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Jiří Jarušek

Igor Bock

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I. Bock and J. Jarušek*

Abstract. The existence of solutions is proved for the unilateral dynamic contact of a von Kármán-Donnell shell with a rigid obstacle. Both purely elastic material and a material with a singular memory are treated.

Keywords. von Kármán-Donnell shell, unilateral dynamic contact, elasticity, singular memory, solvability, penalty approximation

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1 Introduction and notation

The shells belong to rather complicated thin-walled structures because of their curved form. However this form enables wide applicability in the technical practice e.g. roofs, airplane constructions etc. We assume nonlinear strain-displacement relations due to the von Kármán-Donnell theory ([15], [16]) of moderately large deflections :

$$\varepsilon_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i + \partial_i u_3 \partial_j u_3) - k_{ij} u_3 - z\partial_{ij} u_3,\ i,j = 1,2,\ k_{12} = 0$$

with plane displacements u_i, curvatures $k_{ii} > 0,\ i = 1,2$; of a middle surface $\mathscr{S}$ and a bending function u_3.

This model is also called the Donnell-Mushtari-Vlasov model. Donnell [7], Mushtari [11] and Vlasov [14] independently developed a simplified nonlinear engineering theory of thin shells generalizing a similar von Kármán model for plates [10].

A thin isotropic shallow shell occupies the domain

$$G = \{(x,z) \in \mathbb{R}^3 :\ x = (x_1,x_2) \in \Omega,\ |z - \mathscr{S}| < h/2\},$$

where $h > 0$ is the thickness of the shell, Ω is a bounded simply connected domain in $\mathbb{R}^2$ with a sufficiently smooth boundary Γ and a unit outer normal vector $\boldsymbol{n}$. More precise regularity assumptions for Γ will be specified later.

Dynamic contact problems represent the most natural type of contact problems. The studies of their solvability started in the late '70ies of the last century when the elastic strings were treated. Later in '90ies the membranes and bodies were investigated provided the material has some kind of viscosity. For those results cf. [8]. The dynamic contact for thin-walled structures has been studied in the last years ([1], [2], [3], [4], [5], [6]) for

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beams and plates. This paper seams to be the first in the literature concentrated to a model of shells.

In the sequel, fine estimates based on imbedding and interpolation of different kind of Sobolev-type spaces are needed. We use the following notation for them: by $W_p^k(M)$ with $k \geq 0$ and $p \in [1, \infty]$ the Sobolev (for a noninteger k the Sobolev-Slobodetskii) spaces are denoted provided they are defined on a domain or an appropriate manifold M. By $\mathring{W}_p^k(M)$ we denote the spaces with zero traces on ∂M. If $p = 2$ we use the notation $H^k(M)$, $\mathring{H}^k(M)$. For the anisotropic spaces the notation $W_p^k(M)$ $k = (k_1, k_2) \in \mathbb{R}_+^2$ signifies that k_1 is related with the time while k_2 with the space variables (with the obvious consequences for $p = 2$) provided M is a time-space domain. The duals to $\mathring{H}^k(M)$ are denoted by $H^{-k}(M)$. We shall use also the Bochner-type spaces $W_p^k(I; X)$ for a time interval I and a Banach space X. Let us remark that for $k \in (0,1)$ their norm is defined by the relation

$$\|w\|_{W_p^k(I;X)}^p \equiv \int_I \|w(t)\|_X^p dt + \int_I \int_I \frac{\|w(t) - w(s)\|_X^p}{|s-t|^{1+kp}} ds\, dt.$$

By $C(M)$ and $B(M)$ the spaces of continuous and bounded functions on a (possibly relatively) compact manifold M are denoted, respectively. Both are assumed to be equipped with the sup-norm. Analogously the spaces $C(M; X)$, $B(M; X)$ are introduced for a Banach space X.

2 The elastic material

2.1 Problem formulation and preliminaries

We assume the Einstein summation convention through the whole paper. The isotropic elastic stress-strain relations are

$$\sigma_{ij} = \frac{E}{1-\nu^2}[(1-\nu)\varepsilon_{ij} + \nu\delta_{ij}\varepsilon_{kk}], \ i, j \in \{1, 2\}$$

with Young modulus $E > 0$ and Poisson ratio $\nu \in (0, \frac{1}{2})$. The shell is clamped on its boundary and subjected to the load perpendicular to Ω. The system for the deflection $u_3 \equiv u$ of a shell middle surface and the Airy stress function v with an inner obstacle has the form (see [15] for the case without contact)

$$\ddot{u} + \frac{h^2 E}{12\varrho(1-\nu^2)}\Delta^2 u - \Delta_{\boldsymbol{k}}^* v - [u, v] = f + g, \tag{1}$$

$$u - \Psi \geq 0, \ g \geq 0, \ (u - \Psi)g = 0, \tag{2}$$

$$\Delta^2 v = -\frac{E}{\varrho}\left(\tfrac{1}{2}[u, u] + \Delta_{\boldsymbol{k}} u\right) \tag{3}$$

with boundary value conditions

$$u = \partial_n u = v = \partial_n v = 0, \tag{4}$$

and initial conditions

$$u(0, \cdot) = u_0, \ \dot{u}(0, \cdot) = u_1. \tag{5}$$

Here

$$[u, v] \equiv \partial_{11}u\partial_{22}v + \partial_{22}u\partial_{11}v - 2\partial_{12}u\partial_{12}v,$$

$\rho > 0$ is the material density, $\Psi \in B(\Omega)$ is the obstacle function fulfilling $0 < U_0 \leq u_0 - \Psi$ in Ω with a constant U_0 and

$$\Delta_{\boldsymbol{k}} u \equiv \frac{\partial^2}{\partial x_1^2}(k_{22}u) + \frac{\partial^2}{\partial x_2^2}(k_{11}u), \tag{6}$$

$$\Delta_{\boldsymbol{k}}^* v \equiv k_{22}\frac{\partial^2 v}{\partial x_1^2} + k_{11}\frac{\partial^2 v}{\partial x_2^2}, \tag{7}$$

with curvatures $k_{ii} \in H^2(\Omega)$, $i = 1, 2$. We remark that the constant curvatures are considered in [15] and in other references.

