A topological approach to periodic oscillations related to the Liebau phenomenon

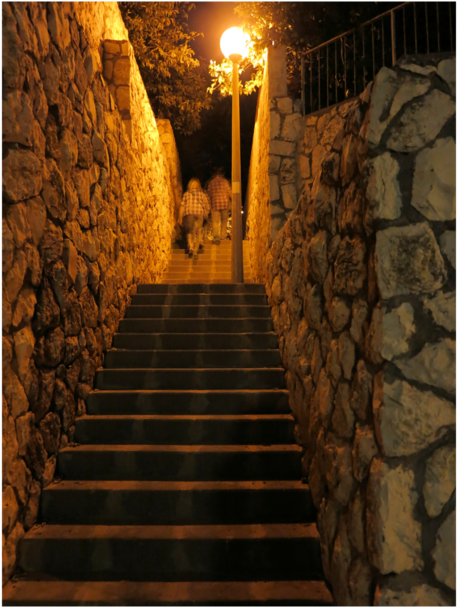
Milan Tvrdý

jointly with

José Angel Cid, Gennaro Infante, Miroslawa Zima

Institute of Mathematics
Academy of Sciences of the Czech Republic

Ariel, August 2014



A topological approach to periodic oscillations related to the Liebau phenomenon

Milan Tvrdý

jointly with

José Angel Cid, Gennaro Infante, Miroslawa Zima

Institute of Mathematics
Academy of Sciences of the Czech Republic

Ariel, August 2014

1. VALVELESS PUMPING

(Liebau phenomena)

In 1954 **G. Liebau** showed experimentally that a periodic compression made on an asymmetric part of a fluid-mechanical model could produce the circulation of the fluid without the necessity of a valve to ensure a preferential direction of the flow.

In 1954 **G. Liebau** showed experimentally that a periodic compression made on an asymmetric part of a fluid-mechanical model could produce the circulation of the fluid without the necessity of a valve to ensure a preferential direction of the flow.

DEFINITION

Let $T > 0$, $g : \mathbb{R}^3 \to \mathbb{R}$ and let $e : \mathbb{R} \to \mathbb{R}$ be nonconstant and T-periodic. Then the equation

$$x'' = g(x, x', e(t))$$

generates a *T-periodically forced pump* if it has a T-periodic solution x such that

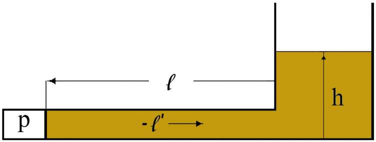
$$g(\bar{x}, 0, \bar{e}) \neq 0,$$

i.e. the mean value $\bar{x}$ of x is not an equilibrium of $x'' = g(x, x', \bar{e})$.

1 tank - 1 pipe model

G. Propst (2006)

ρ	... density of the liquid (constant)
$p(t)$	... T – periodic pressure
g	... acceleration of gravity
r_0	... friction coefficient
ζ	... junction coefficient
A_P/A_T	... cross sections of pipe/tank
V_0	... constant total volume of liquid
$w = -\ell'$	... velocity in the pipe

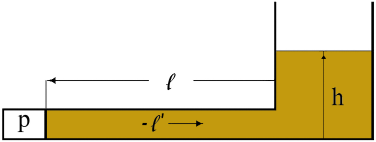


$$A_P\,\ell(t) + A_T\,h(t) \equiv V_0 \quad \Longrightarrow \quad h(t) \equiv \frac{1}{A_T}\big(V_0 - A_P\,\ell(t)\big)\,.$$

Momentum balance with Poiseuille's law and Bernoulli's equation

G. Propst (2006)

ρ	... density of the liquid (constant)
$p(t)$	... T – periodic pressure
g	... acceleration of gravity
r_0	... friction coefficient
ζ	... junction coefficient
A_P/A_T	... cross sections of pipe/tank
V_0	... constant total volume of liquid
$w = -\ell'$	... velocity in the pipe



$$A_P\,\ell(t) + A_T\,h(t) \equiv V_0 \quad \Longrightarrow \quad h(t) \equiv \frac{1}{A_T}\big(V_0 - A_P\,\ell(t)\big)\,.$$

Momentum balance with Poiseuille's law and Bernoulli's equation $\Longrightarrow$

$$\ell\,\ell'' + a\,\ell\,\ell' + b\,(\ell')^2 + c\,\ell = e(t),$$

where

$$T > 0, \quad a = \frac{r_0}{\rho} \geq 0, \quad b = \Big(1 + \frac{\zeta}{2}\Big) \geq 3/2,$$

$$e(t) = \frac{g\,V_0}{A_T} - \frac{p(t)}{\rho} \text{ is } T - \text{periodic}, \quad 0 < c = \frac{g\,A_p}{A_T} < 1.$$

This leads to singular periodic problem:

$$(1) \qquad u'' + a\,u' = \frac{1}{u}\left(e(t) - b\,(u')^2\right) - c\,, \quad u(0) = u(T),\; u'(0) = u'(T)\,,$$

$$T > 0,\; a = \frac{r_0}{\rho} \geq 0,\; b = \left(1 + \frac{\zeta}{2}\right) \geq 3/2,\; 0 < c = \frac{g\,A_p}{A_T} < 1,\; e(t) = \frac{g\,V_0}{A_T} - \frac{p(t)}{\rho}.$$

