



ACADEMY of SCIENCES of the CZECH REPUBLIC

INSTITUTE of MATHEMATICS

**Conditional regularity of very weak
solutions to the Navier-Stokes-Fourier
system**

Eduard Feireisl

Yongzhong Sun

Preprint No. 42-2014

PRAHA 2014

Conditional regularity of very weak solutions to the Navier-Stokes-Fourier system

Eduard Feireisl * Yongzhong Sun[†]

Institute of Mathematics of the Academy of Sciences of the Czech Republic
Žitná 25, 115 67 Praha 1, Czech Republic

Department of Mathematics, Nanjing University, Nanjing, Jiangsu, 210093, China

Abstract

We consider a class of (very) weak solutions to the Navier-Stokes-Fourier system describing the time evolution of the density ϱ , the absolute temperature ϑ , and the macroscopic velocity \mathbf{u} . It is shown that a weak solution emanating from smooth initial data is regular as long as \mathbf{u} and ϑ are bounded and $\|\operatorname{div}_x \mathbf{u}\|_{L^\infty}$ integrable in the existence interval $(0, T)$. Using the method of relative energy we first show that any weak solution enjoying the above mentioned regularity coincides with a strong one as long as the latter exists. In such a way, the proof reduces to showing that the life span of the strong solution can be extended to the desired existence interval $(0, T)$.

Contents

1	Introduction	2
2	Weak solutions, main results	5
2.1	Weak solutions	5
2.2	Main result	7
2.3	Additional regularity of weak solutions	8
2.3.1	Uniform bounds on the density	8
2.3.2	Renormalization of the thermal energy inequality	10

*The research of E.F. leading to these results has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP7/2007-2013)/ ERC Grant Agreement 320078. The Institute of Mathematics of the Academy of Sciences of the Czech Republic is supported by RVO:67985840.

[†]Y. Sun is supported by NSFC Grant No. 11171145

3	Entropy and relative energy inequality	13
3.1	Entropy inequality	14
3.2	Relative energy (entropy)	15
3.3	Weak-strong uniqueness	15
4	Conditional regularity of smooth solutions	18
4.1	First energy bounds for the velocity and temperature	18
4.2	Decomposition of \mathbf{u} and elliptic estimates	19
4.3	Second energy bounds for the velocity	19
4.4	Second energy bounds for the temperature and better regularity of the effective viscous flux .	21
4.5	Estimates of the derivatives of ϱ	23
4.6	Hölder continuity of the temperature	23
4.7	$L^p - L^q$ parabolic estimates	24

1 Introduction

The time evolution of the mass density $\varrho = \varrho(t, x)$, the absolute temperature $\vartheta = \vartheta(t, x)$, and the velocity field $\mathbf{u} = \mathbf{u}(t, x)$ of a compressible, viscous and heat conducting fluid can be described by the *Navier-Stokes-Fourier system* of partial differential equations:

$$\partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{u}) = 0, \tag{1.1}$$

$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = \operatorname{div}_x \mathbb{S}(\nabla_x \mathbf{u}), \tag{1.2}$$

$$c_v \left[\partial_t(\varrho \vartheta) + \operatorname{div}_x(\varrho \vartheta \mathbf{u}) \right] + \operatorname{div}_x \mathbf{q} = \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \vartheta \frac{\partial p(\varrho, \vartheta)}{\partial \vartheta} \operatorname{div}_x \mathbf{u}, \tag{1.3}$$

where $p = p(\varrho, \vartheta)$ is the pressure, $\mathbb{S}(\nabla_x \mathbf{u})$ the viscous stress tensor, c_v denotes the specific heat at constant volume, and \mathbf{q} the heat flux. In addition, we suppose that \mathbb{S} is given by *Newton's law*

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \operatorname{div}_x \mathbf{u} \mathbb{I} \right) + \eta \operatorname{div}_x \mathbf{u} \mathbb{I}, \quad \mu > 0, \quad \eta \geq 0, \tag{1.4}$$

while \mathbf{q} obeys *Fourier's law*

$$\mathbf{q} = -\kappa(\vartheta) \nabla_x \vartheta. \tag{1.5}$$

The fluid is confined to a bounded physical domain $\Omega \subset \mathbb{R}^3$, on the boundary of which

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0, \quad \mathbf{q} \cdot \mathbf{n}|_{\partial\Omega} = 0. \tag{1.6}$$

Furthermore, the velocity field \mathbf{u} satisfies either the no-slip

$$\mathbf{u} \times \mathbf{n}|_{\partial\Omega} = 0, \quad (1.7)$$

or the complete slip

$$(\mathbb{S}(\nabla_x \mathbf{u}) \cdot \mathbf{n}) \times \mathbf{n}|_{\partial\Omega} = 0 \quad (1.8)$$

boundary conditions.

The problem is completed by prescribing the initial conditions

$$\varrho(0, \cdot) = \varrho_0, \quad \vartheta(0, \cdot) = \vartheta_0, \quad \mathbf{u}(0, \cdot) = \mathbf{u}_0. \quad (1.9)$$

Our goal is to study solvability of the Navier-Stokes-Fourier system (1.1-1.9) for sufficiently regular initial data satisfying the physically relevant restriction

$$\varrho_0 > 0, \quad \vartheta_0 > 0 \text{ in } \overline{\Omega}. \quad (1.10)$$

Our principal working hypothesis is that the specific heat at constant volume $c_v > 0$ is constant, therefore the associated specific internal energy reads

$$e(\varrho, \vartheta) = c_v \vartheta + P(\varrho). \quad (1.11)$$

Moreover, we suppose that the heat conductivity $\kappa(\vartheta)$ is an increasing function of the absolute temperature, while the viscosity coefficients $\mu > 0$ and $\eta \geq 0$ remain constant.

We consider the class of (very) weak solutions introduced in the monograph [Fei04], see Section 2 below. Roughly speaking, these are distributional solutions of the equations (1.1), (1.2), while the thermal energy balance (1.3) is replaced by two inequalities:

$$c_v \left[\partial_t(\varrho\vartheta) + \operatorname{div}_x(\varrho\mathbf{u} \otimes \mathbf{u}) \right] - \Delta K(\vartheta) \geq \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \vartheta \frac{\partial p(\varrho, \vartheta)}{\partial \vartheta} \operatorname{div}_x \mathbf{u}, \quad (1.12)$$

with

$$K(\vartheta) = \int_0^{\vartheta} \kappa(z) \, dz,$$

and

$$\int_{\Omega} \left[\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) \right] (\tau, \cdot) \, dx \leq \int_{\Omega} \left[\frac{1}{2} \varrho_0 |\mathbf{u}_0|^2 + \varrho_0 e(\varrho_0, \vartheta_0) \right] \, dx. \quad (1.13)$$

As shown in [Fei04, Chapter 4, Section 4.3.2], both (1.12) and (1.13) reduce to equalities as soon as the corresponding weak solution is smooth enough.

Recently [FN09], an alternative approach has been developed based on the weak formulation of the Navier-Stokes-Fourier system, where the thermal energy balance (1.3) is replaced by the entropy inequality

$$\partial_t(\varrho s(\varrho, \vartheta)) + \operatorname{div}_x(\varrho s(\varrho, \vartheta)\mathbf{u}) + \operatorname{div}_x \left(\frac{\mathbf{q}}{\vartheta} \right) = \sigma, \quad \sigma \geq \frac{1}{\vartheta} \left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right), \quad (1.14)$$

where $s = s(\varrho, \vartheta)$ is the specific entropy determined through Gibbs' relation

$$\vartheta Ds(\varrho, \vartheta) = De(\varrho, \vartheta) + p(\varrho, \vartheta)D\left(\frac{1}{\varrho}\right). \quad (1.15)$$

The relevant existence theory based on the entropy formulation (1.14), developed in [FN09, Chapter 3], requires *all* transport coefficients μ , η , and κ to depend effectively on the temperature, whereas the pressure as well as the internal energy have to be augmented by a radiation component proportional to ϑ^4 to avoid the problem of temperature oscillations on the hypothetical vacuum zones. We note that yet another rather different approach to weak solutions to the Navier-Stokes-Fourier system has been proposed by Bresch and Desjardins [BD07], where the viscosity coefficients μ and λ are functions of the density ϱ interrelated in a specific way, and the pressure contains a singular component unbounded when ϱ approaches zero.

Although the weak formulation based on the entropy balance (1.14) rather than (1.3) is very convenient from the purely theoretical point of view, giving rise to a number of interesting properties of the weak solutions including the *weak-strong uniqueness* (see [FN12]) and *conditional regularity* results (see [FNS14]), the presence of the radiation terms as well as the explicit dependence of the transport coefficients on the temperature makes it too complicated for possible numerical implementations.

Our interest in the weak formulation based on the inequalities (1.12), (1.13) is motivated by the recent development of relevant *numerical schemes* based on a mixed discontinuous Galerkin method, see Karper [Kar13] and [FKN14]. As shown in [FKN14], such a numerical scheme *converges* to a weak solution of the Navier-Stokes-Fourier system specified through (1.12), (1.13). In particular, the conditional regularity based on mere *boundedness* of the numerical solutions may lead to rigorous *error estimates* for the latter. Last but not least, we point out that although the class of weak solutions is apparently larger, the regularity criterion we obtain is considerably stronger than in [FNS14].

Our goal is to show that any weak solution of the Navier-Stokes-Fourier system (1.1-1.8), originating from sufficiently smooth initial data (1.9), (1.10), is regular in $(0, T) \times \Omega$ as soon as

$$\vartheta(t, x) \leq \bar{\vartheta}, |\mathbf{u}(t, x)| \leq \bar{u} \text{ for a.a. } (t, x) \in (0, T) \times \Omega, \int_0^T \|\operatorname{div}_x \mathbf{u}(t, \cdot)\|_{L^\infty(\Omega)} dt \leq L, \quad (1.16)$$

where $\bar{\vartheta}$, \bar{u} , and L are positive constants.

In comparison with the standard blow up criteria applicable to *strong solutions* of the Navier-Stokes-Fourier system (see [FJO10], [HLW13], among others), the problem of conditional regularity for *weak solutions* features an additional difficulty that consists in showing that a weak and strong solution, starting from the same initial data, coincide on their common existence interval. To this end, we adapt the method based on the relative energy functional developed in [FN12]. As this approach relies on the entropy inequality (1.14) rather than the thermal energy balance (1.12), we have to show that any weak solution that complies with (1.12), (1.13), together with the extra regularity properties (1.16), necessarily satisfies also (1.14).

The remaining part of the proof of conditional regularity is then reduced to the class of strong solutions. Here we show that the hypothesis (1.16) can be actually relaxed to

$$\varrho(t, x) \leq \bar{\varrho}, \vartheta(t, x) \leq \bar{\vartheta}, |\mathbf{u}(t, x)| \leq \bar{u} \text{ for any } (t, x) \in (0, T) \times \Omega. \quad (1.17)$$

It is easy to see that (1.16) implies (1.17) and the result may be of independent interest as an alternative to [HLW13].

