Bounded convergence theorem for abstract Kurzweil-Stieltjes integral

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Ariel, August 2014

- $-\infty < a < b < \infty$, X is a Banach space,
- $f\colon [a,b] \to X$ is *regulated* on $[a,b]$, if
  $f(s+) := \lim_{\tau\to s+} f(\tau) \in X$ for $s \in [a,b)$, $f(t-) := \lim_{\tau\to t-} f(\tau) \in X$ for $t \in (a,b]$,
- $\Delta^+ f(s) = f(s+) - f(s)$, $\Delta^- f(t) = f(t) - f(t-)$, $\Delta f(t) = f(t+) - f(t-)$.
- $G = G([a,b],X)$ is the space of functions $f\colon [a,b] \to X$ regulated on $[a,b]$.
  (G is a Banach space with respect to the norm $\|f\|_\infty = \sup_{t\in[a,b]} \|f(t)\|$).
  - regulated functions are uniform limits of finite step functions,
  - regulated functions have at most countably many points of discontinuity.
- $BV = BV([a,b],X) = \left\{ f\colon [a,b] \to X\colon \operatorname{var}_a^b f < \infty \right\}$ is the space of functions with *bounded variation* on $[a,b]$.
- $f\colon [a,b] \to X$ is a *finite step function*, if there is a division of $[a,b]$
  $$a = \alpha_0 < \alpha_1 < \alpha_2 < \ldots < \alpha_m = b$$
  such that f is constant on every (α_{j-1}, α_j), $j = 1, 2, \ldots, m$.
  $S = S([a,b],X)$ is the *set of finite step functions* on $[a,b]$.

- $\mathcal{D}=\Big\{D=\{a{=}\alpha_0{<}\alpha_1{<}\ldots{<}\alpha_m{=}b\}\Big\}$ is the set of divisions of $[a,b]$.
- $L(X)$ is the Banach space of linear bounded mappings $X\to X$.
- For $F:[a,b]\to L(X)$ and $D=\{\alpha_0,\alpha_1,\ldots,\alpha_m\}\in\mathcal{D}$ put

$$V(F,D)=\sup\left\{\Big\|\sum_{j=1}^{m}\big[F(\alpha_j)-F(\alpha_{j-1})\big]\,x_j\Big\|_X : x_j\in X,\ \|x_j\|_X\le 1\right\}.$$

Then $\mathrm{SV}_a^b(F)=\sup_{D\in\mathcal{D}} V(F,D)$ is the semi-variation of F on $[a,b]$ and

$SV=SV([a,b],L(X))$ is the set of $F:[a,b]\to L(X)$ with $\mathrm{SV}_a^b(F)<\infty$.

- $\|F\|_{SV}=\|F(a)\|_{L(X)}+\mathrm{SV}_a^b F \implies$ SV is a Banach space.
- For $g:[a,b]\to X$ and $D=\{\alpha_0,\alpha_1,\ldots,\alpha_m\}\in\mathcal{D}$ put

$$V(g,D)=\sup\left\{\Big\|\sum_{j=1}^{m}F_j\big[g(\alpha_j)-g(\alpha_{j-1})\big]\Big\|_X : F_j\in L(X),\ \|F_j\|_{L(X)}\le 1\right\}$$

and $\mathrm{SV}_a^b(g)=\sup_{D\in\mathcal{D}} V(g,D)$.

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$$V(F,D) = \sup\left\{ \left\| \sum_{j=1}^{m} [F(\alpha_j) - F(\alpha_{j-1})]\, G_j \right\|_X : G_j \in L(X),\ \|G_j\|_{L(X)} \le 1 \right\}$$

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$SV = SV([a, b], X)$ is the set of $g : [a, b] \to X$ with $\mathrm{SV}_a^b(g) < \infty$. But SV=BV

