Mathematical theory of compressible viscous fluids

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Another advantage of a mathematical statement is that it is so definite that it might be definitely wrong. . . Some verbal statements have not this merit.

F.L.Richardson (1881-1953)

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1 Introduction

Despite the concerted effort of generations of excellent mathematicians, the fundamental problems in partial differential equations related to continuum fluid mechanics remain largely open. Solvability of the Navier-Stokes system describing the motion of an incompressible viscous fluid is one in the sample of millenium problems proposed by Clay Institute, see [5]. In contrast with these apparent theoretical difficulties, the Navier-Stokes system became a well established model serving as a reliable basis of investigation in continuum fluid mechanics, including the problems involving turbulence phenomena. An alternative approach to problems in fluid mechanics is based on the concept of weak solutions. As a matter of fact, the balance laws, expressed in classical fluid mechanics in the form of partial differential equations, have their origin in integral identities that seem to be much closer to the modern weak formulation of these problems. Leray [8] constructed the weak solutions to the incompressible Navier-Stokes system as early as in 1930, and his "turbulent solutions" are still the only ones available for investigating large data and/or problems on large time intervals. Recently, the real breakthrough is the work of Lions citeLI4 who generalized Leray's theory to the case of barotropic compressible viscous fluids (see also Vaigant and Kazhikhov [13]). The quantities playing a crucial role in the description of density oscillations as the effective viscous flux were identified and used in combination with a renormalized version of the equation of continuity to obtain first large data/large time existence results in the framework of compressible viscous fluids.

The main goal of the this lecture series is present the mathematical theory of *compressible barotropic fluids* in the framework of Lions [9], together with the extensions developed in [6]. We focus on the crucial question of *stability* of a family of weak solutions that is the core of the abstract theory, with implications to numerical analysis and the associated real world applications. For the sake of clarity of presentation, we discuss first the case, where the pressure term has sufficient growth for large value of the density yielding sufficiently strong energy bonds. We also start with the simplest geometry of the physical space, here represented by a cube, on the boundary of which the fluid satisfies the slip boundary conditions. As is well-known, such a situation may be reduced to studying the purely spatially periodic case, where the additional difficulties connected with the presence of boundary conditions is entirely eliminated.

2 Mathematical model

As the main goal of this lecture series is the *mathematical theory*, we avoid a detailed derivation of the mathematical model of a compressible viscous fluid. Remaining on the platform of *continuum fluid mechanics*, we suppose that the motion of a compressible barotropic fluid is described by means of two basic *fields*:

the mass density .... $\varrho = \varrho(t, x)$,
the velocity field .... $\mathbf{u} = \mathbf{u}(t, x)$,

functions of the time $t \in R$ and the spatial position $x \in R^3$.

2.1 Mass conservation

Let us recall the classical argument leading to the mathematical formulation of the physical principle of *mass conservation*, see Chorin and Marsden [2].

Consider a volume $B \subset R^3$ containing a fluid of density ϱ. The change of the total mass of the fluid contained in B during a time interval $[t_1, t_2]$, $t_1 < t_2$ is given as

$$\int_B \varrho(t_2, x) \, \mathrm{d}x - \int_B \varrho(t_1, x) \, \mathrm{d}x.$$

One of the basic laws of physics incorporated in continuum mechanics as the *principle of mass conservation* asserts that mass is neither created nor destroyed. Accordingly, the change of the fluid mass in B is only because of the mass *flux* through the boundary ∂B, here represented by $\varrho \mathbf{u} \cdot \mathbf{n}$, where $\mathbf{n}$ denotes the *outer* normal vector to ∂B:

$$\int_B \varrho(t_2, x) \, \mathrm{d}x - \int_B \varrho(t_1, x) \, \mathrm{d}x = - \int_{t_1}^{t_2} \int_{\partial B} \varrho(t, x)\mathbf{u}(t, x) \cdot \mathbf{n}(x) \, \mathrm{dS}_x \, dt. \quad (2.1)$$

One should remember formula (2.1) since it contains *all* relevant piece of information provided by *physics*. The following discussion is based on *mathematical* arguments based on the (unjustified) hypotheses of *smoothness* of all field in question. To begin, apply Gauss-Green theorem to rewrite (2.1) in the form:

$$\int_B \varrho(t_2, x) \, \mathrm{d}x - \int_B \varrho(t_1, x) \, \mathrm{d}x = - \int_{t_1}^{t_2} \int_B \mathrm{div}_x \Big(\varrho(t, x)\mathbf{u}(t, x) \Big) \, \mathrm{d}x \, \mathrm{d}t.$$

Furthermore, fixing $t_1 = t$ and performing the limit $t_2 \to t_1$ we may use the mean value theorem to obtain

$$\int_B \partial_t \varrho(t, x) \, \mathrm{d}x = \lim_{t_2 \to t} \frac{1}{t_2 - t} \int_B \varrho(t_2, x) \, \mathrm{d}x - \int_B \varrho(t, x) \, \mathrm{d}x \quad (2.2)$$

$$= - \lim_{t_2 \to t} \frac{1}{t_2 - t} \int_{t_1}^{t_2} \int_B \mathrm{div}_x \Big(\varrho(t, x)\mathbf{u}(t, x) \Big) \, \mathrm{d}x \, \mathrm{d}t$$

$$= - \int_B \mathrm{div}_x \Big(\varrho(t, x)\mathbf{u}(t, x) \Big) \, \mathrm{d}x.$$

Finally, as relation (2.2) should hold for *any* volume element B, we may infer that

$$\partial_t \varrho(t, x) + \mathrm{div}_x \Big(\varrho(t, x)\mathbf{u}(t, x) \Big) = 0. \quad (2.3)$$

Relation (2.3) is a first order partial differential equation called *equation of continuity*.

2.2 Balance of momentum

Using arguments similar to the preceding part, we derive *balance of momentum* in the form

$$\partial_t\Big(\varrho(t,x)\mathbf{u}(t,x)\Big) + \mathrm{div}_x\Big(\varrho(t,x)\mathbf{u}(t,x)\otimes\mathbf{u}(t,x)\Big) = \mathrm{div}_x\mathbb{T}(t,x) + \varrho(t,x)\mathbf{f}(t,x), \tag{2.4}$$

or, equivalently (cf. (2.3),

$$\varrho(t,x)\Big[\partial_t\mathbf{u}(t,x) + \mathbf{u}(t,x)\cdot\nabla_x\mathbf{u}(t,x)\Big] = \mathrm{div}_x\mathbb{T}(t,x) + \varrho(t,x)\mathbf{f}(t,x),$$

where the tensor $\mathbb{T}$ is the Cauchy stress and $\mathbf{f}$ denotes the (specific) external force acting on the fluid.

We adopt the standard *mathematical* definition of fluids in the form of *Stokes' law*

$$\mathbb{T} = \mathbb{S} - p\mathbb{I},$$

where tnS is the viscous stress and p is a scalar function termed pressure. In addition, we suppose that the viscous stress is a *linear* function of the velocity gradient, specifically $\mathbb{S}$ obeys *Newton's rheological law*

$$\mathbb{S} = \mathbb{S}(\nabla_x\mathbf{u}) = \mu\left(\nabla_x\mathbf{u} + \nabla_x^t\mathbf{u} - \frac{2}{3}\mathrm{div}_x\mathbf{u}\mathbb{I}\right) + \eta\,\mathrm{div}_x\mathbf{u}\mathbb{I}, \tag{2.5}$$

with the shear viscosity coefficient μ and the bulk viscosity coefficient η, here assumed constant, $\mu > 0$, $\eta \geq 0$.

In order to close the system, we suppose the fluid is *barotropic*, meaning the pressure p is an explicitly given function of the density $p = p(\varrho)$. Accordingly,

$$\mathrm{div}_x\mathbb{T} = \mu\Delta\mathbf{u} + (\lambda+\mu)\nabla_x\mathrm{div}_x\mathbf{u} - \nabla_x p(\varrho),\ \ \mu > 0,\ \ \lambda \geq -\frac{2}{3}\mu,$$

and equations (2.3), (2.4) can be written in a concise form as

Navier-Stokes system

$$\partial_t\varrho + \mathrm{div}_x(\varrho\mathbf{u}) = 0, \tag{2.6}$$

$$\partial_t(\varrho\mathbf{u}) + \mathrm{div}_x(\varrho\mathbf{u}\otimes\mathbf{u}) + \nabla_x p(\varrho) = \mu\Delta\mathbf{u} + (\lambda+\mu)\nabla_x\mathrm{div}_x\mathbf{u} + \varrho\mathbf{f}. \tag{2.7}$$

The system of equations (2.6), (2.7) should be compared with a "more famous" incompressible Navier-Stokes system, where the density is constant, say $\varrho \equiv 1$, while (2.6), (2.7) "reduces" to

$$\mathrm{div}_x \mathbf{u} = 0, \tag{2.8}$$

$$\partial_t \mathbf{u} + \mathrm{div}_x(\mathbf{u} \otimes \mathbf{u}) + \nabla_x p = \mu \Delta \mathbf{u} + \mathbf{f}. \tag{2.9}$$

Unlike in (2.7), the pressure p in (2.9) is an unknown function determined (implicitly) by the fluid motion! The pressure in the *incompressible* Navier-Stokes system has non-local character and may depend on the far field behavior of the fluid system.