This leads to singular periodic problem:

$$(1) \qquad u'' + a\,u' = \frac{1}{u}\left(e(t) - b\,(u')^2\right) - c\,, \quad u(0) = u(T),\; u'(0) = u'(T)\,,$$

$$T > 0,\; a = \frac{r_0}{\rho} \geq 0,\; b = \left(1 + \frac{\zeta}{2}\right) \geq 3/2,\; 0 < c = \frac{g\,A_p}{A_T} < 1,\; e(t) = \frac{g\,V_0}{A_T} - \frac{p(t)}{\rho}.$$

Multiplying the equation by u and integrating over $[0, T]$ gives

THEOREM 1

(1) has a positive solution only if $\quad \overline{e} \geq 0 \quad$ (i.e. $\quad \overline{p} \leq \rho\,g\,\frac{V_0}{A_T}$).

This leads to singular periodic problem:

$$(1) \qquad u'' + a\,u' = \frac{1}{u}\left(e(t) - b\,(u')^2\right) - c\,, \quad u(0) = u(T),\ u'(0) = u'(T)\,,$$

$$T > 0,\ a = \frac{r_0}{\rho} \geq 0,\ b = \left(1 + \frac{\zeta}{2}\right) \geq 3/2,\ 0 < c = \frac{g\,A_p}{A_T} < 1,\ e(t) = \frac{g\,V_0}{A_T} - \frac{p(t)}{\rho}.$$

Multiplying the equation by u and integrating over $[0, T]$ gives

THEOREM 1

(1) has a positive solution only if $\quad \overline{e} \geq 0 \quad$ (i.e. $\quad \overline{p} \leq \rho\, g\, \frac{V_0}{A_T}$).

THEOREM 2

If (1) has a positive solution, then it generates a T-periodically forced pump.

$$\text{(E)}\ u''+ku=\frac{b}{u^{\lambda}}+e(t),\ u(0)=u(T),\ u'(0)=u'(T)\quad (b>0,\ \lambda>0,\ k\geq 0,\ e\in L_1[0,T])$$

has a solution if:

- $k=0,\ \lambda\geq 1,\ \overline{e}<0$ [**Lazer & Solimini**],
- $k\neq \left(n\,\frac{\pi}{T}\right)^2$ for all $n\in\mathbb{N},\ \lambda\geq 1,\ e\in C$ [**del Pino, Manásevich & Montero**]
- $0<k<\left(\frac{\pi}{T}\right)^2,\ \lambda\geq 1,\ e\in L_\infty$ [**Omari & Ye**],
- $k=0,\quad \overline{e}<0,\quad e_*:=\inf\operatorname{ess}_{t\in[0,T]} e(t)>-\left(\frac{1}{T^2\,\lambda\,b}\right)^{\frac{\lambda}{\lambda+1}}(\lambda+1)\,b,$

  $0<k<\left(\frac{\pi}{T}\right)^2,\quad e_*:=\inf\operatorname{ess}_{t\in[0,T]} e(t)>-\left(\frac{\pi^2-T^2k}{T^2\,\lambda\,b}\right)^{\frac{\lambda}{\lambda+1}}(\lambda+1)\,b$

  [supplementary results by **Torres, Hakl & Torres, Chu & Franco** et al.],

  $k=\left(\frac{\pi}{T}\right)^2,\quad \inf\operatorname{ess}_{t\in[0,T]} e(t)>0$ [**Rachůnková, Tvrdý & Vrkoč**],

  [supplementary results by **Bonheure & De Coster, Chu & Torres** et al.]

2. EXISTENCE OF A PERIODIC SOLUTION

$$(1) \qquad u'' + a\,u' = \frac{1}{u}\left(e(t) - b\,(u')^2\right) - c\,, \quad u(0) = u(T),\ u'(0) = u'(T)\,,$$

THEOREM 3

ASSUME:

- $a \geq 0, \quad b > 1, \quad c > 0,$
- e is continuous and T-periodic on $\mathbb{R}$, $e_* > 0$,
- $\dfrac{(b+1)\,c^2}{4\,e_*} < \left(\dfrac{\pi}{T}\right)^2 + \dfrac{a^2}{4}.$

THEN: (1) has a positive solution.

$$(1) \qquad u'' + a\,u' = \frac{1}{u}\left(e(t) - b\,(u')^2\right) - c\,, \quad u(0) = u(T),\; u'(0) = u'(T)\,,$$

THEOREM 3

ASSUME:

- $a \geq 0, \quad b > 1, \quad c > 0,$
- e is continuous and T-periodic on $\mathbb{R}$, $e_* > 0$,
- $\dfrac{(b+1)\,c^2}{4\,e_*} < \left(\dfrac{\pi}{T}\right)^2 + \dfrac{a^2}{4}.$

THEN: (1) has a positive solution.

DEFINITION

A T-periodic function $\sigma_1 \in C^2[0,T]$ is a *lower function* for

$$u'' + a\,u' = f(t,u), \quad u(0) = u(T),\; u'(0) = u'(T)\,,$$

if

$$\sigma_1''(t) + a\,\sigma_1'(t) \geq f(t, \sigma_1(t)) \quad \text{for } t \in [0,T],$$

while an *upper function* is defined analogously, but with reversed inequality.