The paper is organized as follows. In Section 2, we review the basic results of the existence theory developed in [Fei04] and state our main result. In Section 3, we show that any weak solution enjoying the extra regularity (1.16) satisfies also the entropy inequality (1.14) - a result that may be of independent interest. To this end, we use a variant of the technique developed by Freshe et. al. [FMR10] in the context of non-homogeneous fluids. Then we show the weak-uniqueness properties by means of a straightforward application of the relative energy method. Finally, we complete the proof by showing a blow-up criterion for strong solutions in Section 4. Here we follow the arguments of [SWZ11], and Wen and Zhu [WZ13].

2 Weak solutions, main results

We start by specifying the structural restrictions imposed on the thermodynamics functions. It follows from our basic hypothesis (1.11) combined with the Gibbs' relation (1.15) that the pressure p can be written in the form

$$p(\varrho, \vartheta) = p_e(\varrho) + \vartheta p_{th}(\varrho), \quad (2.1)$$

where p_e denotes the elastic (cold) pressure and p_{th} the thermal pressure component. To simplify the presentation, we shall assume that

$$p(\varrho, \vartheta) = \varrho^\gamma + \varrho\vartheta. \quad (2.2)$$

In addition, the heat conductivity coefficient κ is taken

$$\kappa(\vartheta) = \bar{\kappa}(1 + \vartheta^2). \quad (2.3)$$

As a matter of fact, these assumptions are relevant only to establish the *existence* of weak solutions, whereas the specific form of p and κ does not play any role in the regularity criterion as (1.16) implies that both ϱ and ϑ must be bounded.

2.1 Weak solutions

Definition 2.1. *We say that $[\varrho, \vartheta, \mathbf{u}]$ is a weak solution of the Navier-Stokes-Fourier system (1.1 - 1.9) if:*

- *The functions $[\varrho, \vartheta, \mathbf{u}]$ belong to the regularity class:*

$$\left\{ \begin{array}{l} \varrho \geq 0, \quad \varrho \in C_{\text{weak}}([0, T]; L^\gamma(\Omega)), \quad \gamma > 1, \\ \mathbf{u} \in L^2(0, T; W^{1,2}(\Omega)), \\ (\varrho \mathbf{u}) \in C_{\text{weak}}([0, T]; L^{2\gamma/(\gamma+1)}(\Omega; R^3)), \\ \varrho |\mathbf{u}|^2 \in L^\infty(0, T; L^1(\Omega)), \end{array} \right\}$$

$\mathbf{u}|_{\partial\Omega} = 0$ in the case of no-slip (1.7), $\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega}$ in the case of complete slip (1.8)

$\vartheta > 0$ a.a. in $(0, T) \times \Omega$, $K(\vartheta) \in L^1((0, T) \times \Omega)$, $\varrho\vartheta \in L^2((0, T) \times \Omega)$.

- The equations (1.1-1.3) are replaced by the following integral identities:

$$\int_{\Omega} \varrho\varphi \, dx \Big|_{t=\tau_1}^{t=\tau_2} = \int_{\tau_1}^{\tau_2} \int_{\Omega} [\varrho\partial_t\varphi + \varrho\mathbf{u} \cdot \nabla_x\varphi] \, dx dt, \quad \varrho(0, \cdot) = \varrho_0, \quad (2.4)$$

for any $0 \leq \tau_1 < \tau_2 \leq T$ and any test function $\varphi \in C_c^\infty([0, T] \times \bar{\Omega})$;

$$\begin{aligned} \int_{\Omega} \varrho\mathbf{u} \cdot \varphi \, dx \Big|_{t=\tau_1}^{t=\tau_2} = \\ \int_{\tau_1}^{\tau_2} \int_{\Omega} [\varrho\mathbf{u} \cdot \partial_t\varphi + \varrho\mathbf{u} \otimes \mathbf{u} : \nabla_x\varphi + p(\varrho, \vartheta)\operatorname{div}_x\varphi - \mathbb{S}(\nabla_x\varphi) : \nabla_x\varphi] \, dx \, dt, \\ \varrho\mathbf{u}(0, \cdot) = \varrho_0\mathbf{u}_0, \end{aligned} \quad (2.5)$$

for any $0 \leq \tau_1 < \tau_2 \leq T$, and any test function $\varphi \in C_c^\infty([0, T] \times \Omega; \mathbb{R}^3)$ in the case of the no-slip (1.7), $\varphi \in C_c^\infty([0, T] \times \bar{\Omega}; \mathbb{R}^3)$, $\varphi \cdot \mathbf{n}|_{\partial\Omega} = 0$ in the case of the complete slip (1.8);

$$\begin{aligned} c_v \int_{\Omega} \varrho\vartheta\varphi \, dx \Big|_{t=\tau_1}^{t=\tau_2} \geq \int_{\tau_1}^{\tau_2} \int_{\Omega} [c_v(\varrho\vartheta\partial_t\varphi + \varrho\vartheta\mathbf{u} \cdot \nabla_x\varphi) - K(\vartheta)\Delta\varphi] \, dx \, dt \\ + \int_{\tau_1}^{\tau_2} \int_{\Omega} \left[\mathbb{S}(\nabla_x\mathbf{u}) : \nabla_x\mathbf{u}\varphi - \vartheta \frac{\partial_\vartheta p(\varrho, \vartheta)}{\partial\vartheta} \operatorname{div}_x\mathbf{u}\varphi \right] \, dx \, dt \end{aligned} \quad (2.6)$$

for a.a. $0 \leq \tau_1 < \tau_2 \leq T$ including $\tau_1 = 0$, where $\varrho\vartheta(0, \cdot) = \varrho_0\vartheta_0$, and for any test function $\varphi \in C_c^\infty([0, T] \times \bar{\Omega})$, $\varphi \geq 0$, $\nabla_x\varphi \cdot \mathbf{n}|_{\partial\Omega} = 0$;

$$\int_{\Omega} \left[\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) \right] (\tau, \cdot) \, dx \leq \int_{\Omega} \left[\frac{1}{2}\varrho_0|\mathbf{u}_0|^2 + \varrho_0 e(\varrho_0, \vartheta_0) \right] \, dx \text{ for a.a. } \tau \in [0, T]. \quad (2.7)$$

The *existence* of global-in-time weak solutions was established in the monograph [Fei04, Chapter 7, Theorem 7.1] under the hypotheses (2.2), (2.3) with

$$\gamma > 3, \quad \Omega \subset \mathbb{R}^3 \text{ -a bounded domain of class } C^{2+\nu}.$$

In addition, the temperature ϑ constructed in [Fei04] enjoys higher regularity outside the hypothetical vacuum zone, namely

$$\varrho\vartheta = \varrho\tilde{\vartheta}, \quad \text{where } \tilde{\vartheta}, \log(\tilde{\vartheta}) \in L^2(0, T; W^{1,2}(\Omega)). \quad (2.8)$$

Finally, the initial value of the temperature is attained in the following sense:

$$\left. \begin{aligned} \text{ess lim}_{t \rightarrow 0^+} \int_{\Omega} \varrho \vartheta(t, \cdot) \varphi \, dx &= \int_{\Omega} \varrho_0 \vartheta_0 \varphi \, dx, \\ \text{ess lim inf}_{t \rightarrow 0^+} \int_{\Omega} \varrho \vartheta^\alpha(t, \cdot) \varphi \, dx &\geq \int_{\Omega} \varrho_0 \vartheta_0^\alpha \varphi \, dx \end{aligned} \right\} \text{for any } 0 \leq \alpha < 1, \quad (2.9)$$

and any $\varphi \in C_c^\infty(\overline{\Omega})$, $\varphi \geq 0$,

see [Fei04, Chapter 4].

2.2 Main result

We are ready to state the main result of the present paper.

Theorem 2.1. *Let $\Omega \subset R^3$ be a bounded domain of class $C^{4+\nu}$. Suppose that p and κ are given by (2.2), (2.3), with*

$$\gamma > 3. \quad (2.10)$$

Let the initial data $[\varrho_0, \vartheta_0, \mathbf{u}_0]$ satisfy (1.10),

$$\varrho_0, \vartheta_0 \in W^{3,2}(\Omega), \quad \mathbf{u}_0 \in W^{3,2}(\Omega; R^3), \quad (2.11)$$

with the relevant compatibility conditions. Let $[\varrho, \vartheta, \mathbf{u}]$ be a weak solution of the Navier-Stokes-Fourier system (1.1 - 1.9) in $(0, T) \times \Omega$ satisfying (2.8) and enjoying the extra regularity (1.16).

Then $[\varrho, \vartheta, \mathbf{u}]$ is a regular (classical) solution belonging to the class

$$\varrho, \vartheta \in C([0, T]; W^{3,2}(\Omega)), \quad \mathbf{u} \in C([0, T]; W^{3,2}(\Omega; R^3)), \quad (2.12)$$

$$\vartheta \in L^2(0, T; W^{4,2}(\Omega)), \quad \mathbf{u} \in L^2(0, T; W^{4,2}(\Omega; R^3)), \quad (2.13)$$

$$\partial_t \vartheta \in L^2(0, T; W^{2,2}(\Omega)), \quad \partial_t \mathbf{u} \in L^2(0, T; W^{2,2}(\Omega; R^3)). \quad (2.14)$$

The rest of the paper is devoted to the proof of Theorem 2.1. We note that the local-in-time strong solutions to the Navier-Stokes-Fourier system in the class (2.12 - 2.14) were constructed by Valli [Val82a], [Val82b], see also Valli and Zajackowski [VZ86]. Global-in-time solutions with the data close to an equilibrium were obtained in the seminal papers by Matsumura and Nishida [MN83], [MN80]. Hoff [Hof12] considered a simplified barotropic system supplemented by the complete slip conditions in a slightly weaker framework than (2.12-2.14). Similar results for the full system were obtained by Cho et al. [CCK04]. Finally, in view of the standard parabolic regularity, it is easy to observe that all relevant derivatives of $[\varrho, \vartheta, \mathbf{u}]$ are

continuous in the open set $(0, T) \times \Omega$; whence the solution belonging to the regularity class (2.12-2.14) is classical smooth.

2.3 Additional regularity of weak solutions

We derive certain additional regularity properties of the weak solutions satisfying (1.16) that may be of independent interest.

2.3.1 Uniform bounds on the density

Following DiPerna and Lions [DL89], we say that $[\varrho, \mathbf{u}]$ is a *renormalized solution* of the continuity equation (1.1) if the integral identity

$$\int_{\Omega} b(\varrho)\varphi \, dx \Big|_{t=\tau_1}^{t=\tau_2} = \int_{\tau_1}^{\tau_2} \int_{\Omega} \left[b(\varrho)\partial_t\varphi + b(\varrho)\mathbf{u} \cdot \nabla_x\varphi + (b(\varrho) - b'(\varrho)\varrho)\operatorname{div}_x\mathbf{u}\varphi \right] \, dx \, dt \quad (2.15)$$

holds for any $0 \leq \tau_1 < \tau_2 \leq T$, any test function $\varphi \in C_c^\infty([0, T] \times \overline{\Omega})$, and any continuously differentiable function b with the derivative b' vanishing outside a compact set.