2.3 Spatial domain and boundary conditions

In the real world applications, the fluid is confined to a bounded spatial domain $\Omega \subset R^3$. The presence of the physical boundary $\partial\Omega$ and the associated problem of fluid-structure interaction represent a source of substantial difficulties in the mathematical analysis of fluids in motion. In order to avoid technicalities, we suppose that the motion is *space-periodic*, specifically,

$$\varrho(t,x) = \varrho(t, x + \mathbf{a}),\ \mathbf{u}(t,x) = \mathbf{u}(t, x + \mathbf{a}) \text{ for all } t, x,$$

where the period vector $\mathbf{a} \in R^3$ is given. Equivalently, we may assume that Ω is a flat *torus*,

$$\Omega = [0, a_1]|_{\{0,a_1\}} \times [0, a_2]|_{\{0,a_2\}} \times [0, a_3]|_{\{0,a_3\}}.$$

The space-periodic boundary conditions have a nice physical interpretation in fluid mechanics, see Ebin [4]. Indeed, if we restrict ourselves to the classes of functions defined on the torus Ω and satisfying the extra geometric restrictions:

$$\varrho(t,x) = \varrho(t,-x),\ u_i(t,\cdot,x_i,\cdot) = -u_i(t,\cdot,-x_i,\cdot),\ i = 1,2,3,$$

$$u_i(t,\cdot,x_j,\cdot) = u_i(t,\cdot,-x_j,\cdot) \text{ for } i \neq j,$$

and, similarly,

$$f_i(t,\cdot,x_i,\cdot) = -f_i(t,\cdot,-x_i,\cdot),\ f_i(t,\cdot,x_j,\cdot) = f_i(t,\cdot,-x_j,\cdot) \text{ for } i \neq j,$$

we can check that

- the equations (2.6), (2.7) are invariant with respect to the above transformations;

- the velocity field $\mathbf{u}$ satisfies the so-called *complete slip* conditions

$$\mathbf{u} \cdot \mathbf{n} = 0, \ [\mathbb{S}\mathbf{n}] \times \mathbf{n} = 0 \tag{2.10}$$

  on the boundary of the spatial block $[0, a_1] \times [0, a_2] \times [0, a_3]$.

We remark that the most commonly used boundary conditions for *viscous* fluids confined to a general spatial domain Ω (not necessarily a flat torus) are the *no-slip*

$$\mathbf{u}|_{\partial\Omega} = 0.$$

As a matter of fact, the problem of the choice of correct boundary conditions in the real world applications is rather complex, some parts of the boundaries may consist of a different fluid in motion, or the fluid domain is not *a priori* known (free boundary problems). The interested reader may consult Priezjev and Troian [11] for relevant discussion.

2.4 Initial conditions

Given the initial state at a reference time t_0, say $t_0 = 0$, the time evolution of the fluid is determined as a solution of the Navier-Stokes system (2.6), (2.7). It is convenient to introduce the initial density

$$\varrho(0, x) = \varrho_0(x), \ x \in \Omega, \tag{2.11}$$

together with the initial distribution of the momentum,

$$(\varrho\mathbf{u})(0, x) = (\varrho\mathbf{u})_0(x), \ x \in \Omega, \tag{2.12}$$

as, strictly speaking, the momentum balance (2.7) is an evolutionary equation for $\varrho\mathbf{u}$ rather than $\mathbf{u}$. Such a difference will become clear in the so-called *weak* formulation of the problem discussed in the forthcoming section.

3 Weak solutions

A vast class of *non-linear* evolutionary problems arising in mathematical fluid mechanics is not known to admit classical (differentiable, smooth) solutions for all choices of data and on an arbitrary time interval. On the other hand, most of the real world problems call for solutions defined in-the-large approached in the numerical simulations. In order to perform a rigorous analysis, we have to introduce a concept of *generalized* or *weak* solutions, for which derivatives are interpreted in the sense of distributions. It rn represented by viscosity should provide a strong regularizing effect. Another motivation, at least in the case of the compressible Navier-Stokes system (2.6), (2.7), is the possibility to study the fluid dynamics emanating from irregular initial state, for instance, the density ϱ_0 may not be continuous. As shown by Hoff [7], the singularities incorporated initially will "survive" in the system at any time; thus the weak solutions are necessary in order to describe the dynamics.

3.1 Equation of continuity - weak formulation

We consider equation (2.6) on the space-time cylinder $(0,T) \times \Omega$, where Ω is the flat torus introduced in Section 2.3. Multiplying (2.6) on $\varphi \in C_c^\infty((0,T) \times \Omega)$, integrating the resulting expression over $(0,T) \times \Omega$, and performing by-parts integration, we obtain

$$\int_0^T \int_\Omega \Big(\varrho(t,x) \partial_t \varphi(t,x) + \varrho(t,x) \mathbf{u}(t,x) \cdot \nabla_x \varphi(t,x) \Big) \, \mathrm{d}x \, \mathrm{d}t = 0. \tag{3.1}$$

Definition 3.1 *We say that a pair of functions ϱ, $\mathbf{u}$ is a* weak solution *to equation (2.6) in the space-time cylinder $(0,T) \times \Omega$ if ϱ, $\varrho\mathbf{u}$ are locally integrable in $(0,T) \times \Omega$ and the integral identity (3.1) holds for any test function $\varphi \in C_c^\infty((0,T) \times \Omega)$.*

3.1.1 Weak-strong compatibility

It is easy to see that any classical (smooth) solution of equation (2.6) is also a weak solution. Similarly, any weak solution that is continuously differentiable satisfies (2.6) pointwise. Such a property is called *weak-strong compatibility.*

3.1.2 Weak continuity

Up to now, we have left apart the problem of satisfaction of the initial condition (2.11). Obviously, some kind of *weak continuity* is needed for (2.11) to make sense. To this end, we make an extra hypothesis, namely,

$$\varrho\mathbf{u} \in L^1(0,T;L^1(\Omega;R^3)). \tag{3.2}$$

Taking

$$\varphi(t,x) = \psi(t)\phi(x),\ \psi \in C_c^\infty(0,T),\ \phi \in C_c^\infty(\Omega)$$

as a test function in (3.1) we may infer, by virtue of (3.2), that the function

$$t \mapsto \int_\Omega \varrho(t,x)\phi(x)\ \mathrm{d}x \text{ is absolutely continuous in } [0,T] \tag{3.3}$$

for any $\phi \in C_c^\infty(\Omega)$. In particular, the initial condition (2.11) may be satisfied in the sense that

$$\lim_{t\to 0+} \int_\Omega \varrho(t,x)\phi(x)\ \mathrm{d}x = \int_\Omega \varrho_0(x)\phi(x)\ \mathrm{d}x \text{ for any } \phi \in C_c^\infty(\Omega).$$

Now, take

$$\varphi_\varepsilon(t,x) = \psi_\varepsilon(t)\varphi(t,x),\ \varphi \in C_c^\infty([0,T]\times\Omega),$$

where $\psi_\varepsilon \in C_c^\infty(0,\tau)$,

$$0 \le \psi_\varepsilon \le 1,\ \psi_\varepsilon \nearrow 1_{[0,\tau]} \text{ as } \varepsilon \to 0.$$

Taking φ_ε as a test function in (3.1) and letting $\varepsilon \to 0$, we conclude, making use of (3.3), that

$$\int_\Omega \varrho(\tau,x)\varphi(\tau,x)\ \mathrm{d}x - \int_\Omega \varrho_0(x)\varphi(0,x)\ \mathrm{d}x \tag{3.4}$$

$$= \int_0^\tau \int_\Omega \Big(\varrho(t,x)\partial_t\varphi(t,x) + \varrho(t,x)\mathbf{u}(t,x)\cdot\nabla_x\varphi(t,x) \Big)\ \mathrm{d}x\ \mathrm{d}t$$

for any $\tau \in [0,T]$ and any $\varphi \in C_c^\infty([0,T]\times\Omega)$.

Formula (3.4) can be alternatively used a definition of *weak solution* to problem (2.6), (2.11). It is interesting to compare (3.4) with the original integral formulation of the principle of mass conservation stated in (2.1). To this end, we take

$$\varphi_\varepsilon(t,x) = \phi_\varepsilon(x),$$

with $\phi_\varepsilon \in C_c^\infty(B)$ such that

$$0 \le \phi_\varepsilon \le 1,\ \ \phi_\varepsilon \nearrow 1_B \text{ as } \varepsilon \to 0.$$

It is easy to see that

$$\int_\Omega \varrho(\tau,x)\varphi_\varepsilon(\tau,x)\ \mathrm{d}x - \int_\Omega \varrho_0(x)\varphi_\varepsilon(0,x)\ \mathrm{d}x \to \int_B \varrho(\tau,x)\ \mathrm{d}x - \int_B \varrho_0(x)\ \mathrm{d}x \text{ as } \varepsilon \to 0,$$

which coincides with the expression on the left-hand side of (2.1). Consequently, the right-hand side of (3.4) must posses a limit and we set

$$\int_0^\tau \int_\Omega \varrho(t,x)\mathbf{u}(t,x)\cdot\nabla_x\phi_\varepsilon(x)\ \mathrm{d}x\ \mathrm{d}t \to -\int_0^\tau \int_{\partial B} \varrho(t,x)\mathbf{u}(t,x)\cdot\mathbf{n}\ \mathrm{dS}_x\ \mathrm{d}t.$$

In other words, the weak solutions possess a *normal trace* on the boundary of the cylinder $(0,\tau)\times B$ that satisfies (2.1), see Chen and Frid [1] for more elaborate treatment of the normal traces of solutions to conservation laws.

3.1.3 Total mass conservation

Taking $\varphi = 1$ for $t \in [0,\tau]$ in (3.4) we obtain

$$\int_\Omega \varrho(\tau,x)\ \mathrm{d}x = \int_\Omega \varrho_0(x)\ \mathrm{d}x = M_0 \text{ for any } \tau \ge 0, \tag{3.5}$$

meaning, the total mass M_0 of the fluid is a constant of motion.