We can define the operators $L : H^2(\Omega) \to \mathring{H}^2(\Omega)$, $\Phi : H^2(\Omega) \times H^2(\Omega) \to \mathring{H}^2(\Omega)$ by uniquely solved equations

$$(\Delta L u, \Delta w) \equiv (\Delta_{\boldsymbol{k}} u, w)\,\forall w \in \mathring{H}^2(\Omega), \tag{8}$$

$$(\Delta \Phi(u, v), \Delta w) \equiv ([u, v], w)\,\forall w \in \mathring{H}^2(\Omega). \tag{9}$$

with the inner product $(\cdot, \cdot)$ in the space $L_2(\Omega)$. The operator L is linear and compact. The bilinear operator Φ is symmetric and compact. Moreover due to Lemma 1 from [9] $\Phi : H^2(\Omega)^2 \to W_p^2(\Omega)$, $2 < p < \infty$ and

$$\|\Phi(u, v)\|_{W_p^2(\Omega)} \leq c\|u\|_{H^2(\Omega)}\|v\|_{W_p^1(\Omega)}\ \forall u \in H^2(\Omega),\, v \in W_p^1(\Omega). \tag{10}$$

Remark 1 *In order to apply Lemma 1 from [9] containing the estimate (10) we need the regularity $v \in H^3(\Omega)$ for a weak solution of the Dirichlet problem*

$$\triangle^2 v = g \ \text{ on } \Omega,\ v = \partial_n v = 0 \ \text{ on } \Gamma,\ g \in H^{-1}(\Omega).$$

The regularity result for $C^{3,1}$ domain Ω is due to Theorem 2.2, Chapter 4 from [12]. In the case of convex polygonal domain we apply Theorem 2.1 from [13]. With the use of the translation method in [8] $C^{2+\varepsilon}$ smoothness of the boundary for any $\varepsilon > 0$ is sufficient.

We introduce the space $\mathring{\mathscr{H}} \equiv L_\infty(I; \mathring{H}^2(\Omega))$ and a (shifted) cone $\mathscr{C}$ as

$$\mathscr{C} := \{y \in \mathring{\mathscr{H}};\ y \geq \Psi \text{ for a.e. } t \in I\}. \tag{11}$$

Using the operators L and Φ we can eliminate a function v in the equation (1) and express it in a form

$$\ddot{u} + a\Delta^2 u + b\left(\Delta_{\boldsymbol{k}}^*(Lu + \tfrac{1}{2}\Phi(u, u)) + [u, Lu + \tfrac{1}{2}\Phi(u, u)]\right) = f + g \tag{12}$$

with $a = h^2 E/(12\varrho(1 - \nu^2))$ and $b = E/\varrho$.

For $u,\, y \in L_2(I; H^2(\Omega))$ we define the following bilinear form for $u, y \in L_2(I; H^2(\Omega))$:

$$A : (u, y) \mapsto a\big(\partial_{kk} u \partial_{kk} y + \nu(\partial_{11} u \partial_{22} y + \partial_{22} u \partial_{11} y) + 2(1 - \nu)\partial_{12} u \partial_{12} y\big) \tag{13}$$

almost everywhere on Q. With the use of the form A and of the relations

$$([u, v], w) = (u, [v, w])\ \forall u,\ v,\ w\ \in \mathring{H}^2(\Omega),$$
$$(\Delta_k^* u, v) = (u, \Delta_k v),\ \ \forall u,\ v\ \in \mathring{H}^2(\Omega)$$

this leads to the following variational formulation.

Problem $\mathscr{P}^{\rm e}$. *We look for* $u \in \mathscr{C}$ *such that* $\ddot{u} \in \mathring{\mathscr{H}}^*$, *the initial conditions* (5) *are satisfied in a certain generalized sense, and there holds the inequality*

$$\begin{aligned}
&\langle \ddot{u}, y-u\rangle_Q \\
&\qquad + \int_Q \big(A(u,y-u) + b\Delta(Lu + \tfrac{1}{2}\Phi(u,u))\Delta(L(y-u)+\Phi(u,y-u))\big)\,dx\,dt \\
&\qquad \geq \int_Q f(y-u)\ \forall\, y \in \mathscr{C}.
\end{aligned} \tag{14}$$

Here $\langle\cdot,\cdot\rangle_Q$ denotes the duality pairing between $\mathring{\mathscr{H}}$ and $\mathring{\mathscr{H}}^*$ as an extension of the scalar product in $L_2(Q)$.

In the sequel we shall use the following imbedding and interpolation results from [8], Chapter 2.

Theorem 2 (Embedding theorem) *Let* $M \subset \mathbb{R}^N$ *be a bounded domain with a Lipschitz boundary. Let* $p, q \in (1,\infty)$, $\gamma \in [0,1)$ *and* $\alpha \in (\gamma, 1]$ *be numbers such that the inequality*

$$\frac{1}{\alpha}\left(\frac{N}{p} - \frac{N}{q} + \gamma\right) \leq 1, \tag{15}$$

holds. Then the Sobolev–Slobodetskii space $W_p^\alpha(M)$ *is continuously embedded into* $W_q^\gamma(M)$.

If inequality (15) *is strict, then the embedding is compact for any real* $q \geq 1$. *For* $q = \infty$ *this is true under the convention* $1/q = 0$.

Corollary 3 *Let* M *and* I *be as above. Let* p_i, q_i *belong to* $(1,+\infty)$, α_i *belong to* $(0,1]$ *and* γ_i *to* $[0,\alpha_i)$, $i = 1,2$. *Assume that* (15) *holds with* $i = 1$ *and* N *replaced by* 1 *and that it simultaneously holds for* $i = 2$. *Then* $W_{p_1}^{\alpha_1}\big(I; W_{p_2}^{\alpha_2}(M)\big)$ *can be imbedded into* $W_{q_1}^{\gamma_1}\big(I; W_{q_2}^{\gamma_2}(M)\big)$. *If both inequalities are strict, the imbedding is compact.*

The last assertion still holds if q_i *is infinite, provided we use the convention* $1/q_i = 0$, $i = 1,2$.

Theorem 4 (Interpolation theorem) *Let* M *be as above, let* k_1, k_2 *belong to* $[0,+\infty)$, *let* p_1, p_2 *belong to* $(1,+\infty)$ *and* Θ *to* $[0,1]$. *Then there exists a constant* c *such that for all* $u \in W_{p_1}^{k_1}(M) \cap W_{p_2}^{k_2}(M)$ *the following estimate holds*

$$\|u\|_{W_p^k(M)} \leq c\|u\|^{\Theta}_{W_{p_1}^{k_1}(M)} \|u\|^{1-\Theta}_{W_{p_2}^{k_2}(M)}$$

with $k = \Theta k_1 + (1-\Theta)k_2$ *and* $\dfrac{1}{p} = \dfrac{\Theta}{p_1} + \dfrac{1-\Theta}{p_2}$. *The assertion remains true if* $k_1 = k_2 = 0$ *and* p_1, p_2 *belong to* $[1,+\infty]$.