(1) $$u''+a\,u' = \frac{1}{u}\left(e(t) - b\,(u')^2\right) - c\,, \quad u(0)=u(T),\; u'(0)=u'(T)\,,$$

<u>STEP 1</u>: $u : [0,T]\to\mathbb{R}$ is a positive solution of (1) iff $x = u^{1/\mu}$ is a positive solution of

(2) $$x'' + a\,x'(t) = r(t)\,x^{\alpha} - s(t)\,x^{\beta}\,, \quad x(0)=x(T),\; x'(0)=x'(T)\,,$$

where

$$0<\mu=\frac{1}{b+1}<\frac{2}{5},\; r(t)=\frac{e(t)}{\mu}>0,\; s(t)=\frac{c}{\mu}>0,\; 0<\alpha=1-2\,\mu,\; <\beta=1-\mu<1.$$

$$(1) \qquad u'' + a\,u' = \frac{1}{u}\left(e(t) - b\,(u')^2\right) - c\,, \quad u(0) = u(T),\; u'(0) = u'(T)\,,$$

<u>STEP 1</u>: $u : [0,T] \to \mathbb{R}$ is a positive solution of (1) iff $x = u^{1/\mu}$ is a positive solution of

$$(2) \qquad x'' + a\,x'(t) = r(t)\,x^{\alpha} - s(t)\,x^{\beta}\,, \quad x(0) = x(T),\; x'(0) = x'(T)\,,$$

where

$$0 < \mu = \frac{1}{b+1} < \frac{2}{5},\; r(t) = \frac{e(t)}{\mu} > 0,\; s(t) = \frac{c}{\mu} > 0,\; 0 < \alpha = 1 - 2\,\mu,\; < \beta = 1 - \mu < 1.$$

<u>STEP 2</u>: There are constant lower and upper functions σ_1 and σ_2 of (2) such that

$$0 < \sigma_2 < x_0 = (r_*/s^*)^{1/(\beta-\alpha)} < x_1 = (r^*/s_*)^{1/(\beta-\alpha)} < \sigma_1.$$

$$(1) \qquad u''+a\,u' = \frac{1}{u}\left(e(t)-b\,(u')^2\right)-c\,, \quad u(0)=u(T),\ u'(0)=u'(T)\,,$$

STEP 1: $u:[0,T]\to\mathbb{R}$ is a positive solution of (1) iff $x=u^{1/\mu}$ is a positive solution of

$$(2) \qquad x''+a\,x'(t)=r(t)\,x^{\alpha}-s(t)\,x^{\beta}\,, \quad x(0)=x(T),\ x'(0)=x'(T)\,,$$

where

$$0<\mu=\frac{1}{b+1}<\frac{2}{5},\ r(t)=\frac{e(t)}{\mu}>0,\ s(t)=\frac{c}{\mu}>0,\ 0<\alpha=1-2\,\mu,\ <\beta=1-\mu<1.$$

STEP 2: There are constant lower and upper functions σ_1 and σ_2 of (2) such that

$$0<\sigma_2<x_0=(r_*/s^*)^{1/(\beta-\alpha)}<x_1=(r^*/s_*)^{1/(\beta-\alpha)}<\sigma_1.$$

STEP 3: We show that there is $\delta_0\in(0,\sigma_2)$ such that

$$r(t)\,x^{\alpha}-s(t)\,x^{\beta}<0 \quad \text{for } t\in[0,T],\ x\in(0,\delta_0)$$

and

$$-\left(\left(\frac{\pi}{T}\right)^2+\frac{a^2}{4}\right)x+r(t)\,x^{\alpha}-s(t)\,x^{\beta}<0 \quad \text{for } t\in[0,T],\ x\ge\delta_0.$$

$$(1) \qquad u''+a\,u' = \frac{1}{u}\left(e(t)-b\,(u')^2\right)-c\,, \quad u(0)=u(T),\; u'(0)=u'(T)\,,$$

STEP 1: $u:[0,T]\to\mathbb{R}$ is a positive solution of (1) iff $x=u^{1/\mu}$ is a positive solution of

$$(2) \qquad x''+a\,x'(t)=r(t)\,x^{\alpha}-s(t)\,x^{\beta}\,, \quad x(0)=x(T),\; x'(0)=x'(T)\,,$$

where

$$0<\mu=\frac{1}{b+1}<\frac{2}{5},\; r(t)=\frac{e(t)}{\mu}>0,\; s(t)=\frac{c}{\mu}>0,\; 0<\alpha=1-2\,\mu,\; <\beta=1-\mu<1.$$

STEP 2: There are constant lower and upper functions σ_1 and σ_2 of (2) such that

$$0<\sigma_2<x_0=(r_*/s^*)^{1/(\beta-\alpha)}<x_1=(r^*/s_*)^{1/(\beta-\alpha)}<\sigma_1.$$

STEP 3: We show that there is $\delta_0\in(0,\sigma_2)$ such that

$$r(t)\,x^{\alpha}-s(t)\,x^{\beta}<0 \quad \text{for } t\in[0,T],\; x\in(0,\delta_0)$$

and

$$-\left(\left(\frac{\pi}{T}\right)^2+\frac{a^2}{4}\right)x+r(t)\,x^{\alpha}-s(t)\,x^{\beta}<0 \quad \text{for } t\in[0,T],\; x\geq\delta_0.$$