3.2 Balance of momentum - weak formulation

Similarly to the preceding part, we introduce a weak formulation of the balance of momentum (2.7):

Definition 3.2 *The functions* ϱ, $\mathbf{u}$ *represent a* weak solution *to the momentum equation (2.7) in the set* $(0,T)\times\Omega$ *if the integral identity*

$$\int_0^T \int_\Omega \Big((\varrho\mathbf{u})(t,x)\partial_t\varphi(t,x) + (\varrho\mathbf{u}\otimes\mathbf{u})(t,x) : \nabla_x\varphi(t,x)$$

$$+p(\varrho)(t,x)\mathrm{div}_x\varphi(t,x)\Big)\ \mathrm{d}x\ \mathrm{d}t \tag{3.6}$$

$$= \int_0^T \int_\Omega \Big(\mu\nabla_x\mathbf{u}(t,x) : \nabla_x\varphi(t,x)$$

$$+(\lambda+\mu)\mathrm{div}_x\mathbf{u}(t,x)\mathrm{div}_x\varphi(t,x) - \varrho(t,x)\mathbf{f}(t,x)\cdot\varphi(t,x)\Big)\ \mathrm{d}x\ \mathrm{d}t$$

is satisfied for any test function $\varphi \in C_c^\infty((0,T)\times\Omega;R^3)$.

Of course, we have tacitly assume that all quantities appearing in (3.6) are at least locally integrable in $(0,T)\times\Omega$. In particular, as (3.6) contains explicitely $\nabla_x\mathbf{u}$, we have to assume integrability of this term. As we shall see in the following section, one can expect, given the available *a priori* bounds, $\nabla_x\mathbf{u}$ to be square integrable, specifically,

$$\mathbf{u}\in L^2(0,T;W^{1,2}(\Omega;R^3)).$$

If $\Omega\subset R^3$ is a (bounded) domain with a non-void boundary, we can enforce several kinds of boundary conditions by means of the properties of the test functions. Thus, for instance, the *no-slip* bundary conditions

$$\mathbf{u}|_{\partial\Omega} = 0, \tag{3.7}$$

require the integral identity (3.6) to be satisfied for any *compactly supported* test function φ, while

$$\mathbf{u}\in L^2(0,T;W_0^{1,2}(\Omega;R^3)),$$

where $W_0^{1,2}(\Omega;R^3)$ is the Sobolev space obtained as the closure of $C_c^\infty(\Omega;R^3)$ in the $W^{1,2}-$norm.

4 A priori bounds

A priori bounds are natural constraints imposed on the set of (hypothetical) smooth solutions by the data as well as by the differential equations satisfied. *A priori* bounds determine the function spaces framework the (weak) solutions are looked for. By definition, they are formal, derived under the principal hypothesis of smoothness of all quantities in question.

4.1 Total mass conservation

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The fluid density ϱ satisfies the equation of continuity that may be written in the form

$$\partial_t \varrho + \mathbf{u} \cdot \nabla_x \varrho = -\varrho \mathrm{div}_x \mathbf{u} \tag{4.1}$$

This is a transport equation with the characteristic field defined

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathbf{X}(t, x_0) = \mathbf{u}(t, \mathbf{X}),\ \mathbf{X}(0, x_0) = x_0.$$

Accordingly, (4.1) can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t} \varrho(t, \mathbf{X}(t, \cdot)) = -\varrho(t, \mathbf{X}(t, \cdot)) \mathrm{div}_x \mathbf{u}(t, \mathbf{X}(t, \cdot)).$$

Consequently, we obtain

$$\begin{aligned} &\inf_{x \in \Omega} \varrho(0, x) \exp\left(-t \|\mathrm{div}_x \mathbf{u}\|_{L^\infty((0,T)\times\Omega)} \right) \\ &\leq \varrho(t, x) \leq \\ &\leq \sup_{x \in \Omega} \varrho(0, x) \exp\left(t \|\mathrm{div}_x \mathbf{u}\|_{L^\infty((0,T)\times\Omega)} \right) \end{aligned} \tag{4.2}$$

for any $t \in [0, T]$.

Unfortunately, the bounds established in (4.2) depend on the norm $\|\mathrm{div}_x \mathbf{u}\|_{L^\infty}$ on which we have no information. Thus we may infer only that

$$\varrho(t, x) \geq 0. \tag{4.3}$$

Relation (4.3) combined with the total mass conservation (3.5) yield

$$\|\varrho(t, \cdot)\|_{L^1(\Omega)} = \|\varrho_0\|_{L^1(\Omega)},\ \varrho(0, \cdot) = \varrho_0. \tag{4.4}$$

4.2 Energy balance

Taking the scalar product of the momentum equation (2.4) with $\mathbf{u}$ we deduce the *kinetic energy balance equation*

$$\partial_t\left(\frac{1}{2}\varrho|\mathbf{u}|^2\right)+\mathrm{div}_x\left(\frac{1}{2}\varrho|\mathbf{u}|^2\mathbf{u}\right)+\mathrm{div}_x(p(\varrho)\mathbf{u})-p(\varrho)\mathrm{div}_x\mathbf{u}-\mathrm{div}_x(\mathbb{S}\mathbf{u})+\mathbb{S}:\nabla_x\mathbf{u} \tag{4.5}$$
$$=\varrho\mathbf{f}\cdot\mathbf{u}.$$

Our goal is to integrate (4.5) by parts in order to deduce *a priori* bounds. Imposing the no-slip boundary condition

$$\mathbf{u}|_{\partial\Omega}=0,$$

we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left(\frac{1}{2}\varrho|\mathbf{u}|^2\right)\ \mathrm{d}x-\int_\Omega p(\varrho)\mathrm{div}_x\mathbf{u}\ \mathrm{d}x+\int_\Omega\mathbb{S}:\nabla_x\mathbf{u}\ \mathrm{d}x=\int_\Omega\varrho\mathbf{f}\cdot\mathbf{u}\ \mathrm{d}x,$$

where, in accordance with (2.7),

$$\mathbb{S}:\nabla_x\mathbf{u}=\mu|\nabla_x\mathbf{u}|^2+3(\lambda+\mu)|\mathrm{div}_x\mathbf{u}|^2\geq c|\nabla_x\mathbf{u}|^2,\ c>0, \tag{4.6}$$

provided $\lambda+2/3\mu>0$.

Seeing that

$$\int_\Omega\varrho\mathbf{f}\cdot\mathbf{u}\ \mathrm{d}x\leq\int_\Omega|f|\sqrt{\varrho}\sqrt{\varrho}|\mathbf{u}|\ \mathrm{d}x\leq\frac{1}{2}\|\mathbf{f}\|_{L^\infty((0,T)\times\Omega)}\left(\int_\Omega\varrho\ \mathrm{d}x+\int_\Omega\varrho|\mathbf{u}|^2\ \mathrm{d}x\right)$$

we focus on the integral

$$\int_\Omega p(\varrho)\mathrm{div}_x\mathbf{u}\ \mathrm{d}x.$$

Multiplying the equation of continuity (4.1) by $b'(\varrho)$ we obtain the *renormalized equation of continuity*

$$\partial_t b(\varrho)+\mathrm{div}_x(b(\varrho)\mathbf{u})+\Big(b'(\varrho)\varrho-b(\varrho)\Big)\mathrm{div}_x\mathbf{u}=0. \tag{4.7}$$

Consequently, in particular, the choice

$$b(\varrho)=P(\varrho)\equiv\varrho\int_1^\varrho\frac{p(z)}{z^2}\ \mathrm{d}z$$

leads to

$$b'(\varrho)\varrho - b(\varrho) = p(\varrho).$$

Thus

$$-\int_\Omega p(\varrho)\mathrm{div}_x\mathbf{u}\ \mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega P(\varrho)\ \mathrm{d}x,$$

and we deduce the *total energy* balance

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho)\right)\ \mathrm{d}x + \int_\Omega \mathbb{S} : \nabla_x\mathbf{u}\ \mathrm{d}x = \int_\Omega \varrho\mathbf{f}\cdot\mathbf{u}\ \mathrm{d}x. \tag{4.8}$$

We conclude with

Energy estimates:

$$\sup_{t\in[0,T]} \|\sqrt{\varrho}\mathbf{u}(t,\cdot)\|_{L^2(\Omega;R^3)} \leq c(E_0,T), \tag{4.9}$$

$$\sup_{t\in[0,T]} \int_\Omega P(\varrho)(t,\cdot)\ \mathrm{d}x \leq c(E_0,T), \tag{4.10}$$

$$\int_0^T \|\mathbf{u}(t,\cdot)\|^2_{W_0^{1,2}(\Omega;R^3)}\ \mathrm{d}t \leq c(E_0,T), \tag{4.11}$$

where E_0 denotes the initial energy

$$E_0 = \int_\Omega \left(\frac{1}{2}\varrho_0|\mathbf{u}_0|^2 + P(\varrho_0)\right)\ \mathrm{d}x.$$

4.3 Pressure estimates

A seemingly direct way to pressure estimates is to "compute" the pressure in the momentum balance (2.7):

$$p(\varrho) = -\Delta^{-1}\mathrm{div}_x\partial_t(\varrho\mathbf{u}) - \Delta^{-1}\mathrm{div}_x\mathrm{div}_x(\varrho\mathbf{u}\otimes\mathbf{u}) + \Delta^{-1}\mathrm{div}_x\mathrm{div}_x\mathbb{S} + \Delta^{-1}\mathrm{div}_x(\varrho\mathbf{f}),$$

where Δ^{-1} is an "inverse" of the Laplacean. In order to justify this formal step, we use the so-called *Bogovskii operator* $\mathcal{B} \approx \mathrm{div}_x^{-1}$.

