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Mathematical analysis of variable density flows in porous media

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Abstract

We consider a simple model describing the motion of a two-component mixture through a porous medium. We discuss well-posedness of the associated initial-boundary value problem, in particular, with respect to the choice of boundary and far-field conditions. The existence of global-in-time solutions is proved in the ideal case when the fluid occupies the whole physical space. Finally, similar results are obtained also for the boundary value problems in the simplified 1-D geometry.

Key words: Variable density flow, flows in porous media, global-in-time solutions.

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1 Introduction

Density driven flows occur in numerous real world phenomena. A prominent and well-studied situation describes propagation of salt (or polluted) water in underground deposits and mines, among other interesting applications (see for instance Diersch and Kolditz [3], Kolditz et al. [8]). A simple mathematical description is based on the principal hypothesis that the ***mass density*** ϱ ($\varrho > 0$) and the ***concentration*** c ($c \in \mathbb{R}$) are correlated through the following *constitutive law* represented by a diffeomorphic one-to-one mapping

$$\varrho = \varrho(c) \quad \text{or, equivalently,} \quad c = c(\varrho)\,. \tag{1.1}$$

The *motion* of the fluid is characterized by the velocity field $\mathbf{v}$ and the underlying physical principles of *mass* and *mass fraction* conservation, respectively, expressed through a standard system of conservation laws written in the Eulerian coordinate system as

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{v}) = 0\,, \tag{1.2}$$

$$\partial_t(\varrho c) + \mathrm{div}_x(\varrho c \mathbf{v}) + \mathrm{div}_x \mathbf{J} = 0\,, \tag{1.3}$$

with the *diffusion flux* $\mathbf{J}$. We consider a simple case with $\mathbf{J}$ given by *Fick's law*,

$$\mathbf{J} = -\varrho D \nabla_x c, \quad D \equiv \text{const} > 0\,, \tag{1.4}$$

where the positive constant D stands for the diffusion coefficient.

System (1.2), (1.3) must be completed by specifying an equation of motion. The porous media flows are characterized by *Darcy's law*,

$$\mathbf{v} = -\frac{k}{\mu}\left(\nabla_x p - \varrho \mathbf{g}\right), \tag{1.5}$$

where $k, \mu > 0$ are some physical constants, p is the pressure, and $\mathbf{g}$ the gravitational force.

Problem (1.2) – (1.5) is considered in a physical domain $\Omega \subset \mathbb{R}^N$; $N = 1, 2, 3$, and for time $t \in (0, T)$. Accordingly, in view of the basic hypothesis (1.1), the initial distribution of the density,

$$\varrho(\,\cdot\,, 0) = \varrho_0\,, \tag{1.6}$$

as well as suitable boundary conditions for $\mathbf{v}$ and $\mathbf{J}$ must be prescribed in order to obtain a (formally) well-posed problem for the unknown functions ϱ and p.

The interested reader may consult Dentz et al. [2], Diersch and Kolditz [3], or Johannsen's Habilitation Thesis [7] for a detailed derivation of the model as well as an extensive list of available literature.

1.1 Compatibility

In order to reveal the role of the pressure p, in accordance with the constitutive law (1.1) we assume that the concentration $c = c(\varrho) \in \mathbb{R}$ is a continuously differentiable function of the mass density $\varrho \in (0, \infty)$ with $c'(\varrho) > 0$. Next, we rewrite the continuity equation (1.2) in its *renormalized* form

$$\partial_t(\varrho c(\varrho)) + \mathrm{div}_x(\varrho c(\varrho)\mathbf{v}) = -\varrho^2 c'(\varrho)\mathrm{div}_x\mathbf{v}\,. \tag{1.7}$$

Thus, comparing (1.3) with (1.7), we obtain the *compatibility relation*

$$-\,\varrho^2 c'(\varrho)\mathrm{div}_x\mathbf{v} = \,-\,\mathrm{div}_x\mathbf{J} = \mathrm{div}_x(\varrho D\nabla_x c(\varrho)) \tag{1.8}$$

between the velocity field $\mathbf{v}$ and the diffusion flux $\mathbf{J}$, which, together with eqs. (1.4) and (1.5), gives rise to

$$\varrho^2 c'(\varrho)\left(\frac{k}{\mu}\Delta p - \mathrm{div}_x(\varrho\mathbf{g})\right) = \mathrm{div}_x(\varrho D\nabla_x c(\varrho))\,. \tag{1.9}$$

Relation (1.9) may be viewed as an *elliptic* equation for the (unknown) pressure p in terms of the density ϱ.

A formal inspection of eq. (1.9) reveals immediately one of the principal *stumbling blocks* of the problem: The right-hand side is in the divergence form, while the left-hand side is not unless the following relation holds,

$$\varrho^2 c'(\varrho) \equiv \mathrm{const} = \kappa > 0\,, \quad \text{meaning} \quad c(\varrho) = \overline{c} - \frac{\kappa}{\varrho} \quad \text{for } \ \varrho > 0\,, \tag{1.10}$$

where the limit $\overline{c} = \lim_{\varrho\to+\infty} c(\varrho)$ is assumed to exist, $0 < \overline{c} < \infty$.

It is interesting to note that the constitutive equation (1.10) corresponds to the *ideal* density--concentration relation with constant volume fractions, see [7]. If, in addition, we impose the impermeability (Neumann) boundary conditions

$$\mathbf{v}\cdot\mathbf{n} = \nabla_x\varrho\cdot\mathbf{n}\,|_{\partial\Omega} = 0\,, \tag{1.11}$$

and set, for simplicity, $\mathbf{g} = 0$, System (1.2) – (1.5) reduces to the standard “heat” equation for the density ϱ,

$$\partial_t\varrho = D\Delta\varrho\,.$$

1.2 Density-concentration constitutive relation

Besides (1.10), there are two other basic examples of constitutive equations studied in the literature, see van Duijn, Peletier, and Schotting [4] and Hilhorst et al. [6], namely, *exponential*

$$\varrho = \exp(c) > 0 \quad \text{or} \quad c = \log(\varrho) \in \mathbb{R}\,, \tag{1.12}$$

and *linear*

$$\varrho = \alpha c + \beta\,; \qquad \alpha = \mathrm{const} > 0\,, \quad \beta = \mathrm{const} > 0\,. \tag{1.13}$$