Corollary 5 (Generalization) *Let* M, k_1, k_2, p_1, p_2 *be as above. Let* I *be a bounded interval in* R, *let* κ_1, κ_2 *belong to* $[0,1]$, *let* q_1, q_2 *belong to* $(1,+\infty)$ *and* Θ *to* $[0,1]$. *Then there exists a constant* c *such that for all* $u \in W_{q_1}^{\kappa_1}\big(I; W_{p_1}^{k_1}(M)\big) \cap W_{q_2}^{\kappa_2}\big(I; W_{p_2}^{k_2}(M)\big)$ *it holds*

$$\|u\|_{W_q^\kappa(I;W_p^k(M))} \leq c\|u\|^{\Theta}_{W_{q_1}^{\kappa_1}\left(I;W_{p_1}^{k_1}(M)\right)} \|u\|^{1-\Theta}_{W_{q_2}^{\kappa_2}\left(I;W_{p_2}^{k_2}(M)\right)},$$

where $k = \Theta k_1 + (1-\Theta)k_2$, $\kappa = \Theta\kappa_1 + (1-\Theta)\kappa_2$, $\dfrac{1}{q} = \dfrac{\Theta}{q_1} + \dfrac{1-\Theta}{q_2}$ *and* $\dfrac{1}{p} = \dfrac{\Theta}{p_1} + \dfrac{1-\Theta}{p_2}$. *If* $\kappa_1 = \kappa_2 = 0$ *and* q_1, q_2 *belong to* $[1,+\infty]$, *the assertion still holds.*

2.2 Penalty approximation

We prove the existence of solutions to Problem $\mathscr{P}^{\mathrm{e}}$ using the penalization method. For any $\eta > 0$ we formulate the *penalized*

Problem $\mathscr{P}^{\mathrm{e}}_{\eta}$. *We look for $u \in \mathring{\mathscr{H}}$ such that $\ddot{u} \in L_2(I; \mathring{H}^2(\Omega))^*$, the equation*

$$\begin{aligned}&\int_I \langle \ddot{u}, z\rangle_\Omega \, dt + \int_Q \big(A(u,z) + b\Delta(Lu + \tfrac{1}{2}\Phi(u,u))\Delta(Lz + \Phi(u,z))\big)\, dx\, dt = \\ &\int_Q \big(\eta^{-1}(u-\Psi)^- + f\big)\, z\, dx\, dt.\end{aligned} \tag{16}$$

holds for any $z \in L_2(I; \mathring{H}^2(\Omega))$ and the initial conditions (5) *are satisfied.*

We have denoted by $\langle \cdot, \cdot \rangle_\Omega$ the duality between the spaces $H^{-2}(\Omega)$ and $\mathring{H}^2(\Omega)$ as the extension of the inner product in $L_2(\Omega)$.

Theorem 6 *Let $f \in L_2(Q)$, $u_0 \in \mathring{H}^2(\Omega)$, $u_1 \in L_2(\Omega)$. Then there exists a solution u of the Problem $\mathscr{P}^{e}_{\eta}$.*

Proof. Let us denote by $\{w_i \in \mathring{H}^2(\Omega);\ i \in \mathbb{N}\}$ an orthonormal in $L_2(\Omega)$ basis of $\mathring{H}^2(\Omega)$. We construct the Galerkin approximation u_m of a solution in a form

$$u_m(t) = \sum_{i=1}^{m} \alpha_i(t) w_i,\ \ \alpha_i(t) \in \mathbb{R},\ \ i = 1, ..., m,\ \ m \in \mathbb{N},$$

such that

$$\begin{aligned}&\int_\Omega \big(\ddot{u}_m(t) w_i + A(u_m(t), w_i) \\ &\qquad + b\Delta(Lu_m(t) + \tfrac{1}{2}\Phi(u_m(t), u_m(t)))\Delta(Lw_i + \Phi(u_m(t), w_i))\big)\, dx \\ &= \int_\Omega \big(\eta^{-1}(u_m(t) - \Psi)^- + f(t)\big)\, w_i\, dx,\ \ i = 1, ..., m,\end{aligned} \tag{17}$$

$$u_m(0) = u_{0m},\ \ \dot{u}_m(0) = u_{1m},\ \ u_{0m} \to u_0 \text{ in } \mathring{H}^2(\Omega) \text{ and } u_{1m} \to u_1 \text{ in } L_2(\Omega). \tag{18}$$

The system (17) can be expressed in the form

$$\ddot{\alpha}_i = F_i(t, \alpha_1, ..., \alpha_m),\ i = 1, ..., m.$$

Its right-hand side satisfies the conditions for the local existence of a solution fulfilling the initial conditions corresponding to the functions u_{0m}, u_{1m}. Hence there exists a Galerkin approximation $u_m(t)$ defined on some interval $I_m \equiv [0, t_m]$, $0 < t_m < T$. After multiplying the equation (17) by $\dot{\alpha}_i(t)$, summing up with respect to i and integrating we obtain for $Q_m = I_m \times \Omega$ the relation

$$\begin{aligned}&\int_{Q_m} \tfrac{1}{2}\partial_t\big(\dot{u}_m^2 + (Au_m, u_m) + b\big(\Delta(Lu_m + \tfrac{1}{2}\Phi(u_m, u_m)\big)^2 \\ &\qquad + \eta^{-1}((u_m - \Psi)^-)^2\big)\, dx\, dt = \int_{Q_m} f\dot{u}_m\, dx\, dt\end{aligned} \tag{19}$$

which leads to the estimate

$$\begin{aligned}&\|\dot{u}_m\|_{B(I;L_2(\Omega))} + \|u_m\|_{B(I;\mathring{H}^2(\Omega))} + \|\Phi(u_m, u_m)\|_{B(I;\mathring{H}^2(\Omega))} + \\ &\eta^{-1}\|(u_m - \Psi)^-\|_{B(I;L_2(\Omega))} \le c \equiv c(f, u_0, u_1).\end{aligned} \tag{20}$$

We could set above the whole interval I instead of local I_m as the right-hand side of the estimate (20) does not depend on m. Moreover

$$\|\varPhi(u_m,u_m)\|_{B(I;W_p^2(\varOmega))} \le c_p \equiv c_p(f,u_0,u_1)\,\forall\, p>2 \tag{21}$$

due to (10). The estimate (21) further implies

$$\begin{gathered}[u_m,\varPhi(u_m,u_m)] \in B(I;L_r(\varOmega)),\ r=\frac{2p}{p+2},\\ \big\|[u_m,\varPhi(u_m,u_m)]\big\|_{B(I;L_r(\varOmega))} \le c_r \equiv c_r(f,u_0,u_1).\end{gathered} \tag{22}$$

From the equation (17) we obtain straightforwardly the estimate

$$\|\ddot{u}_m\|^2_{L_2(I;V_m)^*} \le c_\eta,\ \ m\in\mathbb{N}, \tag{23}$$

where $V_m \subset \mathring{H}^2(\varOmega)$ is the linear hull of $\{w_i\}_{i=1}^m$.