Lemma 2.1. *Let $[\varrho, \mathbf{u}]$,*

$$\begin{aligned} \varrho &\in L^\infty(0, T; L^1(\Omega)), \\ \mathbf{u} &\in L^2(0, T; W^{1,2}(\Omega; R^3)), \quad \mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0, \quad \operatorname{div}_x\mathbf{u} \in L^1(0, T; L^\infty(\Omega)), \end{aligned} \quad (2.16)$$

be a renormalized solution of the continuity equation (1.1) such that

$$b(\varrho)(0, \cdot) = b(\varrho_0), \quad 0 < \operatorname{ess\,inf}_{\Omega} \varrho_0 \leq \varrho_0 \leq \operatorname{ess\,sup}_{\Omega} \varrho_0 < \infty.$$

Then

$$\begin{aligned} \operatorname{ess\,inf}_{\Omega} \varrho_0 \exp\left(-\int_0^t \|\operatorname{div}_x\mathbf{u}\|_{L^\infty(\Omega)} \, ds\right) &\leq \varrho(t, \cdot) \\ &\leq \operatorname{ess\,sup}_{\Omega} \varrho_0 \exp\left(\int_0^t \|\operatorname{div}_x\mathbf{u}\|_{L^\infty(\Omega)} \, ds\right) \end{aligned} \quad (2.17)$$

for any $t \in [0, T]$.

Proof. Step 1:

Although the proof seems obvious at the level of uniform bounds, the fact that the equation of continuity is satisfied only in the renormalized form (2.15) requires certain effort. To begin, we take a spatially homogenous φ in (2.15) to deduce that

$$\int_{\Omega} b(\varrho)(\tau, \cdot) \, dx \leq \int_{\Omega} b(\varrho_0) \, dx + \int_0^\tau \int_{\Omega} |b(\varrho) - b'(\varrho)\varrho| |\operatorname{div}_x\mathbf{u}| \, dx \, dt \quad (2.18)$$

for a.a. $\tau \in (0, T)$. Moreover, as the velocity satisfies (2.16), the validity of (2.18) can be extended to any b with uniformly bounded derivative.

Step 2:

Now, we take

$$b_m(\varrho) = \begin{cases} \varrho^\alpha & \text{for } 0 \leq \varrho < m, \\ m^\alpha + \alpha m^{\alpha-1}(\varrho - m) & \text{for } \varrho \geq m. \end{cases}$$

It is easy to check that

$$|b_m(\varrho) - b'_m(\varrho)\varrho| \leq c\alpha b_m(\varrho)$$

for all $\varrho, \alpha, m > 1$, where the constant c is independent of m and α .

Thus, going back to (2.18), we may use the Gronwall's inequality to deduce

$$\int_{\Omega} b_m(\varrho)(\tau, \cdot) \, dx \leq \exp\left(c\alpha \int_0^\tau \|\operatorname{div}_x \mathbf{u}\|_{L^\infty(\Omega)} \, ds\right) \int_{\Omega} b_m(\varrho_0) \, dx;$$

whence, letting $m \rightarrow \infty$,

$$\|\varrho(\tau, \cdot)\|_{L^\alpha(\Omega)} \leq \exp\left(c \int_0^\tau \|\operatorname{div}_x \mathbf{u}\|_{L^\infty(\Omega)} \, ds\right) \|\varrho_0\|_{L^\alpha(\Omega)}.$$

Since the previous estimate holds for *any* $\alpha > 0$, we deduce that ϱ is uniformly bounded in $(0, T) \times \Omega$. Thus, in particular, $\varrho \in C([0, T]; L^1(\Omega))$ and the norm is bounded uniformly for all t .

To complete the proof, we make use of the regularizing procedure of DiPerna and Lions [DL89].

Step 3:

Using the regularization method of DiPerna and Lions [DL89], we deduce that

$$\begin{aligned} & \partial_t B(\varrho + M(t)) + \operatorname{div}_x \left[B(\varrho + M(t)) \mathbf{u} \right] \\ & + \left[B'(\varrho + M(t))(\varrho + M(t)) - B(\varrho + M(t)) \right] \operatorname{div}_x \mathbf{u} \\ & = B'(\varrho + M(t)) \left(\partial_t M(t) + M(t) \operatorname{div}_x \mathbf{u} \right) \end{aligned} \tag{2.19}$$

for any sublinear B , where M is a Lipschitz spatially homogeneous function of time.

Step 4:

Taking

$$M(t) = -\operatorname{ess\,sup}_{\Omega} \varrho_0 \exp\left(\int_0^t \|\operatorname{div}_x \mathbf{u}\|_{L^\infty(\Omega)} \, ds\right)$$

we observe that

$$\left(\partial_t M(t) + M(t) \operatorname{div}_x \mathbf{u} \right) \leq 0;$$

whence the choice $B(z) = z^+$ and integration (2.19) over Ω gives rise to the right inequality in (2.17). The left inequality can be deduced in a similar fashion. \square

It follows from the hypothesis (2.10) and the regularity of the weak solutions considered in Theorem 2.1 that $[\varrho, \mathbf{u}]$ is a renormalized solution of the equation of continuity (1.1). Consequently, under the hypotheses of Theorem 2.1, we have that the density ϱ complies with the bounds (2.17). We note that no extra regularity of $\partial\Omega$ was needed in the proof.

2.3.2 Renormalization of the thermal energy inequality

Multiplying equation (1.3) on $\chi'(\vartheta)$, $\chi' \geq 0$ and passing, formally, to the weak formulation, we obtain a renormalized modification of the thermal energy inequality (2.6):

$$\begin{aligned} & c_v \int_{\Omega} \varrho \chi(\vartheta) \varphi \, dx \Big|_{t=\tau_1}^{t=\tau_2} \\ & \geq \int_{\tau_1}^{\tau_2} \int_{\Omega} [c_v (\varrho \chi(\vartheta) \partial_t \varphi + \varrho \chi(\vartheta) \mathbf{u} \cdot \nabla_x \varphi) + \chi'(\vartheta) \kappa(\vartheta) \nabla_x \vartheta \nabla_x \varphi] \, dx \, dt \\ & + \int_{\tau_1}^{\tau_2} \int_{\Omega} [\chi'(\vartheta) \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \varphi - \kappa(\vartheta) \chi''(\vartheta) |\nabla_x \vartheta|^2 \varphi - \varrho \vartheta \chi'(\vartheta) \operatorname{div}_x \mathbf{u} \varphi] \, dx \, dt \end{aligned} \quad (2.20)$$

for any $\varphi \in C_c^\infty([0, T] \times \overline{\Omega})$, $\varphi \geq 0$.

Lemma 2.2. *Suppose that*

$$\left\{ \begin{array}{l} \mathbf{u} \in L^2(0, T; W^{1,2}(\Omega; \mathbb{R}^3)) \cap L^\infty((0, T) \times \Omega; \mathbb{R}^3), \\ \mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0, \operatorname{div}_x \mathbf{u} \in L^1(0, T; L^\infty(\Omega)), \end{array} \right\} \quad (2.21)$$

and

$$\varrho, \vartheta \in L^\infty((0, T) \times \Omega), \operatorname{ess\,inf} \varrho > 0, \vartheta \in L^2(0, T; W^{1,2}(\Omega)) \quad (2.22)$$

satisfy the weak form of the continuity equation (2.4), together with the thermal energy inequality (2.6).

Then the renormalized thermal energy inequality (2.20) holds for any continuously differentiable function χ .

Proof. To begin, we note that, in accordance with the hypothesis (2.22), any composition $b(\vartheta)$ with a continuously differentiable b belongs to the class

$$b(\vartheta) \in L^\infty((0, T) \times \Omega) \cap L^2(0, T; W^{1,2}(\Omega)).$$

Next, we introduce the space $W_0^{-1,p}(\Omega)$ as the dual to $W^{1,p'}(\Omega)$, $\frac{1}{p} + \frac{1}{p'} = 1$.

Step 1:

It follows from the equation of continuity (2.4) and the hypotheses (2.21), (2.22) that

$$\partial_t \varrho \in L^\infty(0, T; W_0^{-1,p}(\Omega)) \text{ for any } 1 < p < \infty, \quad (2.23)$$

in particular, the density ϱ itself can be interpreted as a Lipschitz mapping of t ranging in the Banach space $W_0^{-1,p}(\Omega)$,

$$\begin{aligned} & \|\varrho(t, \cdot) - \varrho(s, \cdot)\|_{W_0^{-1,p}(\Omega)} \\ &= \sup_{\Phi \in W^{1,p'}(\Omega)} \int_{\Omega} \left(\varrho(t, \cdot) - \varrho(s, \cdot) \right) \Phi \, dx \leq L|t - s|, \quad t, s \in [0, T]. \end{aligned} \quad (2.24)$$

Step 2:

In accordance with the variational inequality (2.6), we may extend

$$\varrho\vartheta(\tau, \cdot) = \operatorname{ess\,lim}_{t \rightarrow 0^+} \varrho\vartheta(t, \cdot) \text{ for all } \tau \leq 0, \quad \varrho\vartheta(\tau, \cdot) = \operatorname{ess\,lim}_{t \rightarrow T^-} \varrho\vartheta(t, \cdot) \text{ for all } \tau \geq 0, \quad (2.25)$$

setting, finally,

$$\vartheta(\tau, \cdot) = \begin{cases} \varrho\vartheta/\varrho_0 & \text{for } \tau \leq 0, \\ \varrho\vartheta/\varrho(T, \cdot) & \text{for } \tau \geq T. \end{cases}$$

Here, relation (2.25) is understood in the weak sense. We note that, thanks to the regularity properties (2.21), (2.22), and the fact that ϱ, \mathbf{u} satisfy the renormalized equation of continuity, the mapping $t \mapsto \varrho(t, \cdot)$ is (strongly) continuous in $L^p(\Omega)$ for any finite p .

Accordingly, we define a regularization $[v]_\delta$ by

$$[v]_\delta(t, \cdot) = h_\delta * v = \int_{-\infty}^{\infty} h_\delta(t-s)v(s, \cdot) \, ds,$$

where $\{h_\delta = h_\delta(t)\}_{\delta>0}$ is a standard family of regularizing kernels in the time variable supported in a δ -neighborhood of zero.

Step 3:

We take the quantity $[\varphi\chi'([\vartheta]_\delta)]_\delta$, $\varphi \in C_c^\infty((0, T) \times \bar{\Omega})$, $\varphi, \chi' \geq 0$, as a test function in (2.6) to obtain

$$\begin{aligned} & - \int_0^T \int_{\Omega} c_v \left(\varrho\vartheta \partial_t [\varphi\chi'([\vartheta]_\delta)]_\delta + \varrho\vartheta \mathbf{u} \cdot \nabla_x [\varphi\chi'([\vartheta]_\delta)]_\delta \right) \, dx \, dt \\ & \quad - \int_0^T \int_{\Omega} \kappa(\vartheta) \nabla_x \vartheta \cdot \nabla_x [\varphi\chi'([\vartheta]_\delta)]_\delta \, dx \, dt \\ & \geq \int_0^T \int_{\Omega} \left[\mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} [\varphi\chi'([\vartheta]_\delta)]_\delta - \varrho\vartheta \operatorname{div}_x \mathbf{u} [\varphi\chi'([\vartheta]_\delta)]_\delta \right] \, dx \, dt \end{aligned} \quad (2.26)$$

as soon as δ is small enough. Note that this step can be fully justified by means of a density argument.