We multiply equation (2.7) by

$$\mathbf{B}[\varrho] = \mathcal{B}\left[b(\varrho) - \frac{1}{|\Omega|}\int_\Omega b(\varrho)\ \mathrm{d}x\right]$$

and integrate by parts to obtain

$$\int_0^T \int_\Omega p(\varrho) b(\varrho) \ \mathrm{d}x \ \mathrm{d}t \tag{4.12}$$

$$= \frac{1}{|\Omega|} \int_0^T \int_\Omega p(\varrho) \ \mathrm{d}x \int_\Omega b(\varrho) \ \mathrm{d}x \ \mathrm{d}t + \int_0^T \int_\Omega \mathbb{S} : \nabla_x \mathbf{B}[\varrho] \ \mathrm{d}x \ \mathrm{d}t$$

$$- \int_0^T \int_\Omega \varrho \mathbf{u} \otimes \mathbf{u} : \nabla_x \mathbf{B}[\varrho] \ \mathrm{d}x - \int_0^T \int_\Omega \varrho \mathbf{u} \cdot \partial_t \mathbf{B}[\varrho] \ \mathrm{d}x \ \mathrm{d}t$$

$$\int_\Omega \left(\varrho \mathbf{u} \cdot \mathbf{B}[\varrho](\tau, \cdot) - \varrho_0 \mathbf{u}_0 \cdot \mathbf{B}[\varrho_0] \right) \ \mathrm{d}x.$$

Furthermore, we have

$$\partial_t \mathbf{B}[\varrho] \tag{4.13}$$

$$= -\mathcal{B} \left[\mathrm{div}_x (b(\varrho) \mathbf{u}) + \Big(b'(\varrho)\varrho - b(\varrho) \Big) \mathrm{div}_x \mathbf{u} - \frac{1}{|\Omega|} \int_\Omega \Big(\big(b'(\varrho)\varrho - b(\varrho) \big) \mathrm{div}_x \mathbf{u} \Big) \ \mathrm{d}x \right].$$

We recall the basic properties of the Bogovskii operator:

BOGOVSKII OPERATOR:

$$\mathrm{div}_x \mathcal{B}[h] = h \text{ for any } h \in L^p(\Omega), \ \int_\Omega h \ \mathrm{d}x = 0, \ 1 < p < \infty, \ \mathcal{B}[h]|_{\partial\Omega} = 0. \tag{4.14}$$

$$\|\mathcal{B}[h]\|_{W_0^{1,p}(\Omega;R^3)} \leq c(p) \|h\|_{L^p(\Omega)}, \ 1 < p < \infty, \tag{4.15}$$

$$\|\mathcal{B}[h]\|_{L^q(\Omega)} \leq \|g\|_{L^q(\Omega;R^3)} \tag{4.16}$$

$$\text{for } h \in L^p(\Omega), \ h = \mathrm{div}_x \mathbf{g}, \ \mathbf{g} \cdot \mathbf{n}|_{\partial\Omega} = 0, \ 1 < q < \infty.$$

5 Complete weak formulation

A *complete* weak formulation of the (compressible) *Navier-Stokes* system takes into account both the renormalized equation of continuity and the energy inequality. Here and hereafter we assume that $\Omega \subset R^3$ is a bounded domain with Lipschitz boundary. For the sake of definiteness, we take the pressure in the form

$$p(\varrho) = a\varrho^\gamma, \text{ with } a > 0 \text{ and } \gamma > 3/2. \tag{5.1}$$

Later, we restrict ourselves to the case when γ is "sufficiently" large.

5.1 Equation of continuity

Let us introduce a class of (nonlinear) functions b such that

$$b \in C^1[0,\infty), \ b(0) = 0, \ b'(r) = 0 \text{ whenever } r \geq M_b. \tag{5.2}$$

We say that ϱ, $\mathbf{u}$ is a (renormalized) solution of the equation of continuity (2.3), supplemented with the initial condition,

$$\varrho(0,\cdot) = \varrho_0,$$

if $\varrho \in C_{\rm weak}([0,T]; L^\gamma(\Omega))$, $\varrho \geq 0$, $\mathbf{u} \in L^2(0,T; W_0^{1,2}(\Omega; R^3))$, and the integral identity

$$\int_0^T \int_\Omega \Big((\varrho + b(\varrho))\, \partial_t\varphi + (\varrho + b(\varrho))\, \mathbf{u}\cdot\nabla_x\varphi + (b(\varrho) - b'(\varrho)\varrho)\, {\rm div}_x \mathbf{u}\varphi \Big)\ {\rm d}x\ {\rm d}t \\ = - \int_\Omega \varrho_0 \varphi(0,\cdot)\ {\rm d}x \tag{5.3}$$

is satisfied for any $\varphi \in C_c^\infty([0,\infty) \times \overline{\Omega})$ and any b belonging to the class specified in (5.2).

In particular, taking $b \equiv 0$ we deduce the standard weak formulation of (2.3) in the form

$$\int_\Omega \Big(\varrho(\tau,\cdot)\varphi(\tau,\cdot) - \varrho_0\varphi(0,\cdot) \Big)\ {\rm d}x = \\ \int_0^\tau \int_\Omega \Big(\varrho\partial_t\varphi + \varrho\mathbf{u}\cdot\nabla_x\varphi \Big)\ {\rm d}x\ {\rm d}t \tag{5.4}$$

for any $\tau \in [0,T]$ and any $\varphi \in C_c^\infty([0,T] \times \overline{\Omega})$.

Note that (5.4) actually holds on the whole physical space R^3 provided ϱ, $\mathbf{u}$ were extended to be zero outside Ω.

5.2 Momentum equation

In addition to the previous assumptions we suppose that

$$\varrho\mathbf{u} \in C_{\rm weak}([0,T]; L^q(\Omega;R^3)) \text{ for a certain } q>1,\ p(\varrho) \in L^1((0,T)\times\Omega).$$

The weak formulation of the momentum equation reads:

$$\int_\Omega \Big(\varrho\mathbf{u}(\tau,\cdot)\cdot\varphi(\tau,\cdot) - (\varrho\mathbf{u})_0\cdot\varphi(0,\cdot)\Big)\ {\rm d}x \tag{5.5}$$
$$= \int_0^\tau \int_\Omega \Big(\varrho\mathbf{u}\cdot\partial_t\varphi + \varrho\mathbf{u}\otimes\mathbf{u} : \nabla_x\varphi + p(\varrho){\rm div}_x\varphi \Big)\ {\rm d}x\ {\rm d}t$$
$$- \int_0^\tau \int_\Omega \Big(\mu\nabla_x\mathbf{u} : \nabla_x\varphi + (\lambda+\mu){\rm div}_x\mathbf{u}{\rm div}_x\varphi - \varrho\mathbf{f}\cdot\varphi \Big)\ {\rm d}x\ {\rm d}t$$

for any $\tau \in [0,T]$ and for any test function $\varphi \in C_c^\infty([0,T]\times\Omega;R^3)$.

Note that (5.5) already includes the satisfaction of the initial condition

$$\varrho\mathbf{u}(0,\cdot) = (\varrho\mathbf{u})_0$$

5.3 Energy inequality

The weak solutions are not known to be uniquely determined by the initial data. Therefore it is desirable to introduce as much physically grounded conditions as allowed by the construction of the weak solutions. One of them is

Energy inequality:

$$\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho)\right)(\tau,\cdot)\ \mathrm{d}x + \int_0^\tau \int_\Omega \mu|\nabla_x \mathbf{u}|^2 + (\lambda+\mu)|\mathrm{div}_x \mathbf{u}|^2\ \mathrm{d}x\ \mathrm{d}t$$
$$\leq \int_\Omega \left(\frac{1}{2\varrho_0}|(\varrho\mathbf{u})_0|^2 + P(\varrho_0)\right)\ \mathrm{d}x + \int_0^\tau \int_\Omega \varrho\mathbf{f}\cdot\mathbf{u}\ \mathrm{d}x\ \mathrm{d}t \tag{5.6}$$

for a.a. $\tau \in (0,T)$, where

$$P(\varrho) = \varrho \int_1^\varrho \frac{p(z)}{z^2}\ \mathrm{d}z.$$

Some remarks are in order. To begin, given the specific choice of the pressure $p(\varrho) = a\varrho^\gamma$ and the fact that the total mass of the fluid is a constant of motion, the function $P(\varrho)$ in (5.6) can be taken as

$$P(\varrho) = \frac{a}{\gamma-1}\varrho^\gamma.$$

Next, we need a kind od *compatibility* condition between ϱ_0 and $(\varrho\mathbf{u})_0$ provided we allow the initial density to vanish on a nonempty set:

$$(\varrho\mathbf{u})_0 = 0 \text{ a.a. on the "vacuum" set } \{x \in \Omega \mid \varrho_0(x) = 0\}. \tag{5.7}$$

6 Weak sequential stability

The problem of *weak sequential stability* may be stated as follows:

Weak sequential stability:

Given a family $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon\}_{\varepsilon>0}$ of weak solutions of the compressible Navier-Stokes system, emanating from the initial data

$$\varrho_\varepsilon(0,\cdot) = \varrho_{0,\varepsilon},\ (\varrho\mathbf{u})(0,\cdot) = (\varrho\mathbf{u})_{0,\varepsilon},$$

we want to show that

$$\varrho_\varepsilon \to \varrho,\ \mathbf{u}_\varepsilon \to \mathbf{u} \text{ as } \varepsilon \to 0$$

in a certain sense and at least for suitable subsequences, where ϱ, $\mathbf{u}$ is another weak solution of the same system.

Although showing *weak sequential stability* does not provide an explicit proof of *existence* of the weak solutions, its verification represents one of the prominents steps towards a rigorous *existence theory* for a given system of equations.

6.1 Uniform bounds

To begin the analysis, we need *uniform* bounds in terms of the data. To this end, we choose the initial data in such a way that

$$\int_\Omega \left(\frac{1}{2\varrho_{0,\varepsilon}} |(\varrho\mathbf{u})_0|^2 + P(\varrho_{0,\varepsilon}) \right) \, \mathrm{d}x \le E_0, \tag{6.1}$$

where the constant E_0 is independent of ε. Moreover, the main and most difficult steps of the proof of weak sequential stability remain basically the same under the simplifying assumption

$$\mathbf{f} \equiv 0.$$

In accordance with the energy inequality (5.6), we get

$$\sup_{t\in(0,T)} \|\varrho_\varepsilon(t,\cdot)\|_{L^\gamma(\Omega)} \le c \tag{6.2}$$

and

$$\operatorname{ess}\sup_{t\in(0,T)} \|\sqrt{\varrho_\varepsilon}\mathbf{u}_\varepsilon(t,\cdot)\|_{L^2(\Omega;R^3))} \le c, \tag{6.3}$$

together with

$$\int_0^T \|\mathbf{u}(t,\cdot)\|^2_{W^{1,2}(\Omega;R^3)}\ \mathrm{d}t \le c, \tag{6.4}$$

where the symbol c stands for a generic constant independent of ε.

