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**On asymptotic isotropy
for a hydrodynamic model
of liquid crystals**

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Abstract

We study a PDE system describing the motion of liquid crystals by means of the Q -tensor description for the crystals coupled with the incompressible Navier-Stokes system. Using the method of Fourier splitting, we show that solutions of the system tend to the isotropic state at the rate $(1+t)^{-\beta}$ as $t \rightarrow \infty$ for a certain $\beta > \frac{1}{2}$.

Key words: Liquid crystal, Q -tensor description, long-time behavior, Fourier splitting

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1 Introduction

We consider a frequently used hydrodynamic model of nematic liquid crystals, where the local configuration of the crystal is represented by the Q -tensor $\mathbb{Q} = \mathbb{Q}(t, x)$, while its motion is described through the Eulerian velocity field $\mathbf{u} = \mathbf{u}(t, x)$, both quantities being functions of the time $t > 0$ and the spatial position $x \in \mathbb{R}^3$. The tensor $\mathbb{Q} \in \mathbb{R}_{\text{sym},0}^{3 \times 3}$ is a symmetric traceless matrix, whose time evolution is described by the equation

$$\partial_t \mathbb{Q} + \text{div}_x(\mathbb{Q}\mathbf{u}) - \mathbb{S}(\nabla_x \mathbf{u}, \mathbb{Q}) = \Delta_x \mathbb{Q} - \mathcal{L}[\partial F(\mathbb{Q})], \quad (1.1)$$

with

$$\mathcal{L}[\mathbb{A}] \equiv \mathbb{A} - \frac{1}{3} \text{tr}[\mathbb{A}] \mathbb{I}$$

denoting the projection onto the space of traceless matrices, and F denoting a potential function which will be described later. The velocity field obeys the Navier-Stokes system

$$\partial_t \mathbf{u} + \text{div}_x(\mathbf{u} \otimes \mathbf{u}) + \nabla_x p = \Delta_x \mathbf{u} + \text{div}_x \Sigma(\mathbb{Q}) \quad (1.2)$$

supplemented with the incompressibility constraint

$$\text{div}_x \mathbf{u} = 0. \quad (1.3)$$

The tensors \mathbb{S} and Σ are taken the form

$$\mathbb{S}(\nabla_x \mathbf{u}, \mathbb{Q}) = (\xi \varepsilon(\mathbf{u}) + \omega(\mathbf{u})) \left(\mathbb{Q} + \frac{1}{3} \mathbb{I} \right) + \left(\mathbb{Q} + \frac{1}{3} \mathbb{I} \right) (\xi \varepsilon(\mathbf{u}) - \omega(\mathbf{u})) - 2\xi \left(\mathbb{Q} + \frac{1}{3} \mathbb{I} \right) \mathbb{Q} : \nabla_x \mathbf{u}, \quad (1.4)$$

$$\Sigma(\mathbb{Q}) = 2\xi \mathbb{H} : \mathbb{Q} \left(\mathbb{Q} + \frac{1}{3} \mathbb{I} \right) - \xi \left[\mathbb{H} \left(\mathbb{Q} + \frac{1}{3} \mathbb{I} \right) - \left(\mathbb{Q} + \frac{1}{3} \mathbb{I} \right) \mathbb{H} \right] - (\mathbb{Q}\mathbb{H} - \mathbb{H}\mathbb{Q}) - \nabla_x \mathbb{Q} \odot \nabla_x \mathbb{Q}, \quad (1.5)$$

where we have denoted

$$\epsilon(\mathbf{u}) = \frac{1}{2}(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u}), \quad \omega(\mathbf{u}) = \frac{1}{2}(\nabla_x \mathbf{u} - \nabla_x^t \mathbf{u}), \quad \mathbb{H} = \Delta_x \mathbb{Q} - \mathcal{L}[\partial F(\mathbb{Q})], \quad \text{and } (\nabla_x \mathbb{Q} \odot \nabla_x \mathbb{Q})_{ij} = \partial_i \mathbb{Q}_{\alpha\beta} \partial_j \mathbb{Q}_{\alpha\beta}.$$

Here and hereafter, we use the summation convention for repeated indices. The number $\xi \geq 0$ is a scalar parameter measuring the ratio between the rotation and the aligning effect that a shear flow exerts over the directors.