We proceed with the convergence of the Galerkin approximation. Applying the estimates (20-23) and the compact imbedding theorem we obtain for any $p\in[1,\infty)$ a subsequence of $\{u_m\}$ (denoted again by $\{u_m\}$), and a function u the convergences

$$\begin{aligned}
u_m &\rightharpoonup^* u && \text{in } \mathring{\mathscr{H}},\\
\dot{u}_m &\rightharpoonup^* \dot{u} && \text{in } L_\infty(I;L_2(\Omega)),\\
\ddot{u}_m &\rightharpoonup \ddot{u} && \text{in } \big(L_2(I;\mathring{H}^2(\Omega))\big)^*,\\
u_m &\to u && \text{in } C(\bar{I};\mathring{H}^{2-\varepsilon}(\Omega)) \text{ for any } \varepsilon>0,\\
Lu_m &\rightharpoonup^* Lu && \text{in } \mathring{\mathscr{H}},\\
Lu_m &\to Lu && \text{in } L_2(I;\mathring{H}^2(\Omega)),\\
\varPhi(u_m,u_m) &\rightharpoonup^* \varPhi(u,u) && \text{in } L_\infty(I;W_p^2(\Omega))\\
\varPhi(u_m,u_m) &\to \varPhi(u,u) && \text{in } L_2(I;W_p^2(\Omega)).
\end{aligned} \tag{24}$$

Indeed, the first two convergences are obvious and imply

$$\begin{aligned}u_m \rightharpoonup u \text{ in } H^{1,2}(Q)) \hookrightarrow\hookrightarrow H^{1/2+\varepsilon'}(I;H^{1-\varepsilon''}(\Omega)) \text{ for } \varepsilon'>0\\ \text{and } 0<\varepsilon''(\varepsilon')\searrow 0 \text{ if } \varepsilon'\searrow 0.\end{aligned} \tag{25}$$

The compact imbedding is based on the use both of an extension operator acting from domains $I,\varOmega$ to respective spaces $\mathbb{R},\mathbb{R}^2$ and of the Fourier transform there. This technique is explained and used in detail on [8]. To prove the fourth convergence we start from the compact imbedding

$$H^{1/2+\varepsilon'}(I;H^{1-\varepsilon''}(\Omega)) \hookrightarrow\hookrightarrow C(\bar{I};H^{1-\varepsilon}(\Omega))$$

valid for any $\varepsilon>\varepsilon''$. Clearly $0<\varepsilon$ can be arbitrarily small again. Since $u_m, m\in\mathbb{N}$ and u belong to $B(I;\mathring{H}^2(\varOmega))$, the reflexivity of $\mathring{H}^2(\varOmega))$ ensures that they are continuous from I to $\mathring{H}^{2-\varepsilon}(\varOmega))$ for any $\varepsilon>0$. Interpolation of the first convergence with the strong convergence in $C(\bar{I},\mathring{H}^{1-\varepsilon}(\varOmega))$ yields the strong convergence in $L_\infty(I;\mathring{H}^{2-\varepsilon}(\varOmega))$. Then the time continuity follows from its validity in weaker spaces by interpolation.

The fifth convergence is enabled by the boundedness of the operator $L:\mathring{\mathscr{H}}\mapsto\mathring{\mathscr{H}}$. The strong convergence of $\{Lu_m\}$ follows from its definition (8). The seventh convergence is the consequence of the first and the fourth convergence and of the inequality (10). For the strong convergence of $\{\Phi(u_m, u_m)\}$ we use the inequalities (10) and

$$\int_I \Big(\|w\|_{H^2(\Omega)}\|u_m - u\|_{W_p^1(\Omega)}\Big)^2\,dt \le \|w\|^2_{L_2(I;H^2(\Omega))}\|u_m - u\|^2_{L_\infty(I;W_p^1(\Omega))}$$

successively with $w = u$ and $w = u_m$.

Let $\mu \in \mathbb{N}$ and $z_\mu = \sum_{i=1}^m \phi_i(t)w_i,\ \phi_i \in \mathscr{D}(0,T),\ i = 1, ..., \mu$. We have

$$\begin{aligned}&\int_\Omega \big(\ddot u_m(t) z_\mu(t) + A(u_m(t), z_\mu(t))\\ &\qquad + b\Delta(Lu_m(t) + \tfrac12\Phi(u_m(t), u_m(t)))\Delta(Lz_\mu(t) + \Phi(u_m(t), z_\mu(t)))\big)\,dx\\ &\qquad = \int_\Omega \big(\eta^{-1}(u_m(t) - \Psi)^- + f(t)\big)\,z_\mu(t)\,dx,\ \forall m \ge \mu,\ t \in I.\end{aligned}$$

The convergence process (24) implies that the function u fulfils

$$\begin{aligned}&\int_I \langle \ddot u, z_\mu\rangle_\Omega\,dt + \int_Q \big(A(u, z_\mu) + b\Delta(Lu + \tfrac12\Phi(u,u))\Delta(Lz_\mu + \Phi(u, z_\mu))\big)\,dx\,dt\\ &\qquad = \int_Q \big(\eta^{-1}(u - \Psi)^- + f(t)\big)\,z_\mu\,dx\,dt\end{aligned}$$

Functions $\{z_\mu;\ \mu \in \mathbb{N}\}$ form a dense subset of the set $L_2(I; \mathring{H}^2(\Omega))$, hence the function u fulfils the identity (16). The initial conditions (5) follow due to (18) and the proof of the existence of a solution is complete.

We remark that the *a priori* estimates (20), (21) imply the estimate

$$\begin{aligned}&\|\dot u_\eta\|_{L_\infty(I;L_2(\Omega))} + \|u_\eta\|_{L_\infty(I;\mathring{H}^2(\Omega))} + \|\Phi(u_\eta, u_\eta)\|_{L_\infty(I;W_p^2(\Omega))} +\\ &\eta^{-1}\|(u_\eta - \Psi)^-\|_{B(I;L_2(\Omega))} \le c_p \equiv c_p(f, u_0, u_1),\ p \ge 2.\end{aligned} \tag{26}$$

2.3 The limit process to the original problem

Let $u_\eta,\ \eta > 0$, be a solution of the penalized Problem $\mathscr{P}^{\mathrm{e}}_\eta$. To get the estimates independent of η we put $z(t,\cdot) = u_0 - u_\eta(t,\cdot),\ t \in I$, in (16). We arrive at the estimate

$$\begin{aligned}0 \le\,& U_0 \int_Q \eta^{-1}(u_\eta - \Psi)^-\,dx\,dt \le \int_Q \eta^{-1}(u_\eta - \Psi)^-(u_0 - \Psi)\,dx\,dt \le\\ &\int_Q \eta^{-1}(u_\eta - \Psi)^-[(u_0 - \Psi) - (u_\eta - \Psi)]\,dx\,dt = \int_Q \eta^{-1}(u_\eta - \Psi)^-(u_0 - u_\eta)\,dx\,dt\\ =& \int_Q \big(\dot u_\eta^2 + A(u_\eta, u_0 - u_\eta) +\\ &\qquad b\Delta(Lu_\eta + \tfrac12\Phi(u_\eta, u_\eta))\Delta(L(u_0 - u_\eta) + \Phi(u_\eta, u_0 - u_\eta))\big)\,dx\,dt\\ &- \int_Q f(u_0 - u_\eta)\,dx\,dt + \int_\Omega \dot u_\eta(u_0 - u_\eta)(T,\cdot)\,dx.\end{aligned}$$