Interpolating (6.2), (6.3), we get

$$\|\varrho_\varepsilon \mathbf{u}_\varepsilon\|_{L^q(\Omega;R^3)} = \|\sqrt{\varrho_\varepsilon}\sqrt{\varrho_\varepsilon}\mathbf{u}_\varepsilon\|_{L^q(\Omega;R^3)} \le \|\sqrt{\varrho_\varepsilon}\|_{L^{2\gamma}(\Omega)}\|\sqrt{\varrho_\varepsilon}\mathbf{u}_\varepsilon\|_{L^2(\Omega;R^3)},$$

with

$$q = \frac{2\gamma}{\gamma+1} > 1 \text{ provided } \gamma > 1.$$

We conclude that

$$\mathrm{sup}_{t\in[0,T]}\|\varrho_\varepsilon \mathbf{u}_\varepsilon(t,\cdot)\|_{L^q(\Omega;R^3)},\ q = \frac{2\gamma}{\gamma+1}. \tag{6.5}$$

Next, applying a similar treatment to the convective term in the momentum equation, we have

$$\|\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon\|_{L^q(\Omega;R^{3\times 3})} = \|\varrho_\varepsilon \mathbf{u}_\varepsilon\|_{L^{2\gamma/(\gamma+1)}(\Omega;R^3)}\|\mathbf{u}_\varepsilon\|_{L^6(\Omega;R^3)},\ \text{with } q = \frac{6\gamma}{4\gamma+3}.$$

Using the standard *embedding relation*

$$W^{1,2}(\Omega) \hookrightarrow L^6(\Omega), \tag{6.6}$$

we may therefore conclude that

$$\int_0^T \int_\Omega \|\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon\|^2_{L^q(\Omega;R^{3\times 3})}\ \mathrm{d}x\ \mathrm{d}t \le c,\ q = \frac{6\gamma}{4\gamma+3}. \tag{6.7}$$

Note that

$$\frac{6\gamma}{4\gamma+3} > 1 \text{ as long as } \gamma > \frac{3}{2}.$$

Finally, we have the pressure estimates established in the previous part:

$$\int_0^T \int_\Omega p(\varrho_\varepsilon)\varrho_\varepsilon^\alpha\ \mathrm{d}x\ \mathrm{d}t = a\int_0^T \int_\Omega \varrho_\varepsilon^{\gamma+\alpha}\ \mathrm{d}x\ \mathrm{d}t \le c \text{ for } \alpha = \frac{2}{3}\gamma - 1. \tag{6.8}$$

6.2 Limit passage

In view of the uniform bounds established in the previous section, we may assume that

$$\varrho_\varepsilon \to \varrho \text{ weakly-(*) in } L^\infty(0,T;L^\gamma(\Omega)), \tag{6.9}$$

$$\mathbf{u}_\varepsilon \to \mathbf{u} \text{ weakly in } L^2(0,T;W_0^{1,2}(\Omega;R^3)) \tag{6.10}$$

passing to suitable subsequences as the case may be. Moreover, since ϱ_ε satisfies the equation of continuity (5.4), (6.9) may be strengthened to

$$\varrho_\varepsilon \to \varrho \text{ in } C_{\rm weak}([0,T];L^\gamma(\Omega)). \tag{6.11}$$

Let us recall that, in view of (6.9), relation (6.11) simply means

$$\left\{ t \mapsto \int_\Omega \varrho_\varepsilon(t,\cdot)\varphi \ {\rm d}x \right\} \to \left\{ t \mapsto \int_\Omega \varrho(t,\cdot)\varphi \ {\rm d}x \right\} \text{ in } C[0,T]$$

for any $\varphi \in C_c^\infty(\Omega)$.

6.3 Compactness of the convective term

Our next goal is to establish convergence of the convective terms. Recall that, in view of the estimate (6.5), we may suppose that

$$\varrho_\varepsilon \mathbf{u}_\varepsilon \to \overline{\varrho \mathbf{u}} \text{ weakly-(*) in } L^\infty(0,T;L^{2\gamma/(\gamma+1)}(\Omega;R^3))$$

and even

$$\varrho_\varepsilon \mathbf{u}_\varepsilon \to \overline{\varrho \mathbf{u}} \text{ in } C_{\rm weak}([0,T]:L^{2\gamma/(\gamma+1)}(\Omega;R^3)), \tag{6.12}$$

where the bar denotes (and will always denote in the future) a weak limit of a composition.

Our goal is to show that

$$\overline{\varrho \mathbf{u}} = \varrho \mathbf{u}.$$

This can be observed in several ways. Seeing that

$$W_0^{1,2}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega) \text{ compactly for } 1 \le q < 6,$$

we deduce that

$$L^p(\Omega) \hookrightarrow\hookrightarrow W^{-1,2}(\Omega) \text{ compactly whenever } p > \frac{6}{5}. \tag{6.13}$$

In particular, relation (6.12) yields

$$\varrho_\varepsilon \mathbf{u}_\varepsilon \to \overline{\varrho \mathbf{u}} \text{ in } C([0,T];W^{-1,2}(\Omega))),$$

which, combined with (6.10), gives rise to the desired conclusion

$$\overline{\varrho \mathbf{u}} = \varrho \mathbf{u}.$$

6.3.1 Compactness via Div-Curl lemma

Div-Curl lemma, developed by Murat and Tartar [10], [12], represents an efficient tool for handling compactness in non-linear problems, where the classical Rellich-Kondraschev argument is not applicable.

DIV-CURL LEMMA:

Lemma 6.1 *Let $B \subset R^M$ be an open set. Suppose that*

$$\mathbf{v}_n \to \mathbf{v} \text{ weakly in } L^p(B; R^3),$$

$$\mathbf{w}_n \to \mathbf{w} \text{ weakly in } L^q(B; R^3)$$

as $n \to \infty$, where

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} < 1.$$

Let, moreover,

$$\{\mathrm{div}[\mathbf{v}]\}_{n=1}^{\infty} \text{ be precompact in } W^{-1,s}(B),$$

$$\{\mathrm{curl}[\mathbf{w}]\}_{n=1}^{\infty} \text{ be precompact in } W^{-1,s}(B, R^{M\times M})$$

for a certain $s > 1$.

Then

$$\mathbf{v}_n \cdot \mathbf{w}_n \to \mathbf{v} \cdot \mathbf{w} \text{ weakly in } L^r(B).$$

We give the proof only for a very special case that will be needed in the future, namely, we assume that

$$\mathrm{div}\mathbf{v}_n = 0, \ \mathbf{w}_n = \nabla_x \Phi_n, \ \int_{R^M} \Phi_n \ \mathrm{d}y = 0. \tag{6.14}$$

Moreover, given the local character of the weak convergence, it is enough to show the result for $B = R^M$. By the same token, we may assume that all functions are compactly supported. We recall that a (scalar) sequence

$\{g_n\}_{n=1}^\infty$ is precompact in $W^{-1,s}(R^M)$ if

$$g_n = \mathrm{div}[\mathbf{h}_n], \text{ with } \{\mathbf{h}_n\}_{n=1}^\infty \text{ precompact in } L^s(R^M;R^M).$$

Now, it follows from the standard compactness arguments that

$$\Phi_n \to \Phi \text{ (strongly) in } L^q(R^M), \ \nabla_x \Phi = \mathbf{v}.$$

Taking $\varphi \in C_c^\infty(R^M)$ we have

$$\int_{R^M} \mathbf{v}_n \cdot \mathbf{w}_n \varphi \ \mathrm{d}y = \int_{R^M} \mathbf{v}_n \cdot \nabla_x \Phi_n \varphi \ \mathrm{d}y$$

$$= -\int_{R^M} \mathbf{v}_n \cdot \nabla_x \varphi \Phi_n \ \mathrm{d}y \to -\int_{R^M} \mathbf{v} \cdot \nabla_x \varphi \Phi \ \mathrm{d}y$$

$$= \int_{R^M} \mathbf{v} \cdot \mathbf{w} \varphi \ \mathrm{d}y,$$

which completes the proof under the symplifying hypothesis (6.14).

Now, compactness of the product term $\varrho_\varepsilon \mathbf{u}_\varepsilon$ can be viewed by a direct application of Div-Curl lemma in the space-time R^4, with the choice

$$\mathbf{v}_\varepsilon = [\varrho_\varepsilon, \varrho_\varepsilon \mathbf{u}_\varepsilon], \ \mathbf{w}_\varepsilon = [u_{j,\varepsilon}, 0, 0, 0], \ j = 1, 2, 3.$$

6.4 Passing to the limit - step 1

Now, combining (6.12), compactness of the embedding (6.13), and the fact that $\gamma > 3/2$, we may infer that

$$\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon \to \varrho \mathbf{u} \otimes \mathbf{u} \text{ weakly in } L^q((0,T) \times \Omega; R^3) \text{ for a certain } q > 1. \quad (6.15)$$

Summing up the previous discussion we deduce that the limit functions ϱ, $\mathbf{u}$ satisfy the equation of continuity

$$\int_\Omega \Big(\varrho(\tau, \cdot) \varphi(\tau, \cdot) - \varrho_0 \varphi(0, \cdot) \Big) \ \mathrm{d}x \quad (6.16)$$

$$= \int_0^\tau \int_\Omega \Big(\varrho \partial_t \varphi + \varrho \mathbf{u} \cdot \nabla_x \varphi \Big) \ \mathrm{d}x \ \mathrm{d}t$$

for any $\tau \in [0,T]$ and any $\varphi \in C_c^\infty([0,T] \times \overline{\Omega})$, together with a relation for the momentum

$$\int_\Omega \Big(\varrho \mathbf{u}(\tau, \cdot) \cdot \varphi(\tau, \cdot) - (\varrho \mathbf{u})_0 \cdot \varphi(0, \cdot) \Big) \ \mathrm{d}x \quad (6.17)$$

$$= \int_0^\tau \int_\Omega \Big(\varrho \mathbf{u} \cdot \partial_t \varphi + \varrho \mathbf{u} \otimes \mathbf{u} : \nabla_x \varphi + \overline{p(\varrho)} \mathrm{div}_x \varphi \Big) \ \mathrm{d}x \ \mathrm{d}t$$

$$- \int_0^\tau \int_\Omega \Big(\mu \nabla_x \mathbf{u} : \nabla_x \varphi + (\lambda + \mu) \mathrm{div}_x \mathbf{u} \mathrm{div}_x \varphi \Big) \ \mathrm{d}x \ \mathrm{d}t$$

for any test function $\varphi \in C_c^\infty([0,T] \times \Omega; R^3)$.

