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Topology design of elastic structures for a contact model

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TOPOLOGY DESIGN OF ELASTIC STRUCTURES FOR A CONTACT MODEL

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Abstract. In this paper we employ topological derivatives for optimum design problems in solid mechanics. A nonlinear contact model governed by a variational inequality is considered. Beside the theoretical developments, some computational examples are included. The numerical results show that the proposed method of optimum design can be applied to a broad class of engineering problems.

1. Introduction

An asymptotic expansion of a given shape functional, when a geometrical domain is singularly perturbed by the insertion of holes, can be obtained by performing a topological asymptotic analysis. This analysis is applied in the mathematical model that represents the physical phenomena under consideration. Asymptotic analysis of linear and nonlinear models in solid mechanics is considered in details in the recent monograph [1]. The related results can be also found in [14, 13, 23, 26, 7, 18, 27, 11, 6].

Classical shape optimization for contact problems is considered in [28] for the variational inequalities of the first and the second kind. The shape and material derivatives are determined in the framework of the conical differentiability of solutions to variational inequalities. Another branch of applied models with unilateral constraints are the crack models with nonlinear nonpenetration conditions on the crack faces (lips) [21, 20, 22]. For such models the elastic energy is differentiated with respect to the crack length [10]. The stability of solutions to the evolution variational inequalities is analyzed in [19]. A new class of variational inequalities arises when a finite interpenetration is allowed in the potential contact region of the body with a rigid foundation, as proposed in [8].

In this work we present a closed form for the topological derivative when a small circular disc, with a material different than the surrounding medium, is introduced in an arbitrary point of the elastic body. We consider the energy shape functional associated to the frictionless contact problem allowing a finite interpenetration between an elastic body and a rigid foundation [8].

In order to apply the theoretical results, we present a computational procedure for topological optimization based on the topological derivative concept. The optimization procedure consists in minimizing the structural compliance for a given amount of material. This constraint in the volume of the optimized structure is introduced in the formulation of the optimization problem by means of an exact quadratic scheme. The robustness of the topological optimization technique presented in this work is demonstrated by a set of numerical examples, related to the topology design of elastic structures under this particular nonlinear contact condition. On the other hand, the formulation of the problem of topology optimization of structures in unilateral contact, with computational approaches such as SIMP (Solid Isotropic Microstructure with Penalization) and ESO (Evolutionary Structural Optimization), can be found in [16, 24, 9, 29].

This paper is organized as follows. Section 2 describes the frictionless contact model for finite interpenetration in two-dimensional elasticity. The topological derivative associated to this problem is presented in Section 3, where a simple and analytical formula is given. The compliance topology optimization procedure for elastic structures subjected to a volume constraint is outlined in Section 4. A set of numerical experiments is presented in Section 5. The paper ends in Section 6 with some concluding remarks.

Key words and phrases. Topological derivative, frictionless contact problem, asymptotic analysis, topological optimization, optimum design problems.

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2. Static contact model for finite interpenetration

We consider the problem of an elastic body having contact with a rigid foundation. The domain of the body is denoted by $\Omega \subset \mathbb{R}^2$. The boundary $\partial\Omega$ of the body consists of three mutually disjoint parts with positive measures Γ_D, Γ_N and Γ_C, where different boundary conditions are prescribed. On the boundary Γ_D we prescribe Dirichlet boundary conditions (displacement), on Γ_N Neumann boundary conditions (traction) and, finally, on Γ_C the contact condition with the rigid foundation that admits an interpenetration, see Figure 1(a). For the contact model, we consider only a normal compliance law of the type

$$\sigma_n(u) = -p(u_n - g)\,, \tag{2.1}$$

where $u_n := u \cdot n$ denotes the normal component of the displacement field u, n is the unit outward normal vector to the boundary $\partial\Omega$ and g the gap on the potential contact zone. Moreover, in (2.1), $\sigma_n(u)$ represents the normal component to the boundary of the stress tensor $\sigma(u)$, i.e. $\sigma_n(u) = \sigma(u)n{\cdot}n$. The Cauchy stress tensor $\sigma(u)$ is defined as:

$$\sigma(u) := \mathbb{C}\varepsilon(u)\,, \tag{2.2}$$

where $\varepsilon(u)$ is the symmetric part of the gradient of the displacement field u, i.e.

$$\varepsilon(u) := \frac{1}{2}(\nabla u + (\nabla u)^\top)\,, \tag{2.3}$$

and $\mathbb{C}$ denotes the fourth-order elastic tensor. For an isotropic elastic body, this tensor is given by:

$$\mathbb{C} = 2\mu\mathbb{I} + \lambda(\mathrm{I} \otimes \mathrm{I}), \tag{2.4}$$

with μ and λ denoting the Lamé coefficients. In the above expression, we use $\mathbb{I}$ and I to denote, respectively, the identities of fourth and second order. In terms of the engineering constant E (Young's modulus) and υ (Poisson's ratio) the above constitutive response can be written as:

$$\mathbb{C} = \frac{E}{1-\upsilon^2}[(1-\upsilon)\mathbb{I} + \upsilon(\mathrm{I} \otimes \mathrm{I})]\,. \tag{2.5}$$