Thus, letting $\delta \rightarrow 0$ in (2.26), we deduce

$$\begin{aligned}
& - \lim_{\delta \rightarrow 0} \int_0^T \int_{\Omega} c_v \varrho \vartheta \partial_t [\varphi \chi'([\vartheta]_{\delta})]_{\delta} \, dx \, dt \\
& - \int_0^T \int_{\Omega} [c_v \varrho \vartheta \mathbf{u} \cdot \nabla_x (\varphi \chi'(\vartheta)) + \kappa(\vartheta) \chi'(\vartheta) \nabla_x \vartheta \cdot \nabla_x \varphi] \, dx \, dt \\
& \geq \int_0^T \int_{\Omega} \left[\mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \chi'(\vartheta) \varphi - \kappa(\vartheta) \chi''(\vartheta) |\nabla_x \vartheta|^2 \varphi - \varrho \vartheta \operatorname{div}_x \mathbf{u} \chi'(\vartheta) \varphi \right] \, dx \, dt.
\end{aligned} \tag{2.27}$$

Step 4:

It remains to identify the limit

$$\begin{aligned}
\lim_{\delta \rightarrow 0} \int_0^T \int_{\Omega} \varrho \vartheta \partial_t [\varphi \chi'([\vartheta]_{\delta})]_{\delta} \, dx \, dt &= \lim_{\delta \rightarrow 0} \int_0^T \int_{\Omega} [\varrho \vartheta]_{\delta} \partial_t (\varphi \chi'([\vartheta]_{\delta})) \, dx \, dt \\
&= - \lim_{\delta \rightarrow 0} \int_0^T \int_{\Omega} \partial_t [\varrho \vartheta]_{\delta} \chi'([\vartheta]_{\delta}) \varphi \, dx \, dt.
\end{aligned}$$

To this end, we estimate the commutator

$$\partial_t [\varrho \vartheta]_{\delta} - \partial_t (\varrho [\vartheta]_{\delta}).$$

Assume, for a moment, that we can show

$$\partial_t [\varrho \vartheta]_{\delta} - \partial_t (\varrho [\vartheta]_{\delta}) \rightarrow 0 \text{ in } L^2(0, T; W_0^{-1,2}(\Omega)). \tag{2.28}$$

Taking (2.28) for granted and with (2.23) in mind, we deduce

$$\begin{aligned}
\lim_{\delta \rightarrow 0} \int_0^T \int_{\Omega} \partial_t [\varrho \vartheta]_{\delta} \chi'([\vartheta]_{\delta}) \varphi \, dx \, dt &= \lim_{\delta \rightarrow 0} \int_0^T \int_{\Omega} \partial_t (\varrho [\vartheta]_{\delta}) \chi'([\vartheta]_{\delta}) \varphi \, dx \, dt \\
&= \int_0^T \int_{\Omega} \varrho \mathbf{u} \cdot \nabla_x (\vartheta \chi'(\vartheta) \varphi) \, dx \, dt - \int_0^T \int_{\Omega} \varrho \mathbf{u} \cdot \nabla_x (\chi(\vartheta) \varphi) \, dx \, dt \\
&= \int_0^T \int_{\Omega} \varrho \mathbf{u} \cdot \nabla_x (\vartheta \chi'(\vartheta) \varphi) \, dx + \int_0^T \int_{\Omega} \varrho \chi(\vartheta) \partial_t \varphi \, dx \, dt \\
&\quad + \lim_{\delta \rightarrow 0} \int_0^T \int_{\Omega} \int_{\Omega} \varrho \partial_t (\chi([\vartheta]_{\delta}) \varphi) \, dx \, dt \\
&= \int_0^T \int_{\Omega} \varrho \mathbf{u} \cdot \nabla_x (\vartheta \chi'(\vartheta) \varphi) \, dx + \int_0^T \int_{\Omega} \varrho \chi(\vartheta) \partial_t \varphi \, dx \, dt
\end{aligned}$$

$$- \int_0^T \int_0^T \int_{\Omega} \varrho \mathbf{u} \cdot \nabla_x (\chi(\vartheta)\varphi) \, dx \, dt,$$

which, combined with (2.27), yields the desired conclusion (2.20), at least for a.a. $0 < \tau_1 < \tau_2 < T$. Finally, under present circumstances, the relations (2.9) imply *strong* continuity of the temperature at the time $t = 0$; whence $\tau_1 = 0$ may be included by standard arguments.

Step 5:

To conclude the proof, we have to show (2.28). To this end, we employ the arguments similar to those for the so-called Friedrichs lemma. Note that similar strategy was employed by Frehse et al. [FMR10] in a slightly different context. Seeing that (2.28) obviously holds for a dense set of (smooth) functions ϑ in $L^2(0, T; W^{1,2}(\Omega))$, it is enough to establish the estimate

$$\|\partial_t [\varrho\vartheta]_{\delta} - \partial_t (\varrho[\vartheta]_{\delta})\|_{L^2(0,T;W_0^{-1,2}(\Omega))} \leq c(\varrho, \mathbf{u}) \|\vartheta\|_{L^2(0,T;W^{1,2}(\Omega))}. \quad (2.29)$$

We write

$$\begin{aligned} & \int_0^T \int_{\Omega} \left(\partial_t [\varrho\vartheta]_{\delta} - \partial_t (\varrho[\vartheta]_{\delta}) \right) \Phi(x) \, dx \, dt \\ &= \int_0^T \int_{\Omega} \Phi(x) \int_{-\infty}^{\infty} \left(\varrho(s, x) - \varrho(t, x) \right) \vartheta(s, x) h'_{\delta}(t-s) \, ds \, dx \, dt \\ & \quad - \int_0^T \int_{\Omega} \varrho \mathbf{u} \cdot \nabla_x ([\vartheta]_{\delta} \Phi) \, dx \, dt. \end{aligned}$$

Using (2.23) and taking the supremum of the last integral over Φ belonging to a unit ball in $W^{1,2}(\Omega)$ we observe that this term complies with (2.29). Finally, writing

$$\begin{aligned} & \int_0^T \int_{\Omega} \Phi(x) \int_{-\infty}^{\infty} \left(\varrho(s, x) - \varrho(t, x) \right) \vartheta(s, x) h'_{\delta}(t-s) \, ds \, dx \, dt \\ &= \int_0^T \int_{\Omega} \Phi(x) \int_{-\infty}^{\infty} \frac{\varrho(s, x) - \varrho(t, x)}{s-t} \vartheta(s, x) (s-t) h'_{\delta}(t-s) \, ds \, dx \, dt \end{aligned}$$

we use (2.24) to conclude the proof of (2.29). □

We remark that, similarly to Lemma 2.1, no regularity properties of $\partial\Omega$ were needed in the proof.

3 Entropy and relative energy inequality

We start by deriving the standard entropy inequality from (2.20). To this end, we first show that, under the hypotheses of Theorem 2.1, the temperature remains bounded below away from zero.

Lemma 3.1. *Let $[\varrho, \vartheta, \mathbf{u}]$ be a weak solution of the Navier-Stokes-Fourier system in $(0, T) \times \Omega$ belonging to the class (1.16). Suppose that*

$$\operatorname{ess\,inf}_{\Omega} \vartheta_0 > 0.$$

Then there exists a constant $\underline{\vartheta} > 0$, depending only on the quantities $\bar{\vartheta}$, L from (1.16) and on T , such that

$$\vartheta(t, \cdot) \geq \underline{\vartheta} > 0 \text{ for a.a. } t \in [0, T]. \quad (3.1)$$

Proof. As shown in Lemma 2.2, the solution $[\varrho, \vartheta, \mathbf{u}]$ satisfies the renormalized thermal energy balance (2.20). Taking

$$\chi_{\delta}(\vartheta) = -\frac{1}{(\vartheta + \delta)^{\alpha}}, \quad \alpha > 1, \quad \delta > 0$$

in (2.20), we deduce that

$$\int_{\Omega} \frac{1}{(\vartheta + \delta)^{\alpha}}(\tau, \cdot) \, dx \leq \int_{\Omega} \frac{1}{(\vartheta_0 + \delta)^{\alpha}} \, dx + \alpha \bar{\varrho} \bar{\vartheta} \int_0^{\tau} \int_{\Omega} |\operatorname{div}_x \mathbf{u}| \frac{1}{(\vartheta + \delta)^{\alpha}} \, dx \, dt$$

for a.a. $\tau \in [0, T]$.

Consequently, applying Gronwall's lemma and letting $\delta \rightarrow 0$, we obtain

$$\sup_{\alpha > 1} \left\{ \operatorname{ess\,sup}_{\tau \in (0, T)} \left\| \frac{1}{\vartheta}(\tau, \cdot) \right\|_{L^{\alpha}(\Omega)} \right\} < \infty,$$

which yields the desired lower bound for ϑ . □

In view of the results obtained in Lemmas 2.1 - 3.1 we may infer that, under the hypotheses of Theorem 2.1,

$$0 < \underline{\varrho} \leq \varrho(t, x) \leq \bar{\varrho}, \quad 0 < \underline{\vartheta} \leq \vartheta(t, x) \leq \bar{\vartheta} \text{ for a.a. } (t, x) \in (0, T) \times \Omega. \quad (3.2)$$

3.1 Entropy inequality

The temperature being bounded below, we can take $\chi(\vartheta) = \log(\vartheta)$ in (2.20) to obtain the *entropy inequality*

$$\begin{aligned} & \int_{\Omega} \varrho s(\varrho, \vartheta) \varphi \, dx \Big|_{t=\tau_1}^{t=\tau_2} \\ & \geq \int_{\tau_1}^{\tau_2} \int_{\Omega} \left[(\varrho s(\varrho, \vartheta)) \partial_t \varphi + \varrho s(\varrho, \vartheta) \mathbf{u} \cdot \nabla_x \varphi + \frac{\kappa(\vartheta)}{\vartheta} \nabla_x \vartheta \cdot \nabla_x \varphi \right] \, dx \, dt \\ & \quad + \int_{\tau_1}^{\tau_2} \int_{\Omega} \frac{1}{\vartheta} \left[\mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \varphi + \frac{\kappa(\vartheta)}{\vartheta} |\nabla_x \vartheta|^2 \varphi \right] \, dx \, dt \end{aligned} \quad (3.3)$$

for any $\varphi \in C_c^{\infty}([0, T] \times \bar{\Omega})$, $\varphi \geq 0$, where

$$s(\varrho, \vartheta) = c_v \log(\vartheta) - \log(\varrho).$$

3.2 Relative energy (entropy)

Having collected all the necessary tools we are able to use the technique of relative entropies adapted to the Navier-Stokes-Fourier system as in [FN12]. We introduce the *relative energy functional*

$$\mathcal{E} \left(\varrho, \vartheta, \mathbf{u} \mid r, \Theta, \mathbf{U} \right) = \int_{\Omega} \left[\frac{1}{2} \varrho |\mathbf{u} - \mathbf{U}|^2 + H_{\Theta}(\varrho, \vartheta) - \frac{\partial H_{\Theta}(r, \Theta)}{\partial \vartheta} - H_{\Theta}(r, \Theta) \right] dx,$$

where

$$H_{\Theta}(\varrho, \vartheta) = \varrho e(\varrho, \vartheta) - \varrho \Theta s(\varrho, \vartheta) = c_v \varrho \vartheta + \varrho P(\varrho) - \varrho \Theta \log \left(\frac{\vartheta^{c_v}}{\varrho} \right).$$