- $\mathcal{G} = \{\delta \colon [a,b] \to (0,1)\}$ are gauges on $[a,b]$.
- $\mathcal{P} = \{P = (D,\xi),\ D = \{a = \alpha_0 < \alpha_1 < \ldots < \alpha_m = b\},\ \xi = (\xi_1, \ldots, \xi_m) \in [a,b]^m,\ \xi_j \in [\alpha_{j-1}, \alpha_j]\}$ are tagged divisions of $[a,b]$.
- $P = (D,\xi) \in \mathcal{P}$ is δ-fine if $[\alpha_{j-1}, \alpha_j] \subset (\xi_j - \delta(\xi_j), \xi_j + \delta(\xi_j))$ for all j.
- For $F : [a,b] \to L(X)$, $g : [a,b] \to X$, $P = (D,\xi) \in \mathcal{P}$ define

$$S(F \Delta g, P) = \sum_{j=1}^{m} F(\xi_j)\,[g(\alpha_j) - g(\alpha_{j-1})]\,.$$

Definition

$$I = \int_a^b F\, d\,[g] \quad \Longleftrightarrow \quad \begin{cases} \text{for each } \varepsilon > 0 \text{ there is a gauge } \delta \in \mathcal{G} \text{ such that} \\ \qquad \big|S(F\Delta g, P) - I\big| < \varepsilon \\ \text{for every } \delta - \text{fine tagged division } P. \end{cases}$$

$$\int_c^c F\, d\,[g] = 0.$$

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$$\int_c^c d\,[F]\,g = 0.$$

- RS $\subset$ KS, $X = \mathbb{R} \implies$ KS = PS.
- KS-integral has usual linear properties and it is additive function of intervals.
- $F : [a,b] \to L(X)$ and $g : [a,b] \to X$ are regulated $\implies$

  $$\int_a^b F\, d[g] \text{ and } \int_a^b d[F]\, g \quad \text{exist whenever}$$

  one of the functions F, g is a finite step function.
- $F \in SV$ and $\int_a^b d[F]\, g$ exists $\implies \left\| \int_a^b d[F]\, g \right\|_X \le SV_a^b(F)\, \|g\|_\infty$.
- $g \in SV$ and $\int_a^b d[F]\, g$ exists $\implies \left\| \int_a^b d[F]\, g \right\|_X \le 2\, \|F\|_\infty\, SV_a^b(g)$.
- $F \in SV$ and $\int_a^b F\, d[g]$ exists $\implies \left\| \int_a^b F\, d[g] \right\|_X \le 2\, SV_a^b(F)\, \|g\|_\infty$.
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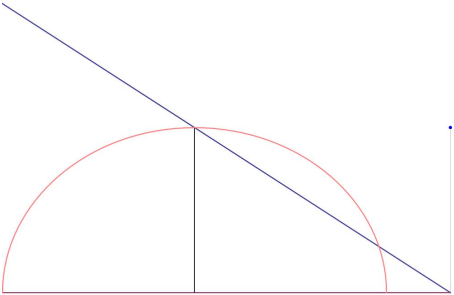
- $F(t) \equiv C \in L(X),\ g : [a,b] \to X \implies \displaystyle\int_a^b F\, d[g] = C\,[g(b) - g(a)].$
- $F : [a,b] \to L(X),\ g(t) \equiv c \in X \implies \displaystyle\int_a^b F\, d[g] = 0.$
- $g : [a,b] \to X$ semi-regulated, $\tau \in [a,b]$, and $F(t) = \chi_{[\tau,b]}(t)\,C$ for some $C \in L(X)$

  $$\implies \int_\tau^b F\, d[g] = C\,[g(b) - g(\tau)].$$

  Let $\delta(x) = \begin{cases} \frac{1}{4}\,(\tau - x) & \text{pro } x < \tau\,, \\ \eta & \text{pro } x = \tau \end{cases}$

  and $(D,\xi) = \big(\{\alpha_0, \alpha_1, \ldots, \alpha_m\}, (\xi_1, \xi_2, \ldots, \xi_m)\big)$ is δ-fine. Then