The function $p : \mathbb{R} \to \overline{\mathbb{R}_+} = [0, +\infty]$ in (2.1) is used to model the interpenetration condition between the body and the foundation. This function p is monotone with the following properties:

$$\begin{cases} p(y) &= 0 & \text{for } y \le a, \text{ with } a \text{ constant} \\ \lim\limits_{y\to b-} p(y) &= +\infty & \text{for } y > a, \text{ with } b \text{ constant and } b > a \\ p(y) &= +\infty & \text{for } y \ge b \end{cases}. \tag{2.6}$$

The parameter a indicates the initial contact and the value of b describes a limit such that no further interpenetration is possible, see Figure 1(b).

The strong form of the equilibrium equation under this contact condition is given by: find the displacement field $u : \Omega \to \mathbb{R}^2$ such that

$$\begin{cases} -\operatorname{div}\sigma(u) &= 0 & \text{in} \quad \Omega \\ u &= \overline{u} & \text{on} \quad \Gamma_D \\ \sigma(u)n &= \overline{t} & \text{on} \quad \Gamma_N \\ \sigma_n(u) &= -p(u_n - g) & \text{on} \quad \Gamma_C \\ \sigma_\tau(u) &= 0 & \text{on} \quad \Gamma_C \end{cases}. \tag{2.7}$$

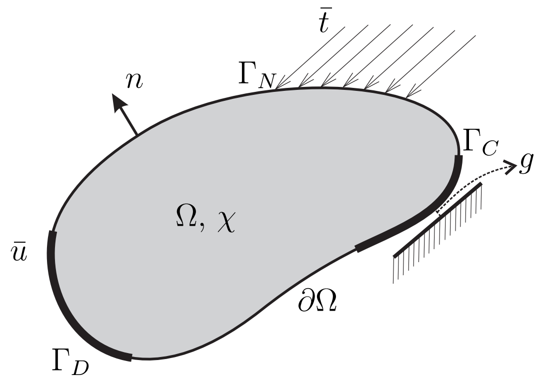
The last condition in (2.7) indicates that the contact is without friction, where $\sigma_\tau(u) = \sigma(u)n - \sigma_n(u)n$ denotes the tangential component of the stress tensor $\sigma(u)$.

The weak formulation of the problem stated in (2.7) is given by the following variational equation: find $u \in \mathcal{U}$ with $(u_n - g) \in \operatorname{dom}(p)$, such that:

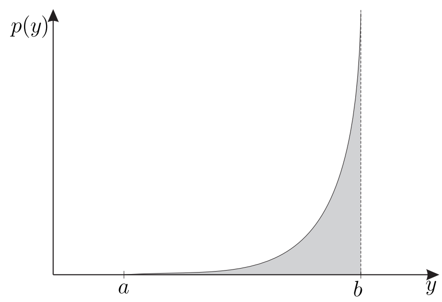
$$\int_\Omega \sigma(u) \cdot (\varepsilon(v) - \varepsilon(u)) + \int_{\Gamma_C} p(u_n - g)(v_n - u_n) = \int_{\Gamma_N} \bar{t} \cdot (v - u) \quad \forall v \in \mathcal{U}\,, \tag{2.8}$$

where the set of admissible functions $\mathcal{U}$ is given by:

$$\mathcal{U} := \{\varphi \in H^1(\Omega; \mathbb{R}^2) \,:\, \varphi = \bar{u} \quad \text{on} \quad \Gamma_D\}, \tag{2.9}$$



(a) Contact problem.



(b) example of function $p(y)$.

Figure 1. Contact problem formulation.

and the domain of definition of the function p, namely $\text{dom}(p)$, is:

$$\text{dom}(p) := \left\{ \varphi \in L^1(\Gamma_C) \, : \, p(\varphi) \in L^1(\Gamma_C), \exists C > 0 \, : \, \int_{\Gamma_C} p(\varphi) v \leq C \|v\|_{H^{1/2}(\Gamma_C)} \right\} . \tag{2.10}$$

For a detailed description of this model, we refer the reader to [8].

3. Topological derivative

In this section we obtain an asymptotic expansion for the energy shape functional when a small disc of radius ρ, with different constitutive property, is introduced in an arbitrary point $\widehat{x}$ of the domain Ω, far enough from the potential contact region Γ_C, and denoted by $\mathcal{B}_\rho := \{x \in \mathbb{R}^2 \, : \, |x - \widehat{x}| < \rho\}$, see Figure 2. Thus, introducing a characteristic function $\chi = \mathbb{1}_\Omega$, associated to the unperturbed domain, it is possible to define the characteristic function associated to the topological perturbed domain χ_ρ. Particularly, when the topological perturbation is an inclusion, we have $\chi_\rho(\widehat{x}) = \mathbb{1}_\Omega - (1-\gamma)\mathbb{1}_{\overline{\mathcal{B}_\rho(\widehat{x})}}$, where $\gamma \in \mathbb{R}^+$ is the contrast parameter in the material property of the medium. Then we assume that a given shape functional $\psi(\chi_\rho(\widehat{x}))$, associated to the topological perturbed domain Ω_ρ, admits the following topological asymptotic expansion