Applying the *a priori* estimates (26) we obtain the estimate

$$\|\eta^{-1}(u_\eta - \Psi)^-\|_{L_1(Q)} \le c(f, u_0, u_1, U_0) \tag{27}$$

Using the embedding $\mathring{H}^2(\Omega) \hookrightarrow L_\infty(\Omega)$ we obtain from (16), (27) the crucial dual estimate of the acceleration term

$$\|\ddot{u}_\eta\|_{\mathscr{H}^*} \leq C \tag{28}$$

with C η-independent. The *a priori* estimates (26), (28) and the property (10) yield the existence of a sequence $\eta_k \searrow 0$ such that for $u_k \equiv u_{\eta_k}$ following convergences hold:

$$\begin{aligned}
u_k &\rightharpoonup^* u && \text{in } \mathring{\mathscr{H}},\\
\dot{u}_k &\rightharpoonup^* \dot{u} && \text{in } L_\infty(I; L_2(\Omega)),\\
\ddot{u}_k &\rightharpoonup^* \ddot{u} && \text{in } \mathring{\mathscr{H}}^*,\\
u_k &\to u && \text{in } C(\bar{I}; H^{2-\varepsilon}(\Omega)) \text{ for any } \varepsilon > 0,\\
Lu_k &\rightharpoonup^* Lu && \text{in } \mathring{\mathscr{H}},\\
\Phi(u_k, u_k) &\to \Phi(u, u) && \text{in } L_2(I; \mathring{H}^2(\Omega)),\\
\Phi(u_k, u_k) &\rightharpoonup^* \Phi(u, u) && \text{in } L_\infty(I; W_p^2(\Omega))\\
\eta^{-1}(u_k - \Psi)^- &\rightharpoonup^* g && \text{in } (L_\infty(Q))^*,
\end{aligned} \tag{29}$$

where g is the contact force between the shell and the obstacle.

The fourth convergence in (29) implies

$$u_k \to u \ \text{ in } C(\bar{Q}). \tag{30}$$

The L_1 estimate (27) then implies

$$(u_k - \Psi)^- \to 0 \ \text{ in } B(Q). \tag{31}$$

and

$$u \geq \Psi \ \text{ in } Q. \tag{32}$$

The contact force g is nonnegative in the dual sense

$$\langle g, y\rangle_Q \geq 0 \ \forall y \in L_\infty(Q),\ y \geq 0. \tag{33}$$

It fulfils further the complementarity condition

$$\langle g, u - \Psi\rangle_Q = 0 \tag{34}$$

due to the relations

$$\begin{aligned}
\langle g, u - \Psi\rangle_Q &= \lim_{k\to\infty} \langle \eta^{-1}(u_k - \Psi)^-, u - \Psi\rangle_Q\\
&\leq \sup_{k\in\mathbb{N}} \|\eta^{-1}(u_k - \Psi)^-\|_{L_1(Q)} \lim_{k\to\infty} \|(u_k - \Psi)^-\|_{B(Q)} = 0.
\end{aligned}$$

In order to verify the inequality (14) we express the penalized equation (16) with $u \equiv u_k$ in an operator form

$$\ddot{u}_k + \mathscr{B}(u_k) - \eta^{-1}(u_k - \Psi)^- = f \tag{35}$$

with

$$\begin{aligned}
&\mathscr{B} : \mathring{H}^2(\Omega) \to H^{-2}(\Omega),\\
&\langle \mathscr{B}(u), z\rangle = \int_\Omega \big(A(u, z) + b\Delta(Lu + \tfrac{1}{2}\Phi(u, u))\Delta(Lz + \Phi(u, z))\big)\ dx\, dt.
\end{aligned}$$

The convergences (29) and the properties (33), (34) imply

$$\begin{aligned}&\langle \ddot{u} + \mathscr{B}(u) - f, y - u\rangle_Q = \langle g, y - u\rangle_Q \\ &= \langle g, y - \Psi\rangle_Q - \langle g, u - \Psi\rangle_Q = \langle g, y - \Psi\rangle_Q \geq 0 \ \forall y \in \mathscr{C}\end{aligned}$$

and the inequality (14) follows. The initial condition $u(0) = u_0$ is satisfied due to the strong convergence of $\{u_k\}$ in $C(\bar{I}; H^{2-\varepsilon}(\Omega))$. The initial condition for velocities is satisfied in a weak sense due to a weak convergence of $\{\dot{u}_k\}$ in $L_2(\Omega)$.

Hence we have proved

Theorem 7 *Let the domain Ω be convex polygonal or $C^{3,1}$ domain in $\mathbb{R}^2$. Let $u_0 \in \mathring{H}^2(\Omega)$, $u_1 \in L_2(\Omega)$, $f \in L_2(Q)$ and let $\Psi \in B(\Omega)$ fulfilling $0 < U_0 \leq u_0 - \Psi$ in Ω. Then there exists a solution of Problem $\mathscr{P}^e$.*

Remark 8 *With the use of the standard translation method ([8]) the assumption $\Gamma \in C^{3,1}$ can be weakened. Most probably the assumption $\Gamma \in C^{1,1}$ is sufficient.*

3 The material with singular memory

3.1 Assumptions and problem formulation

We assume a model with a time variable elastic coefficient and a singular kernel in the memory part. A shell is simply supported. The initial-boundary value problem to be solved is

$$\ddot{u} - d\Delta\ddot{u} + \mathfrak{E}\Delta^2 u + E(t)\Delta^2 u - \Delta_{\boldsymbol{k}}^* v - [u, v] = f + g, \tag{36}$$

$$(u - \Psi) \geq 0,\ g \geq 0,\ (u - \Psi)g = 0, \tag{37}$$

$$\Delta^2 v = -e\left(\mathfrak{E}\left(\tfrac{1}{2}[u, u] + \Delta_{\boldsymbol{k}} u\right) + E(t)\left(\tfrac{1}{2}[u, u] + \Delta_{\boldsymbol{k}} u\right)\right), \tag{38}$$

$$u = \mathscr{M}(u) = v = \partial_n v = 0, \tag{39}$$

$$u(0, \cdot) = u_0,\ \dot{u}(0, \cdot) = u_1 \tag{40}$$

with constants

$$d = \frac{h^2}{12},\ e = \frac{12(1 - \mu^2)}{h^2}$$

and the bending moment

$$\begin{aligned}&\mathscr{M}(u) = \mathfrak{E}m(u) + E(t)m(u), \\ &m(u) = \triangle u + (1 - \nu)\big(2n_1 n_2 \partial_{1,2} u - n_1^2 \partial_{2,2} u - n_2^2 \partial_{1,1} u\big).\end{aligned}$$

We remark that $-d\Delta\ddot{u}$ expresses the rotational inertia of the shell which was neglected in the purely elastic case.

The time variable elasticity coefficient $t \mapsto E(t)$ fulfils

$$E \in H^1(I),\ E' \leq 0,\ 0 < e_0 \leq E(t) \leq e_1,\ \forall t \in \bar{I} \tag{41}$$

The memory operator has the form

$$\mathfrak{E} : v \mapsto \int_0^t K(t - s)\big(v(t, \cdot) - v(s, \cdot)\big)\, ds. \tag{42}$$

The kernel K of the singular memory term is assumed to be integrable over $\mathbb{R}_+$, to fulfil the estimate

$$\int_0^\infty K(s)ds < 2e_0 \tag{43}$$

and to have the form

$$\begin{aligned} &K : t \mapsto t^{-2\alpha} q(t) + r(t),\ t \in \mathbb{R}_+ \equiv (0, +\infty) \text{ with } \alpha \in \left(0, \tfrac{1}{2}\right), \\ &K : t \mapsto 0,\ t \le 0. \end{aligned} \tag{44}$$

Both q and r belong to $C^1(\mathbb{R}_+)$; they are non-negative and non-increasing functions. Moreover, $q(t) > 0$ for t in a right neighborhood of the origin.