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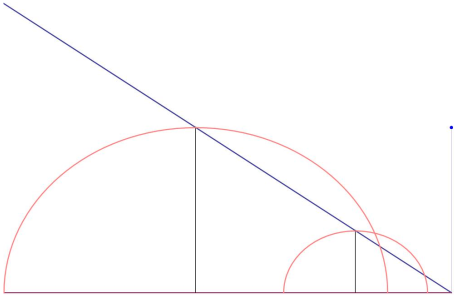

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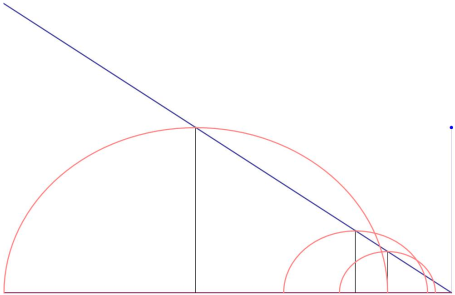

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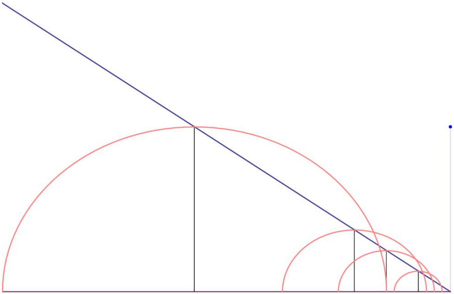

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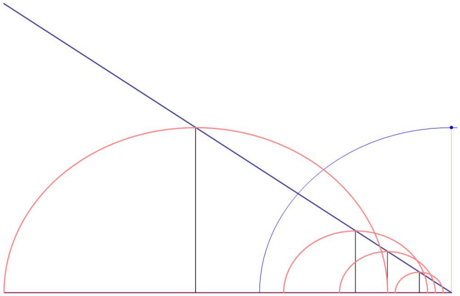

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- $g : [a,b] \to X$ semi-regulated, $C \in L(X),\ \tau \in [a,b], \implies$

$$\int_a^b \chi_{[\tau,b]} C\; d[g] = C\,g(b) - C\,g(\tau-), \quad \int_a^b \chi_{(\tau,b]} C\; d[g] = C\,g(b) - C\,g(\tau+).$$

$$\int_a^b \chi_{[a,\tau]} C\; d[g] = C\,g(\tau+) - C\,g(a), \quad \int_a^b \chi_{[a,\tau)} C\; d[g] = C\,g(\tau-) - C\,g(a).$$

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$$\int_a^b \chi_{[a,\tau]} C\ d[g] = C\,g(\tau+) - C\,g(a), \quad \int_a^b \chi_{[a,\tau)} C\ d[g] = C\,g(\tau-) - C\,g(a).$$

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- $F : [a,b] \to L(X),\ \widetilde{x} \in X,\, \tau \in [a,b] \implies$

$$\int_a^b F\, d[\chi_{[a,\tau]}\,\widetilde{x}] = \int_a^b F\, d[\chi_{[a,\tau)}\,\widetilde{x}] = -F(\tau)\,\widetilde{x},$$

$$\int_a^b F\, d[\chi_{[\tau,b]}\,\widetilde{x}] = \int_a^b F\, d[\chi_{(\tau,b]}\,\widetilde{x}] = \quad F(\tau)\,\widetilde{x},$$

$$\int_a^b F\, d[\chi_{[\tau]}\,\widetilde{x}\,] = \begin{cases} -F(a)\,\widetilde{x} & \text{for } \tau = a\,, \\ 0 & \text{for } \tau \in (a,b)\,, \\ F(b)\,\widetilde{x} & \text{for } \tau = b\,. \end{cases}$$

Schwabik

Let $F:[a,b]\to L(X)$ and $g:[a,b]\to X$.