Here, we have also to assume at least weak convergence of the initial data, specifically,

$$\varrho_{0,\varepsilon} \to \varrho_0 \text{ weakly in } L^\gamma(\Omega), \tag{6.18}$$

$$(\varrho \mathbf{u})_{0,\varepsilon} \to (\varrho \mathbf{u})_0 \text{ weakly in } L^1(\Omega; R^3).$$

Thus it remains to show the crucial relation

$$\overline{p(\varrho)} = p(\varrho)$$

or, equivalently,

$$\varrho_\varepsilon \to \varrho \text{ a.a. in } (0,T) \times \Omega. \tag{6.19}$$

This will be carried over in a series of steps specified in the remaining part of this section.

6.5 Strong convergence of the densities

In order to simplify presentation and to highlight the leading ideas, we assume that

$$\gamma > 5,$$

in particular

$$\varrho_\varepsilon \to \varrho \text{ in } C_{\mathrm{weak}}(0,T; L^\gamma(\Omega)), \ \gamma > 5.$$

6.5.1 Renormalized equation

We start with the renormalized equation (5.3) with $b(\varrho) = \varrho \log(\varrho) - \varrho$:

$$\int_0^T \int_\Omega \Big((\varrho_\varepsilon \log(\varrho_\varepsilon) \, \partial_t \psi + \varrho_\varepsilon \mathrm{div}_x \mathbf{u}_\varepsilon \psi \Big) \ \mathrm{d}x \ \mathrm{d}t = - \int_\Omega \varrho_{0,\varepsilon} \log(\varrho_{0,\varepsilon}) \ \mathrm{d}x \tag{6.20}$$

for any $\psi \in C_c^\infty[0,T)$, $\psi(0)=1$. Clearly, relation (6.20) can be deduced from (5.3) by means of Lebesgue convergence theorem.

Passing to the limit for $\varepsilon \to 0$ in (6.20) and making use of (6.18) we get

$$\int_0^T \int_\Omega \Big(\Big(\overline{\varrho \log(\varrho)}\Big) \partial_t \psi + \overline{\varrho \mathrm{div}_x \mathbf{u}} \psi \Big) \ \mathrm{d}x \ \mathrm{d}t = - \int_\Omega \varrho_0 \log(\varrho_0) \ \mathrm{d}x. \tag{6.21}$$

Our next goal is to show that the limit functions ϱ, $\mathbf{u}$, besides (6.16), satisfy also its renormalized version. To this end, we use the procedure proposed by DiPerna and Lions [3], specifically, we regularize (6.16) by a family of regularizing kernels $\kappa_\delta(x)$ to obtain:

$$\partial_t \varrho_\delta + \mathrm{div}_x(\varrho_\delta \mathbf{u}) = \mathrm{div}_x(\varrho_\delta \mathbf{u}) - [\mathrm{div}_x(\varrho \mathbf{u})]_\delta,$$

with

$$v_\delta = \kappa_\delta * v, \ \text{where } * \text{ stands for spatial convolution.}$$

We easily deduce that

$$\partial_t b(\varrho_\delta) + \mathrm{div}_x(b(\varrho_\delta)\mathbf{u}) + \Big(b'(\varrho_\delta)\varrho_\delta - b(\varrho_\delta)\Big)\mathrm{div}_x \mathbf{u}$$

$$= b'(\varrho_\delta)\Big(\mathrm{div}_x(\varrho_\delta \mathbf{u}) - [\mathrm{div}_x(\varrho \mathbf{u})]_\delta\Big).$$

Taking the limit $\delta \to 0$ and using Friedrich's lemma, we get

$$\int_0^T \int_\Omega \Big(\varrho \log(\varrho) \partial_t \psi + \varrho \mathrm{div}_x \mathbf{u} \psi\Big) \ \mathrm{d}x \ \mathrm{d}t = - \int_\Omega \varrho_0 \log(\varrho_0) \ \mathrm{d}x;$$

whence, in combination with (6.21),

$$\int_0^T \int_\Omega \Big(\overline{\varrho \log(\varrho)} - \varrho \log(\varrho)\Big)\partial_t \psi + \Big(\overline{\varrho \mathrm{div}_x \mathbf{u}} - \varrho \mathrm{div}_x \mathbf{u}\Big)\psi\Big) \ \mathrm{d}x \ \mathrm{d}t = 0. \tag{6.22}$$

Assume, for a moment, that we can show

$$\int_0^\tau \int_\Omega \overline{\varrho \mathrm{div}_x \mathbf{u}} \ \mathrm{d}x \ \mathrm{d}t \geq \int_0^\tau \int_\Omega \varrho \mathrm{div}_x \mathbf{u} \ \mathrm{d}x \ t \text{ for any } \tau > 0, \tag{6.23}$$

which, together with lower semi-continuity of convex functionals, yields

$$\overline{\varrho \log(\varrho)} = \varrho \log(\varrho). \tag{6.24}$$

In order to continue, we need the following (standard) result:

Lemma 6.2 *Suppose that*

$$\varrho_\varepsilon \to \varrho \text{ weakly in } L^3(\Omega),$$

where

$$\overline{\varrho \log(\varrho)} = \varrho \log(\varrho).$$

Then

$$\varrho_\varepsilon \to \varrho \text{ in } L^1(\Omega).$$

Proof:

Suppose that

$$0 < \delta \leq \varrho, \ \ \varrho_\varepsilon \leq M.$$

Consequently, because of convexity of $z \mapsto z\log(z)$, we have

$$\varrho_\varepsilon \log(\varrho_\varepsilon) - \varrho \log(\varrho) = (\log(\varrho) - 1)\,(\varrho_\varepsilon - \varrho) + \alpha(M)|\varrho_\varepsilon - \varrho|^2, \ \ \alpha(\delta) > 0,$$

therefore

$$\int_{\{\delta<\varrho\}} |\varrho_\varepsilon - \varrho|^2 \ \mathrm{d}x \ \mathrm{d}t \leq \int_{\{\delta<\varrho,\varrho_\varepsilon<M\}} |\varrho_\varepsilon - \varrho|^2 \ \mathrm{d}x \ \mathrm{d}t + \int_{\{\delta<\varrho,\varrho_\varepsilon\geq M\}} |\varrho_\varepsilon - \varrho|^2 \ \mathrm{d}x \ \mathrm{d}t$$

$$\leq \int_{\{\delta<\varrho,\varrho_\varepsilon<M\}} |\varrho_\varepsilon - \varrho|^2 \ \mathrm{d}x \ \mathrm{d}t + h(M), \ \ h(M) \to 0 \text{ as } M \to \infty.$$

Thus we conclude that

$$\varrho_\varepsilon \to \varrho \text{ a.a. on the set } \{\varrho > \delta\} \text{ for any } \delta > 0.$$

Now, since

$$\varrho_\varepsilon \to \varrho \text{ a.a. on the set } \{\varrho = 0\}$$

and

$$|\{0 < \varrho < \delta\}| \to 0 \text{ as } \delta \to 0,$$

we obtain the desired conclusion.

Q.E.D.

In accordance with the previous discussion, the proof of strong (pointwise) convergence of $\{\varrho_\varepsilon\}_{\varepsilon>0}$ reduces to showing (6.23). This will be done in the next section.

6.5.2 The effective viscous flux

The effective viscous flux

$$(2\mu + \lambda)\mathrm{div}_x\mathbf{u} - p(\varrho)$$

is a remarkable quantity that enjoys better regularity and compactness properties than its components separately. To see this, we start with the momentum equation

$$\int_\Omega \Big(\varrho_\varepsilon \mathbf{u}_\varepsilon(\tau,\cdot)\cdot\varphi(\tau,\cdot) - (\varrho\mathbf{u})_{0,\varepsilon}\cdot\varphi(0,\cdot)\Big)\ \mathrm{d}x \tag{6.25}$$

$$= \int_0^\tau \int_\Omega \Big(\varrho_\varepsilon\mathbf{u}_\varepsilon\cdot\partial_t\varphi + \varrho_\varepsilon\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon : \nabla_x\varphi + p(\varrho_\varepsilon)\mathrm{div}_x\varphi\Big)\ \mathrm{d}x\ \mathrm{d}t$$

$$-\int_0^\tau \int_\Omega \Big(\mu\nabla_x\mathbf{u}_\varepsilon : \nabla_x\varphi + (\lambda+\mu)\mathrm{div}_x\mathbf{u}_\varepsilon\mathrm{div}_x\varphi - \varrho_\varepsilon\mathbf{f}\cdot\varphi\Big)\ \mathrm{d}x\ \mathrm{d}t,$$

together with its weak limit

$$\int_\Omega \Big(\varrho\mathbf{u}(\tau,\cdot)\cdot\varphi(\tau,\cdot) - (\varrho\mathbf{u})_0\cdot\varphi(0,\cdot)\Big)\ \mathrm{d}x \tag{6.26}$$

$$= \int_0^\tau \int_\Omega \Big(\varrho\mathbf{u}\cdot\partial_t\varphi + \varrho\mathbf{u}\otimes\mathbf{u} : \nabla_x\varphi + \overline{p(\varrho)}\mathrm{div}_x\varphi\Big)\ \mathrm{d}x\ \mathrm{d}t$$

$$-\int_0^\tau \int_\Omega \Big(\mu\nabla_x\mathbf{u} : \nabla_x\varphi + (\lambda+\mu)\mathrm{div}_x\mathbf{u}\mathrm{div}_x\varphi - \varrho\mathbf{f}\cdot\varphi\Big)\ \mathrm{d}x\ \mathrm{d}t.$$