Now, exactly as in [FN12] we may deduce the *relative energy inequality* in the form:

$$\begin{aligned} & \left[\mathcal{E} \left(\varrho, \vartheta, \mathbf{u} \mid r, \Theta, \mathbf{U} \right) \right]_{t=0}^{t=\tau} + \int_0^{\tau} \int_{\Omega} \frac{\Theta}{\vartheta} \left(\mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} + \frac{\kappa(\vartheta) |\nabla_x \vartheta|^2}{\vartheta} \right) dx dt \\ & \leq \int_0^{\tau} \int_{\Omega} \left(\varrho (\mathbf{U} - \mathbf{u}) \cdot \partial_t \mathbf{U} + \varrho (\mathbf{U} - \mathbf{u}) \otimes \mathbf{u} : \nabla_x \mathbf{U} - p(\varrho, \vartheta) \operatorname{div}_x \mathbf{U} + \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{U} \right) dx dt \\ & \quad - \int_0^{\tau} \int_{\Omega} \left(\varrho (s(\varrho, \vartheta) - s(r, \Theta)) \partial_t \Theta + \varrho (s(\varrho, \vartheta) - s(r, \Theta)) \mathbf{u} \cdot \nabla_x \Theta \right) dx dt \\ & \quad + \int_0^{\tau} \int_{\Omega} \frac{\kappa(\vartheta) \nabla_x \vartheta}{\vartheta} \cdot \nabla_x \Theta dx dt \\ & \quad + \int_0^{\tau} \int_{\Omega} \left(\left(1 - \frac{\varrho}{r} \right) \partial_t p(r, \Theta) - \frac{\varrho}{r} \mathbf{u} \cdot \nabla_x p(r, \Theta) \right) dx dt \end{aligned} \quad (3.4)$$

for any trio of sufficiently regular test functions $[r, \Theta, \mathbf{U}]$, where \mathbf{U} satisfies the same boundary conditions as \mathbf{u} . Note that, in contrast with [FN12], the weak solution $[\varrho, \vartheta, \mathbf{u}]$ satisfying the hypotheses of Theorem 2.1 is already quite regular, in particular bounded, so that the class of test functions can be extended considerably. As a matter of fact, we only need $[r, \Theta, \mathbf{U}]$ to be bounded with first derivatives square integrable.

3.3 Weak-strong uniqueness

Following further [FN12], we suppose that $r = \tilde{\varrho}$, $\vartheta = \tilde{\vartheta}$, $\mathbf{U} = \tilde{\mathbf{u}}$ is a strong solution of the Navier-Stokes-Fourier system emanating from the same initial data. After some manipulation, making use of the bounds already obtained, we arrive at

$$\begin{aligned} & \left[\mathcal{E} \left(\varrho, \vartheta, \mathbf{u} \mid \tilde{\varrho}, \tilde{\vartheta}, \tilde{\vartheta} \right) \right]_{t=0}^{t=\tau} \\ & + \int_0^{\tau} \int_{\Omega} \left(\frac{\tilde{\vartheta}}{\vartheta} - 1 \right) \left(\mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} + \frac{\kappa(\vartheta) |\nabla_x \vartheta|^2}{\vartheta} \right) dx dt \\ & + \int_0^{\tau} \int_{\Omega} \left(\mathbb{S}(\nabla_x \mathbf{u}) - \mathbb{S}(\nabla_x \tilde{\mathbf{u}}) \right) : \left(\nabla_x \mathbf{u} - \nabla_x \tilde{\mathbf{u}} \right) dx dt \end{aligned} \quad (3.5)$$

$$\begin{aligned}
& + \int_0^\tau \int_\Omega \frac{\kappa(\vartheta) \nabla_x \vartheta}{\vartheta} \cdot (\nabla_x \vartheta - \nabla_x \tilde{\vartheta}) \, dx \, dt \\
\leq & c \int_0^\tau \left[\|\partial_t \tilde{\vartheta}\|_{L^\infty(\Omega)} + \|\partial_t \tilde{\mathbf{u}}\|_{L^\infty(\Omega; \mathbb{R}^3)} + \|\nabla_x \tilde{\vartheta}\|_{L^\infty(\Omega; \mathbb{R}^3)} \right. \\
& \left. + \|\nabla_x \tilde{\varrho}\|_{L^\infty(\Omega; \mathbb{R}^3)} + \|\nabla_x \tilde{\mathbf{u}}\|_{L^\infty(\Omega; \mathbb{R}^{3 \times 3})} \right] \times \\
& \times \left[\|\varrho - \tilde{\varrho}\|_{L^2(\Omega)}^2 + \|\vartheta - \tilde{\vartheta}\|_{L^2(\Omega)}^2 + \|\mathbf{u} - \tilde{\mathbf{u}}\|_{L^2(\Omega; \mathbb{R}^3)}^2 \right] \, dt \\
& + \int_0^\tau \int_\Omega \tilde{\varrho} (s(\tilde{\varrho}, \tilde{\vartheta}) - s(\varrho, \vartheta)) (\partial_t \tilde{\vartheta} + \tilde{\mathbf{u}} \cdot \nabla_x \tilde{\vartheta}) \, dx \, dt \\
& + \int_0^\tau \int_\Omega (p(\varrho, \vartheta) - p(\tilde{\varrho}, \tilde{\vartheta})) \operatorname{div}_x \tilde{\mathbf{u}} \, dx \, dt \\
& + \int_0^\tau \int_\Omega \left(1 - \frac{\varrho}{\tilde{\varrho}}\right) (\partial_t p(\tilde{\varrho}, \tilde{\vartheta}) + \tilde{\mathbf{u}} \cdot \nabla_x p(\tilde{\varrho}, \tilde{\vartheta})) \, dx \, dt
\end{aligned}$$

where the constant depends only on the L^∞ -norms of both the strong and the weak solution considered.

Next, we observe that the above inequality keeps the same form if we replace

$$\begin{aligned}
s(\varrho, \vartheta) - s(\tilde{\varrho}, \tilde{\vartheta}) & \approx \frac{\partial s(\tilde{\varrho}, \tilde{\vartheta})}{\partial \varrho} (\varrho - \tilde{\varrho}) + \frac{\partial s(\tilde{\varrho}, \tilde{\vartheta})}{\partial \vartheta} (\vartheta - \tilde{\vartheta}), \\
p(\varrho, \vartheta) - p(\tilde{\varrho}, \tilde{\vartheta}) & \approx \frac{\partial p(\tilde{\varrho}, \tilde{\vartheta})}{\partial \varrho} (\varrho - \tilde{\varrho}) + \frac{\partial p(\tilde{\varrho}, \tilde{\vartheta})}{\partial \vartheta} (\vartheta - \tilde{\vartheta}).
\end{aligned}$$

Using the fact that $[\tilde{\varrho}, \tilde{\vartheta}, \tilde{\mathbf{u}}]$ solves the equations we can reduce (3.5) to the following inequality:

$$\begin{aligned}
& \left[\mathcal{E} \left(\varrho, \vartheta, \mathbf{u} \mid \tilde{\varrho}, \tilde{\vartheta}, \tilde{\mathbf{u}} \right) \right]_{t=0}^{t=\tau} \tag{3.6} \\
& + \int_0^\tau \int_\Omega \left(\frac{\tilde{\vartheta}}{\vartheta} - 1 \right) \left(\mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} + \frac{\kappa(\vartheta) |\nabla_x \vartheta|^2}{\vartheta} \right) \, dx \, dt \\
& + \int_0^\tau \int_\Omega \left(\frac{\vartheta}{\tilde{\vartheta}} - 1 \right) \left(\mathbb{S}(\nabla_x \tilde{\mathbf{u}}) : \nabla_x \tilde{\mathbf{u}} + \frac{\kappa(\tilde{\vartheta}) |\nabla_x \tilde{\vartheta}|^2}{\tilde{\vartheta}} \right) \, dx \, dt \\
& + \int_0^\tau \int_\Omega \left(\mathbb{S}(\nabla_x \mathbf{u}) - \mathbb{S}(\nabla_x \tilde{\mathbf{u}}) \right) : (\nabla_x \mathbf{u} - \nabla_x \tilde{\mathbf{u}}) \, dx \, dt \\
& + \int_0^\tau \int_\Omega \left(\frac{\kappa(\vartheta) \nabla_x \vartheta}{\vartheta} - \frac{\kappa(\tilde{\vartheta}) \nabla_x \tilde{\vartheta}}{\tilde{\vartheta}} \right) \cdot (\nabla_x \vartheta - \nabla_x \tilde{\vartheta}) \, dx \, dt
\end{aligned}$$

$$\begin{aligned}
&\leq c \int_0^\tau \left[\left\| \partial_t \tilde{\vartheta} \right\|_{L^\infty(\Omega)} + \left\| \partial_t \tilde{\mathbf{u}} \right\|_{L^\infty(\Omega; R^3)} + \left\| \nabla_x \tilde{\vartheta} \right\|_{L^\infty(\Omega; R^3)} \right. \\
&\quad \left. + \left\| \nabla_x \tilde{\varrho} \right\|_{L^\infty(\Omega; R^3)} + \left\| \nabla_x \tilde{\mathbf{u}} \right\|_{L^\infty(\Omega; R^{3 \times 3})} \right] \times \\
&\quad \times \left[\left\| \varrho - \tilde{\varrho} \right\|_{L^2(\Omega)}^2 + \left\| \vartheta - \tilde{\vartheta} \right\|_{L^2(\Omega)}^2 + \left\| \mathbf{u} - \tilde{\mathbf{u}} \right\|_{L^2(\Omega; R^3)}^2 \right] dt.
\end{aligned}$$

Finally, eliminating the terms with transport coefficients on the left hand side of (3.6) in the same way as in [FNS14, Section 6], we may infer that

$$\begin{aligned}
&\left[\mathcal{E} \left(\varrho, \vartheta, \mathbf{u} \mid \tilde{\varrho}, \tilde{\vartheta}, \tilde{\mathbf{u}} \right) \right]_{t=0}^{t=\tau} \tag{3.7} \\
&\leq c \int_0^\tau \left[\left\| \partial_t \tilde{\vartheta} \right\|_{L^\infty(\Omega)} + \left\| \partial_t \tilde{\mathbf{u}} \right\|_{L^\infty(\Omega; R^3)} + \left\| \nabla_x \tilde{\vartheta} \right\|_{L^\infty(\Omega; R^3)} \right. \\
&\quad \left. + \left\| \nabla_x \tilde{\varrho} \right\|_{L^\infty(\Omega; R^3)} + \left\| \nabla_x \tilde{\mathbf{u}} \right\|_{L^\infty(\Omega; R^{3 \times 3})} \right] \mathcal{E} \left(\varrho, \vartheta, \mathbf{u} \mid \tilde{\varrho}, \tilde{\vartheta}, \tilde{\mathbf{u}} \right) dt.
\end{aligned}$$

Applying Gronwall's lemma we obtain the desired conclusion $\varrho = \tilde{\varrho}$, $\vartheta = \tilde{\vartheta}$, $\mathbf{u} = \tilde{\mathbf{u}}$.