(i) Let $F\in\mathrm{SV}$, $g_k:[a,b]\to X$, $\int_a^b d[F]\,g_k$ exists for all $n\in\mathbb{N}$ and $g_k\rightrightarrows g$ on $[a,b]$. Then

$$\int_a^b d\,[F]\,g \quad \text{exists and} \quad \int_a^b d\,[F]\,g=\lim_{k\to\infty}\int_a^b d\,[F]\,g_k\,.$$

(ii) Let $F\in\mathrm{SV}$ be semi-regulated and $g\in\mathrm{G}$. Then $\int_a^b d\,[F]\,g$ exists.

(iii) Let $F\in\mathrm{SV}$ be semi-regulated and $g\in BV$. Then $\int_a^b F\,d\,[g]$ and $\int_a^b d\,[F]\,g$ exist,

the sum $\sum_{a\le\tau<b}\Delta^+F(\tau)\,\Delta^+g(\tau)-\sum_{a<\tau\le b}\Delta^-F(\tau)\,\Delta^-g(\tau)$ converges in X and

$$\int_a^b F\,d\,[g]+\int_a^b d\,[F]\,g$$
$$=F(b)\,g(b)-F(a)\,g(a)-\sum_{a\le t<b}\Delta^+F(t)\,\Delta^+g(t)+\sum_{a<t\le b}\Delta^-F(t)\,\Delta^-g(t)\,.$$

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(i) Let $g\in \mathrm{SV}$, $F_k:[a,b]\to X$, $\int_a^b F_k\,d[g]$ exists for all $n\in\mathbb{N}$ and $F_k\rightrightarrows F$ on $[a,b]$. Then

$$\int_a^b F\,d[g] \quad \text{exists and} \quad \int_a^b F\,d[g]=\lim_{k\to\infty}\int_a^b F_k\,d[g]\,.$$

(ii) Let $g\in \mathrm{SV}$ be semi-regulated and $F\in \mathrm{G}$. Then $\int_a^b F\,d[g]$ exists.

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the sum $\sum_{a\le\tau<b}\Delta^+F(\tau)\,\Delta^+g(\tau)-\sum_{a<\tau\le b}\Delta^-F(\tau)\,\Delta^-g(\tau)$ converges in X and

$$\int_a^b F\,d[g]+\int_a^b d[F]\,g$$

$$=F(b)\,g(b)-F(a)\,g(a)-\sum_{a\le t<b}\Delta^+F(t)\,\Delta^+g(t)+\sum_{a<t\le b}\Delta^-F(t)\,\Delta^-g(t)\,.$$

Let $F:[a,b]\to L(X)$ and $g:[a,b]\to X$.

(i) If $F\in\mathrm{G}$, $g\in\mathrm{G}$ and at least one of them has a bounded semi-variation on $[a,b]$, then both integrals $\int_a^b F\,d[g]$ and $\int_a^b d[F]\,g$ exist and

$$\int_a^b F\,d[g]+\int_a^b d[F]\,g$$
$$=F(b)\,g(b)-F(a)\,g(a)-\sum_{a\le t<b}\Delta^+F(t)\,\Delta^+g(t)+\sum_{a<t\le b}\Delta^-F(t)\,\Delta^-g(t)\,.$$

(ii) If $F\in\mathrm{BV}$, $g\in\mathrm{G}$,

then $\quad \left|\int_a^b d[F]\,g\right|\le \operatorname{var}_a^b F\,\|g\|_\infty \quad$ and $\quad \left|\int_a^b F\,d[g]\right|\le 2\operatorname{var}_a^b F\,\|g\|_\infty.$

(iii) If $g\in\mathrm{BV}$, $F_k\in\mathrm{G}$ for $k\in\mathbb{N}$ and $F_k\rightrightarrows F$,

then $\quad \int_a^t d[F_k]\,g\rightrightarrows\int_a^t d[F]\,g.$

(iv) If $F\in\mathrm{BV}$, $g_k\in\mathrm{G}$ for $k\in\mathbb{N}$ and $g_k\rightrightarrows g$,

then $\quad \int_a^t F\,d[g_k]\rightrightarrows\int_a^t F\,d[g].$

ASSUME:

- $F, F_k \in G$ for $n \in \mathbb{N}$, $g \in SV[a,b]$ is semi-regulated,
- $F_k \rightrightarrows F$.