$$\psi(\chi_\rho(\widehat{x})) = \psi(\chi) + f(\rho)\mathcal{T}\psi(\widehat{x}) + o(f(\rho)) \, , \tag{3.1}$$

where $\psi(\chi)$ it is the shape functional associated to the unperturbed domain, $f(\rho)$ it is a function such that $f(\rho) \to 0$, with $\rho \to 0^+$. The function $\widehat{x} \mapsto \mathcal{T}\psi(\widehat{x})$ is the so-called topological derivative of ψ in the point $\widehat{x}$. Thus, the topological derivative can be seen as a first order correction factor over $\psi(\chi)$ to approximate $\psi(\chi_\rho(\widehat{x}))$. In fact, after rearranging (3.1), we have

$$\frac{\psi(\chi_\rho(\widehat{x})) - \psi(\chi)}{f(\rho)} = \mathcal{T}\psi(\widehat{x}) + \frac{o(f(\rho))}{f(\rho)} \, . \tag{3.2}$$

Taking the limit $\rho \to 0^+$ in the above expression, we have the classical definition of the topological derivative [25] given by

$$\mathcal{T}\psi(\widehat{x}) = \lim_{\rho \to 0^+} \frac{\psi(\chi_\rho(\widehat{x})) - \psi(\chi)}{f(\rho)} \, . \tag{3.3}$$

Note that, the shape functionals $\psi(\chi_\rho(\widehat{x}))$ and $\psi(\chi)$ are associated to domains with different topologies. Then, to calculate the limit $\rho \to 0^+$ in (3.3) it is necessary to perform a asymptotic expansion of the functional $\psi(\chi_\rho(\widehat{x}))$ with respect to the parameter ρ.

In this work we are interested in the asymptotic expansion for the energy shape functional associated to the contact problem (2.8), given by [8]:

$$\mathcal{J}_\chi(u) := \frac{1}{2}\int_\Omega \sigma(u) \cdot \varepsilon(u) - \int_{\Gamma_N} \bar{t} \cdot u + \int_{\Gamma_C} P(u_n - g), \tag{3.4}$$

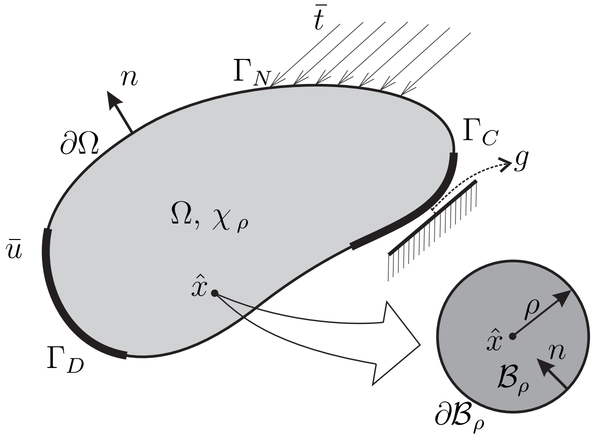


FIGURE 2. Perturbed contact problem.

where the function $P(y)$ is given by:

$$P(y) := \int_{-\infty}^{y} p(z). \tag{3.5}$$

Considering the singular perturbation described above and denoted by $\mathcal{B}_\rho$, the energy shape functional associated to the perturbed domain is given by:

$$\mathcal{J}_{\chi_\rho}(u_\rho) := \frac{1}{2}\int_\Omega \sigma_\rho(u_\rho)\cdot\varepsilon(u_\rho) - \int_{\Gamma_N} \bar{t}\cdot u_\rho + \int_{\Gamma_C} P(u_{\rho n} - g), \tag{3.6}$$

where u_ρ is the solution of the problem in the singularly perturbed domain given by: find the displacement field $u_\rho : \Omega \to \mathbb{R}^2$ such that

$$\left\{\begin{array}{rcll} -\operatorname{div}\sigma_\rho(u_\rho) & = & 0 & \text{in} \quad \Omega \\ u_\rho & = & \overline{u} & \text{on} \quad \Gamma_D \\ \sigma(u_\rho)n & = & \overline{t} & \text{on} \quad \Gamma_N \\ \sigma_n(u_\rho) & = & -p(u_{\rho n} - g) & \text{on} \quad \Gamma_C \\ \sigma_\tau(u_\rho) & = & 0 & \text{on} \quad \Gamma_C \\ [\![u_\rho]\!] & = & 0 & \text{on} \quad \partial\mathcal{B}_\rho \\ [\![\sigma_\rho(u_\rho)]\!]n & = & 0 & \text{on} \quad \partial\mathcal{B}_\rho \end{array}\right. , \tag{3.7}$$

since $u_{\rho n} := u_\rho \cdot n$ is used to denote the normal component of the displacement field u_ρ on the boundary Γ_C. The symbol $[\![(\cdot)]\!]$ in (3.7) denotes the jump of function $(\cdot)$ across the boundary $\partial\mathcal{B}_\rho$ and the stress operator $\sigma_\rho(\cdot)$ is defined as:

$$\sigma_\rho(\phi) := \gamma_\rho \mathbb{C}\varepsilon(\phi), \tag{3.8}$$

where the parameter γ_ρ is defined as:

$$\gamma_\rho := \left\{\begin{array}{ll} 1 & \text{in} \quad \Omega \setminus \overline{\mathcal{B}_\rho} \\ \gamma & \text{in} \quad \mathcal{B}_\rho \end{array}\right. . \tag{3.9}$$