We have proved the following result:

Lemma 3.2. *Under the hypotheses of Theorem 2.1, let $[\varrho, \vartheta, \mathbf{u}]$ be a weak solution enjoying the extra regularity properties (1.16), and $[\tilde{\varrho}, \tilde{\vartheta}, \tilde{\mathbf{u}}]$ a strong solution to the Navier-Stokes-Fourier system belonging to the class (2.12 - 2.14) and such that*

$$\varrho(0, \cdot) = \tilde{\varrho}(0, \cdot), \quad \vartheta(0, \cdot) = \tilde{\vartheta}(0, \cdot), \quad \mathbf{u}(0, \cdot) = \tilde{\mathbf{u}}(0, \cdot).$$

Then

$$\varrho = \tilde{\varrho}, \quad \vartheta = \tilde{\vartheta}, \quad \mathbf{u} = \tilde{\mathbf{u}}$$

on their joint interval of existence.

Remark 3.1. *Lemma 3.2 is not a standard weak-strong uniqueness result in the spirit of [FNS14] as the weak solution is required to satisfy (1.16). As we shall see below such a weak solution is in fact strong as claimed in Theorem 2.1.*

Remark 3.2. *As a matter of fact, the relative entropy inequality (3.7) requires very mild assumptions concerning the strong solution $[\tilde{\varrho}, \tilde{\vartheta}, \tilde{\mathbf{u}}]$, namely*

$$\left\{ \begin{array}{l} \tilde{\varrho}, \tilde{\vartheta} \in L^\infty((0, T) \times \Omega), \nabla_x \tilde{\varrho}, \nabla_x \tilde{\vartheta} \in L^1(0, T; L^\infty(\Omega; R^3)) \cap L^2((0, T) \times \Omega; R^3), \\ \partial_t \tilde{\vartheta} \in L^1(0, T; L^\infty(\Omega)) \end{array} \right\} \tag{3.8}$$

$$\left\{ \begin{array}{l} \tilde{\mathbf{u}} \in L^\infty((0, T) \times \Omega; R^3), \partial_t \tilde{\mathbf{u}} \in L^1(0, T; L^\infty(\Omega; R^3)), \\ \nabla_x \tilde{\mathbf{u}} \in L^1(0, T; L^\infty(\Omega; R^{3 \times 3})) \cap L^2((0, T) \times \Omega; R^{3 \times 3}). \end{array} \right\} \tag{3.9}$$

4 Conditional regularity of smooth solutions

Our ultimate goal is to show that the weak solution $[\varrho, \vartheta, \mathbf{u}]$ enjoys the regularity claimed in (2.12 - 2.14). In view of the weak-strong uniqueness result established in Lemma 3.2, it is enough to work with the (local-in-time) strong solution of the same problem, the existence of which was established by Valli [Val82b], and Valli and Zajackowski [VZ86]. We follow step by step the arguments of [FNS14], performing the necessary modifications using the ideas of [SWZ11], and Wen and Zhu [WZ13]. In what follows, $[\varrho, \vartheta, \mathbf{u}]$ will denote the local-in-time strong solution emanating from the regular initial data (2.11).

We claim the following blow-up criterion that may be of independent interest.

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class $C^{4+\nu}$, and let $[\varrho, \vartheta, \mathbf{u}]$ be a strong solution of the Navier-Stokes-Fourier system (1.1 - 1.9) in $(0, T) \times \Omega$, $0 < T < \bar{T}$ satisfying (1.17) in $(0, T)$.*

Then $[\varrho, \vartheta, \mathbf{u}]$ can be extended beyond T . More specifically, the solution can be extended to a time interval $[0, T + \delta)$, where $\delta > 0$ depends only on the constants $\bar{\varrho}, \bar{\vartheta}, \bar{\mathbf{u}}$, the initial data $\varrho_0, \vartheta_0, \mathbf{u}_0$, and on \bar{T} .

Obviously, Theorem 4.1, together with Lemma 3.2, complete the proof of Theorem 2.1. The remaining part of this section is devoted to the proof of Theorem 4.1. Our goal is to show that (1.17) forces the strong solution to remain bounded in the spaces specified through (2.12 - 2.14). For definiteness, we focus on the case of the no-slip boundary conditions (1.6), (1.7) noticing that the complete slip conditions (1.6), (1.8) can be treated in a similar way by obvious modifications.

To begin, we observe that smooth solutions to the Navier-Stokes-Fourier system satisfy the total energy balance

$$\partial_t(\varrho E) + \operatorname{div}_x(\varrho \mathbf{u} E) + \operatorname{div}_x(p(\varrho, \vartheta) \mathbf{u}) - \operatorname{div}_x(\mathbb{S}(\nabla_x \mathbf{u}) \mathbf{u}) + \operatorname{div}_x \mathbf{q} = 0, \quad (4.1)$$

where

$$E = \frac{1}{2} |\mathbf{u}|^2 + c_v \vartheta + \frac{1}{\gamma - 1} \varrho^{\gamma-1}.$$

4.1 First energy bounds for the velocity and temperature

We claim that the hypothesis (1.17) implies

$$\|\nabla_x \mathbf{u}\|_{L^2((0, T) \times \Omega; \mathbb{R}^{3 \times 3})} + \|\nabla_x \vartheta\|_{L^2((0, T) \times \Omega; \mathbb{R}^3)} \leq c, \quad (4.2)$$

where c denotes a generic constant depending only on the parameters $\bar{\varrho}, \bar{\vartheta}, \bar{\mathbf{u}}$, the initial data $\varrho_0, \vartheta_0, \mathbf{u}_0$, and the time T as the case may be. Indeed the estimate (4.2) follows by multiplying (1.3) by 1 and ϑ respectively and integrating by parts over the space-time cylinder $(0, T) \times \Omega$.

4.2 Decomposition of \mathbf{u} and elliptic estimates

For every fixed $t \in [0, T]$ we solve the following Lamé system for \mathbf{v}

$$\operatorname{div}_x \mathbb{S}(\nabla_x \mathbf{v}) = \nabla_x p(\varrho, \vartheta) \text{ in } \Omega, \quad \mathbf{v}|_{\partial\Omega} = 0. \quad (4.3)$$

According to the elliptic estimates and the hypothesis (1.17), we have

$$\left\{ \begin{array}{l} \|\mathbf{v}\|_{W^{1,q}(\Omega; \mathbb{R}^3)} \leq c(q) \|p\|_{L^q(\Omega)} \leq c, \\ \|\mathbf{v}\|_{W^{2,q}(\Omega; \mathbb{R}^3)} \leq \|p\|_{W^{1,q}(\Omega)} \leq c(1 + \|\nabla_x \varrho\|_{L^q(\Omega; \mathbb{R}^3)} + \|\nabla_x \vartheta\|_{L^q(\Omega; \mathbb{R}^3)}) \end{array} \right\} \quad (4.4)$$

for any $1 < q < \infty$.

We then define

$$\mathbf{w} = \mathbf{u} - \mathbf{v},$$

the so-called *effective viscous flux*, satisfying

$$\operatorname{div}_x \mathbb{S}(\nabla_x \mathbf{w}) = \varrho \partial_t \mathbf{u} + \varrho \mathbf{u} \cdot \nabla_x \mathbf{u}, \quad \mathbf{w}|_{\partial\Omega} = 0. \quad (4.5)$$

Again by elliptic estimates we have

$$\begin{aligned} \|\mathbf{w}\|_{W^{2,2}(\Omega; \mathbb{R}^3)} &\leq c \|\varrho \partial_t \mathbf{u} + \varrho \mathbf{u} \cdot \nabla_x \mathbf{u}\|_{L^2(\Omega; \mathbb{R}^3)} \\ &\leq c \left(\|\sqrt{\varrho} \partial_t \mathbf{u}\|_{L^2(\Omega; \mathbb{R}^3)} + \|\nabla_x \mathbf{u}\|_{L^2(\Omega; \mathbb{R}^{3 \times 3})} \right). \end{aligned} \quad (4.6)$$

Finally, we note that $\partial_t \mathbf{v}$ satisfies

$$\operatorname{div}_x \mathbb{S}(\nabla_x \partial_t \mathbf{v}) = \nabla_x \partial_t p, \quad \partial_t \mathbf{v}|_{\partial\Omega} = 0. \quad (4.7)$$

4.3 Second energy bounds for the velocity

Rewrite the momentum equation (1.2) in the form

$$\varrho \partial_t \mathbf{u} + \operatorname{div}_x \mathbb{S}(\nabla_x \mathbf{u}) = -\varrho \mathbf{u} \cdot \nabla_x \mathbf{u} - \nabla_x p(\varrho, \vartheta), \quad (4.8)$$

Taking the scalar product with $\partial_t \mathbf{u}$ of both sides of (4.8) and integrating on Ω , we find

$$\begin{aligned} &\int_{\Omega} \varrho |\partial_t \mathbf{u}(t, x)|^2 \, dx + \frac{1}{2} \frac{d}{dt} \int_{\Omega} \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \, dx \\ &\leq c \int_{\Omega} |\nabla_x \mathbf{u}(t, x)|^2 \, dx - \int_{\Omega} \nabla_x p(t, x) \cdot \partial_t \mathbf{u}(t, x) \, dx. \end{aligned} \quad (4.9)$$

Moreover,

$$-\int_{\Omega} \nabla_x p(t, x) \cdot \partial_t \mathbf{u}(t, x) \, dx = \int_{\Omega} p \partial_t \operatorname{div}_x \mathbf{u} \, dx = \frac{d}{dt} \int_{\Omega} p \operatorname{div}_x \mathbf{u} \, dx - \int_{\Omega} \partial_t p \operatorname{div}_x \mathbf{u} \, dx, \quad (4.10)$$

where we have

$$-\int_{\Omega} \partial_t p \operatorname{div}_x \mathbf{u} \, dx = -\int_{\Omega} \partial_t p \operatorname{div}_x \mathbf{v} \, dx - \int_{\Omega} \partial_t p \operatorname{div}_x \mathbf{w} \, dx.$$

It follows from (4.7) that

$$\begin{aligned} -\int_{\Omega} \partial_t p \operatorname{div}_x \mathbf{v} \, dx &= \int_{\Omega} \mathbf{v} \cdot \nabla \partial_t p \, dx = \int_{\Omega} \mathbf{v} \cdot \operatorname{div}_x \mathbb{S}(\nabla_x \partial_t \mathbf{v}) \, dx \\ &= -\int_{\Omega} \nabla_x \mathbf{v} : \mathbb{S}(\partial_t \nabla_x \mathbf{v}) \, dx = -\frac{1}{2} \frac{d}{dt} \int_{\Omega} \mathbb{S}(\nabla_x \mathbf{v}) : \nabla_x \mathbf{v} \, dx. \end{aligned} \quad (4.11)$$

Seeing that

$$\int_{\Omega} \partial_t p \operatorname{div}_x \mathbf{w} \, dx = \int_{\Omega} \partial_t \varrho^\gamma \operatorname{div}_x \mathbf{w} \, dx + \int_{\Omega} \partial_t (\varrho \vartheta) \operatorname{div}_x \mathbf{w} \, dx, \quad (4.12)$$

we may use the relations $\partial_t \varrho^\gamma = -\operatorname{div}_x(\varrho^\gamma \mathbf{u}) - (\gamma - 1)\varrho^\gamma \operatorname{div}_x \mathbf{u}$ and $\operatorname{div}_x \mathbf{u} = \operatorname{div}_x \mathbf{v} + \operatorname{div}_x \mathbf{w}$ to handle the first integral

$$\begin{aligned} \int_{\Omega} \partial_t \varrho^\gamma \operatorname{div}_x \mathbf{w} \, dx &\leq c \int_{\Omega} |\nabla_x \operatorname{div}_x \mathbf{w}| + |\operatorname{div}_x \mathbf{u} \operatorname{div}_x \mathbf{w}| \, dx \\ &\leq c(\varepsilon) \left(1 + \|\nabla_x \mathbf{u}\|_{L^2(\Omega)}^2 \right) + \varepsilon \|\nabla_x^2 \mathbf{w}\|_{L^2(\Omega; R^{27})}^2, \end{aligned} \quad (4.13)$$

where ε is a small positive number to be determined below.