THEN: $\displaystyle\int_a^t F_k\, d[g] \rightrightarrows \int_a^t F\; d[g]$ on $[a,b]$.

ASSUME:

- $F \in \mathrm{SV}$, $g, g_k \in G$ for $n \in \mathbb{N}$,
- $g_k \rightrightarrows g$.

THEN: $\displaystyle\int_a^t F\, d[g_k] \rightrightarrows \int_a^t F\, d[g]$ on $[a,b]$.

ASSUME:

- $F, F_k \in G$, $g, g_k \in \mathrm{BV}$ for $n \in \mathbb{N}$,
- $F_k \rightrightarrows F$, $g_k \rightrightarrows g$,
- $\alpha^* := \sup\{\mathrm{var}_a^b\, g_k : n \in \mathbb{N}\} < \infty$.

THEN: $\displaystyle\int_a^t F_k\, d[g_k] \rightrightarrows \int_a^t F\, d[g]$ on $[a,b]$.

Let $A \in \mathrm{BV}$. Put $(\mathcal{A}\,x)(t) = \displaystyle\int_a^t d[A]\,x$ for $x \in \mathrm{G}$ and $t \in [a,b]$. Then

$$|\mathcal{A}x| \le \mathrm{var}_a^b A\,\|x\|_\infty \le \mathrm{var}_a^b\ \|x\|_{\mathrm{BV}} \quad \text{for } x \in \mathrm{G},$$

i.e. both $\mathcal{A} : \mathrm{G} \to \mathrm{BV}$ and $\mathcal{A} : \mathrm{BV} \to \mathrm{BV}$ are linear bounded operators.

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$$|z_\ell(t)| \le 4\,\varkappa, \qquad \lim_{\ell\to\infty} z_\ell(t) = 0 \quad \text{for } t \in [a,b] \quad \text{and}$$

$$V(\mathcal{A}z_\ell, D) = \sum_{j=1}^m \big|(\mathcal{A}z_\ell)(\alpha_j) - (\mathcal{A}z_\ell)(\alpha_{j-1})\big| = \sum_{j=1}^m \Big|\int_{\alpha_{j-1}}^{\alpha_j} d[A]\,z_k\Big| \le \sum_{j=1}^m \int_{\alpha_{j-1}}^{\alpha_j} d\left[\mathrm{var}_a^s A\right] |z_\ell(s)|$$

$$\le \int_a^b d\left[\mathrm{var}_a^s A\right] |z_\ell(s)| \qquad \text{for } D = \{\alpha_0, \alpha_2, \dots, \alpha_m\} \in \mathcal{D}[a,b] \text{ and } \ell \in \mathbb{N}.$$

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Hence

$$\mathrm{var}_a^b\,(\mathcal{A}\,z_\ell) \le \int_a^b d\,[\mathrm{var}_a^s\, A]\,|z_\ell(s)| \quad \text{for } \ell \in \mathbb{N}.$$

Let $A \in \mathrm{BV}$. Put $(\mathcal{A}x)(t) = \int_a^t d[A]\,x$ for $x \in \mathrm{G}$ and $t \in [a,b]$. Then

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Hence

$$\mathrm{var}_a^b\,(\mathcal{A}z_\ell) \le \int_a^b d\left[\mathrm{var}_a^s A\right] |z_\ell(s)| \quad \text{for } \ell \in \mathbb{N}.$$

TO HAVE $\|\mathcal{A}z_\ell\|_{\mathrm{BV}} \to 0$ as $\ell \to 0$ WE NEED $\int_a^b d\left[\mathrm{var}_a^s A\right] |z_\ell(s)| \to 0$ as $\ell \to 0$.

BOUNDED CONVERGENCE THEOREM (for $X = \mathbb{R}$)

(i) ASSUME:

- $F \in \mathrm{BV}, \quad g, g_k \in \mathrm{G}$ for $k \in \mathbb{N}$,
- $g_k(t) \to g(t)$ on $[a,b]$,
- $\|g_k\|_\infty \le K < \infty$ for $k \in \mathbb{N}$.