In the present paper, we focus on the *exponential case* that is mathematically more interesting. Setting $D = k = \mu = 1$ for simplicity, we easily check that, while assuming (1.12), eq. (1.3) can be rewritten as

$$\partial_t \varrho + \mathbf{v} \cdot \nabla_x \varrho = \Delta \varrho \,, \tag{1.14}$$

while (1.5) and (1.9), respectively, read

$$\mathbf{v} = -\nabla_x p + \varrho \mathbf{g} \quad \text{and} \quad \Delta p = \frac{1}{\varrho} \Delta \varrho + \mathrm{div}_x (\varrho \mathbf{g}) \,. \tag{1.15}$$

From the mathematical viewpoint, problem (1.14), (1.15) is a *strongly* coupled elliptic--parabolic system. Besides the explicit solutions constructed in van Duijn et al. [4], and the studies devoted to the so-called Boussinesq approximation, see Clément, van Duijn, and Li [1], Efendiev, Fuhrmann, and Zelik [5], we do not know about any rigorous result concerning well-posedness of problem (1.14), (1.15). Our main goal is to fill this gap, at least partially, by proving a general global-in-time existence theorem under very mild assumptions on the given data entering our problem. As a matter of fact, we only need that the initial density ϱ_0 does not oscillate too much relative to its amplitude (see Theorem 2.1 below). We focus on two principal geometric settings: The Cauchy problem in the whole physical space $\mathbb{R}^3$ and a boundary value problem in the simplified 1-D geometry.

At present, we are not able to formulate "reasonable" standard *boundary conditions* for the initial-boundary value problem for (1.14), (1.15) in a smooth ***bounded*** domain $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) that would render a well-posed evolutionary problem (see Section 2 below).

The paper is organized as follows. In Section 2 we state the main results and shortly discuss compatibility of the boundary and/or initial conditions. Section 3 is devoted to (formal) *a priori* bounds for the problem posed in the whole space $\mathbb{R}^3$. In Section 4 we introduce an approximate system, discuss its solvability, and, finally, prove the existence result for the Cauchy problem. In Section 4.5 we deal with the boundary-initial value problem in a bounded space interval (1-D). The difficulties when dealing with a general choice of boundary conditions are illustrated by a specific example of a (1-D) motion discussed in Section 5.

2 Preliminaries, main results

As a matter of fact, a proper choice of the boundary conditions for problem (1.14), (1.15) presents a rather delicate issue. Indeed, the simplest and most popular *periodic* boundary conditions seem to be incompatible with the elliptic problem for p in (1.15) that can be rewritten as

$$\Delta p = \Delta \log(\varrho) + |\nabla_x \log(\varrho)|^2 + \mathrm{div}_x(\varrho \mathbf{g}) \,,$$

which, when integrated over the periodic spatial domain $\Omega \approx \mathbb{R}^N/(2\pi\mathbb{Z})^N \equiv (\mathbb{R}/2\pi\mathbb{Z})^N$, gives rise to

$$\int_\Omega |\nabla_x \log(\varrho)|^2 \, \mathrm{d}x = 0 \,, \quad \text{thus yielding} \quad \varrho \equiv \text{const in } \Omega \,.$$

Similarly, the homogeneous Neumann boundary conditions (1.11) entail a rather unnatural compatibility constraint

$$\int_\Omega |\nabla_x \log(\varrho)|^2 \, \mathrm{d}x = -\int_{\partial\Omega} \varrho \mathbf{g} \cdot \mathbf{n} \, \mathrm{d}S_x$$

to be satisfied at all times $t \in (0, T)$.

In general, imposing the homogeneous ***Neumann*** boundary conditions

$$\nabla_x \varrho(x,t) \cdot \mathbf{n}(x) = 0, \ x \in \partial\Omega \,, \tag{2.1}$$

for the density may lead to unsurmountable difficulties and overdetermined problems. A simple example in the 1-D geometry is discussed in §5 below.

In the light of the previous arguments, an admissible choice of boundary behavior should be of *mixed* type, combining the Dirichlet and Neumann boundary conditions for the fields ϱ and p; for instance, the homogeneous Neumann boundary conditions for ϱ and the homogeneous Dirichlet boundary conditions for p, cf. §2.2 below, eq. (2.14), in the simplified 1-D geometry with $\Omega = (0,1)$.

2.1 The Cauchy problem

In order to avoid the aforementioned difficulties related to the presence of a kinematic boundary, we consider the Cauchy problem for system (1.14), (1.15) in $\mathbb{R}^3$. Accordingly, we impose the far field conditions in the form

$$p \to 0, \ \varrho \to \overline{\varrho} \text{ as } |x| \to \infty, \tag{2.2}$$

where $\overline{\varrho} > 0$.

Replacing

$$\varrho \approx \overline{\varrho} + \varrho \qquad \text{(and } \varrho_0 \approx \overline{\varrho} + \varrho_0\text{)}$$

in (1.14), (1.15), we obtain the system

$$\partial_t \varrho - \nabla_x p \cdot \nabla_x \varrho - \Delta \varrho = -(\varrho + \overline{\varrho})\mathbf{g} \cdot \nabla_x \varrho, \tag{2.3}$$

$$\Delta p = \frac{1}{\overline{\varrho} + \varrho} \Delta \varrho + \mathrm{div}_x \left((\varrho + \overline{\varrho})\mathbf{g} \right), \tag{2.4}$$

supplemented by the initial condition

$$\varrho(\cdot, 0) = \varrho_0 \text{ in } R^3. \tag{2.5}$$

Our main result concerning the (Cauchy) initial value problem (2.3) – (2.5) reads:

Theorem 2.1 *Let* $\mathbf{g} \in R^3$ *be a given (constant) vector, and let the initial data* ϱ_0 *satisfy*

$$\varrho_0 \in W^{1,2}(R^3), \tag{2.6}$$

$$-\overline{\varrho} < \varrho_{\min} \leq \varrho_0 \leq \varrho_{\max}, \tag{2.7}$$

$$\frac{\max\{-\varrho_{\min},\ \varrho_{\max}\}}{\overline{\varrho}} \leq \delta(3), \tag{2.8}$$

where $\delta(N)$ $(N = 3)$ *is a dimensional constant specified in* (3.10) *below.*

Then problem (2.3) - (2.5) *possesses a (global) weak solution* $\{\varrho, p\}$ *in* $(0,T) \times R^3$ *belonging to the following class:*

$$\varrho \in C([0,T]; W^{1,2}(R^3)),\ \partial_t \varrho,\ \Delta \varrho \in L^2((0,T) \times R^3), \tag{2.9}$$

$$\varrho \in L^\infty((0,T) \times R^3),\ \text{ess} \inf_{(0,T)\times R^3} (\overline{\varrho} + \varrho) > 0, \tag{2.10}$$

$$\nabla_x p \in L^2(0,T; D^{1,2}(R^3; R^3)). \tag{2.11}$$

The proof of Theorem 2.1 will be given in Section 4.