We refer to Beris and Edwards [3] for the physical background, and to Zarnescu et al. [7], [8], [6] for mathematical aspects of the problem.

1.1 Energy balance

The problem (1.1 - 1.5) admits a natural *energy functional*, namely

$$E = \frac{1}{2}|\mathbf{u}|^2 + \frac{1}{2}|\nabla_x \mathbb{Q}|^2 + F(\mathbb{Q}),$$

where

$$F : \mathbb{R}_{\text{sym}}^{3 \times 3} \rightarrow (-\infty, \infty]$$

is a given (generalized) function.

We assume that $F \in C^2(\mathcal{O})$, where $\mathcal{O} \subset \mathbb{R}_{\text{sym}}^{3 \times 3}$ is an open set containing the isotropic state $\mathbb{Q} \equiv 0$, and there are two balls B_{r_1}, B_{r_2} such that

$$\mathbb{Q} = 0 \in B_{r_1} \equiv \{|\mathbb{Q}| < r_1\} \subset B_{r_2} \equiv \{|\mathbb{Q}| \leq r_2\} \subset \mathcal{O}.$$

In addition, we suppose that $\mathbb{Q} = 0$ is the (unique) global minimum of F in \mathcal{O} , specifically,

$$F(0) = 0, \quad F(\mathbb{Q}) > 0 \text{ for any } \mathbb{Q} \in \mathcal{O} \setminus \{0\} \tag{1.6}$$

and

$$\partial F(\mathbb{Q}) : \mathbb{Q} \geq 0 \text{ whenever } \mathbb{Q} \in B_{r_1} \text{ or } \mathbb{Q} \in \mathcal{O} \setminus B_{r_2}. \tag{1.7}$$

Let us note here that the polynomial potentials considered by Paicu and Zarnescu [7]:

$$F(\mathbb{Q}) = \frac{a}{2}|\mathbb{Q}|^2 + \frac{b}{3}\text{trace}[\mathbb{Q}^3] + \frac{c}{4}|\mathbb{Q}|^4, \tag{1.8}$$

at least in case $a > 0$ in a neighborhood of 0, fit this conditions (cf. Section 6 for further comments on this point).

Taking the scalar product of equation (1.1) with \mathbb{H} , the scalar product of equation (1.2) with \mathbf{u} , adding the resulting expressions and integrating over the physical space \mathbb{R}^3 (cf. [8, Proof of Prop. 1] for details), we obtain the total energy balance

$$\frac{d}{dt} \int_{\mathbb{R}^3} E \, dx + \int_{\mathbb{R}^3} |\nabla_x \mathbf{u}|^2 + \left| \Delta_x \mathbb{Q} - \mathcal{L}[\partial F(\mathbb{Q})] \right|^2 \, dx = 0 \quad (1.9)$$

provided that

$$\mathbf{u} \rightarrow 0, \quad \mathbb{Q} \rightarrow 0 \text{ as } |x| \rightarrow \infty \quad (1.10)$$

sufficiently fast.

The presence of the dissipative term

$$\int_{\mathbb{R}^3} |\nabla_x \mathbf{u}|^2 + \left| \Delta_x \mathbb{Q} - \mathcal{L}[\partial F(\mathbb{Q})] \right|^2 \, dx$$

suggests that

$$\nabla_x \mathbf{u}(t, \cdot) \rightarrow 0, \quad \mathbb{Q}(t, \cdot) \rightarrow \tilde{\mathbb{Q}} \text{ as } t \rightarrow \infty \text{ in a certain sense,} \quad (1.11)$$

where $\tilde{\mathbb{Q}}$ is a static distribution of the \mathbb{Q} -tensor density, namely it satisfies

$$-\Delta_x \tilde{\mathbb{Q}} + \mathcal{L} \left[\partial F(\tilde{\mathbb{Q}}) \right] = 0. \quad (1.12)$$