THEN : $\displaystyle \int_a^b d[F]\, g_k \to \int_a^b d[F]\, g.$

(ii) ASSUME:

- $g \in \mathrm{BV}, \quad F, F_k \in \mathrm{G}$ for $k \in \mathbb{N}$,
- $F_k(t) \to F(t)$ on $[a,b]$,
- $\|F_k\|_\infty \le K < \infty$ for $k \in \mathbb{N}$.

THEN : $\displaystyle \int_a^b F_k\, d[g] \to \int_a^b F\, d[g].$

LEBESGUE INTEGRAL: Lebesgue Dominated Convergence Theorem
RIEMANN or STIELTJES INTEGRAL: Arzelà-Osgood Theorem.

Available proofs can not be extended to the abstract setting !!

Moreover, deep Arzelà's lemma is needed.

BOUNDED CONVERGENCE THEOREM

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- $F \in \mathrm{BV}, \quad g, g_k \in \mathrm{G} \;$ for $\; k \in \mathbb{N}$,
- $g_k(t) \to g(t) \;$ on $\; [a,b]$,
- $\|g_k\|_\infty \le \gamma^* < \infty \;$ for $k \in \mathbb{N}$.

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(ii) ASSUME:

- $g \in \mathrm{BV}, \quad F, F_k \in \mathrm{G} \;$ for $\; k \in \mathbb{N}$,
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THEN: $\displaystyle\int_a^b F_k\, d[g] \to \int_a^b F\, d[g].$

LEMMA (Arzelà) *Let* $\{\{J_{k,j}\} : k \in \mathbb{N},\, j \in U_k\}$ *be the set of subintervals of* $[a,b]$ *such that: for each* $k \in \mathbb{N}$, *the set of indices* U_k *is finite, the intervals from* $\{J_{k,j} : j \in U_k\}$ *are mutually disjoint and*

$$\sum_{j \in U_k} |J_{k,j}| > c > 0.$$

Then there exist sequences of indices $\{k_\ell\}$ *and* $\{j_\ell\}$ *such that*

$$j_\ell \in U_{k_\ell} \;\; \textit{for} \;\; \ell \in \mathbb{N} \;\; \textit{and} \;\; \bigcap_{\ell \in \mathbb{N}} J_{k_\ell, j_\ell} \ne \emptyset.$$

DEFINITIONS

- $J \subset \mathbb{R}$ is an interval if $\alpha, \beta \in J,\ \alpha < \beta,\ \alpha < x < \beta \implies x \in J \quad (\{a\} = [a])$.
- For intervals $J \subset [a, b]$, sets $D = \{\alpha_0, \alpha_1, \ldots, \alpha_m\}$ such that
$$\alpha_0 < \alpha_1 < \cdots < \alpha_m \quad \text{and} \quad \alpha_j \in J \quad \text{for } j = 0, 1, \ldots, m$$
are divisions of J. $\mathcal{D}(J)$ is the set of all divisions of J.
- For $f : J \to X$ $\quad var_J\, f = \sup\{V(f, D) : D \in \mathcal{D}(J)\}$ is its variation over J, $\text{var}_{\emptyset} f = \text{var}_{[c]} f = 0$ for any $c \in [a, b]$.
- A bounded subset E of $\mathbb{R}$ is an elementary set if it is a finite union of intervals. For $A \subset \mathbb{R}$, $\mathcal{E}(A)$ is the set of all elementary subsets of A.
- A collection of intervals $\{J_k : k = 1, 2, \ldots, m\}$, is a minimal decomposition of E if
$$E = \bigcup_{k=1}^{m} J_k, \text{ while } J_k \cup J_\ell \text{ is not an interval whenever } k \neq \ell.$$
- For $f : [a, b] \to X$ and an elementary subset E of $[a, b]$ with a minimal decomposition $\{J_k : k = 1, \ldots, m\}$, we define $\text{var}(f, E) = \sum_{k=1}^{m} \text{var}_{J_k} f$.

