fact that the test function $\boldsymbol{\varphi}$ is solenoidal. Next, we use equations (4.8), (4.9) to handle the first term

$$-\int_{I_l}\int_{\Omega_t}\partial_k\Psi_{\varepsilon,l}\Delta\Psi_{\varepsilon,l}\varphi_k\,\mathrm{d}\mathbf{x}\mathrm{d}t=\varepsilon\int_{I_l}\int_{\Omega_t}(\partial_t d_{\varepsilon,l}-f_{\varepsilon,l})\partial_k\Psi_{\varepsilon,l}\varphi_k\,\mathrm{d}\mathbf{x}\mathrm{d}t=$$
$$\varepsilon\int_{I_l}\int_{\Omega_t}\left(\partial_t(d_{\varepsilon,l}\partial_k\Psi_{\varepsilon,l})\varphi_k-d_\varepsilon\partial_t(\partial_k\Psi_{\varepsilon,l})\varphi_k-f_{\varepsilon,l}\partial_k\Psi_{\varepsilon,l}\varphi_k\right)\mathrm{d}\mathbf{x}\mathrm{d}t=$$
$$=\varepsilon\int_{I_l}\int_{\Omega_t}\left(-d_{\varepsilon,l}\partial_k\Psi_{\varepsilon,l}\partial_t\varphi_k+\frac{1}{2\varepsilon}p'(\overline{\varrho})\partial_k\left|d_{\varepsilon,l}\right|^2\varphi_k-d_{\varepsilon,l}\mathbf{g}_{\varepsilon,l}\cdot\boldsymbol{\varphi}-f_{\varepsilon,l}\nabla_x\Psi_{\varepsilon,l}\cdot\boldsymbol{\varphi}\right)\mathrm{d}\mathbf{x}\mathrm{d}t.$$

The second term is again equal to zero and the proof of the desired relation (5.1) is finished as soon as we show that remaining terms tend to 0 as $\varepsilon\to 0$, i.e. that integrals

$$\int_{I_l}\int_{\Omega_t}d_{\varepsilon,l}\nabla_x\Psi_{\varepsilon,l}\cdot\partial_t\boldsymbol{\varphi}\,\mathrm{d}\mathbf{x}\mathrm{d}t,$$

$$\int_{I_l}\int_{\Omega_t}d_{\varepsilon,l}\mathbf{g}_{\varepsilon,l}\cdot\boldsymbol{\varphi}\,\mathrm{d}\mathbf{x}\mathrm{d}t,$$

and

$$\int_{I_l} \int_{\Omega_t} f_{\varepsilon,l} \nabla_x \Psi_{\varepsilon,l} \cdot \boldsymbol{\varphi} \, \mathrm{d}\mathbf{x}\mathrm{d}t$$

are bounded uniformly with respect to l and $\varepsilon \to 0$. But this is a consequence of the uniform estimates, properties of $\mathbf{V}$ and smoothness of the eigenprojections P_{M_l}, $\mathbf{Q}_{M_l}$ established in the preceding part of the paper.

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