2.2 The initial-boundary value problem

Next, we consider the initial-boundary value problem (2.3), (2.4) in the simplified 1-D geometry $\Omega = (0,1)$,

$$\partial_t \varrho - \partial_x p \cdot \partial_x \varrho - \partial^2_{x,x} \varrho = -(\overline{\varrho} + \varrho) g \cdot \partial_x \varrho \,, \tag{2.12}$$

$$\partial^2_{x,x} p = \frac{1}{\overline{\varrho} + \varrho} \cdot \partial^2_{x,x} \varrho + \partial_x \left((\overline{\varrho} + \varrho) g \right) \,, \tag{2.13}$$

respectively, with *mixed*-type boundary conditions

$$\partial_x \varrho(0,t) = \partial_x \varrho(1,t) = 0 \quad \text{and} \quad p(0,t) = p(1,t) = 0 \quad \text{for } t \in (0,T)\,, \tag{2.14}$$

and the initial condition

$$\varrho(\,\cdot\,, 0) = \varrho_0 \quad \text{in } (0,1)\,. \tag{2.15}$$

We claim the following result about the well-posedness of this problem.

Theorem 2.2 *Let b be a given constant, and let the initial data ϱ_0 satisfy*

$$\varrho_0 \in W^{1,2}(0,1), \tag{2.16}$$

$$-\overline{\varrho} < \varrho_{\min} \leq \varrho_0 \leq \varrho_{\max}, \tag{2.17}$$

$$\frac{\max\{-\varrho_{\min},\ \varrho_{\max}\}}{\overline{\varrho}} \leq \delta(1) = 2. \tag{2.18}$$

Then problem (2.12) – (2.15) *possesses a (global) weak solution* $\{\varrho, p\}$ *in* $(0,1) \times (0,T)$ *belonging to the following class:*

$$\varrho \in C\left([0,T] \to W^{1,2}(0,1)\right),\ \partial_t \varrho,\ \partial^2_{x,x}\varrho \in L^2((0,1)\times(0,T)), \tag{2.19}$$

$$\min_{[0,1]\times[0,T]} (\overline{\varrho} + \varrho) > 0, \tag{2.20}$$

$$p \in L^2\left((0,T) \to W^{2,2}(0,1)\right). \tag{2.21}$$

Remark 2.1 *A standard bootstrap argument can be used to show that the solution (ϱ, p) is smooth as soon as suitable smoothness and compatibility conditions are imposed on the data.*

The proof of Theorem 2.2 is given in Section 4.5.

3 A priori estimates

We start with a short discussion of the *a priori* bounds available for the (hypothetical) smooth solutions of the problem. These will be exploited in the subsequent sections, where the rigorous existence proofs are given. We specify the arguments in the more complicated $3D$-case pointing out the necessary modifications to accommodate the boundary conditions (2.14).

3.1 Comparison principle

Applying the standard comparison argument to (2.3), we deduce from hypothesis (2.7) that

$$\varrho_{\min} \leq \varrho(\cdot, t) \leq \varrho_{\max} \text{ for any } t \geq 0. \tag{3.1}$$

The same holds in the case of boundary conditions for ϱ as long as constants represent sub an super solutions.

3.2 Second order gradient estimates

A natural second step when dealing with a parabolic equation like (2.3) is to derive the energy estimates. Here, however, we prefer to exploit directly the "maximal" regularity available.

3.2.1 The Cauchy problem in $\mathbb{R}^3$

We start with $\Omega = \mathbb{R}^3$ - the case without boundary conditions. Multiplying eq. (2.3) by $-\Delta\varrho$ and integrating the resulting expression we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{R^3} |\nabla_x \varrho|^2 \ \mathrm{d}x + \int_{R^3} |\Delta\varrho|^2 \ \mathrm{d}x = -\int_{R^3} \nabla_x p \cdot \nabla_x \varrho \Delta\varrho \ \mathrm{d}x + \int_{R^3} (\overline{\varrho} + \varrho)\mathbf{g} \cdot \nabla_x \varrho \Delta\varrho \ \mathrm{d}x, \tag{3.2}$$

where the last integral can be estimated as

$$\left| \int_{R^3} (\overline{\varrho} + \varrho)\mathbf{g} \cdot \nabla_x \varrho \Delta\varrho \ \mathrm{d}x \right| \leq \delta \int_{R^3} |\Delta\varrho|^2 \ \mathrm{d}x + c(\delta)\int_{R^3} |\nabla_x \varrho|^2 \ \mathrm{d}x$$

provided

$$\mathbf{g} \in L^\infty(R^3; R^3). \tag{3.3}$$

As for the first integral on the right-hand side of (3.2), we get

$$\int_{R^3} \nabla_x p \cdot \nabla_x \varrho \Delta\varrho \ \mathrm{d}x = \int_{R^3} \nabla_x p \cdot \mathrm{div}_x \left(\nabla_x \varrho \otimes \nabla_x \varrho\right) \ \mathrm{d}x - \frac{1}{2}\int_{R^3} \nabla_x p \cdot \nabla_x |\nabla_x \varrho|^2 \ \mathrm{d}x,$$

where, furthermore,

$$\int_{R^3} \nabla_x p \cdot \mathrm{div}_x \left(\nabla_x \varrho \otimes \nabla_x \varrho\right) \ \mathrm{d}x = -\int_{R^3} \nabla_x^2 p : \left(\nabla_x \varrho \otimes \nabla_x \varrho\right) \ \mathrm{d}x;$$

whence

$$\left| \int_{R^3} \nabla_x p \cdot \mathrm{div}_x \left(\nabla_x \varrho \otimes \nabla_x \varrho\right) \ \mathrm{d}x \right| \leq \|\nabla_x^2 p\|_{L^2(R^3; R^{3\times 3})} \|\nabla_x \varrho\|^2_{L^4(R^3;R^3)} \tag{3.4}$$

$$\leq \delta_1(3)\|\Delta p\|_{L^2(R^3)} \|\nabla_x \varrho\|^2_{L^4(R^3;R^3)},$$

where $\delta_1(N) = N$ is a dimensional constant from the inequality

$$\|\nabla_x^2 p\|_{L^2(R^N; R^{N\times N})} \leq \delta_1(N)\|\Delta p\|_{L^2(R^N)}.$$