As we shall see below (Lemma 4.1), the hypothesis (1.6) implies that $\tilde{\mathbb{Q}} \equiv 0$; more specifically, any solution $\tilde{\mathbb{Q}}$ of (1.12) belonging to the class

$$\nabla_x \tilde{\mathbb{Q}} \in L^2(\mathbb{R}^3; \mathbb{R}^3), \quad F(\tilde{\mathbb{Q}}) \in L^1(\mathbb{R}^3), \quad \tilde{\mathbb{Q}}(x) \in B_{r_2} \text{ for all } x \in \mathbb{R}^3$$

necessarily vanishes identically in \mathbb{R}^3 , in particular (1.11) reduces to

$$\nabla_x \mathbf{u}(t, \cdot) \rightarrow 0, \quad \mathbb{Q}(t, \cdot) \rightarrow 0 \text{ as } t \rightarrow \infty. \quad (1.13)$$

1.2 Asymptotic isotropy

Our goal is to justify (1.13) in the class of *weak* solutions to the system (1.1 - 1.3). To this end, we need a simplifying assumption setting the parameter $\xi = 0$. Hence, (1.4), (1.5) reduce to

$$\mathbb{S}(\nabla_x \mathbf{u}, \mathbb{Q}) = \omega(\mathbf{u})\mathbb{Q} - \mathbb{Q}\omega(\mathbf{u}), \quad \Sigma(\mathbb{Q}) = -\mathbb{Q}\Delta_x \mathbb{Q} + \Delta_x \mathbb{Q}\mathbb{Q} - \nabla_x \mathbb{Q} \odot \nabla_x \mathbb{Q}, \quad (1.14)$$

where we have used

$$\mathbb{Q}\mathcal{L}[\partial F(\mathbb{Q})] - \mathcal{L}[\partial F(\mathbb{Q})]\mathbb{Q} = \mathbb{Q}\partial F(\mathbb{Q}) - \partial F(\mathbb{Q})\mathbb{Q} = 0.$$

Such an assumption simplifies considerably the analysis of the Q -tensor equation (1.1), in particular we may use its renormalized version in order to deduce stability of the isotropic state in the space L^∞ .

Our aim is to show that

$$\|\mathbf{u}(t, \cdot)\|_{L^2(\mathbb{R}^3; \mathbb{R}^3)} + \|\mathbb{Q}(t, \cdot)\|_{H^1(\mathbb{R}^3; \mathbb{R}^{3 \times 3})} \leq c(1+t)^{-\beta}, \quad t > 0 \quad (1.15)$$

for *any* weak solution of the problem (1.1 - 1.3), (1.14), where the constant c depends only on the initial data, and the decay rate $\beta > \frac{1}{2}$ in general, $\beta = \frac{3}{4}$ provided F is strictly convex in the neighborhood of the isotropic state $\mathbb{Q} \equiv 0$. Such a result seems optimal, at least in the latter case, as the decay coincides with that for the linear heat equation. We would like to point out that our hypotheses (cf. (1.6), in particular) are also optimal for *unconditional* convergence to an equilibrium. Indeed one may conjecture, by analogy with the nowadays standard existence theory for semilinear elliptic problems developed by Berestycki and Lions [1], [2], that the stationary problem (1.12) may admit a non-zero solution if $F < 0$ at some point. Under these circumstances, convergence to a single stationary state is in general not expected, cf. [5, Theorem 4.1].

In order to show (1.15) we make use of the method of *Fourier splitting* developed in [9], [10], [11] and later used in [4] to study the long-time behavior of a liquid crystal model based on the description via the director field. Besides the higher complexity of the Q -tensor model reflected through the constitutive relations (1.14), the main difference between [4] and this paper is that the present result is *unconditional* and applies to all weak solutions of the problem satisfying an energy inequality, while [4] requires the initial data to be small and regular. As is well known, the ultimate regularity of the Navier-Stokes and related problems is based on the so-called Ladyzhenskaya estimates (cf. [4]) available for the present problem only in the $2D$ -geometry, see Paicu and Zarnescu [7].