Proposition

Let $c, d \in [a, b],\ c < d$. Then

- $\text{var}_{[c,d]} f = \text{var}_c^d f, \qquad \text{var}_{[c,d)} f = \lim_{\delta \to 0+} \text{var}_c^{d-\delta} f = \sup_{t \in [c,d)} \text{var}_c^t f,$
- $\text{var}_{(c,d)} f = \lim_{\delta \to 0+} \text{var}_{c+\delta}^{d-\delta} f, \quad \text{var}_{(c,d]} f = \lim_{\delta \to 0+} \text{var}_{c+\delta}^{d} f = \sup_{t \in (c,d]} \text{var}_t^d f.$
- If $f \in BV((c, d), X)$ and $f(c+), f(d-)$ exist, then $f \in BV([c, d], X)$ and
$$\text{var}_c^d f = \text{var}_{(c,d)} f + \|\Delta^+ f(c)\|_X + \|\Delta^- f(d)\|_X.$$

DEFINITION

Let $F\colon[a,b]\to L(X)$, $g\colon[a,b]\to X$ and let $E\in\mathcal{E}([a,b])$. Then we define

$$\int_E d[F]\,g=\int_a^b d[F]\,(g\,\chi_E)\quad\text{and}\quad\int_E F\;d[g]=\int_a^b (F\,\chi_E)\;d[g]$$

provided the integrals on the right-hand sides exist.

Propositions

- Let $E_1,\,E_2\in\mathcal{E}([a,b])$, $E_1\cap E_2=\emptyset$, $F\colon[a,b]\to L(X)$, $g\colon[a,b]\to X$ and let the integrals $\displaystyle\int_{E_j} d[F]\,g, j=1,2$, exist. Then
$$\int_{E_1\cup E_2} d[F]\,g=\int_{E_1} d[F]g+\int_{E_2} d[F]g.$$
- Let $J=(c,d)$ and let $\int_J d[F]\,g$ exists. Then
$$\Big\|\int_{(c,d)} d[F]\,g\Big\|_X\le\Big(\operatorname{var}_{(c,d)}F\Big)\Big(\sup_{t\,\in\,(c,d)}\|g(t)\|_X\Big).$$
- Let $J=[c,d)$, and let $\int_J d[F]\,g$ and $F(c-)$ exist. Then
$$\Big\|\int_{[c,d)} d[F]g\Big\|_X\le\Big(\operatorname{var}_{[c,d)}\Big)F\Big(\sup_{t\,\in\,[c,d)}\|g(t)\|_X\Big)+\|\Delta^-F(c)\|_{L(X)}\,\|g(c)\|_X.$$

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- $F \in \mathrm{BV}, \quad g_k \in \mathrm{G} \;$ for $\; k \in \mathbb{N}$,
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THEN: $\displaystyle\int_a^b d[F]\, g_k \to 0.$

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Lewin (1986)

Let $\{A_n\}$ be a sequence of bounded subsets of $[a, b]$ such that

$$A_{n+1} \subset A_n \quad \text{and} \quad \bigcap A_n = \emptyset.$$

Put

$$\alpha_n = \sup\{\, m(E) : E \text{ elementary subset of } A_n \,\} \quad \text{for } n \in \mathbb{N}.$$

Then $\lim_{n \to \infty} \alpha_n = 0.$

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Proof.

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Proof. $\{\alpha_n\}$ is decreasing.

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Proof. $\{\alpha_n\}$ is decreasing. Assume that $\alpha_n \not\to 0$. Then, there is $\varepsilon > 0$ such that

$$\alpha_n > \varepsilon \text{ for every } n \in \mathbb{N}.$$

Hence, for each $n \in \mathbb{N}$, there is an elementary subset E_n of A_n such that

$$\alpha_n - \frac{\varepsilon}{2^n} < \mathrm{var}(f, E_n).$$

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By Cantor's intersection theorem we get $\bigcap_n H_n \neq \emptyset$. This contradicts our assumption $\bigcap_n A_n = \emptyset$.