Similarly,

$$\left| \int_{R^3} \nabla_x p \cdot \nabla_x |\nabla_x \varrho|^2 \ \mathrm{d}x \right| \leq \|\Delta p\|_{L^2(R^3)} \|\nabla_x \varrho\|^2_{L^4(R^3;R^3)}. \tag{3.5}$$

At this stage, we make use of the *Gagliardo-Nirenberg interpolation inequality*:

$$\|\nabla_x \varrho\|^2_{L^4(R^3;R^3)} \leq c(N)\|\varrho\|_{L^\infty(R^3)} \|\Delta\varrho\|_{L^2(R^3)} \tag{3.6}$$

yielding

$$\|\Delta p\|_{L^2(R^3)} \|\nabla_x \varrho\|^2_{L^4(R^3;R^3)} \leq \delta_2(3)\|\varrho\|_{L^\infty(R^3)} \|\Delta p\|_{L^2(R^3)} \|\Delta\varrho\|_{L^2(R^3)}.$$

Finally, we deduce from equation (2.4) that

$$\|\Delta p\|_{L^2(R^3)} \leq \frac{1}{\underline{\varrho}} \|\Delta\varrho\|_{L^2(R^3)} + \|\mathrm{div}_x \left((\varrho + \overline{\varrho})\mathbf{g}\right)\|_{L^2(R^3)} \tag{3.7}$$

$$\leq \frac{1}{\underline{\varrho}} \|\Delta\varrho\|_{L^2(R^3)} + \|\mathbf{g}\|_{L^\infty(R^3;R^3)} \|\nabla_x \varrho\|_{L^2(R^3;R^3)} + \|\varrho + \overline{\varrho}\|_{L^\infty(R^3)} \|\mathrm{div}_x \mathbf{g}\|_{L^2(R^3)},$$

where the last term remains bounded as soon as

$$\mathrm{div}_x \mathbf{g} \in L^2(R^3). \tag{3.8}$$

Summing up the previous estimates we may infer that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{R^3}|\nabla_x\varrho|^2\ \mathrm{d}x+\lambda\int_{R^3}|\Delta\varrho|^2\ \mathrm{d}x\le c\left(\int_{R^3}|\nabla_x\varrho|^2\ \mathrm{d}x+1\right),\ \text{with } \lambda>0 \tag{3.9}$$

provided

$$\frac{\|\varrho\|_{L^\infty((0,T)\times R^3)}}{\overline{\varrho}}\le\delta(3), \tag{3.10}$$

where $\delta(N)$ is a dimensional constant. However, this is exactly the stated in hypothesis (2.8)

Thus the relations (3.9), (3.10) give rise to *a priori* bounds

$$\sup_{t\in(0,T)}\|\nabla_x\varrho(\cdot,t)\|_{L^2(R^3;R^3)}\le c, \tag{3.11}$$

and

$$\int_0^T\int_{R^3}\left|\nabla_x^2\varrho\right|^2\ \mathrm{d}x\ \mathrm{d}t\le c. \tag{3.12}$$

Moreover, going back to (2.4), we get

$$\int_0^T\int_{R^3}|\nabla_x^2 p|^2\ \mathrm{d}x\ \mathrm{d}t\le c, \tag{3.13}$$

which yields, in particular,

$$\int_0^T\|\nabla_x p\|^2_{L^6(R^3;R^3)}\ \mathrm{d}t\le c. \tag{3.14}$$

Finally, combining (3.11), (3.14) we obtain

$$\nabla_x p\cdot\nabla_x\varrho \text{ bounded in } L^2(0,T;L^{3/2}(R^3)),$$

and, applying the standard maximal regularity estimates to the parabolic equation (2.3), we may conclude that

$$\partial_t\varrho \text{ bounded in } L^2(0,T;L^{3/2}(R^3))+L^q(0,T;L^2(R^3)), \tag{3.15}$$

$$\nabla_x^2\varrho \text{ bounded in } L^2(0,T;L^{3/2}(R^3;R^{3\times3}))+L^q(0,T;L^2(R^3;R^{3\times3})) \text{ for any } 1<q<\infty; \tag{3.16}$$

in particular,

$$\sup_{t\in(0,T)}\|\varrho(\cdot,t)\|_{L^2(R^3)}\le c. \tag{3.17}$$

3.2.2 The boundary value problem

The above estimates can be repeated without any modification to the problem with boundary conditions as soon as we are able to handle the boundary terms when using Green's theorem in (3.4), (3.5). To this end we need either the "full" Neumann boundary conditions (1.11) or $\nabla_x\varrho$ to vanish on $\partial\Omega$. As we have seen in Section 2, imposing the boundary conditions (1.11) may result in unsurmountable difficulties on a bounded domain. On the other hand, the stipulation $\nabla_x\varrho|_{\partial\Omega}=0$ leads to an over-determined problem unless $N=1$. Thus, the *a priori* bounds established in the previous part remain valid in the 1-D-case with the mixed boundary conditions (2.14).

4 Global existence for the Cauchy problem

Our goal in this section is to prove Theorem 2.1. To this end, we introduce a suitable family of *approximate* problems solvable by Schauder's fixed point theorem.

We start with the standard definition of *smoothing operators* acting in the $x-$variable:

$$[h]_\varepsilon = \kappa_\varepsilon * h,$$

where $\kappa_\varepsilon = \kappa_\varepsilon(x)$ is a family of regularizing kernels,

$$\kappa_\varepsilon \in C_c^\infty(R^3), \kappa_\varepsilon \geq 0, \ \int_{R^3} \kappa_\varepsilon \ \mathrm{d}x = 1, \ \kappa_\varepsilon \text{ radially symmetric, } \mathrm{supp}[\kappa_\varepsilon] \subset \{|x| < \varepsilon\}.$$

Furthermore, we introduce cut-off functions ψ_ε,

$$\psi_\varepsilon = \psi(\varepsilon|x|), \ \psi \in C_c^\infty(R), \ 0 \leq \psi \leq 1, \ \psi \equiv 1 \text{ in a neighborhood of } 0 \in R.$$

4.1 Approximate solutions

Keeping in mind our hypothesis that $\mathbf{g}$ is a constant vector, we introduce a regularized system

$$\partial_t \varrho - \nabla_x [p]_\varepsilon \cdot \nabla_x \varrho - \Delta \varrho = - \left([\varrho]_\varepsilon + \overline{\varrho} \right) \mathbf{g} \cdot \nabla_x \varrho, \tag{4.1}$$

$$\Delta p = \frac{\psi_\varepsilon}{\overline{\varrho} + \varrho} \Delta [\varrho]_\varepsilon + \psi_\varepsilon \mathrm{div}_x \left([\varrho]_\varepsilon \mathbf{g} \right). \tag{4.2}$$

$$\varrho, \nabla_x p \to 0 \text{ as } |x| \to \infty, \tag{4.3}$$

$$\varrho(\cdot, 0) = \varrho_{0,\varepsilon}, \tag{4.4}$$

where the initial datum $\varrho_{0,\varepsilon} \in C_c^\infty(R^3)$ satisfies

$$-\overline{\varrho} < \varrho_{\min} \leq \varrho_{0,\varepsilon} \leq \varrho_{\max}, \tag{4.5}$$

uniformly for $\varepsilon \searrow 0$.