We introduce the Hilbert space $V = H^2(\Omega) \cap \mathring{H}^1(\Omega)$ and the convex set

$$\mathscr{K} = \{y \in L_2(I, V) \cap H^1(Q) :\ y \ge \Psi \ \text{ for a.e. } t \in I\}. \tag{45}$$

Using the operators L and Φ we state a variational formulation of the problem (36)-(40):

Problem $\mathscr{P}^{\mathrm{s}}$ *We look for $u \in \mathscr{K}$ such that $\dot{u} \in L_2(I; H^1(\Omega))$ and the inequality*

$$\begin{aligned} &\int_Q \big(A(\mathfrak{C}u + Eu, y - u) - d\nabla\dot{u} \cdot \nabla(\dot{y} - \dot{u}) - \dot{u}(\dot{y} - \dot{u})\big)\, dx\, dt + \\ &\int_Q e\Delta(\mathfrak{C}(Lu + \tfrac{1}{2}\Phi(u, u)) + E(t)(Lu + \tfrac{1}{2}\Phi(u, u)))\Delta(L(y - u) + \Phi(u, y - u))\, dx\, dt \\ &+ \int_\Omega \big(d\nabla\dot{u} \cdot \nabla(y - u) + \dot{u}(y - u)\big)(T, \cdot)\, dx \\ &\ge \int_\Omega \big(d\nabla u_1 \cdot (\nabla y(0, \cdot) - \nabla u_0) + u_1(y(0, \cdot) - u_0)\big)\, dx + \int_Q f(y - u)\, dx\, dt \end{aligned} \tag{46}$$

holds for any $y \in \mathscr{K}$.

The obstacle Ψ fulfils the same assumptions as in the elastic case i.e.

$$\Psi \in B(\Omega),\ 0 < U_0 \le u_0 - \Psi \text{ in } \Omega. \tag{47}$$

Let us remark that a weak formulation (46) yields that the initial condition for u is satisfied in the classical sense while that for $\dot{u}$ is fulfilled in the weak sense only.

3.2 Solving a penalized problem

We penalize the unilateral condition in the same way as in the first part. The penalized problem has the form

Problem $\mathscr{P}^{\mathrm{s}}_\eta$. *We look for $u \in L_2(I, V)$ such that $\ddot{u} \in L_2(Q)$, the equation*

$$\begin{aligned} &\int_Q \big(\ddot{u}(z - d\Delta z) + A(\mathfrak{C}u + Eu, z)\big)\, dx\, dt + \\ &\int_Q e\Delta(\mathfrak{C}(Lu + \tfrac{1}{2}\Phi(u, u)) + E(t)(Lu + \tfrac{1}{2}\Phi(u, u)))\Delta(Lz + \Phi(u, z)))\, dx\, dt \\ &= \int_Q ((\eta^{-1}(u - \Psi)^- + f)z\, dx\, dt \end{aligned} \tag{48}$$

holds for any $z \in L_2(I, V)$ and the initial conditions (40) *remain valid.*

Theorem 9 *Let $f \in L_1(I; L_2(\Omega))$, $u_0 \in V$, $u_1 \in \overset{\circ}{H}{}^1(\Omega)$. Then there exists a solution u of the problem $\mathscr{P}^s_\eta$.*

Proof. Let us denote by $\{w_i \in V;\, i \in \mathbb{N}\}$ an orthonormal with respect to the inner product

$$(.,.)_d : (v,w) \mapsto \int_\Omega (vw + d\,\nabla v \cdot \nabla w)\,dx \tag{49}$$

basis of V. We construct the Galerkin approximation u_m of a solution in the form

$$u_m(t) = \sum_{i=1}^m \alpha_i(t) w_i,\ \ \alpha_i(t) \in \mathbb{R},\ \ i = 1, ..., m,\ \ m \in \mathbb{N}$$

given by the solution of the approximate problem

$$\begin{aligned}
&\int_\Omega \big(\ddot u_m(t) w_i + d\nabla \ddot u_m(t)\cdot \nabla w_i + A(\mathfrak{E} u_m(t) + E(t) u_m(t), w_i)\big)\,dx + \\
&\int_\Omega \big(e\Delta(\mathfrak{E}(Lu_m(t) + \tfrac12 \Phi(u_m,u_m)(t)) + E(t)(Lu_m(t) + \tfrac12\Phi(u_m,u_m)(t))\big) \\
&\quad \times\ \Delta(Lw_i + \Phi(u_m(t), w_i)) - \eta^{-1}(u_m(t) - \Psi)^- w_i\big)\,dx \\
=&\int_\Omega f(t) w_i\,dx,\ \ i = 1, ..., m,
\end{aligned} \tag{50}$$

$$u_m(0) = u_{0m},\ \ \dot u_m(0) = u_{1m},\ \ u_{im} \to u_i \text{ in } H^{2-i}(\Omega),\ i = 0,1. \tag{51}$$

We have

$$\int_\Omega \big(w_i w_j + d\,\nabla w_i \cdot \nabla w_j\big)\,dx = \delta_{ij},\ \ i,j = 1, ..., m.$$

The system (50) can then be expressed in the form

$$\ddot\alpha_i = F_i(t, \alpha_1, ..., \alpha_m),\ i = 1, ..., m.$$

Its right-hand side satisfies the conditions for the local existence of a solution fulfilling the initial conditions corresponding to the functions u_{0m}, u_{1m}. Hence there exists a Galerkin approximation $u_m(t)$ defined on some interval $[0, t_m]$, $0 < t_m \le T$. To derive the *a priori* estimates for solutions of (50), (51) we multiply the equation (50) by $\dot\alpha_i(t)$, add with respect to i and integrate on the interval $[0,s]$, $s \le t_m$. Using the integration by parts and properties of the functions K, E, Ψ we get

$$\begin{aligned}
&\int_{Q_s} \Big(\partial_t\Big(|\dot u_m|^2 + d|\nabla \dot u_m|^2 + E\big(A(u_m,u_m) + e(\triangle(Lu_m + \frac12\Phi(u_m,u_m)))^2\big) \\
&\quad + \eta^{-1}((u_m - \Psi)^-)^2\Big) - E'\Big(A(u_m,u_m) + e(\triangle(Lu_m + \frac12\Phi(u_m,u_m)))^2\Big) \\
&\quad + K(s-t)\Big(A(u_m(s) - u_m(t), u_m(s) - u_m(t)) \\
&\quad + e\big(\triangle\big(Lu_m(s) + \frac12\Phi(u_m,u_m)(s) - Lu_m(t) - \frac12\Phi(u_m,u_m)(t)\big)\big)^2\Big)\Big)\,dx\,dt \\
&- \int_{Q_s}\int_0^t K'_t(t-r) A(u_m(t) - u_m(r), u_m(t) - u_m(r))\,dr\,dt\,dx \\
&- \int_{Q_s}\int_0^t K'_t(t-r)\, e(\triangle\big(Lu_m(t) + \frac12\Phi(u_m,u_m)(t) \\
&\qquad - Lu_m(r) - \frac12\Phi(u_m,u_m)(r))\big)^2\,dr\,dt\,dx = 2\int_{Q_s} f\dot u_m\,dx\,dt.
\end{aligned} \tag{52}$$