STEP 4: We choose $\delta\in(0,\delta_0)$, put $\lambda^*=\left(\frac{\pi}{T}\right)^2+\frac{a^2}{4}$,

$$\widetilde{f}(t,x)=\begin{cases} r(t)\,\delta^{\alpha}-s(t)\,\delta^{\beta}-\lambda^*\,(x-\delta) & \text{for } x<\delta\,,\\ r(t)\,x^{\alpha}-s(t)\,x^{\beta} & \text{for } x\geq\delta\end{cases}$$

and consider auxiliary problem

$$(\text{Aux}) \qquad x''+a\,x'(t)=\widetilde{f}(t,x)\,, \qquad x(0)=x(T),\; x'(0)=x'(T)\,,$$

$$(1) \qquad u'' + a\,u' = \frac{1}{u}\left(e(t) - b\,(u')^2\right) - c\,, \quad u(0) = u(T),\; u'(0) = u'(T)\,,$$

<u>STEP 1</u>: $u : [0,T] \to \mathbb{R}$ is a positive solution of (1) iff $x = u^{1/\mu}$ is a positive solution of

$$(2) \qquad x'' + a\,x'(t) = r(t)\,x^{\alpha} - s(t)\,x^{\beta}\,, \quad x(0) = x(T),\; x'(0) = x'(T)\,,$$

where

$$0 < \mu = \frac{1}{b+1} < \frac{2}{5},\; r(t) = \frac{e(t)}{\mu} > 0,\; s(t) = \frac{c}{\mu} > 0,\; 0 < \alpha = 1 - 2\,\mu,\; < \beta = 1 - \mu < 1.$$

<u>STEP 2</u>: There are constant lower and upper functions σ_1 and σ_2 of (2) such that

$$0 < \sigma_2 < x_0 = (r_*/s^*)^{1/(\beta-\alpha)} < x_1 = (r^*/s_*)^{1/(\beta-\alpha)} < \sigma_1.$$

<u>STEP 3</u>: We show that there is $\delta_0 \in (0, \sigma_2)$ such that

$$r(t)\,x^{\alpha} - s(t)\,x^{\beta} < 0 \quad \text{for } t \in [0,T],\; x \in (0, \delta_0)$$

and

$$-\left(\left(\frac{\pi}{T}\right)^2 + \frac{a^2}{4}\right) x + r(t)\,x^{\alpha} - s(t)\,x^{\beta} < 0 \quad \text{for } t \in [0,T],\; x \ge \delta_0.$$

<u>STEP 4</u>: We choose $\delta \in (0, \delta_0)$, put $\lambda^* = \left(\frac{\pi}{T}\right)^2 + \frac{a^2}{4}$,

$$\widetilde{f}(t,x) = \begin{cases} r(t)\,\delta^{\alpha} - s(t)\,\delta^{\beta} - \lambda^*\,(x - \delta) & \text{for } x < \delta\,, \\ r(t)\,x^{\alpha} - s(t)\,x^{\beta} & \text{for } x \ge \delta \end{cases}$$

and consider auxiliary problem

$$(\text{Aux}) \qquad x'' + a\,x'(t) = \widetilde{f}(t,x)\,, \qquad x(0) = x(T),\; x'(0) = x'(T)\,,$$

Method of non-ordered lower and upper functions (BONHEURE & De COSTER)

$\Longrightarrow$ (Aux) has a solution x.

STEPS 1–4:

$$(1) \qquad u'' + a\,u' = \frac{1}{u}\left(e(t) - b\,(u')^2\right) - c\,, \quad u(0) = u(T),\ u'(0) = u'(T)\,,$$

$$\Updownarrow$$

$$(2) \qquad x'' + a\,x'(t) = r(t)\,x^{\alpha} - s(t)\,x^{\beta}0\,, \quad x(0) = x(T),\ x'(0) = x'(T)\,,$$

where

$$0 < \mu = \frac{1}{b+1} < \frac{2}{5},\ r(t) = \frac{e(t)}{\mu} > 0,\ s(t) = \frac{c}{\mu} > 0,\ 0 < \alpha = 1 - 2\,\mu,\ < \beta = 1 - \mu < 1.$$

We have a solution x to

$$(\mathrm{Aux}) \qquad x'' + a\,x'(t) = \widetilde{f}(t,x)\,, \qquad x(0) = x(T),\ x'(0) = x'(T)\,,$$

where

$$\widetilde{f}(t,x) = \begin{cases} r(t)\,\delta^{\alpha} - s(t)\,\delta^{\beta} - \lambda^*\,(x-\delta) & \text{for } x < \delta\,, \\ r(t)\,x^{\alpha} - s(t)\,x^{\beta} & \text{for } x \ge \delta \end{cases}$$

STEP 5: Put $v = x - \delta$. Then

$$v''(t) + a\,v'(t) + \lambda^*\,v(t) = h(t) \text{ for } t \in [0,T],\ v(0) = v(T),\ v'(0) = v'(T),$$

where (by Step 3) $h(t) := \lambda^*\,(x(t) - \delta) - \widetilde{f}(t, x(t)) \ge 0$ on $[0,T]$.