4.2 A priori bounds for the approximate system

Let $\{\varrho_\varepsilon, p_\varepsilon\}$ be a solution of the approximate problem (4.1 - 4.4). Our goal is to show that $\{\varrho_\varepsilon, p_\varepsilon\}$ admit the *a priori* bounds identified in Section 3 for the original system. Moreover, we show that these bounds are independent of the parameter $\varepsilon \searrow 0$.

4.2.1 Maximum principle

Applying the standard comparison principle to the parabolic equation (4.1), we may use the hypothesis (4.5) to deduce that

$$\overline{\varrho} + \varrho_\varepsilon \geq \overline{\varrho} + \varrho_{\min} > 0 \text{ in } [0, T] \times R^3. \tag{4.6}$$

Moreover, we easily check that ϱ_ε satisfies (3.10), specifically,

$$\frac{\|\varrho_\varepsilon(\cdot, t)\|_{L^\infty(R^3)}}{\overline{\varrho}} \leq \delta(3) \text{ for all } t \in [0, T]. \tag{4.7}$$

4.2.2 Second order gradient estimates

We are going to modify step by step the estimates derived in Section 3.2. Accordingly, we recover the inequality (3.2) in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{R^3} |\nabla_x \varrho_\varepsilon|^2 \ \mathrm{d}x + \int_{R^3} |\Delta \varrho_\varepsilon|^2 \ \mathrm{d}x = -\int_{R^3} \nabla_x [p_\varepsilon]_\varepsilon \cdot \nabla_x \varrho_\varepsilon \Delta \varrho_\varepsilon \ \mathrm{d}x + \int_{R^3} (\overline{\varrho} + [\varrho_\varepsilon]_\varepsilon) \mathbf{g} \cdot \nabla_x \varrho_\varepsilon \Delta \varrho_\varepsilon \ \mathrm{d}x,$$

where, exactly as in (3.3),

$$\left| \int_{R^3} (\overline{\varrho} + [\varrho_\varepsilon]_\varepsilon) \mathbf{g} \cdot \nabla_x \varrho_\varepsilon \Delta \varrho_\varepsilon \ \mathrm{d}x \right| \leq \delta \int_{R^3} |\Delta \varrho_\varepsilon|^2 \ \mathrm{d}x + c(\delta) \int_{R^3} |\nabla_x \varrho_\varepsilon|^2 \ \mathrm{d}x,$$

where the estimate is uniformly for $\varepsilon \searrow 0$ as ϱ_ε satisfies (4.7).

Furthermore, it is easy to check that the relations (3.4), (3.5) remain valid for ϱ_ε, p_ε, with the *same* structural constant $\delta_1(3)$. Consequently, in order to deduce (3.11), (3.12), we have to handle the estimates on $\Delta [p_\varepsilon]_\varepsilon$ using (4.2):

$$\|\Delta [p_\varepsilon]_\varepsilon\|_{L^2(R^3)} \leq \|\Delta p_\varepsilon\|_{L^2(R^3)} \leq \frac{1}{\underline{\varrho}} \|\Delta [\varrho_\varepsilon]_\varepsilon\|_{L^2(R^3)} + \|\mathrm{div}_x \left([\varrho_\varepsilon]_\varepsilon \mathbf{g} \right)\|_{L^2(R^3)} \tag{4.8}$$

$$\leq \frac{1}{\underline{\varrho}} \|\Delta \varrho_\varepsilon\|_{L^2(R^3)} + \|\mathrm{div}_x \left(\varrho_\varepsilon \mathbf{g} \right)\|_{L^2(R^3)},$$

which is exactly the same as (3.7). Thus, in view of (4.7), we may infer that

$$\mathrm{ess} \sup_{t \in (0,T)} \|\nabla_x \varrho_\varepsilon(\cdot, t)\|_{L^2(R^3;R^3)} \leq c, \tag{4.9}$$

$$\int_0^T \int_{R^3} |\nabla_x^2 \varrho_\varepsilon|^2 \ \mathrm{d}x \ \mathrm{d}t \leq c, \tag{4.10}$$

and

$$\int_0^T \int_{R^3} |\nabla_x^2 [p_\varepsilon]_\varepsilon|^2 \leq \int_0^T \int_{R^3} |\nabla_x^2 p_\varepsilon|^2 \ \mathrm{d}x \ \mathrm{d}t \leq c, \tag{4.11}$$

where the constants are independent of ε.

With this piece of information at hand, we may repeat the arguments of Section 3 to conclude that

$$\nabla_x [p_\varepsilon]_\varepsilon \cdot \nabla_x \varrho_\varepsilon \text{ is bounded in } L^2(0,T; L^{3/2}(R^3; R^3)),$$

yielding

$$\partial_t \varrho_\varepsilon \text{ bounded in } L^2(0,T; L^{3/2}(R^3)) + L^q(0,T; L^2(R^3)), \tag{4.12}$$

$$\nabla_x^2 \varrho_\varepsilon \text{ bounded in } L^2(0,T; L^{3/2}(R^3; R^{3\times 3})) + L^q(0,T; L^2(R^3; R^{3\times 3})) \text{ for any } 1 < q < \infty. \tag{4.13}$$

and

$$\mathrm{ess} \sup_{t \in (0,T)} \|\varrho_\varepsilon(\cdot, t)\|_{L^2(R^3)} \leq c, \tag{4.14}$$

where the bound in the last estimates depends on the initial datum $\varrho_{0,\varepsilon}$.