Next, using the argument of Wen and Zhu [WZ13], we calculate the second integral in (4.12):

$$\begin{aligned} &c_v \int_{\Omega} \partial_t (\varrho \vartheta) \operatorname{div}_x \mathbf{w} \, dx \\ &= \int_{\Omega} \partial_t (\varrho E) \operatorname{div}_x \mathbf{w} \, dx - \frac{1}{2} \int_{\Omega} \partial_t (\varrho |\mathbf{u}|^2) \operatorname{div}_x \mathbf{w} \, dx - \frac{1}{\gamma - 1} \int_{\Omega} \partial_t \varrho^\gamma \operatorname{div}_x \mathbf{w} \, dx, \end{aligned}$$

where we may use the total energy balance (4.1) to obtain

$$\begin{aligned} &\int_{\Omega} \partial_t (\varrho E) \operatorname{div}_x \mathbf{w} \, dx \\ &\leq c(\varepsilon) \left(1 + \|\nabla_x \mathbf{u}\|_{L^2(\Omega; R^{3 \times 3})}^2 + \|\nabla_x \vartheta\|_{L^2(\Omega; R^3)}^2 \right) + \varepsilon \|\nabla_x^2 \mathbf{w}\|_{L^2(\Omega; R^{27})}^2. \end{aligned} \quad (4.14)$$

Furthermore,

$$\begin{aligned} -\frac{1}{2} \int_{\Omega} \partial_t (\varrho |\mathbf{u}|^2) \operatorname{div}_x \mathbf{w} \, dx &= -\frac{1}{2} \int_{\Omega} \partial_t \varrho |\mathbf{u}|^2 \operatorname{div}_x \mathbf{w} \, dx - \int_{\Omega} \varrho \mathbf{u} \cdot \partial_t \mathbf{u} \operatorname{div}_x \mathbf{w} \, dx \\ &= \frac{1}{2} \int_{\Omega} \operatorname{div}_x (\varrho \mathbf{u}) |\mathbf{u}|^2 \operatorname{div}_x \mathbf{w} \, dx - \int_{\Omega} \varrho \mathbf{u} \cdot \partial_t \mathbf{u} \operatorname{div}_x \mathbf{w} \, dx \end{aligned} \quad (4.15)$$

$$\leq c(\varepsilon) \left(1 + \|\nabla_x \mathbf{u}\|_{L^2(\Omega; \mathbb{R}^{3 \times 3})}^2 \right) + \frac{1}{2} \|\sqrt{\varrho} \partial_t \mathbf{u}\|_{L^2(\Omega; \mathbb{R}^3)}^2 + \varepsilon \|\nabla_x^2 \mathbf{w}\|_{L^2(\Omega; \mathbb{R}^{27})}^2.$$

Substituting (4.11)-(4.15) in (4.9) and (4.10) and taking ε small to absorb the constant appearing in (4.6), we may infer, by Gronwall's argument, that

$$\sup_{t \in (0, T)} \|\mathbf{u}(t, \cdot)\|_{W^{1,2}(\Omega; \mathbb{R}^3)}^2 + \int_0^T \|\sqrt{\varrho} \partial_t \mathbf{u}\|_{L^2(\Omega; \mathbb{R}^3)}^2 dt \leq c, \quad \int_0^T \|\mathbf{w}\|_{W^{2,2}(\Omega; \mathbb{R}^3)}^2 \leq c. \quad (4.16)$$

Thus, by Sobolev's embedding theorem and (4.4), we get

$$\int_0^T \|\mathbf{u}(t, \cdot)\|_{W^{1,q}(\Omega; \mathbb{R}^3)}^2 \leq c \text{ for any } 1 \leq q \leq 6. \quad (4.17)$$

4.4 Second energy bounds for the temperature and better regularity of the effective viscous flux

We denote $\dot{f} = \partial_t f + \mathbf{u} \cdot \nabla_x f$ the material derivative of f . To begin, we use the arguments of [SWZ11], [WZ13], together with (4.16) to control $\|\nabla_x \mathbf{u}\|_{L^2(\Omega; \mathbb{R}^{3 \times 3})}^2$, to obtain

$$\frac{d}{dt} \int_{\Omega} \varrho |\dot{\mathbf{u}}|^2 dx + \int_{\Omega} |\nabla_x \dot{\mathbf{u}}|^2 dx \leq C \left(1 + \|\nabla_x \mathbf{u}\|_{L^4(\Omega; \mathbb{R}^3)}^4 + \|\Delta_x K(\vartheta)\|_{L^2(\Omega)}^2 \right). \quad (4.18)$$

We recall that $K(\vartheta) = \int_0^\vartheta \kappa(z) dz$,

$$c_v \varrho \dot{\vartheta} - \Delta K(\vartheta) = \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \varrho \vartheta \operatorname{div}_x \mathbf{u}. \quad (4.19)$$

In particular, by means of the standard elliptic estimates,

$$\begin{aligned} \|\nabla_x^2 K(\vartheta)\|_{L^2(\Omega; \mathbb{R}^9)}^2 &\leq c \left(1 + \|\Delta_x K(\vartheta)\|_{L^2(\Omega)}^2 \right) \\ &\leq c \left(1 + \|\sqrt{\varrho} \dot{\vartheta}\|_{L^2(\Omega)}^2 + \|\nabla_x \mathbf{u}\|_{L^4(\Omega; \mathbb{R}^{3 \times 3})}^4 \right). \end{aligned} \quad (4.20)$$

Therefore we get from (4.18) that

$$\frac{d}{dt} \int_{\Omega} \varrho |\dot{\mathbf{u}}|^2 dx + \int_{\Omega} |\nabla_x \dot{\mathbf{u}}|^2 dx \leq c \left(1 + \|\nabla_x \mathbf{u}\|_{L^4(\Omega; \mathbb{R}^{3 \times 3})}^4 + \|\sqrt{\varrho} \dot{\vartheta}\|_{L^2(\Omega)}^2 \right). \quad (4.21)$$

On the other hand, multiplying the equation (4.19) by $\kappa(\vartheta) \dot{\vartheta}$ and integrating over Ω , we get

$$\begin{aligned} &\frac{d}{dt} \int_{\Omega} |\nabla_x K(\vartheta)|^2 dx + \int_{\Omega} \varrho \kappa(\vartheta) |\dot{\vartheta}|^2 dx \\ &\leq c \int_{\Omega} |\nabla_x \mathbf{u}|^2 + |\nabla_x \vartheta|^2 dx + \int_{\Omega} \kappa(\vartheta) \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \dot{\vartheta} dx + \int_{\Omega} \kappa(\vartheta) \Delta K(\vartheta) \mathbf{u} \cdot \nabla_x \vartheta dx, \end{aligned}$$

where $\Delta_x K(\vartheta)$ appearing in the last integral can be absorbed by the left-hand side by means of (4.20). More specifically, we may use similar arguments to [WZ13, formula (4.50)] to deduce that

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} |\nabla_x K(\vartheta)|^2 dx + \int_{\Omega} \varrho \kappa(\vartheta) |\dot{\vartheta}|^2 dx \\ & \leq c \int_{\Omega} |\nabla_x \vartheta|^2 + |\nabla_x \mathbf{u}|^2 + |\nabla_x \mathbf{u}|^3 + |\nabla_x \mathbf{u}|^4 + |\nabla_x \mathbf{u}| |\nabla_x \dot{\mathbf{u}}| dx \\ & \leq c(\varepsilon) \left(1 + \|\nabla_x \vartheta\|_{L^2(\Omega)}^2 + \|\nabla_x \mathbf{u}\|_{L^4(\Omega; \mathbb{R}^{3 \times 3})}^4 \right) + \varepsilon \|\nabla_x \dot{\mathbf{u}}\|_{L^2(\Omega; \mathbb{R}^{3 \times 3})}^2. \end{aligned} \quad (4.22)$$

Remark 4.1. *As a matter of fact, the argument in the present situation is considerably simpler than in [WZ13] as we assume a uniform bound on the velocity field.*

We now divide (4.21) by $\frac{1}{2c}$ and add the resulting expression to (4.22) to absorb the term $\|\sqrt{\varrho} \dot{\vartheta}\|_{L^2(\Omega)}^2$. Next, choosing $\varepsilon = \frac{1}{4c}$ to control $\|\nabla_x \dot{\mathbf{u}}\|_{L^2(\Omega; \mathbb{R}^{3 \times 3})}^2$, we arrive at

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} |\nabla_x K(\vartheta)|^2 + \varrho |\dot{\mathbf{u}}|^2 dx + \int_{\Omega} \varrho \kappa(\vartheta) |\dot{\vartheta}|^2 + |\nabla_x \dot{\mathbf{u}}|^2 dx \\ & \leq c \left(1 + \|\nabla_x \vartheta\|_{L^2(\Omega; \mathbb{R}^3)}^2 + \|\nabla_x \mathbf{u}\|_{L^4(\Omega; \mathbb{R}^{3 \times 3})}^4 \right). \end{aligned} \quad (4.23)$$