The paper is organized as follows. In Section 2, we introduce the concept of *finite energy weak solution* to the problem (1.1 - 1.3), (1.14) and collect some preliminary material, including the energy inequality and its immediate implications. Section 3 states rigorously our main result. Section 4 deals with the Q -tensor equation, in particular, we deduce decay estimates for \mathbb{Q} assuming higher integrability of the initial data. The proof of the decay of the velocity field is completed in Section 5 by means of the Fourier splitting method. Finally, we discuss the implications of our results for a special class of polynomial potentials in Section 6.

2 Preliminaries, weak solutions, energy inequality

The expected regularity of the weak solutions is basically determined by the energy balance (1.9). More specifically, we consider the weak solutions (\mathbb{Q}, \mathbf{u}) belonging to the following class:

(a)

$$\mathbb{Q} \in C_{\text{weak}}([0, T]; L^2(\mathbb{R}^3; \mathbb{R}^{3 \times 3})), \quad \sup_{t \in [0, T]} (\|\mathbb{Q}(t, \cdot)\|_{L^1 \cap L^\infty(\mathbb{R}^3; \mathbb{R}^{3 \times 3})} + \|\mathbb{Q}\|_{W^{1,2}(\mathbb{R}^3; \mathbb{R}^{3 \times 3})}) < \infty,$$

$$\mathbb{Q} \in L^2(0, T; W^{2,2}(\mathbb{R}^3; \mathbb{R}^{3 \times 3})), \quad \mathbb{Q}(t, x) \in B_{r_2} \text{ for all } t \in [0, T], \text{ a.a. } x \in \mathbb{R}^3,$$

for any $T > 0$;

(b)

$$\mathbf{u} \in C_{\text{weak}}(0, T; L^2(\mathbb{R}^3; \mathbb{R}^3)), \quad \nabla_x \mathbf{u} \in L^2(0, T; L^2(\mathbb{R}^3; \mathbb{R}^{3 \times 3}))$$

for any $T > 0$.

The last condition in (a) states that \mathbb{Q} remains separated from the boundary of the domain \mathcal{O} of F (if F is allowed to explode near $\partial\mathcal{O}$). This property is often referred to as *strict physicality* of the Q -tensor configuration. Such a property has been recently proved by Wilkinson [12] in the case where system (1.1 - 1.5) is settled in the unit torus and complemented with periodic boundary conditions. The estimates performed below (cf. in particular Subsec. 4.1) will imply, as a byproduct, that the same property holds also in the present case. A rigorous proof of *existence* for weak solutions to (1.1 - 1.5) in the whole euclidean space \mathbb{R}^3 was established by Paicu and Zarnescu [7], [8] for a certain class of smooth potentials F . Actually, the uniform estimates we are going to detail below can give some idea on the highlights of their argument; moreover, they will show that *singular potentials* satisfying (1.6), (1.7) can be dealt with by the same method. We finally note that the regularity conditions stated in (a) are fully consistent with the a-priori estimates.

2.1 Weak solutions for the Q -tensor equation

If \mathbb{Q} , \mathbf{u} belong to the regularity class specified above, it is easy to check that

$$\|\mathbb{S}(\nabla_x \mathbf{u}, \mathbb{Q})\|_{L^2(0, T; L^1 \cap L^2(\mathbb{R}^3; \mathbb{R}^{3 \times 3}))} \leq c, \quad (2.1)$$

$$\text{ess sup}_{t \in (0, T)} \|\mathcal{L}[\partial F(\mathbb{Q})](t, \cdot)\|_{L^1 \cap L^\infty(\mathbb{R}^3; \mathbb{R}^{3 \times 3})} \leq c \quad (2.2)$$