4.2.3 Decay estimates for $|x| \to \infty$

We conclude this part by showing a decay estimate for solutions of the approximate system ensuring compactness with respect to the spatial variable. To this end, we multiply (4.1) on $\varrho \exp(|x|)$ to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{R^3} |\varrho_\varepsilon|^2 \exp(|x|) \ \mathrm{d}x + \int_{R^3} |\nabla_x \varrho_\varepsilon|^2 \exp(|x|) \ \mathrm{d}x \tag{4.15}$$

$$\leq \int_{R^3} |\nabla_x \varrho_\varepsilon||\varrho_\varepsilon| \exp(|x|) \ \mathrm{d}x + \frac{1}{2} \int_{R^3} \nabla_x [p_\varepsilon]_\varepsilon \cdot \nabla_x \varrho_\varepsilon^2 \exp(|x|) \ \mathrm{d}x - \int_{R^3} ([\varrho_\varepsilon]_\varepsilon - \overline{\varrho}) \mathbf{g} \cdot \nabla_x \varrho_\varepsilon \varrho_\varepsilon \exp(|x|) \ \mathrm{d}x,$$

where, furthermore,

$$\int_{R^3} \nabla_x [p_\varepsilon]_\varepsilon \cdot \nabla_x \varrho_\varepsilon^2 \exp(|x|) \ \mathrm{d}x \leq - \int_{R^3} \Delta [p_\varepsilon]_\varepsilon \cdot \varrho_\varepsilon^2 \exp(|x|) \ \mathrm{d}x - \int_{R^3} |\nabla_x [p_\varepsilon]_\varepsilon| \varrho_\varepsilon^2 \exp(|x|) \ \mathrm{d}x,$$

and

$$\left| \int_{R^3} ([\varrho_\varepsilon]_\varepsilon - \overline{\varrho}) \mathbf{g} \cdot \nabla_x \varrho_\varepsilon \varrho_\varepsilon \exp(|x|) \ \mathrm{d}x \right| \leq \delta \int_{R^3} |\nabla_x \varrho_\varepsilon|^2 \exp(|x|) \ \mathrm{d}x + c(\delta) \int_{R^3} |\varrho_\varepsilon|^2 \exp(|x|) \ \mathrm{d}x.$$

Thus we may infer form (4.15) and the standard Gronwall argument that

$$\operatorname*{ess\ sup}_{t \in (0,T)} \int_{R^3} |\varrho_\varepsilon|^2 \exp(|x|) \ \mathrm{d}x + \int_0^T \int_{R^3} |\nabla_x \varrho_\varepsilon|^2 \exp(|x|) \ \mathrm{d}x \ \mathrm{d}t \leq c(\varepsilon), \tag{4.16}$$

where this bound *depends* on the parameter ε, and, in particular, on the initial datum $\varrho_{0,\varepsilon}$.

4.3 Existence of approximate solutions

In view of the *a priori* bounds obtained in Section 4.2, the existence of the approximate solutions $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon\}$ is a routine matter. We consider the set

$$X = \left\{ \varrho \in BUC([0,T] \times R^3) \ \middle| \ - \overline{\varrho} < \varrho_{\min} \leq \varrho \leq \varrho_{\max}; \ \varrho_{\min}, \ \varrho_{\max} \text{ satisfying (2.8)} \right\}$$

viewed as a closed convex subset of the Banach space $BUC([0,T] \times R^3)$.

Next, we introduce a mapping

$$\mathcal{T} : X \to X,$$

where $Y = \mathcal{T}[\varrho]$ is the unique solution of the (linear) parabolic problem

$$\partial_t Y - \nabla_x [p]_\varepsilon \cdot \nabla_x Y - \Delta Y = - \left([\varrho]_\varepsilon + \overline{\varrho} \right) \mathbf{g} \cdot \nabla_x Y \tag{4.17}$$

$$Y \to 0 \text{ as } |x| \to \infty, \tag{4.18}$$

$$Y(\cdot, 0) = \varrho_{0,\varepsilon}, \tag{4.19}$$

where $p = p[\varrho]$ is solves (4.2), (4.3).

After revisiting the bounds established in Section 4.2 we convince ourselves easily that $\mathcal{T}$ is a compact mapping on X and as such admits a fixed point in X - an approximate solution ϱ_ε with the associated pressure p_ε. Note that the desired uniform spatial decay is obtained interpolating (4.16) with the standard parabolic estimates for (4.1).

4.4 Proof of Theorem 2.1

We are ready to complete the proof of Theorem 2.1 in the following two steps:

- We construct a family of solutions $\{\varrho_\varepsilon, p_\varepsilon\}$ of the approximate problem (4.1 - 4.4), with the initial datum $\varrho_{0,\varepsilon}$ - a suitable mollification of ϱ_0.
- In accordance with the *a priori* bounds on the family $\{\varrho_\varepsilon, p_\varepsilon\}_{\varepsilon>0}$ established in Section 4.2, we can pass to the limit for $\varepsilon \searrow 0$ to obtain a solution of the original problem the existence of which is claimed in Theorem 2.1. Indeed, passing to a suitable subsequence if necessary, we obtain, in particular,

  $$\varrho_\varepsilon \to \varrho,\ \nabla_x\varrho_\varepsilon \to \nabla_x\varrho \text{ pointwise (a.a.) in } (0,T)\times R^3,\ p_\varepsilon \to p \text{ weakly in } L^2(0,T;D^{1,2}(R^3)).$$

  Thus the only thing is to observe that p_ε and $[p_\varepsilon]_\varepsilon$ possess the same (weak) limit p. Indeed we have

  $$\int_{R^3} ([p_\varepsilon]_\varepsilon - p_\varepsilon)\,\varphi\ \mathrm{d}x = \int_{R^3} ([\varphi]_\varepsilon - \varphi)\,p\ \mathrm{d}x,$$

  where the expression $[\varphi]_\varepsilon - \varphi$ can be made uniformly small for small ε and all φ belonging to a dense subset of $L^2(R^3)$.

We have proved Theorem 2.1

4.5 Proof of Theorem 2.2 - sketch

The arguments presented in Section 4 can be easily modified to accommodate the initial-boundary value problem (2.12 - 2.15). Moreover, the *a priori* bounds are considerably better than for R^3 thanks to better embeddings of the related Sobolev spaces. The decay estimates (4.16) are not need on a *bounded* physical space.

Note that the estimates (3.4), (3.5) simplify as $\Delta\varrho$ coincides with the full "Hessian" $\partial^2_{x,x}$. One checks easily that $\delta(1) = 2$ in (3.10).