By virtue of properties of the kernel K and the function E, the identity (52) leads to the *a priori* estimates independent of the penalty parameter η, of $m \in \mathbb{N}$ as well as of $t_m \in I$:

$$\begin{aligned}&\|u_m\|^2_{H^\alpha(I;V)} + \|\dot{u}_m\|^2_{B(I;\mathring{H}^1(\Omega))} + \|u_m\|^2_{B(I;V)}\\ &+ \|\Phi(u_m,u_m)\|^2_{H^\alpha(I;\mathring{H}^2(\Omega))} + \eta^{-1}\|(u_m-\Psi)^-\|^2_{B(I;L_2(\Omega))} \le c \equiv c(f,u_0,u_1).\end{aligned} \tag{53}$$

The estimates

$$\|\Phi(u_m,u_m)\|_{B(I;W^2_p(\Omega))} \le c_p \equiv c_p(f,u_0,u_1)\,\forall\, p>2 \tag{54}$$

$$\big\|[u_m,\Phi(u_m,u_m)]\big\|_{B(I;L_r(\Omega))} \le c_r \equiv c_r(f,u_0,u_1),\ r=\frac{2p}{p+2} \tag{55}$$

derived in the previous section still hold. The solution u_m then exists on the whole interval $[0,T]$.

Moreover, using (53) we arrive at the important *dual* estimate

$$\|\ddot{u}_m\|^2_{L_2(Q)} \le c_\eta,\ m\in\mathbb{N}. \tag{56}$$

Indeed, we have just proved that the sequence of remainders $a\triangle\ddot{u}_m - \ddot{u}_m$ is bounded in $L_2(I;W^*)$, $W=\bigcup_{m\in\mathbb{N}} W_m$, where W_m is the linear hull of $\{w_i\}_{i=1}^m$. We get via integration by parts

$$\begin{aligned}\|\ddot{u}_m\|_{L_2(Q)} &= \sup_{\|\varphi\|_{L_2(Q)}\le 1} (\ddot{u}_m,\varphi)_Q \le c \sup_{\|v\|_{L_2(I;V)}\le 1} (\ddot{u}_m, v - a\triangle v)_Q\\ &= c \sup_{\|v\|_{L_2(I;V)}\le 1} \int_Q (\ddot{u}_m v + a\nabla\ddot{u}_m\cdot\nabla v)\,dx\,dt \le k,\end{aligned}$$

where we employ the orthonormality (49) and also the properties of the Green operator for the elliptic problem $v - a\triangle v = f$ with the homogeneous Dirichlet boundary condition and the right-hand side in $L_2(\Omega)$ for Ω of the class C^2 or convex polygonal as well as the fact that $\eta>0$ is fixed.

We continue with the convergence. Applying the estimates (53-56) we obtain for any $p>1,\ q\ge 2$ a subsequence of $\{u_m\}$ (denoted again by $\{u_m\}$), a small $\theta\equiv\theta(\alpha)>0$ and a function u the following convergences

$$\begin{aligned} u_m &\rightharpoonup u && \text{in } H^\alpha(I;V),\\ u_m &\rightharpoonup^* u && \text{in } L_\infty(I;V),\\ \dot{u}_m &\rightharpoonup^* \dot{u} && \text{in } L_\infty(I;\mathring{H}^1(\Omega)),\\ \ddot{u}_m &\rightharpoonup \ddot{u} && \text{in } L_2(Q),\\ \dot{u}_m &\to \dot{u} && \text{in } L_q(I;W^1_{2+\theta}(\Omega)),\\ u_m &\to u && \text{in } C(\bar{I}:H^{2-\varepsilon}(\Omega)),\\ Lu_m &\to Lu && \text{in } L_2(I;\mathring{H}^2(\Omega)),\\ \Phi(u_m,u_m) &\to \Phi(u,u) && \text{in } L_2(I;W^2_p(\Omega)).\end{aligned} \tag{57}$$

To get the fifth convergence we first interpolate the first and third one via Corollary 5 with $\Theta = 1/(2-\alpha)+\theta$, $0<\theta$ arbitrarily small and we use Corollary 3 to the result. Then we interpolate this result once again with the sixth convergence, where we replace ∞ by an arbitrarily large real $\tilde{p}$. The fifth convergence and the strong convergences of $\{Lu_m\}$

and $\{\Phi(u_m, u_m)\}$ can be derived in the same way as in the elastic case in the previous section.

Let $\mu \in \mathbb{N}$ and $z_\mu(t) = \sum_{i=1}^{m} \phi_i(t) w_i,\ \phi_i \in \mathscr{D}(0,T),\ i = 1, ..., \mu$. We have for arbitrary $m \in \mathbb{N}$ and $t \in I$ the relation

$$\begin{aligned}
&\int_\Omega \big(\ddot{u}_m(t)(z_\mu - d\Delta z_\mu) + A(\mathfrak{E} u_m(t) + E(t)u_m(t), z_\mu)\big)\,dx + \\
&\int_\Omega \big(e\Delta(\mathfrak{E}(Lu_m(t) + \tfrac{1}{2}\Phi(u_m,u_m)(t)) + E(t)(Lu_m(t) + \tfrac{1}{2}\Phi(u_m,u_m)(t))\big) \\
&\qquad \times\ \Delta(Lz_\mu + \Phi(u_m(t), z_\mu)) - \eta^{-1}(u_m(t) - \Psi)^- z_\mu\big)\,dx = \int_\Omega f(t) z_\mu\,dx.
\end{aligned}$$

The convergence process (57) implies for the function u the relation

$$\begin{aligned}
&\int_Q \big(\ddot{u}(z_\mu - d\Delta z_\mu) + A(\mathfrak{E} u + E(t)u, z_\mu)\big)\,dx\,dt + \\
&\int_Q e\Delta(\mathfrak{E}(Lu + \tfrac{1}{2}\Phi(u,u)) + E(t)(Lu + \tfrac{1}{2}\Phi(u,u))\big)\Delta(Lz_\mu + \Phi(u, z_\mu))\,dx\,dt \\
&- \int_Q \eta^{-1}(u - \Psi)^- z_\mu\,dx\,dt = \int_Q f(t) z_\mu\,dx\,dt.
\end{aligned}$$

The functions $\{z_\mu\}$ form a dense subset of the set $L_2(I;V)$, hence the function u fulfils the identity (48). The initial conditions (40) follow due to (51) and the proof of the existence of a solution is complete.