Finally note that

$$\begin{aligned} \|\nabla_x \mathbf{u}\|_{L^4(\Omega)}^4 & \leq c(\|\nabla_x \mathbf{v}\|_{L^4(\Omega; \mathbb{R}^{3 \times 3})}^4 + \|\nabla_x \mathbf{w}\|_{L^4(\Omega; \mathbb{R}^{3 \times 3})}^4) \\ & \leq c(1 + \|\nabla_x \mathbf{w}\|_{L^4(\Omega; \mathbb{R}^{3 \times 3})}^4) \\ & \leq c(1 + \|\nabla_x \mathbf{w}\|_{L^2(\Omega; \mathbb{R}^{3 \times 3})}^2 \|\nabla_x^2 \mathbf{w}\|_{L^2(\Omega; \mathbb{R}^{27})}^4) \\ & \leq c(1 + \|\nabla_x^2 \mathbf{w}\|_{L^2(\Omega; \mathbb{R}^{27})}^2 \|\sqrt{\varrho} \dot{\mathbf{u}}\|_{L^2(\Omega; \mathbb{R}^3)}^2) \end{aligned}$$

where we have used (4.6) to control $\|\nabla_x^2 \mathbf{w}\|_{L^2(\Omega; \mathbb{R}^{27})}^2$ by $\|\sqrt{\varrho} \dot{\mathbf{u}}\|_{L^2(\Omega; \mathbb{R}^3)}^2$. Moreover, according to (4.16), we have

$$\int_0^T \|\nabla_x^2 \mathbf{w}\|_{L^2(\Omega)}^2 dt \leq c.$$

Thus we deduce from (4.23) by Gronwall's inequality that

$$\left\{ \begin{array}{l} \|\sqrt{\varrho} \dot{\mathbf{u}}\|_{L^\infty(0, T; L^2(\Omega; \mathbb{R}^3))}^2 + \|\nabla_x \dot{\mathbf{u}}\|_{L^2(0, T; L^2(\Omega; \mathbb{R}^{3 \times 3}))}^2 \leq c, \\ \|\nabla_x \vartheta\|_{L^\infty(0, T; L^2(\Omega; \mathbb{R}^3))}^2 + \|\sqrt{\varrho} \partial_t \vartheta\|_{L^2(0, T; L^2(\Omega))}^2 \leq c; \end{array} \right\} \quad (4.24)$$

whence by Sobolev's embedding theorem

$$\|\partial_t \mathbf{u}\|_{L^2(0, T; L^6(\Omega; \mathbb{R}^3))} \leq c.$$

Next, by virtue of (4.5) and elliptic estimates,

$$\|\mathbf{w}\|_{L^2(0,T;W^{2,6}(\Omega;R^3))} \leq c. \quad (4.25)$$

Again by elliptic estimates, we also find

$$\|\nabla_x^2 K(\vartheta)\|_{L^2(0,T;L^2(\Omega;R^6))} \leq c,$$

which, together with the Gagliardo-Nirenberg inequality,

$$\|\nabla_x f\|_{L^4(\Omega;R^3)}^2 \leq c\|f\|_{L^\infty(\Omega)}\|\nabla_x^2 f\|_{L^2(\Omega)}$$

gives rise to

$$\|\nabla_x K(\vartheta)\|_{L^4(0,T;L^4(\Omega;R^3))} \leq c \text{ yielding } \|\nabla_x \vartheta\|_{L^4(0,T;L^4(\Omega;R^3))} \leq c,$$

and

$$\|\nabla_x^2 \vartheta\|_{L^2(0,T;L^2(\Omega;R^9))} \leq c. \quad (4.26)$$

Remark 4.2. *As already pointed out, the main idea of this part comes from [WZ13]. If we replace upper bound on ϱ by the stipulation $\int_0^T \|\operatorname{div}_x \mathbf{u}\|_{L^\infty(\Omega)} dt \leq L$ as in (1.16), we get immediately the upper and lower bounds on ϱ . In this case, the argument of this subsection can be considerably simplified, see [SWZ11].*

4.5 Estimates of the derivatives of ϱ

With the previous estimates at hand, in particular (4.25) and (4.26), we can proceed exactly as in [SWZ11, Section 5] to obtain

$$\|\partial_t \varrho\|_{L^\infty(0,T;L^6(\Omega))} + \|\nabla_x \varrho\|_{L^\infty(0,T;L^6(\Omega;R^3))} \leq c. \quad (4.27)$$

Going back to the momentum equation (1.2) we then deduce a bound for the second order derivatives of \mathbf{u} , specifically,

$$\int_0^T \|\mathbf{u}\|_{W^{2,q}(\Omega;R^3)}^2 dt \leq c, \quad 1 \leq q \leq 6. \quad (4.28)$$

In particular, relation (4.28) implies boundedness of $\operatorname{div}_x \mathbf{u}$ in $L^1(0,T;L^\infty(\Omega))$, and, accordingly, a lower bound for ϱ ,

$$\varrho(t, x) \geq \underline{\varrho} > 0 \text{ for all } t, x.$$

4.6 Hölder continuity of the temperature

Since the principal part of the thermal energy balance (1.3) is non-linear, the higher order estimates of ϑ require a refined technique based on the L^p maximal regularity. To this end, we show that ϑ is Hölder continuous. We remark that ϱ enjoys Hölder regularity as a consequence of (4.27).

We rewrite the thermal energy balance in the form

$$\begin{aligned} c_v \partial_t \vartheta + \left(c_v \mathbf{u} - \frac{\kappa(\vartheta)}{\varrho^2} \nabla_x \varrho \right) \cdot \nabla_x \vartheta - \operatorname{div}_x \left(\frac{\kappa(\vartheta)}{\varrho} \nabla_x \vartheta \right) \\ = \frac{1}{\varrho} \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \vartheta \operatorname{div}_x \mathbf{u} \end{aligned} \quad (4.29)$$

that can be viewed as a linear parabolic equation with non-constant coefficients.

Now, by virtue of (4.4), (4.25), and (4.27), we have

$$\frac{1}{\varrho} \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \vartheta \operatorname{div}_x \mathbf{u} \in L^\infty(0, T; L^3(\Omega)), \quad \varrho \in L^\infty(0, T; W^{1,6}(\Omega)); \quad (4.30)$$

whence the nowadays standard parabolic theory (see e.g. Ladyzhenskaya et al. [LSU68]) yields

$$\vartheta \text{ Hölder continuous in } [0, T] \times \overline{\Omega}. \quad (4.31)$$

4.7 $L^p - L^q$ parabolic estimates

Seeing that both ϱ and ϑ are Hölder continuous, we are ready to apply the machinery of the $L^p - L^q$ estimates, specifically Denk, Hieber, and Prüss [DHP07, Theorem 2.3] (see also Amann [Ama93], Krylov [Kry07]) to the parabolic problems (4.8) and (4.29).

First, in accordance with (4.30), (4.31), we obtain

$$\vartheta \in L^p(0, T; W^{2,3}(\Omega)), \quad \partial_t \vartheta \in L^p(0, T; L^3(\Omega)), \quad \text{for any } 1 < p < \infty$$

in particular,

$$\nabla_x \vartheta \in L^p(0, T; L^q(\Omega; \mathbb{R}^3)) \quad \text{for all } 1 < p, q < \infty. \quad (4.32)$$

Now, using (4.32) and applying the same treatment to (4.8), we deduce

$$\mathbf{u} \in L^p(0, T; W^{2,6}(\Omega; \mathbb{R}^3)), \quad \partial_t \mathbf{u} \in L^p(0, T; L^6(\Omega)) \quad \text{for all } 1 < p < \infty. \quad (4.33)$$

Finally, since (4.33) implies

$$\nabla_x \mathbf{u} \in L^\infty((0, T) \times \Omega), \quad (4.34)$$

we are in the situation treated in [FNS14]. Thus the proof of Theorem 4.1 is completed by the same arguments as in [FNS14, Section 4.6].

Acknowledgement

We would like to express our thanks to Zhifei Zhang for a fruitful and stimulating discussion concerning the blow-up result.

References

- [Ama93] H. Amann, *Nonhomogeneous linear and quasilinear elliptic and parabolic boundary value problems*, Function spaces, differential operators and nonlinear analysis (Friedrichroda, 1992), Teubner-Texte Math., vol. **133**, Teubner, Stuttgart, 1993, pp. 9–126.
- [BD07] D. Bresch and B. Desjardins, *On the existence of global weak solutions to the Navier-Stokes equations for viscous compressible and heat conducting fluids*, J. Math. Pures Appl. **87** (2007), 57–90.
- [CCK04] Y. Cho, H.J. Choe, and H. Kim, *Unique solvability of the initial boundary value problems for compressible viscous fluids*, J. Math. Pures Appl. **83** (2004), 243–275.
- [DHP07] R. Denk, M. Hieber, and J. Prüss, *Optimal $L^p - L^q$ -estimates for parabolic boundary value problems with inhomogenous data*, Math. Z. **257** (2007), 193–224.
- [DL89] R.J. DiPerna and P.-L. Lions, *Ordinary differential equations, transport theory and Sobolev spaces*, Invent. Math. **98** (1989), 511–547.
- [Fei04] E. Feireisl, *Dynamics of viscous compressible fluids*, Oxford University Press, Oxford, 2004.
- [FJO10] J. Fan, S. Jiang, and Y. Ou, *A blow-up criterion for compressible viscous heat-conductive flows*, Ann. Inst. H. Poincaré Anal. Non Linéaire **27** (2010), no. 1, 337–350.
- [FKN14] E. Feireisl, T. Karper, and A. Novotný, *On a convergent numerical scheme for the full navier-stokes-fourier system*, In preparation.
- [FMR10] J. Frehse, J. Málek, and M. Růžička, *Large data existence result for unsteady flows of inhomogeneous shear-thickening heat-conducting incompressible fluids*, Comm. Partial Differential Equations **35** (2010), 1891–1919.
- [FN09] E. Feireisl and A. Novotný, *Singular limits in thermodynamics of viscous fluids*, Birkhauser, Basel, 2009.
- [FN12] E. Feireisl and A. Novotný, *Weak-strong uniqueness property for the full Navier-Stokes-Fourier system*, Arch. Rational Mech. Anal. **204** (2012), 683–706.
- [FNS14] E. Feireisl, A. Novotný, and Y. Sun, *A regularity criterion for the weak solutions to the Navier-Stokes-Fourier system*, Arch. Ration. Mech. Anal. **212** (2014), no. 1, 219–239.
- [HLW13] X. Huang, J. Li, and Y. Wang, *Serrin-type blowup criterion for full compressible Navier-Stokes system*, Arch. Ration. Mech. Anal. **207** (2013), no. 1, 303–316.
- [Hof12] D. Hoff, *Local solutions of a compressible flow problem with Navier boundary conditions in general three-dimensional domains*, SIAM J. Math. Anal. **44** (2012), 633–650.

- [Kar13] T. K. Karper, *A convergent FEM-DG method for the compressible Navier-Stokes equations*, Numer. Math. **125** (2013), no. 3, 441–510.
- [Kry07] N. V. Krylov, *Parabolic equations with VMO coefficients in Sobolev spaces with mixed norms*, J. Funct. Anal. **250** (2007), no. 2, 521–558.
- [LSU68] O. A. Ladyzhenskaya, V.A. Solonnikov, and N. N. Uraltseva, *Linear and quasilinear equations of parabolic type*, Trans. Math. Monograph 23, Amer. Math. Soc., Providence, 1968.
- [MN80] A. Matsumura and T. Nishida, *The initial value problem for the equations of motion of viscous and heat-conductive gases*, J. Math. Kyoto Univ. **20** (1980), 67–104.
- [MN83] ———, *The initial value problem for the equations of motion of compressible and heat conductive fluids*, Comm. Math. Phys. **89** (1983), 445–464.
- [SWZ11] Y. Sun, C. Wang, and Z. Zhang, *A Beale-Kato-Majda criterion for three dimensional compressible viscous heat-conductive flows*, Arch. Ration. Mech. Anal. **201** (2011), no. 2, 727–742.
- [Val82a] A. Valli, *A correction to the paper: “An existence theorem for compressible viscous fluids”* [Ann. Mat. Pura Appl. (4) **130** (1982), 197–213]; Ann. Mat. Pura Appl. (4) **132** (1982), 399–400 (1983).
- [Val82b] ———, *An existence theorem for compressible viscous fluids*, Ann. Mat. Pura Appl. (4) **130** (1982), 197–213.
- [VZ86] A. Valli and M. Zajaczkowski, *Navier-Stokes equations for compressible fluids: Global existence and qualitative properties of the solutions in the general case*, Commun. Math. Phys. **103** (1986), 259–296.
- [WZ13] H. Wen and C. Zhu, *Blow-up criterions of strong solutions to 3D compressible Navier-Stokes equations with vacuum*, Adv. Math. **248** (2013), 534–572.