STEPS 1–4:

(1) $$u'' + a\,u' = \frac{1}{u}\left(e(t) - b\,(u')^2\right) - c\,, \quad u(0) = u(T),\ u'(0) = u'(T)\,,$$

$$\Updownarrow$$

(2) $$x'' + a\,x'(t) = r(t)\,x^{\alpha} - s(t)\,x^{\beta}0\,, \quad x(0) = x(T),\ x'(0) = x'(T)\,,$$

where

$$0 < \mu = \frac{1}{b+1} < \frac{2}{5},\ r(t) = \frac{e(t)}{\mu} > 0,\ s(t) = \frac{c}{\mu} > 0,\ 0 < \alpha = 1 - 2\,\mu,\ < \beta = 1 - \mu < 1.$$

We have a solution x to

(Aux) $$x'' + a\,x'(t) = \widetilde{f}(t,x)\,, \qquad x(0) = x(T),\ x'(0) = x'(T)\,,$$

where

$$\widetilde{f}(t,x) = \begin{cases} r(t)\,\delta^{\alpha} - s(t)\,\delta^{\beta} - \lambda^*\,(x - \delta) & \text{for } x < \delta\,, \\ r(t)\,x^{\alpha} - s(t)\,x^{\beta} & \text{for } x \geq \delta \end{cases}$$

STEP 5: Put $v = x - \delta$. Then

$$v''(t) + a\,v'(t) + \lambda^*\,v(t) = h(t) \ \text{ for } t \in [0,T],\ \ v(0) = v(T),\ v'(0) = v'(T),$$

where (by Step 3) $h(t) := \lambda^*\,(x(t) - \delta) - \widetilde{f}(t, x(t)) \geq 0$ on $[0,T]$.

Antimaximum principle (OMARI & TROMBETTA or HAKL & ZAMORA) $\Longrightarrow$ $v \geq 0$, i.e. $x \geq \delta$ $\square$

(2) $$u'' + a\,u' = r(t)\,u^{\alpha} - s(t)\,u^{\beta}, \quad u(0) = u(T),\ u'(0) = u'(T),$$

THEOREM 4

ASSUME:

- $a \geq 0, \quad b > 1, \quad c > 0,\ 0 < \alpha < \beta < 1,$
- $r_* > 0,\ s_* > 0,$
- there is $\delta_0 > 0$ such that

$$r(t)\,u^{\alpha} - s(t)\,u^{\beta} < 0 \quad \text{for } t \in [0,T],\ x \in (0,\delta_0)$$

and

$$-\left(\left(\frac{\pi}{T}\right)^2 + \frac{a^2}{4}\right) x + r(t)\,x^{\alpha} - s(t)\,x^{\beta} < 0 \quad \text{for } t \in [0,T],\ x \geq \delta_0.$$

THEN: (2) has a positive solution.

3. ASYMPTOTIC STABILITY

(3) $$x'' + a\,x'(t) = f(t,x)\,, \qquad x(0) = x(T),\ x'(0) = x'(T)$$

Lemma (Omari & Njoku, 2003)

ASSUME: $a > 0$,

- σ_1 is a strict lower function, σ_2 is a strict upper function of (3) and $\sigma_2 < \sigma_1$ on $[0, T]$.
- $\frac{\partial}{\partial x} f(t,x) \geq -\left(\frac{\pi}{T}\right)^2 + \frac{a^2}{4}$ for $t \in [0, T],\ x \in [\sigma_2(t), \sigma_1(t)]$,
- there is a continuous $\gamma : [0, T] \to [0, \infty)$ such that $\bar{\gamma} > 0$ and

$$\frac{\partial}{\partial x} f(t,x) \leq -\gamma(t) \quad \text{for } t \in [0, T],\ x \in [\sigma_2(t), \sigma_1(t)].$$

Then (3) has at least one asymptotically stable T-periodic solution x fulfilling

$$\sigma_2 \leq x \leq \sigma_1 \quad \text{on } [0, T]\,.$$

(3) $$x'' + a x'(t) = f(t,x), \qquad x(0) = x(T),\ x'(0) = x'(T)$$

THEOREM 5

ASSUME: $a > 0,\ f(t,x) = r(t)x^{\alpha} - s(t)x^{\beta}$,

- r, s are continuous and positive on $[0,T]$, $0 < \alpha < \beta < 1$,
- $\beta s^* \left(\frac{s^*}{r_*}\right)^{(1-\beta)/(\beta-\alpha)} - \alpha r_* \left(\frac{s_*}{r^*}\right)^{(1-\alpha)/(\beta-\alpha)} < \left(\frac{\pi}{T}\right)^2 + \frac{a^2}{4}$,
- $\frac{\alpha}{\beta}\frac{r^*}{s_*} < \frac{r_*}{s^*}$.

THEN: (3) has at least one asymptotically stable positive solution.

(3) $$x'' + a x'(t) = f(t,x)\,, \qquad x(0) = x(T),\ x'(0) = x'(T)$$

THEOREM 5

ASSUME: $a > 0,\ f(t,x) = r(t)\,x^{\alpha} - s(t)\,x^{\beta}$,

- r, s are continuous and positive on $[0, T]$, $0 < \alpha < \beta < 1$,
- $$\beta s^* \left(\frac{s^*}{r_*}\right)^{(1-\beta)/(\beta-\alpha)} - \alpha r_* \left(\frac{s_*}{r^*}\right)^{(1-\alpha)/(\beta-\alpha)} < \left(\frac{\pi}{T}\right)^2 + \frac{a^2}{4},$$
- $$\frac{\alpha}{\beta}\frac{r^*}{s_*} < \frac{r_*}{s^*}.$$

THEN: (3) has at least one asymptotically stable positive solution.