Our goal is to take

$$\varphi = \varphi_\varepsilon = \phi\nabla_x\Delta^{-1}[1_\Omega\varrho_\varepsilon],\ \phi\in C^\infty_c(\Omega)$$

as a test function in (6.25), and

$$\varphi = \phi\nabla_x\Delta^{-1}[1_\Omega\varrho],\ \phi\in C^\infty_c(\Omega).$$

Here, Δ^{-1} represents the inverse of the Laplacean on R^3, specifically,

$$\partial_{x_j}\Delta^{-1}[v] = \mathcal{F}_{\xi\to x}\left[\frac{\mathrm{i}\xi_j}{|\xi|^2}\mathcal{F}_{x\to\xi}[v]\right]$$

Since $\Omega\subset R^3$ is a bounded domain, we have

$$\nabla_x\Delta^{-1}[1_\Omega\varrho_\varepsilon] \text{ bounded in } L^\infty(0,T;W_0^{1,\gamma}(\Omega;R^3)),\ \gamma>3.$$

Moreover, as $1_\Omega\varrho_\varepsilon$ as well as $1_\Omega\varrho$ satisfy the equation of continuity on the whole physical space R^3 provided $\mathbf{u}_\varepsilon$, $\mathbf{u}$ were extended to be zero outside Ω, we have

$$\partial_t\nabla_x\Delta^{-1}[1_\Omega\varrho_\varepsilon] = -\nabla_x\Delta^{-1}\mathrm{div}_x[\varrho_\varepsilon\mathbf{u}_\varepsilon],\ \partial_t\nabla_x\Delta^{-1}[1_\Omega\varrho] = -\nabla_x\Delta^{-1}\mathrm{div}_x[\varrho\mathbf{u}].$$

Step 1:

As

$$\varrho_\varepsilon\to\varrho\in C_{\mathrm{weak}}([0,T];L^\gamma(\Omega)),$$

we have, in accordance with the standard Sobolev embedding relation

$$W^{1,\gamma}(\Omega)\hookrightarrow\hookrightarrow C(\overline{\Omega}),$$

$$\nabla_x\Delta^{-1}[1_\Omega\varrho_\varepsilon]\to\nabla_x\Delta^{-1}[1_\Omega\varrho] \text{ in } C([0,T]\times\overline{\Omega}).$$

In particular, we deduce from (6.25), (6.26),

$$\begin{aligned}
&\lim_{\varepsilon\to0}\int_0^\tau\int_\Omega\Big(\varrho_\varepsilon\mathbf{u}_\varepsilon\cdot\partial_t\varphi_\varepsilon+\varrho_\varepsilon\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon:\nabla_x\varphi_\varepsilon+p(\varrho_\varepsilon)\mathrm{div}_x\varphi_\varepsilon\Big)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad-\int_0^\tau\int_\Omega\Big(\mu\nabla_x\mathbf{u}_\varepsilon:\nabla_x\varphi_\varepsilon+(\lambda+\mu)\mathrm{div}_x\mathbf{u}_\varepsilon\mathrm{div}_x\varphi_\varepsilon\Big)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad=\int_0^\tau\int_\Omega\Big(\varrho\mathbf{u}\cdot\partial_t\varphi+\varrho\mathbf{u}\otimes\mathbf{u}:\nabla_x\varphi+\overline{p(\varrho)}\mathrm{div}_x\varphi\Big)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad-\int_0^\tau\int_\Omega\Big(\mu\nabla_x\mathbf{u}:\nabla_x\varphi+(\lambda+\mu)\mathrm{div}_x\mathbf{u}\mathrm{div}_x\varphi\Big)\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}$$

with

$$\varphi=\phi\nabla_x\Delta^{-1}[1_\Omega\varrho],$$

meaning

$$\lim_{\varepsilon\to0}\int_0^\tau\int_\Omega\phi p(\varrho_\varepsilon)\varrho_\varepsilon-p(\varrho_\varepsilon)\nabla_x\phi\cdot\nabla_x\Delta^{-1}[1_\Omega\varrho_\varepsilon]\,\mathrm{d}x\,\mathrm{d}t \qquad (6.27)$$

$$-\lim_{\varepsilon\to 0}\int_0^\tau\int_\Omega \phi\left(\mu\nabla_x\mathbf{u}_\varepsilon : \nabla_x^2\Delta^{-1}[1_\Omega\varrho_\varepsilon] + (\lambda+\mu)\mathrm{div}_x\mathbf{u}_\varepsilon\varrho_\varepsilon\right)\ \mathrm{d}x\ \mathrm{d}t$$

$$-\lim_{\varepsilon\to 0}\int_0^\tau\int_\Omega \left(\mu\nabla_x\mathbf{u}_\varepsilon\cdot\nabla\phi\cdot\nabla_x\Delta^{-1}[1_\Omega\varrho_\varepsilon] + (\lambda+\mu)\mathrm{div}_x\mathbf{u}_\varepsilon\nabla_x\varphi\cdot\nabla_x\Delta^{-1}[1_\Omega\varrho_\varepsilon]\right)\mathrm{d}x\,\mathrm{d}t$$

$$=\int_0^\tau\int_\Omega \phi\overline{p(\varrho)}\varrho - \overline{p(\varrho)}\nabla_x\phi\cdot\nabla_x\Delta^{-1}[\varrho]\ \mathrm{d}x\ \mathrm{d}t$$

$$-\int_0^\tau\int_\Omega \phi\left(\mu\nabla_x\mathbf{u} : \nabla_x^2\Delta^{-1}[1_\Omega\varrho] + (\lambda+\mu)\mathrm{div}_x\mathbf{u}\varrho\right)\ \mathrm{d}x\ \mathrm{d}t$$

$$-\int_0^\tau\int_\Omega \left(\mu\nabla_x\mathbf{u}\cdot\nabla\phi\cdot\nabla_x\Delta^{-1}[1_\Omega\varrho] + (\lambda+\mu)\mathrm{div}_x\mathbf{u}\nabla_x\varphi\cdot\nabla_x\Delta^{-1}[1_\Omega\varrho]\right)\ \mathrm{d}x\ \mathrm{d}t$$

$$+\lim_{\varepsilon\to 0}\int_0^\tau\int_\Omega \left(\phi\varrho_\varepsilon\mathbf{u}_\varepsilon\cdot\nabla_x\Delta^{-1}[\mathrm{div}_x(\varrho_\varepsilon\mathbf{u}_\varepsilon)] - \varrho_\varepsilon\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon : \nabla_x\left(\phi\nabla_x\Delta^{-1}[1_\Omega\varrho_\varepsilon]\right)\right)\mathrm{d}x\,\mathrm{d}t$$

$$-\int_0^\tau\int_\Omega \left(\phi\varrho\mathbf{u}\cdot\nabla_x\Delta^{-1}[\mathrm{div}_x(\varrho\mathbf{u})] - \varrho\mathbf{u}\otimes\mathbf{u} : \nabla_x\left(\phi\nabla_x\Delta^{-1}[1_\Omega\varrho]\right)\right)\ \mathrm{d}x\ \mathrm{d}t.$$

Step 2:

We have

$$\int_\Omega \phi\nabla_x\mathbf{u}_\varepsilon : \nabla_x^2\Delta^{-1}[1_\Omega\varrho_\varepsilon]\ \mathrm{d}x = \int_\Omega\phi\sum_{i,j=1}^3\left(\partial_{x_j}u_\varepsilon^i[\partial_{x_i}\Delta^{-1}\partial_{x_j}][1_\Omega\varrho_\varepsilon]\right)\ \mathrm{d}x$$

$$=\int_\Omega\sum_{i,j=1}^3\left(\partial_{x_j}(\phi u_\varepsilon^i)[\partial_{x_i}\Delta^{-1}\partial_{x_j}][1_\Omega\varrho_\varepsilon]\right)\ \mathrm{d}x - \int_\Omega\sum_{i,j=1}^3\left(\partial_{x_j}\phi u_\varepsilon^i[\partial_{x_i}\Delta^{-1}\partial_{x_j}][1_\Omega\varrho_\varepsilon]\right)\ \mathrm{d}x$$

$$=\int_\Omega\phi\mathrm{div}_x\mathbf{u}_\varepsilon\varrho_\varepsilon\ \mathrm{d}x - \int_\Omega\nabla_x\phi\cdot\mathbf{u}\varrho_\varepsilon\ \mathrm{d}x - \int_\Omega\sum_{i,j=1}^3\left(\partial_{x_j}\phi u_\varepsilon^i[\partial_{x_i}\Delta^{-1}\partial_{x_j}][1_\Omega\varrho_\varepsilon]\right)\ \mathrm{d}x.$$