Hence, $\lim\limits_{n\to\infty} \alpha_n = 0$.

Sketch of the proof of Bounded Convergence Theorem

Let $\|g_n\| \leq K < \infty$ for $n \in \mathbb{N}$ and $g_n(t) \to 0$ on $[a, b]$.

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Let $n \geq N$ be given and let h_n be such that $\|h_n - g_n\| < \min\left\{K, \dfrac{\varepsilon}{6 \operatorname{var}_a^b F}\right\}$.

Denote $U_n = \left\{ t \in [a,b] : \|h_n(t)\| \geq \dfrac{\varepsilon}{3 \operatorname{var}_a^b F} \right\}$ and $V_n = [a,b] \setminus U_n$.

For $t \in U_n$ we have

$$\|g_n(t)\| > \|h_n(t)\| - \frac{\varepsilon}{6 \operatorname{var}_a^b F} \geq \frac{\varepsilon}{3 \operatorname{var}_a^b F} - \frac{\varepsilon}{6 \operatorname{var}_a^b F} = \frac{\varepsilon}{6 \operatorname{var}_a^b F}, \qquad \text{i.e. } t \in A_n,$$

i.e. $U_n \subset A_n$.

Sketch of the proof of Bounded Convergence Theorem

Let $\|g_n\| \leq K < \infty$ for $n \in \mathbb{N}$ and $g_n(t) \to 0$ on $[a,b]$.

a) $\operatorname{var}_a^b F = 0 \implies \int_a^b d[F]\, g_n = \int_a^b d[F]\, g = 0$ for all $n \in \mathbb{N}$.

b) Let $\operatorname{var}_a^b F \neq 0$, $\varepsilon > 0$ and $A_n = \left\{ t \in [a,b] : \exists m \geq n \text{ such that } \|g_n(t)\| \geq \dfrac{\varepsilon}{6 \operatorname{var}_a^b F} \right\}$.

Then $A_{n+1} \supset A_n$, $\bigcap_n A_n = \emptyset$ and $\alpha_n = \sup\{\operatorname{var}(F,E) : E \text{ elementary subset of } A_n\}$.

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We have $U_n \subset A_n$

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Then $A_{n+1} \supset A_n$, $\bigcap_n A_n = \emptyset$ and $\alpha_n = \sup\{\mathrm{var}(F,E) : E \text{ elementary subset of } A_n\}$.

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We have $U_n \subset A_n$ and hence, by (1),

$$\left\| \int_a^b d[F]\, h_n \right\| \le \left\| \int_{U_n} d[F]\, h_n \right\| + \left\| \int_{V_n} d[F]\, h_n \right\| \le \mathrm{var}(F, U_n)\, \|h_n\|_{U_n} + \mathrm{var}(F, V_n)\, \|h_n\|_{V_n}$$

$$\le \frac{\varepsilon}{6\,K}(K + K) + \mathrm{var}_a^b F\, \frac{\varepsilon}{3\, \mathrm{var}_a^b F} = \frac{2}{3}\,\varepsilon$$

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Sketch of the proof of Bounded Convergence Theorem

Let $\|g_n\| \le K < \infty$ for $n \in \mathbb{N}$ and $g_n(t) \to 0$ on $[a,b]$.

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We have $\left\| \int_a^b d[F]\, h_n \right\| < \dfrac{2}{3}\,\varepsilon.$ Therefore,

$$\left\| \int_a^b d[F]\, g_n \right\| \le \left\| \int_a^b d[F]\, h_n \right\| + \left\| \int_a^b d[F]\, (h_n - g_n) \right\| \le \frac{2}{3}\,\varepsilon + (\operatorname{var}_a^b F)\, \frac{\varepsilon}{6 \operatorname{var}_a^b F} < \varepsilon.$$

Sketch of the proof of Bounded Convergence Theorem

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