Note that the domain Ω is topologically perturbed by the introduction of an inclusion $\mathcal{B}_\rho(\widehat{x})$ of the same nature as the bulk material, but with contrast γ. Finally, the variational problem associated to (3.7) can be written as: find $u_\rho \in \mathcal{U}_\rho$ with $(u_{\rho n} - g) \in \operatorname{dom}(p)$, such that:

$$\int_\Omega \sigma_\rho(u_\rho)\cdot(\varepsilon(v) - \varepsilon(u_\rho)) + \int_{\Gamma_C} p(u_{\rho n} - g)(v_n - u_{\rho n}) = \int_{\Gamma_N} \bar{t}\cdot(v - u_\rho) \quad \forall v \in \mathcal{U}_\rho\,, \tag{3.10}$$

where the set of admissible functions $\mathcal{U}_\rho$ is given by:

$$\mathcal{U}_\rho := \{\varphi \in \mathcal{U} \,:\, [\![\varphi]\!] = 0 \quad \text{on} \quad \partial\mathcal{B}_\rho\}. \tag{3.11}$$

For an explicit and analytical formula for the topological derivative $\mathcal{T}_\mathcal{J}(\hat{x})$ of the functional (3.4) associated to the problem (2.7), we introduce the following result:

Theorem 1. *The energy shape functional of an elastic solid with a disc of radius ρ, centered at point $\widehat{x} \in \Omega$ and with constitutive property characterized by the parameter γ, admits for $\rho \to 0^+$ the following asymptotic expansion:*

$$\mathcal{J}_{\chi_\rho}(u_\rho) = \mathcal{J}_\chi(u) + \rho^2 \pi \mathbb{H}_\gamma \sigma(u(\widehat{x})) \cdot \varepsilon(u(\widehat{x})) + o(\rho^2) \quad \forall \widehat{x} \in \Omega, \tag{3.12}$$

where $u(\widehat{x})$ is the solution of the problem (2.7) evaluated at $\widehat{x}$ and $\mathbb{H}_\gamma$ is the fourth-order tensor defined as:

$$\mathbb{H}_\gamma := \frac{1}{4}\frac{(1-\gamma)^2}{1+\beta\gamma}\left(2\frac{1+\beta}{1-\gamma}\mathbb{I} + \frac{\alpha-\beta}{1+\alpha\gamma}\mathrm{I}\otimes\mathrm{I}\right), \tag{3.13}$$

where I and $\mathbb{I}$ are the identities tensors of second- and fourth-order, respectively, and the parameters α and β depend exclusively on the Poisson's ratio of the elastic medium, given by

$$\alpha := \frac{1+\upsilon}{1-\upsilon} \qquad \text{and} \qquad \beta := \frac{3-\upsilon}{1+\upsilon}. \tag{3.14}$$

Proof. The reader interested in the proof of this result may refer to [15, 17, 1]. □

Corollary 2. *From the asymptotic expansion presented in Theorem 1, we can recognize the topological derivative of the functional $\mathcal{J}_\chi(u)$ given by:*

$$\mathcal{T}_{\mathcal{J}}(\widehat{x}) := \mathbb{H}_\gamma \sigma(u(\widehat{x})) \cdot \varepsilon(u(\widehat{x})). \tag{3.15}$$

4. Topological optimization procedure

In order to illustrate the applicability of the topological asymptotic expansion (3.15), here we present an optimization procedure for elastic structures under the contact condition described in Section 2. The optimization procedure is based on the domain representation in a bi-material fashion, whose constituents properties are characterized by the Young modulus E and the phase contrast γ^*. Thus, as in (3.8) and (3.9), we have

$$E(x) = \begin{cases} E & \forall x \in \Omega^h, \\ \gamma^* E & \forall x \in \Omega^w, \end{cases} \tag{4.1}$$

where Ω^h and Ω^w denote the domains occupied by the two materials, the *hard* and *weak* materials, respectively.