Consequently, going back ot (6.27) and dropping the compact terms, we obtain

$$\lim_{\varepsilon\to 0}\int_0^\tau\int_\Omega \phi\Big(p(\varrho_\varepsilon)\varrho_\varepsilon - (\lambda+2\mu)\mathrm{div}_x\mathbf{u}_\varepsilon\varrho_\varepsilon\Big)\ \mathrm{d}x\ \mathrm{d}t \tag{6.28}$$

$$\int_0^\tau\int_\Omega \phi\Big(\overline{p(\varrho)}\varrho - (\lambda+2\mu)\mathrm{div}_x\mathbf{u}\varrho\Big)\ \mathrm{d}x\ \mathrm{d}t$$

$$=\lim_{\varepsilon\to 0}\int_0^\tau\int_\Omega \phi\left(\varrho_\varepsilon\mathbf{u}_\varepsilon\cdot\nabla_x\Delta^{-1}[\mathrm{div}_x(\varrho_\varepsilon\mathbf{u}_\varepsilon)] - \varrho_\varepsilon\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon : \nabla_x\Delta^{-1}\nabla_x[1_\Omega\varrho_\varepsilon]\right)\mathrm{d}x\,\mathrm{d}t$$

$$-\int_0^\tau\int_\Omega \left(\phi\varrho\mathbf{u}\cdot\nabla_x\Delta^{-1}[\mathrm{div}_x(\varrho\mathbf{u})] - \varrho\mathbf{u}\otimes\mathbf{u} : \nabla_x\Delta^{-1}\nabla_x[1_\Omega\varrho]\right)\ \mathrm{d}x\ \mathrm{d}t.$$

Step 3

Our ultimate goal is to show that the right-hand side of (6.28) vanishes. To this end, we write

$$\varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla_x \Delta^{-1}[\mathrm{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon)] - \varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : \nabla_x \Delta^{-1} \nabla_x [1_\Omega \varrho_\varepsilon]$$

$$= \mathbf{u}_\varepsilon \cdot \Big(\varrho_\varepsilon \nabla_x \Delta^{-1}[\mathrm{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon)] - \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla_x \Delta^{-1} \nabla_x [1_\Omega \varrho_\varepsilon] \Big) .$$

Consider the bilinear form

$$[\mathbf{v}, \mathbf{w}] = \sum_{i,j=1}^{3} \Big(v^i \mathcal{R}_{i,j}[w^j] - w^i \mathcal{R}_{i,j}[v^j] \Big),\ \mathcal{R}_{i,j} = \partial_{x_i} \Delta^{-1} \partial_{x_j},$$

where we may write

$$\sum_{i,j=1}^{3} \Big(v^i \mathcal{R}_{i,j}[w^j] - w^i \mathcal{R}_{i,j}[v^j] \Big)$$

$$\sum_{i,j=1}^{3} \Big((v^i - \mathcal{R}_{i,j}[v^j]) \mathcal{R}_{i,j}[w^j] - (w^i - \mathcal{R}_{i,j}[w^j]) \mathcal{R}_{i,j}[v^j] \Big)$$

$$= \mathbf{U} \cdot \mathbf{V} - \mathbf{W} \cdot \mathbf{Z},$$

where

$$U^i = \sum_{j=1} (v^i - \mathcal{R}_{i,j}[v^j]),\ W^i = \sum_{j=1} (w^i - \mathcal{R}_{i,j}[w^j]),\ \mathrm{div}_x \mathbf{U} = \mathrm{div}_x \mathbf{W} = 0,$$

and

$$V^i = \partial_{x_i} \left(\sum_{j=1}^{3} \Delta^{-1} \partial_{x^j} w^j \right),\ Z^i = \partial_{x_i} \left(\sum_{j=1}^{3} \Delta^{-1} \partial_{x^j} v^j \right),\ i = 1, 2, 3.$$

Thus a direct application of Div-Curl lemma (Lemma 6.1) yields

$$[\mathbf{v}_\varepsilon, \mathbf{w}_\varepsilon] \to [\mathbf{v}, \mathbf{w}] \text{ weakly in } L^s(R^3)$$

whenever $\mathbf{v}_\varepsilon \to \mathbf{v}$ weakly in $L^p(R^3; r^3)$, $\mathbf{w}_\varepsilon \to \mathbf{w}$ weakly in $L^q(R^3; R^3)$,

and

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{s} < 1.$$

Seeing that

$$\varrho_\varepsilon \to \varrho \text{ in } C_{\rm weak}([0,T];L^\gamma(\Omega)),\ \ \varrho_\varepsilon \mathbf{u}_\varepsilon \to \mathbf{u} \text{ in } C_{\rm weak}([0,T];L^{2\gamma/(\gamma+1)}(\Omega))$$

we conclude that

$$1_\Omega \varrho_\varepsilon(t,\cdot)\nabla_x\Delta^{-1}[{\rm div}_x(\varrho_\varepsilon\mathbf{u}_\varepsilon)(t,\cdot)] - (\varrho_\varepsilon\mathbf{u}_\varepsilon)(t,\cdot)\cdot\nabla_x\Delta^{-1}\nabla_x[1_\Omega\varrho_\varepsilon(t,\cdot)] \tag{6.29}$$
$$\to$$
$$\varrho(t,\cdot)\nabla_x\Delta^{-1}[{\rm div}_x(\varrho\mathbf{u})(t,\cdot)] - (\varrho\mathbf{u})(t,\cdot)\cdot\nabla_x\Delta^{-1}\nabla_x[1_\Omega\varrho(t,\cdot)]$$
$$\text{in } L^s(\Omega) \text{ for all } t\in[0,T],$$

with

$$s = \frac{2\gamma}{\gamma+3} > \frac{6}{5} \text{ since } \gamma > 5.$$

Thus we conclude that the convergence in (6.29) takes place in the space

$$L^q(0,T;W^{-1,2}(\Omega)) \text{ for any } 1\le q<\infty;$$

whence, going back to (6.28), we conclude

$$\lim_{\varepsilon\to 0}\int_0^\tau\int_\Omega \phi\Big(p(\varrho_\varepsilon)\varrho_\varepsilon - (\lambda+2\mu){\rm div}_x\mathbf{u}_\varepsilon\varrho_\varepsilon\Big)\ {\rm d}x\ {\rm d}t \tag{6.30}$$
$$\int_0^\tau\int_\Omega \phi\Big(\overline{p(\varrho)}\varrho - (\lambda+2\mu){\rm div}_x\mathbf{u}\varrho\Big)\ {\rm d}x\ {\rm d}t.$$

As a matter of fact, using exactly same method and localizing also in the space variable, we could prove that

$$\overline{p(\varrho)\varrho} - (\lambda+2\mu)\overline{{\rm div}_x\mathbf{u}\varrho} = \overline{p(\varrho)}\varrho - (\lambda+2\mu){\rm div}_x\mathbf{u}\varrho, \tag{6.31}$$

which is the celebrated relation on "weak continuity" of the effective viscous pressure discoverd by Lions [9].

Since p is a non-decreasing function of ϱ_ε, we have

$$\int_0^\tau\int_\Omega \phi\Big(p(\varrho_\varepsilon)-p(\varrho)\Big)(\varrho_\varepsilon-\varrho)\ {\rm d}x\ {\rm d}t \ge 0;$$

where relation (6.30) yield the desired conclusion (6.23), namely

$$\int_0^\tau\int_\Omega \Big(\overline{{\rm div}_x\mathbf{u}\varrho} - {\rm div}_x\mathbf{u}\varrho\Big)\ {\rm d}x\ {\rm d}t \ge 0.$$

Thus we get (6.24); whence

$$\varrho_\varepsilon \to \varrho \text{ a.a. in } (0,T)\times\Omega. \tag{6.32}$$

References

[1] G.-Q. Chen and H. Frid. On the theory of divergence-measure fields and its applications. *Bol. Soc. Brasil. Mat. (N.S.)*, 32(3):401–433, 2001. Dedicated to Constantine Dafermos on his 60th birthday.

[2] A. J. Chorin and J. E. Marsden. *A mathematical introduction to fluid mechanics*. Springer-Verlag, New York, 1979.

[3] R.J. DiPerna and P.-L. Lions. Ordinary differential equations, transport theory and Sobolev spaces. *Invent. Math.*, **98**:511–547, 1989.

[4] D. B. Ebin. Viscous fluids in a domain with frictionless boundary. *Global Analysis - Analysis on Manifolds, H. Kurke, J. Mecke, H. Triebel, R. Thiele Editors, Teubner-Texte zur Mathematik 57, Teubner, Leipzig*, pages 93–110, 1983.

[5] C. L. Fefferman. Existence and smoothness of the Navier-Stokes equation. In *The millennium prize problems*, pages 57–67. Clay Math. Inst., Cambridge, MA, 2006.

[6] E. Feireisl. *Dynamics of viscous compressible fluids*. Oxford University Press, Oxford, 2004.

[7] D. Hoff. Dynamics of singularity surfaces for compressible viscous flows in two space dimensions. *Commun. Pure Appl. Math.*, **55**:1365–1407, 2002.

[8] J. Leray. Sur le mouvement d'un liquide visqueux emplissant l'espace. *Acta Math.*, **63**:193–248, 1934.

[9] P.-L. Lions. *Mathematical topics in fluid dynamics, Vol.2, Compressible models*. Oxford Science Publication, Oxford, 1998.

[10] F. Murat. Compacité par compensation. *Ann. Sc. Norm. Sup. Pisa, Cl. Sci. Ser. 5*, IV:489–507, 1978.

[11] N. V. Priezjev and S.M. Troian. Influence of periodic wall roughness on the slip behaviour at liquid/solid interfaces: molecular versus continuum predictions. *J. Fluid Mech.*, **554**:25–46, 2006.

[12] L. Tartar. Compensated compactness and applications to partial differential equations. *Nonlinear Anal. and Mech., Heriot-Watt Sympos., L.J. Knopps editor, Research Notes in Math 39, Pitman, Boston*, pages 136–211, 1975.

[13] V. A. Vaigant and A. V. Kazhikhov. On the existence of global solutions to two-dimensional Navier-Stokes equations of a compressible viscous fluid (in Russian). *Sibirskij Mat. Z.*, **36**(6):1283–1316, 1995.