(1) $$u'' + a u' = \frac{1}{u}\left(e(t) - b\,(u')^2\right) - c\,, \quad u(0) = u(T),\ u'(0) = u'(T)$$

COROLLARY

(1) has at least one asymptotically stable positive solution if

$$\frac{c^2\,(b\,(e^*)^2 - (b-1)\,(e_*)^2)}{e_*\,(e^*)^2} < \left(\frac{\pi}{T}\right)^2 + \frac{a^2}{4} \quad \text{and} \quad \frac{e^* - e_*}{e^*} < \frac{1}{b}.$$

4. APPLICATION OF KRASNOSELSKII COMPRESION/EXPANSION THEOREM

(4) $$x'' + a x' + m^2 x = 0,\ x(0) - x(T),\ x'(0) = x'(T) \quad \left[a \geq 0,\ 0 < m^2 < \left(\frac{\pi}{T}\right)^2 + \left(\frac{a}{2}\right)^2 \right]$$

(4) $x'' + a\,x' + m^2 x = 0,\ x(0) - x(T),\ x'(0) = x'(T)$ $\left[a \geq 0,\ 0 < m^2 < \left(\frac{\pi}{T}\right)^2 + \left(\frac{a}{2}\right)^2 \right]$

has Green's function $G_m(t,s)$ such that

- $G_m(t,s) > 0$ for all $t, s \in [0,T]$,
- there exists $c_m \in (0,1)$ such that $G_m(s,s) \geq c_m\, G_m(t,s)$ for all $t, s \in [0,T]$,

(4) $\quad x'' + a\,x' + m^2 x = 0,\ x(0) - x(T),\ x'(0) = x'(T) \quad \left[a \geq 0,\ 0 < m^2 < \left(\frac{\pi}{T}\right)^2 + \left(\frac{a}{2}\right)^2\right]$

has Green's function $G_m(t,s)$ such that

- $G_m(t,s) > 0$ for all $t, s \in [0,T]$,
- there exists $c_m \in (0,1)$ such that $G_m(s,s) \geq c_m\, G_m(t,s)$ for all $t, s \in [0,T]$,

Put $\quad (F\,x)(t) = \int_0^T G_m(t,s) \left[r(s)\,x^{\alpha}(s) - s(t)\,x^{\beta}(s) + m^2\,x(s)\right] ds$

(4) $x'' + a x' + m^2 x = 0,\ x(0) - x(T),\ x'(0) = x'(T)$ $\left[a \geq 0,\ 0 < m^2 < \left(\frac{\pi}{T}\right)^2 + \left(\frac{a}{2}\right)^2 \right]$

has Green's function $G_m(t,s)$ such that

- $G_m(t,s) > 0$ for all $t, s \in [0,T]$,
- there exists $c_m \in (0,1)$ such that $G_m(s,s) \geq c_m\, G_m(t,s)$ for all $t, s \in [0,T]$,

Put $(F x)(t) = \displaystyle\int_0^T G_m(t,s) \left[r(s)\, x^{\alpha}(s) - s(t)\, x^{\beta}(s) + m^2 x(s) \right] ds$

Then x is a solution to

(2) $x'' + a x' = r(t)\, x^{\alpha} - s(t)\, x^{\beta},$ $x(0) = x(T),\ x'(0) = x'(T)$

iff $x = F x.$

$$(4) \qquad x'' + a\,x' + m^2 x = 0, \;\; x(0) - x(T), \;\; x'(0) = x'(T) \qquad \left[a \geq 0, \; 0 < m^2 < \left(\frac{\pi}{T}\right)^2 + \left(\frac{a}{2}\right)^2 \right]$$

has Green's function $G_m(t,s)$ such that

- $G_m(t,s) > 0$ for all $t, s \in [0,T]$,
- there exists $c_m \in (0,1)$ such that $G_m(s,s) \geq c_m\, G_m(t,s)$ for all $t, s \in [0,T]$,

Put $$(F\,x)(t) = \int_0^T G_m(t,s) \left[r(s)\, x^{\alpha}(s) - s(t)\, x^{\beta}(s) + m^2\, x(s) \right] ds$$

Then x is a solution to

$$(2) \qquad x'' + a\,x' = r(t)\,x^{\alpha} - s(t)\,x^{\beta}, \qquad x(0) = x(T), \;\; x'(0) = x'(T)$$

iff $\quad x = F\,x.$

Krasnoselskii Fixed Point Theorem

Let P be a cone in X, Ω_1 and Ω_2 be bounded open sets in X such that $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$.
Let $F : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ be a completely continuous operator such that one of the following conditions holds:

- $\|F\,x\| \geq \|x\|$ for $x \in P \cap \partial\Omega_1$ and $\|F\,x\| \leq \|x\|$ for $x \in P \cap \partial\Omega_2$,
- $\|F\,x\| \leq \|x\|$ for $x \in P \cap \partial\Omega_1$ and $\|F\,x\| \geq \|x\|$ for $x \in P \cap \partial\Omega_2$.