The optimization problem consists in minimizing the structural compliance for a given amount of material. It can be written as

$$\begin{cases} \text{Minimize} & \psi(\chi) &=& -\mathcal{J}_\chi(u), \\ \text{Subjected to} & |\Omega^h| &\leq& V, \end{cases} \tag{4.2}$$

where $|\Omega^h|$ is the Lebesgue measure of the domain Ω^h and V is the required volume at the end of the optimization process. In order to solve the above problem, we use an exact quadratic penalization scheme. Thus, problem (4.2) is re-written as following

$$\underset{\Omega\subset\mathbb{R}^2}{\text{Minimize}}\ \mathcal{F}_\Omega(u) = -\mathcal{J}_\chi(u) + \lambda s_\Omega^2, \tag{4.3}$$

where λ is a positive parameter and the function s_Ω is defined as

$$s_\Omega := 1 - \frac{|\Omega^h|}{V}. \tag{4.4}$$

By considering the linearity property of the topological derivative operator, the topological derivative of the functional $\mathcal{F}_\Omega$ can be written as

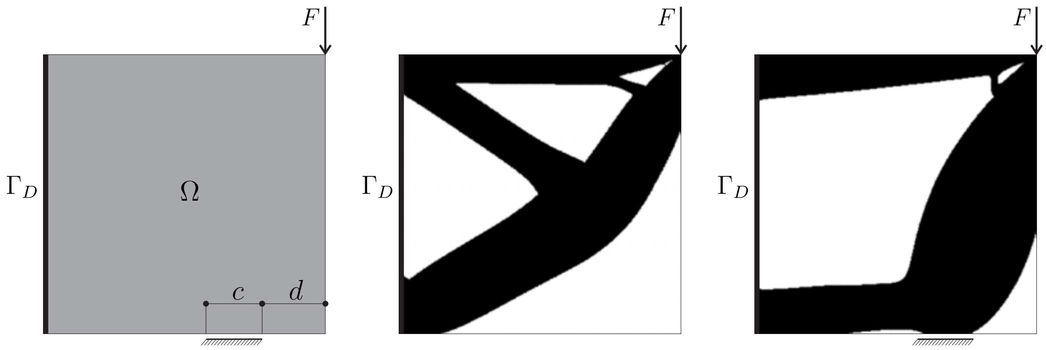
$$\mathcal{T}_{\mathcal{F}}(\widehat{x}) = -\mathcal{T}_{\mathcal{J}}(\widehat{x}) - \frac{2\lambda}{V}s_\Omega. \tag{4.5}$$

From the definition of the Young modulus (4.1), we remark that (4.5) always measures the sensitivity of $\mathcal{T}_{\mathcal{F}}$ when the two materials are interchanged within the domain. Then, the computation of (4.5) is carried out using the expressions (3.15) with $\gamma = \gamma^*$ if $x \in \Omega^h$; and $\gamma = 1/\gamma^*$ if $x \in \Omega^w$. Having made the previous consideration and in order to solve the optimization problem (4.3), we use the topology optimization algorithm proposed in [3]. This algorithm is based on the concept of level-set domain representation and uses the topological derivative (4.5) as a feasible descent direction to minimize the cost function. This class of algorithms has been successfully applied in research areas related to topological optimization such as: microstructure of materials [4], load bearing structures [3], thermal conductors [12] and load bearing structures subjected to pointwise stress constraint [2, 5]. For a detailed development of the algorithm we refer to the previous references.

5. Numerical examples

Here we present five numerical examples associated to the topological optimization procedure outlined in the previous section. In all examples we set the Young modulus $E = 2.1\ GPa$, Poisson's ratio $\nu = 0.3$, the contrast parameter $\gamma = 1 \times 10^{-3}$ and the force $F = 1 \times 1^9\ N$. In the figures, the topology is identified by the strong material distribution (in black) and the inclusions of weak material (in white) are used to mimic the holes. Furthermore, the thick lines that appear on the figures are used to denote clamped boundary conditions ($u|_{\Gamma_D} = 0$). The volume constraint is imposed with an exact quadratic penalization scheme. The function $p(y)$ used in the examples has the same behavior as presented in Fig.(1(b)). The variational equation (2.8) was solved using standard finite element technique. In particular, the three-node triangles are used to discretize the domain.

5.1. **Example 1.** In this first example we consider a unit square panel submitted to a force F applied on its right upper corner, as shown in fig. 3(a). The volume constraint is of 50% of the initial volume. In fig. 3(b) we show the optimal topology without the contact condition. Then, a contact condition is applied in the bottom side with a gap of $g = 0.10$, see fig. 3(a) where $c = 0.20$ and $d = 0.20$, and the parameter b is such that the function p reaches the value of $p(y) = 1 \times 10^{15}$. In fig. 3(c) is presented the obtained topology, where the effect of the contact condition is evident.



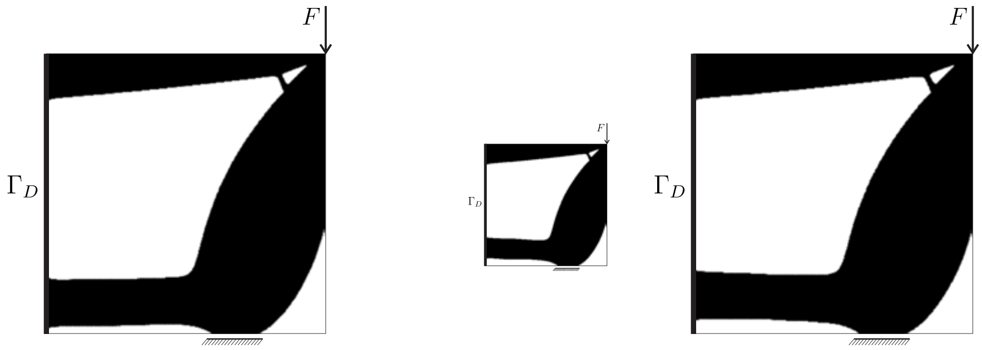
(a) Contact problem. (b) Without contact. (c) With contact.