and

$$\text{div}_x(\mathbb{Q}\mathbf{u}) = \mathbf{u} \cdot \nabla_x \mathbb{Q} \in L^\infty(0, T; L^1(\mathbb{R}^3; \mathbb{R}^{3 \times 3})) \cap L^2(0, T; L^{3/2}(\mathbb{R}^3; \mathbb{R}^{3 \times 3})) \cap L^1(0, T; L^3(\mathbb{R}^3; \mathbb{R}^{3 \times 3})) \quad (2.3)$$

for any $T > 0$, where we have used the embedding relation $W^{1,2} \hookrightarrow L^6$ in three dimension.

Consequently, in view of the standard parabolic $L^p - L^q$ estimates, all partial derivatives appearing in (1.1) exist in the strong sense and the equation is satisfied a.e. in the space time cylinder $[0, \infty) \times \mathbb{R}^3$.

2.2 The Navier-Stokes system

As for the Navier-Stokes system (1.2), we have

$$\mathbf{u} \otimes \mathbf{u} \in L^\infty(0, T; L^1(\mathbb{R}^3; \mathbb{R}^{3 \times 3})) \cap L^2(0, T; L^{3/2}(\mathbb{R}^3; \mathbb{R}^{3 \times 3})) \cap L^1(0, T; L^3(\mathbb{R}^3; \mathbb{R}^{3 \times 3})), \quad (2.4)$$

$$\mathbb{Q} \Delta_x \mathbb{Q}, \Delta_x \mathbb{Q} \mathbb{Q} \in L^2(0, T; L^1 \cap L^2(\mathbb{R}^3; \mathbb{R}^{3 \times 3})), \quad (2.5)$$

$$\nabla_x \mathbb{Q} \odot \nabla_x \mathbb{Q} \in L^\infty(0, T; L^1(\mathbb{R}^3; \mathbb{R}^{3 \times 3})) \cap L^2(0, T; L^2(\mathbb{R}^3; \mathbb{R}^{3 \times 3})), \quad (2.6)$$

where we have used the Gagliardo-Nirenberg interpolation inequality

$$\|\nabla_x v\|_{L^4}^2 \leq c \|\Delta_x v\|_{L^2} \|v\|_{L^\infty}. \quad (2.7)$$

Thus, applying the standard *Helmholtz projection* \mathbf{P} onto the space of solenoidal functions, the system (1.2) may be interpreted as a linear parabolic equation

$$\partial_t \mathbf{u} - \Delta_x \mathbf{u} = \mathbf{P} \operatorname{div}_x \left[-\mathbb{Q} \Delta_x \mathbb{Q} + \Delta_x \mathbb{Q} \mathbb{Q} - \nabla_x \mathbb{Q} \odot \nabla_x \mathbb{Q} - \mathbf{u} \otimes \mathbf{u} \right], \quad (2.8)$$

with the right-hand side ranging in a Sobolev space $L^p(0, T; W^{-1,r}(\mathbb{R}^3; \mathbb{R}^3))$ for certain p, r .

3 Main result

We are ready to state the main result of the present paper.

Theorem 3.1. *Let the potential F satisfy the hypotheses (1.6), (1.7). Let (\mathbb{Q}, \mathbf{u}) be a global-in-time weak solution of the system (1.1 - 1.3), (1.14) satisfying the energy inequality*

$$\begin{aligned} & \int_{\mathbb{R}^3} \left[\frac{1}{2} |\mathbf{u}|^2 + \frac{1}{2} |\nabla_x \mathbb{Q}|^2 + F(\mathbb{Q}) \right] (t, \cdot) \, dx + \int_s^t \int_{\mathbb{R}^3} \left[|\nabla_x \mathbf{u}|^2 + \left| \Delta_x \mathbb{Q} - \mathcal{L}[\partial F(\mathbb{Q})] \right|^2 \right] \, dx \quad (3.1) \\ & \leq \int_{\mathbb{R}^3} \left[\frac{1}{2} |\mathbf{u}|^2 + \frac{1}{2} |\nabla_x \mathbb{Q}|^2 + F(\mathbb{Q}) \right] (s, \cdot) \, dx \end{aligned}$$