Then F has a fixed point in the set $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

(2) $$x'' + a x' = r(t) x^{\alpha} - s(t) x^{\beta}, \qquad x(0) = x(T),\ x'(0) = x'(T)$$

- $G_m(t,s) > 0$ for all $t, s \in [0,T]$,
- there exists $c_m \in (0,1)$ such that $G_m(s,s) \geq c_m\, G_m(t,s)$ for all $t, s \in [0,T]$,

Put

- $P = \{x \in C[0,T] : x(t) \geq 0 \text{ on } [0,T] \text{ and } x(t) \geq c_m \|x\| \text{ on } [0,T]\}$,
- $\Omega_1 = \{x \in C[0,T] \colon \|x\| < R_1\}, \quad \Omega_2 = \{x \in C[0,T] \colon \|x\| < R_2\}$.

(2) $$x'' + a x' = r(t) x^{\alpha} - s(t) x^{\beta}, \qquad x(0) = x(T),\ x'(0) = x'(T)$$

- $G_m(t,s) > 0$ for all $t,s \in [0,T]$,
- there exists $c_m \in (0,1)$ such that $G_m(s,s) \geq c_m G_m(t,s)$ for all $t,s \in [0,T]$,

Put

- $P = \{x \in C[0,T] : x(t) \geq 0 \text{ on } [0,T] \text{ and } x(t) \geq c_m \|x\| \text{ on } [0,T]\}$,
- $\Omega_1 = \{x \in C[0,T]: \|x\| < R_1\}, \quad \Omega_2 = \{x \in C[0,T]: \|x\| < R_2\}$.

THEOREM 6

ASSUME: $a \geq 0,\ r,s \in C[0,T],\ 0 < \alpha < \beta < 1$,

there exist $m > 0$ and $0 < R_1 < R_2$ such that $m^2 < \left(\frac{\pi}{T}\right)^2 + \left(\frac{a}{2}\right)^2$,

$$\begin{aligned}
&r(t)x^{\alpha} - s(t)x^{\beta} + m^2 x \geq 0 && \text{for } t \in [0,T],\ x \in [c_m R_1, R_2],\\
&r(t)x^{\alpha} - s(t)x^{\beta} + m^2 x \geq m^2 R_1 && \text{for } t \in [0,T],\ x \in [c_m R_1, R_1],\\
&r(t)x^{\alpha} - s(t)x^{\beta} + m^2 x \leq m^2 R_2 && \text{for } t \in [0,T],\ x \in [c_m R_2, R_2],
\end{aligned}$$

THEN: (2) has a positive solution $x \in [c_m R_1, R_2]$.

Application of Krasnoselskii compresion/expansion theorem

$$(2) \qquad x'' + a x' = r(t) x^{\alpha} - s(t) x^{\beta}, \qquad x(0) = x(T),\ x'(0) = x'(T)$$

COROLLARY=THEOREM 3

ASSUME:

- $a \geq 0, \quad b > 1, \quad c > 0,$
- e is continuous and T-periodic on $\mathbb{R}$, $e_* > 0$,
- $\dfrac{(b+1)\,c^2}{4\,e_*} < \left(\dfrac{\pi}{T}\right)^2 + \dfrac{a^2}{4}.$

THEN: (1) has a positive solution.

Remark

Compare conditions:

- Theorem 3: there is $\delta > 0$ such that

$$\left(\left(\frac{\pi}{T}\right)^2 + \left(\frac{a}{2}\right)^2\right) x - f(t,x) \geq \left(\left(\frac{\pi}{T}\right)^2 + \left(\frac{a}{2}\right)^2\right) \delta \quad \text{for } t \in [0,T],\ x \geq \delta,$$

- Theorem 6: there is $m \in \left(0, \left(\frac{\pi}{T}\right)^2 + \left(\frac{a}{2}\right)^2\right)$, such that

$$m^2 x - f(t,x) \geq 0 \quad \text{for } t \in [0,T],\ x \in [c_m R_1, R_2]$$

REFERENCES

- J.A. Cid, G. Infante, M. Tvrdý and M. Zima: Topological approach to periodic oscillations related to the Liebau phenomenon, arXiv:1408.0130 [math.CA].
- J.A. Cid, G. Propst and M. Tvrdý: On the pumping effect in a pipe/tank flow configuration with friction. *Physica D* **273-274** (2014), 28-33.

- J.A. Cid, G. Infante, M. Tvrdý and M. Zima: Topological approach to periodic oscillations related to the Liebau phenomenon, arXiv:1408.0130 [math.CA].
- J.A. Cid, G. Propst and M. Tvrdý: On the pumping effect in a pipe/tank flow configuration with friction. *Physica D* **273-274** (2014), 28-33.

- **G. Liebau**: Über ein ventilloses Pumpprinzip. *Naturwissenschaften* **41** (1954), 327.
- **G. Propst**: Pumping effects in models of periodically forced flow configurations. *Physica D* **217** (2006), 193–201.

- J.A. Cid, G. Infante, M. Tvrdý and M. Zima: Topological approach to periodic oscillations related to the Liebau phenomenon, arXiv:1408.0130 [math.CA].
- J.A. Cid, G. Propst and M. Tvrdý: On the pumping effect in a pipe/tank flow configuration with friction. *Physica D* **273-274** (2014), 28-33.

- **G. Liebau**: Über ein ventilloses Pumpprinzip. *Naturwissenschaften* **41** (1954), 327.
- **G. Propst**: Pumping effects in models of periodically forced flow configurations. *Physica D* **217** (2006), 193–201.