Figure 3. Example 1. Results.

In fig. 4, we present the obtained results for three different values for the gap, i.e. $g = \{0.15, 0.20, 0.25\}$ (the result for the gap $g = 0.10$ is shown in fig. 3(c)).

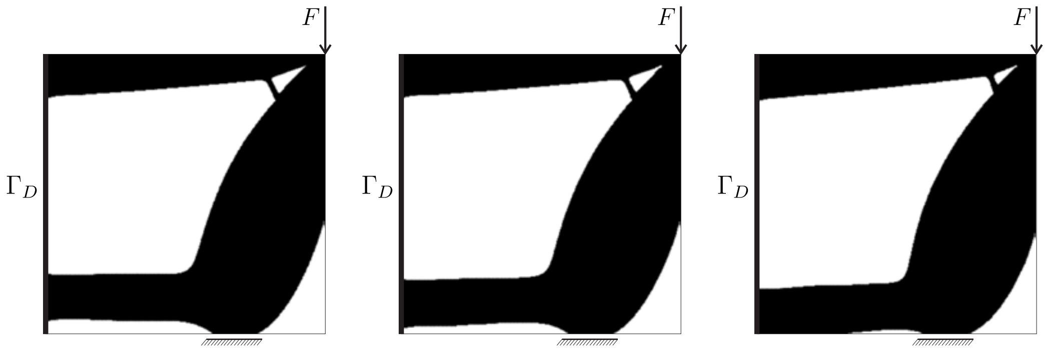
In order to evaluate the effect of the function $p(y)$, in fig. 5 the optimal topologies for different values of function $p(y)$ are presented. For this example, we set the gap in $g = 0.10$ and the parameter b, in each case, is such that the function p reaches the values $p = \{8 \times 10^{12}, 1 \times 10^{13}, 1 \times 10^{20}\}$ (the result for $p = 1 \times 10^{15}$ is shown in fig. 3(c)).

5.2. **Example 2.** In this example we present the optimal topology design of a cantilever beam with a load F applied in the middle right side of its rectangular domain. The domain of the beam is a rectangular plane with dimensions of 2.00×1.00. The contact region is located in the bottom of the plane with length c, as shown in fig. 6(a). The volume constraint is of 40% of the initial volume, the gap is $g = 0.1$ and the parameter b is such that the function p reaches the value of $p = 1 \times 10^{15}$. In this example, we study the influence of the length of the contact region in the optimal topology. In fig. (6(b)), we present the result without considering the contact condition. In figs. 6(c)–6(e) is shown the results for three different values of parameter $c = \{0.20, 0.50, 0.70\}$.



(a) $g = 0.15$. (b) $g = 0.20$. (c) $g = 0.25$.

FIGURE 4. Example 1. Results for different values of the gap.