Last but not least, a standard bootstrap argument shows that the solutions obtained in Theorem 2.2 are smooth (classical) as soon as ϱ_0 is smooth enough satisfying the obvious compatibility condition

$$\partial_x\varrho_0(0) = \partial_x\varrho_0(1) = 0.$$

5 The initial-boundary value problem for $c = c(x,t)$

In order to illustrate the difficulties related to a proper choice of the boundary conditions, we consider the initial-boundary value problem (2.3), (2.4) rewritten equivalently for the pair of *unknown functions* $\{c, v\}$ in place of $\{\varrho, p\}$ in $(0,1)\times(0,T)$, hence, in the simplified 1-D geometry $\Omega = (0,1)$, again. Recall from eq. (1.12) that

$$\overline{\varrho} + \varrho = \exp(c) > 0 \quad \text{or} \quad c = \log(\overline{\varrho} + \varrho) \in \mathbb{R}\,, \tag{1.12'}$$

and from eq. (1.15) that

$$v = -\,\partial_x p + (\overline{\varrho} + \varrho) g \quad \text{and} \quad \partial^2_{x,x} p = \frac{1}{\overline{\varrho} + \varrho}\,\partial^2_{x,x}\varrho + \partial_x(\varrho\cdot g)\,. \tag{1.15'}$$

We combine the last two equations to get

$$- \partial_x v = \frac{1}{\overline{\varrho} + \varrho} \, \partial^2_{x,x} \varrho \, . \tag{5.1}$$

Applying further eq. (1.12'), we arrive at

$$- \partial_x v = (\partial_x c)^2 + \partial^2_{x,x} c \, . \tag{5.2}$$

Finally, applying eqs. (1.15'), (5.1), and (5.2) to (2.12), we obtain the following parabolic equation for the concentration $c = c(\overline{\varrho} + \varrho) \in \mathbb{R}$:

$$\partial_t c + v \cdot \partial_x c = (\partial_x c)^2 + \partial^2_{x,x} c \, , \tag{5.3}$$

supplemented by the initial condition

$$c(\,\cdot\,, 0) = c_0 \quad \text{ in } \; (0,1) \tag{5.4}$$

and the homogeneous Neumann boundary conditions for c,

$$\partial_x c(0,t) = \partial_x c(1,t) = 0 \quad \text{ for all } \; t \in (0,T) \, . \tag{5.5}$$

Now, going back to the equation (5.2), we realize that the velocity v can be computed in terms of c by imposing a *single* boundary condition, say,

$$v(0,t) = 0 \quad \text{ for all } \; t \in (0,T) \, , \tag{5.6}$$

to make a simple, but still realistic choice. Prescribing another boundary restrictions, such as $v(1,t) = 0$ for all $t \in (0,T)$, for instance, would apparently lead to an overdetermined problem.

To see that the resulting problem is mathematically tractable, we derive *a priori* bounds for the concentration c. Multiplying eq. (5.3) by $c(x,t)$ and integrating with respect to $x \in (0,1)$, we calculate

$$\begin{aligned} &\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^1 c(x,t)^2 \, \mathrm{d}x + \frac{1}{2} \int_0^1 v(x,t) \cdot \partial_x [c(x,t)^2] \, \mathrm{d}x \\ &\quad = \int_0^1 (\partial_x c)^2 \, c(x,t) \, \mathrm{d}x + \int_0^1 \partial^2_{x,x} c \cdot c(x,t) \, \mathrm{d}x \, . \end{aligned}$$

Applying the homogeneous Neumann boundary conditions for $\partial_x c$ and the boundary conditions (5.6) for v, we arrive at

$$\begin{aligned} &\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^1 c(x,t)^2 \, \mathrm{d}x - \frac{1}{2} \int_0^1 \partial_x v(x,t) \cdot c(x,t)^2 \, \mathrm{d}x \\ &= \frac{1}{2} \int_0^1 \partial_x c \cdot \partial_x [c(x,t)^2] \, \mathrm{d}x - \int_0^1 (\partial_x c)^2 \, \mathrm{d}x + c^2(1,t) \int_0^1 (\partial_x c)^2 \, \mathrm{d}x \, , \end{aligned}$$

whence

$$\begin{aligned} &\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^1 c(x,t)^2 \, \mathrm{d}x + \int_0^1 (\partial_x c)^2 \, \mathrm{d}x \\ &= \frac{1}{2} \int_0^1 \left[\partial_x v - \partial^2_{x,x} c \right] c(x,t)^2 \, \mathrm{d}x + c^2(1,t) \int_0^1 (\partial_x c)^2 \, \mathrm{d}x \, . \end{aligned}$$

Applying also eq. (5.2), we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 c(x,t)^2\,\mathrm{d}x + \int_0^1 (\partial_x c)^2\,\mathrm{d}x$$
$$= -\frac{1}{2}\int_0^1 \left[(\partial_x c)^2 + 2\,\partial^2_{x,x}c\right] c(x,t)^2\,\mathrm{d}x + c^2(1,t)\int_0^1 (\partial_x c)^2\,\mathrm{d}x\,.$$

With a help from $\partial_x c(0,t) = \partial_x c(0,t) = 0$ for all $t \in (0,T)$, the last equation yields

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 c(x,t)^2\,\mathrm{d}x + \int_0^1 (\partial_x c)^2\,\mathrm{d}x + \frac{1}{8}\int_0^1 \left\{\partial_x[c(x,t)^2]\right\}^2\,\mathrm{d}x \\ = \int_0^1 \partial_x c\cdot\partial_x[c(x,t)^2]\,\mathrm{d}x + c^2(1,t)\int_0^1 (\partial_x c)^2\,\mathrm{d}x\,.\end{aligned} \tag{5.7}$$

Finally, we estimate the integrand in the integral on the right-hand side by Cauchy's inequality,

$$\partial_x c\cdot\partial_x[c(x,t)^2] \le 2\,(\partial_x c)^2 + \frac{1}{8}\left\{\partial_x[c(x,t)^2]\right\}^2\,,$$

thus arriving at

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 c(x,t)^2\,\mathrm{d}x \le \left(1 + c^2(1,t)\right)\int_0^1 (\partial_x c)^2\,\mathrm{d}x \tag{5.8}$$
$$\le \int_0^1 (\partial_x c)^2\,\mathrm{d}x + \int_0^1 (\partial_x c)^2\,\mathrm{d}x\left(\int_0^1 c^2\,\mathrm{d}x + \int_0^1 (\partial_x c)^2\,\mathrm{d}x\right).$$

The integral on the right-hand side above is estimated next.