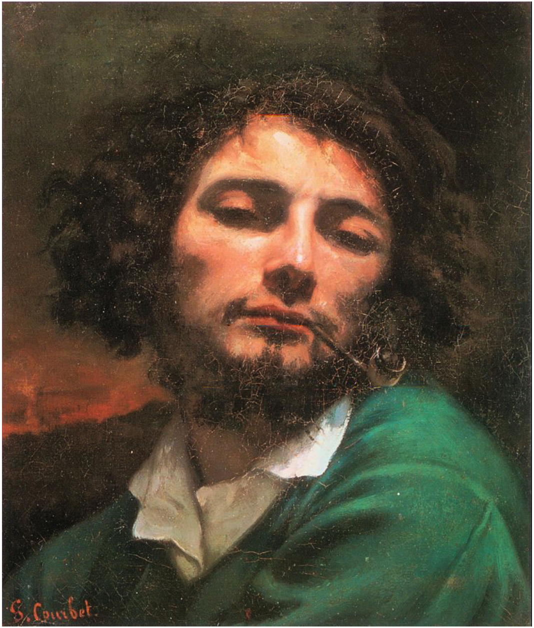
- **D. Bonheure and C. De Coster**. Forced singular oscillators and the method of lower and upper solutions, *Topological Methods in Nonlinear Analysis* **22** (2003), 297–317.
- **F.I. Njoku & P. Omari**. Stability properties of periodic solutions of a Duffing equation in the presence of lower and upper solutions. *Appl. Math. Comput.* **135** (2003), 471–490.
- **P. Omari & M. Trombetta**. Remarks on the lower and upper solutions method for the second and third-order periodic boundary value problems. *Appl. Math. Comput.* **50** (1992), 1–21.
- **I. Rachůnková, M. Tvrdý and I. Vrkoč**. Existence of nonnegative and nonpositive solutions for second order periodic boundary value problems. *J. Differential Equations* **176** (2001), 445–469.

- J.A. Cid, G. Infante, M. Tvrdý and M. Zima: Topological approach to periodic oscillations related to the Liebau phenomenon, arXiv:1408.0130 [math.CA].
- J.A. Cid, G. Propst and M. Tvrdý: On the pumping effect in a pipe/tank flow configuration with friction. *Physica D* **273-274** (2014), 28-33.

- **G. Liebau**: Über ein ventilloses Pumpprinzip. *Naturwissenschaften* **41** (1954), 327.
- **G. Propst**: Pumping effects in models of periodically forced flow configurations. *Physica D* **217** (2006), 193–201.

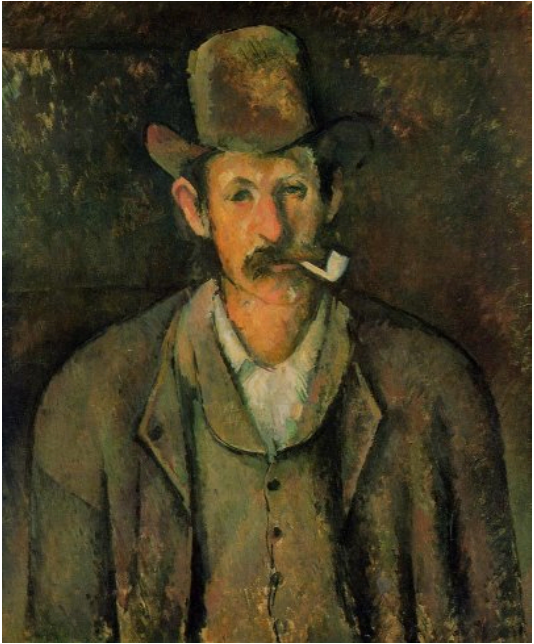
- **D. Bonheure and C. De Coster**. Forced singular oscillators and the method of lower and upper solutions, *Topological Methods in Nonlinear Analysis* **22** (2003), 297–317.
- **F.I. Njoku & P. Omari**. Stability properties of periodic solutions of a Duffing equation in the presence of lower and upper solutions. *Appl. Math. Comput.* **135** (2003), 471–490.
- **P. Omari & M. Trombetta**. Remarks on the lower and upper solutions method for the second and third-order periodic boundary value problems. *Appl. Math. Comput.* **50** (1992), 1–21.
- **I. Rachůnková, M. Tvrdý and I. Vrkoč**. Existence of nonnegative and nonpositive solutions for second order periodic boundary value problems. *J. Differential Equations* **176** (2001), 445–469.
- **I. Rachůnková, S. Staněk & M. Tvrdý**. *Solvability of Nonlinear Singular Problems for Ordinary Differential Equations*.
  Hindawi [Contemporary Mathematics and Its Applications, Vol.5], 2009. ISBN: 9789774540400 (Paperback), ISBN: 9789774540967 (PDF)

JOAN MIRÓ. The man with a pipe. 1925.



GUSTAVE COURBAT. The man with a pipe. 1849.

James McNeill Whistler. The man with a pipe. 1859.



PAUL CÉZANNE. The man with a pipe. 1892.

PABLO PICASSO. The man with a pipe. 1915.

JOAN MIRÓ. The man with a pipe. 1928.

ROYALTY FREE STOCK PHOTO. The man with a pipe. 1954.

PRAY FOR PEACE OF JERUZALEM.