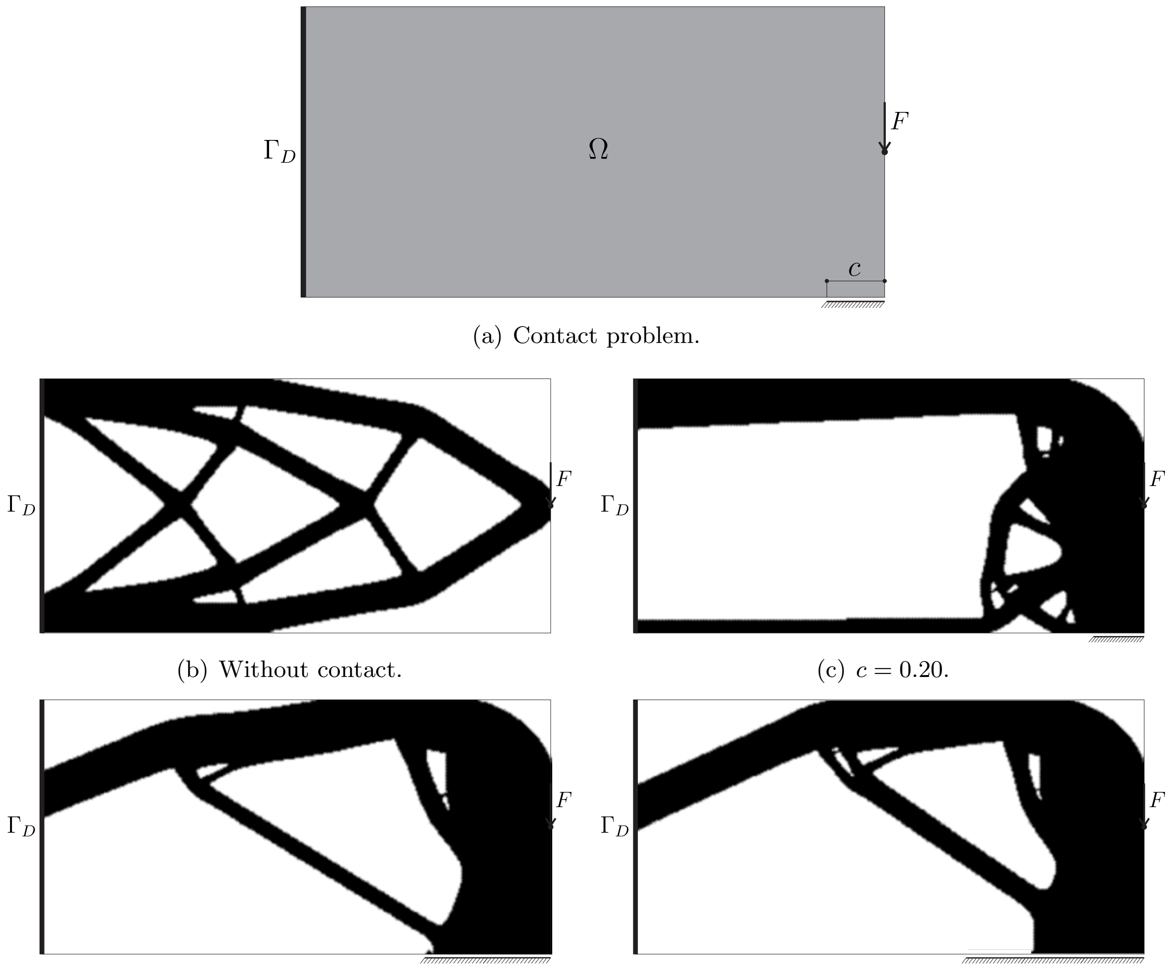
(a) $p(y) = 8^{12}$. (b) $p(y) = 1^{13}$. (c) $p(y) = 1^{20}$.

FIGURE 5. Example 1. Results for different values of function $p(y)$.

5.3. **Example 3.** Now we consider the same domain and boundary conditions as in the previous example. Here we create a square hole of size 0.25×0.25 centered at the rectangular panel and the contact region is located on the top side of the hole, see fig. 7(a). The volume constraint is of 40% of the initial volume and the gap is $g = 1 \times 10^{-5}$. The result for the case without the contact condition is presented in fig. 7(b). In fig. 7(c), we show the obtained topology considering the contact problem. Note the similarity in the results, without the boundary condition in the contact region, between this example and the previous (fig. 6(b)).

5.4. **Example 4.** In this example, the design of a unit square panel subjected to two forces F applied at the corners of the top side with a volume constraint of 30% of the initial volume is presented. The contact region is also in the top side of the panel, located a distance $d = 0.25$ from the right side and length $c = 0.50$. The gap considered is $g = 1 \times 10^{-3}$ and the parameter b is such that the function p reaches the value of $p = 1 \times 10^{15}$. The aim of this example is show the influence of the contact condition in the complexity of the final topology. The results with and without considering the contact condition are presented in figs. 8(c) and 8(b), respectively. As can be seen, topology changes from a very simple (two bars in the directions of the applied forces) to a more complex, characterized by a structure of bars similar to a small bridge.

5.5. **Example 5.** In this last example, we consider the topology design of a rectangular panel with height = 1.2 and width = 1.0, with a square hole in the right side of the domain. The design domain, boundary condition and the system of applied forces are presented in fig. 9(a), where $c = 0.20$ and $d = 0.40$. This example can be seen as the classical case of topology design of a gripping mechanism. On the potential contact region the gap is $g = 1 \times 1^{-5}$ and the function p reaches the value of 1×1^{15}.



(a) Contact problem.

(b) Without contact.

(c) $c = 0.20$.

(d) $c = 0.50$.

(e) $c = 0.70$.

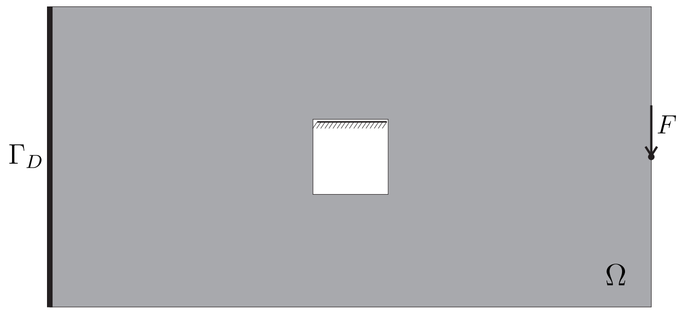
FIGURE 6. Example 2. Results for different lengths of contact region.

The volume constraint imposed is of 50% of the initial volume. The results are presented in figs. 9(c) and 9(b).