Again, using first the boundary conditions (5.5), i.e., $\partial_x c(0,t) = \partial_x c(0,t) = 0$ for all $t \in (0,T)$, then eq. (5.3), we calculate

$$\begin{aligned}&-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 (\partial_x c)^2\,\mathrm{d}x - \frac{1}{2}\int_0^1 \partial_x v(x,t)\cdot(\partial_x c)^2\,\mathrm{d}x \\ &= \int_0^1 \partial_t c\cdot\partial^2_{x,x}c(x,t)\,\mathrm{d}x + \frac{1}{2}\int_0^1 v(x,t)\cdot\partial_x[(\partial_x c)^2]\,\mathrm{d}x \\ &= \int_0^1 (\partial_x c)^2\cdot\partial^2_{x,x}c(x,t)\,\mathrm{d}x + \int_0^1 \left[\partial^2_{x,x}c(x,t)\right]^2\,\mathrm{d}x\,,\end{aligned}$$

whence

$$\frac{1}{2}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 (\partial_x c)^2\,\mathrm{d}x + \int_0^1 \left[\partial^2_{x,x}c(x,t)\right]^2\,\mathrm{d}x = -\frac{1}{2}\int_0^1 \left[\partial_x v + 2\,\partial^2_{x,x}c\right](\partial_x c)^2\,\mathrm{d}x\,.$$

Applying also eq. (5.2), we obtain

$$\frac{1}{2}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 (\partial_x c)^2\,\mathrm{d}x + \int_0^1 \left[\partial^2_{x,x}c(x,t)\right]^2\,\mathrm{d}x = -\frac{1}{2}\int_0^1 \left[\partial^2_{x,x}c - (\partial_x c)^2\right](\partial_x c)^2\,\mathrm{d}x$$

which yields

$$\begin{aligned}\frac{1}{2}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 (\partial_x c)^2\,\mathrm{d}x + \int_0^1 \left[\partial^2_{x,x}c(x,t)\right]^2\,\mathrm{d}x \\ = \frac{1}{2}\int_0^1 (\partial_x c)^4\,\mathrm{d}x - \frac{1}{2}\int_0^1 \partial^2_{x,x}c\cdot(\partial_x c)^2\,\mathrm{d}x\,.\end{aligned} \tag{5.9}$$

We use Cauchy's inequality to estimate the integrand in the second integral on the right-hand side,

$$\left|\partial^2_{x,x}c\right| \cdot (\partial_x c)^2 \le \frac{1}{4}\,\left[\partial^2_{x,x}c\right]^2 + (\partial_x c)^4\,,$$

thus arriving at

$$\frac{1}{2}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 (\partial_x c)^2\,\mathrm{d}x + \frac{7}{8}\int_0^1 \left[\partial^2_{x,x}c(x,t)\right]^2\,\mathrm{d}x \le \int_0^1 (\partial_x c)^4\,\mathrm{d}x\,. \tag{5.10}$$

The last integral is easily estimated from above by the Gagliardo-Nirenberg inequality,

$$\|\partial_x c\|_{L^4(0,1)} \le C\,\|\partial^2_{x,x}c\|^{1/4}_{L^4(0,1)}\cdot\|\partial_x c\|^{3/4}_{L^2(0,1)}\,,\qquad \text{which implies}$$

$$\int_0^1 (\partial_x c)^4\,\mathrm{d}x \le \frac{1}{2}\int_0^1 (\partial^2_{x,x}c)^2\,\mathrm{d}x + \frac{1}{2}\,C^8\left(\int_0^1 (\partial_x c)^2\,\mathrm{d}x\right)^3\,, \tag{5.11}$$

by Cauchy's inequality. Indeed, applying (5.11) to (5.10) we obtain

$$\frac{1}{2}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 (\partial_x c)^2\,\mathrm{d}x + \frac{3}{8}\int_0^1 \left[\partial^2_{x,x}c(x,t)\right]^2\,\mathrm{d}x \le \frac{1}{2}\,C^8\left(\int_0^1 (\partial_x c)^2\,\mathrm{d}x\right)^3\,. \tag{5.12}$$

Let us abbreviate

$$J(t) \stackrel{\text{def}}{=} \int_0^1 [\partial_x c(x,t)]^2\,\mathrm{d}x \qquad \text{for }\; t\in[0,T]\,.$$

Assuming $J(0)>0$, from ineq. (5.12) we derive the following a priori estimate,

$$J(t) \le \left(J(0)^{-2} - 2C^8 t\right)^{-1/2} < \infty \qquad \text{for every }\; t\in[0,T_0)\,, \tag{5.13}$$

where $T_0 = \left(2C^8 J(0)^2\right)^{-1} > 0$.

In these a priori estimates, ***absolutely no*** uniform bounds on the ***concentration*** $c(x,t)$ $(c\in\mathbb{R})$ have been imposed or assumed. If such bounds are available, say,

$$|c(x,t)| \le M = \text{const} < \infty \qquad \text{for all }\; (x,t)\in(0,1)\times(0,T)\,,$$

then one may use another Gagliardo-Nirenberg inequality,

$$\|\partial_x c\|_{L^4(0,1)} \le \left(3\,\|c\|_{L^\infty(0,1)}\right)^{1/2}\cdot\|\partial^2_{x,x}c\|^{1/2}_{L^2(0,1)}\,,\qquad \text{which implies}$$

$$\int_0^1 (\partial_x c)^4\,\mathrm{d}x \le \left(3\,\|c\|_{L^\infty(0,1)}\right)^2\int_0^1 \left[\partial^2_{x,x}c(x,t)\right]^2\,\mathrm{d}x\,. \tag{5.14}$$

Consequently, applying (5.14) to (5.10) we obtain

$$\frac{1}{2}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 (\partial_x c)^2\,\mathrm{d}x \le \left[\left(3\,\|c(\,\cdot\,,t)\|_{L^\infty(0,1)}\right)^2 - \frac{7}{8}\right]\cdot\int_0^1 (\partial^2_{x,x}c)^2\,\mathrm{d}x\,. \tag{5.15}$$

Hence, if $\|c(\,\cdot\,,t)\|_{L^\infty(0,1)} \le M \le (1/6)\sqrt{7/2}$ for all $t\in(0,T)$, then also

$$\|\partial_x c(\,\cdot\,,t)\|_{L^2(0,1)} \le \|\partial_x c(\,\cdot\,,0)\|_{L^2(0,1)} = \|\partial_x c_0\|_{L^2(0,1)} \qquad \text{holds for every }\; t\in(0,T)\,.$$

With these *a priori* bounds at hand, it is a routine matter to show the existence of weak solutions in the spirit of Theorem 2.2. We leave the details to an interested reader.

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