3.3 Solution of the original problem

In order to perform the limit process $\eta \searrow 0$ we need a new η-independent estimate for the solutions $u \equiv u_\eta$ of problems $\mathscr{P}^{\mathrm{s}}_\eta$. The estimates (53) and the convergences (57) imply the estimates

$$\|u_\eta\|^2_{H^\alpha(I;V)} + \|\dot{u}_\eta\|^2_{L_\infty(I;\mathring{H}^1(\Omega))} + \|u_\eta\|^2_{L_\infty(I;V)} + \|\Phi(u_\eta,u_\eta)\|^2_{H^\alpha(I;\mathring{H}(\Omega))} \le c \equiv c(f, u_0, u_1). \quad (58)$$

We need further an η-independent estimate of the penalty and the acceleration term. Applying the obstacle property (47) we obtain the inequalities

$$\begin{aligned}
0 \le{}& U_0 \int_Q \eta^{-1}(u_\eta - \Psi)^-\,dx\,dt \le \int_Q \eta^{-1}(u_\eta - \Psi)^-(u_0 - \Psi)\,dx\,dt \le \\
&\int_Q \eta^{-1}(u_\eta - \Psi)^-[(u_0 - \Psi) - (u_\eta - \Psi)]\,dx\,dt = \int_Q \eta^{-1}(u_\eta - \Psi)^-(u_0 - u_\eta)\,dx\,dt.
\end{aligned}$$

We put the test function $z = u_0 - u_\eta$ in (48) and obtain

$$\begin{aligned}
0 \le U_0 &\int_Q \eta^{-1}(u_\eta - \Psi)^-\,dx\,dt \le \int_Q \big(\dot{u}_\eta^2 + d|\nabla \dot{u}_\eta|^2 + A(\mathfrak{E} u_\eta + E u_\eta, u_0 - u_\eta)\big)\,dx\,dt \\
&+ \int_Q \big(e\Delta(\mathfrak{E}(Lu_\eta + \tfrac{1}{2}\Phi(u_\eta,u_\eta)) + E(Lu_\eta + \tfrac{1}{2}\Phi(u_\eta,u_\eta)))\Delta(L(u_0 - u_\eta) + \Phi(u_\eta, u_0 - u_\eta)) \\
&\qquad - f(u_0 - u_\eta)\big)\,dx\,dt + \int_\Omega (\dot{u}_\eta(u_0 - u_\eta) + d\nabla \dot{u}_\eta \cdot \nabla(u_0 - u_\eta))(T,\cdot)\,dx.
\end{aligned}$$

The *a priori* estimates (58) then imply the penalty estimate

$$\|\eta^{-1}(u_\eta - \Psi)^-\|_{L_1(Q)} \le c. \tag{59}$$

Using the imbedding $L_1(Q) \hookrightarrow L_1(I, V^*)$, the relation (48) and the reasoning to (56) we arrive at the estimates

$$\|\ddot{u}_\eta\|_{L_1(I;L_2(\Omega))} = \sup_{\varphi \in L_\infty(I;L_2(\Omega))} \int_Q \ddot{u}_\eta \varphi \, dx \, dt \le c \sup_{v \in L_\infty(I;V)} \int_Q \ddot{u}_\eta (v - a\Delta v) \, dx \, dt$$
$$= c \sup_{v \in L_\infty(I;V)} \int_Q \ddot{u}_\eta v + a \nabla u_\eta \cdot \nabla v) \, dx \, dt \le \tilde{c}.$$

Due to an appropriate imbedding $\dot{u}_\eta$ is bounded in $W^{1-\theta}_{1+\theta'}(I; L_2(\Omega))$ for any $\theta \in (0,1)$ and for $\theta' \equiv \theta'(\theta) \searrow 0$ if $\theta \searrow 0$. Interpolating this space with the space $L_p(I; H^1(\Omega))$ for $p \ge 1 + 1/\theta'$, we get that

$$\|\dot{u}_\eta\|_{H^{1/2-\theta}(I;H^{1/2}(\Omega))} \le C \text{ with } 0 < \theta \text{ arbitrarily small.}$$

Interpolating this result with the fact that u_η is bounded in $H^\alpha(I, H^2(\Omega))$, we get that u_η is again bounded in some $H^{1+\theta_1}(I; H^{1+\theta_2}(\Omega))$ for some $\theta_1,\ \theta_2 > 0$ dependent on α, i.e. $\dot{u}_\eta$ is bounded in $H^{\theta_1}(I; H^{1+\theta_2}(\Omega))$. This space is compactly imbedded to $L_2(I, H^1(\Omega))$.

Hence there exist sequences $\eta_k \searrow 0,\ u_{\eta_k} \equiv u_k$ and a function u such that the following convergences

$$\begin{aligned}
u_k &\rightharpoonup u && \text{in } H^\alpha(I;V), && (60)\\
\dot{u}_k &\to \dot{u} && \text{in } L_2(I;H^1(\Omega)), && (61)\\
u_k &\to u && \text{in } C(\bar{I};H^{2-\varepsilon}(\Omega)), && (62)\\
Lu_k &\to Lu && \text{in } L_2(I;\mathring{H}^2(\Omega)), && (63)\\
\Phi(u_k,u_k) &\to \Phi(u,u) && \text{in } L_2(I;W^2_p(\Omega)),\ p \in (1,\infty) && (64)
\end{aligned}$$

are valid. We put $u \equiv u_k, z = u_k - u$ in (48) and integrate by parts in the terms where $\ddot{u}_k$ occurs. By adding $A(-\mathfrak{E}u - Eu, u_k - u)$ to both sides of (48) and using the convergences (63), (64) and the assumption (43) we obtain the strong convergence

$$u_k \to u \in L_2(I;V). \tag{65}$$

With the use of all above proved convergences, it is easy to verify that a function u is a solution of the Problem $\mathscr{P}^s$. Hence the following existence theorem holds.

Theorem 10 *Let the assumptions* (41), (43), (44) *hold. Then there exists a solution of the Problem* $\mathscr{P}^s$*.*

4 Conclusion

In the paper we have studied the problems with definite boundary value conditions for the sake of simplicity and the length of paper. Observe that if we interchange the boundary conditions between sections 2 and 3 the existence theorems remain valid. These are even valid for the case of shells free on the boundary which seam to us of minor importance from the practical point of view.

The same results can be obtained for a shell made of a Kelvin-Voigt viscoelastic material.

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Author's addresses: Igor Bock: Institute of Computer Sciences and Mathematics,
Faculty of Electrical Engineering and Information Technology,
Slovak University of Technology, Ilkovičova 3, 812 19 Bratislava,
Slovak Republic;
e-mail: igor.bock@stuba.sk
Jiří Jarušek: Mathematical Institute,
Academy of Sciences of the Czech Republic, Žitná 25,
115 67 Praha 1, Czech Republic;
e-mail: jarusek@math.cas.cz