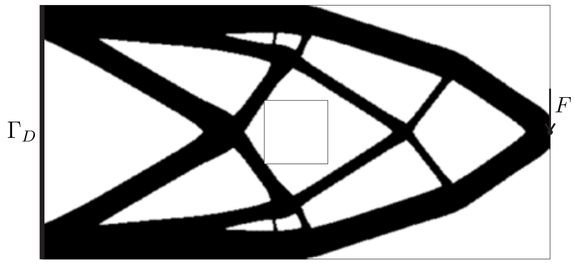
Again, in this example the effect of the contact model is manifested in the complexity of the optimal topology.

6. Final remarks

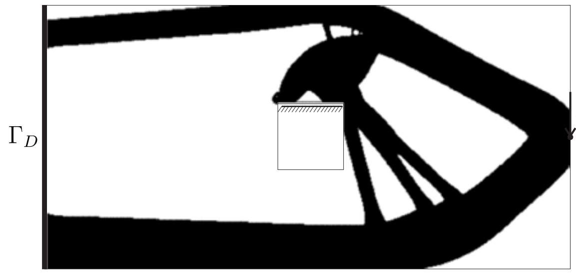
An analytical expression for the topological derivative of the energy shape functional associated to a frictionless contact model that allows a finite interpenetration between a two-dimensional elastic body and a rigid foundation has been presented. As topological perturbation, a disc with a different material has been considered in the analysis. The final formula is a general simple analytical expression in terms of the solution of the state equation and the constitutive parameters evaluated in each point of the unperturbed domain. The associated topological sensitivity has been used in a structural design algorithm based on the topological derivative and a level-set domain representation method. The robustness of the optimization procedure has been analyzed through some numerical experiments of compliance topology optimization of elastic structures subjected to volume constraint. Finally, we remark that the optimization procedure is conditioned by the contact model to produce more complex topologies that obtained by considering a unilateral contact condition and approaches such as SIMP-model.



(a) Contact problem.

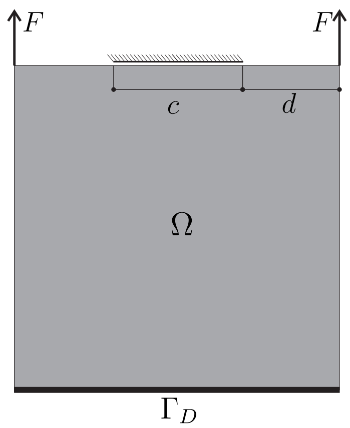


(b) Without contact.

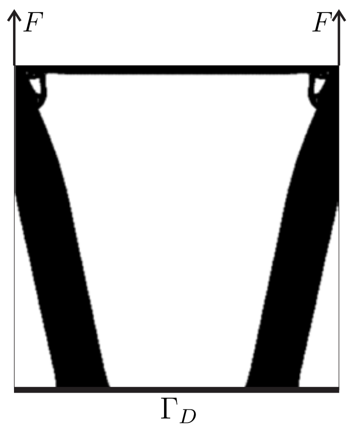


(c) With contact.

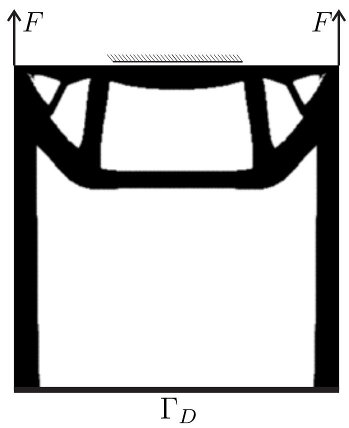
Figure 7. Example 3. Results.



(a) Contact problem.



(b) Without contact.



(c) With contact.

Figure 8. Example 4. Results.

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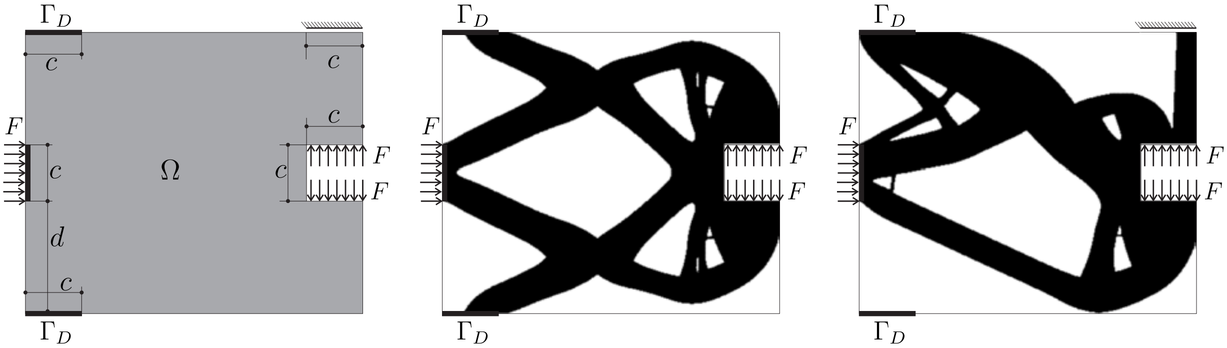

(a) Contact problem.

(b) Without contact.

(c) With contact.

FIGURE 9. Example 5. Results.

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