for all $t > s$ and a.a. $s \in [0, \infty)$ including $s = 0$, emanating from the initial data

$$\mathbf{u}(0, \cdot) = \mathbf{u}_0, \quad \mathbb{Q}(0, \cdot) = \mathbb{Q}_0,$$

$$\mathbf{u}_0 \in L^1 \cap L^2(\mathbb{R}^3; \mathbb{R}^3), \quad \operatorname{div}_x \mathbf{u}_0 = 0, \quad \mathbb{Q}_0 \in L^1 \cap W^{1,2}(\mathbb{R}^3; \mathbb{R}_{\text{sym},0}^{3 \times 3}), \quad |\mathbb{Q}_0(x)| \leq r_2 \text{ for a.a. } x \in \mathbb{R}^3, \quad (3.2)$$

where r_2 has been introduced in (1.7).

Then there exist a constant $c > 0$ depending solely on the initial data $[\mathbf{u}_0, \mathbb{Q}_0]$ such that

$$\|\mathbf{u}(t, \cdot)\|_{L^2(\mathbb{R}^3; \mathbb{R}^3)} + \|\mathbb{Q}(t, \cdot)\|_{W^{1,2}(\mathbb{R}^3; \mathbb{R}^{3 \times 3})} \leq c(1+t)^{-\frac{15}{28}} \quad (3.3)$$

for all $t > 0$.

If, in addition,

$$F(\mathbb{Q}) \geq \lambda |\mathbb{Q}|^2 \text{ in } B_{r_1}, \quad \lambda > 0, \quad (3.4)$$

then the decay exponent $\frac{15}{28}$ in (3.3) may be improved up to $\frac{3}{4}$.

Remark 3.1. As already pointed out above, the existence of the finite energy weak solutions satisfying the energy inequality (3.1) was proved by Paicu and Zarnescu [8, Prop. 2] for certain potentials F .

Remark 3.2. The exponent $-15/28$ is not likely to be optimal. Indeed one could get sharper estimates with technical bootstrap arguments (cf. Remark 5.1 for further details on this topic).

The rest of the present paper is devoted to the proof of Theorem 3.1.

4 Decay for the \mathbb{Q} -tensor

We start by deriving decay estimates for solutions to the \mathbb{Q} -tensor equation, which we rewrite as

$$\partial_t \mathbb{Q} + \mathbf{u} \cdot \nabla_x \mathbb{Q} - \Delta_x \mathbb{Q} = -\mathcal{L}[\partial F(\mathbb{Q})] + \omega(\mathbf{u})\mathbb{Q} - \mathbb{Q}\omega(\mathbf{u}), \quad \mathbb{Q}(0, \cdot) = \mathbb{Q}_0. \quad (4.1)$$

The class of weak solutions considered in Theorem 3.1 has the \mathbb{Q} -component in $L^1 \cap L^\infty(\mathbb{R}^3)$ at least on compact time intervals, therefore we may take the scalar product of (4.1) with $2G'(|\mathbb{Q}|^2)\mathbb{Q}$, where $G' \in C[0, \infty)$, and integrate over the physical space to obtain

$$\frac{d}{dt} \int_{\mathbb{R}^3} G(|\mathbb{Q}|^2) dx + \int_{\mathbb{R}^3} \left[2G'(|\mathbb{Q}|^2) |\nabla_x \mathbb{Q}|^2 + G''(|\mathbb{Q}|^2) |\nabla_x |\mathbb{Q}|^2|^2 \right] dx = -2 \int_{\mathbb{R}^3} G'(|\mathbb{Q}|^2) \partial F(\mathbb{Q}) : \mathbb{Q} dx, \quad (4.2)$$

where we have used that

$$[\omega(\mathbf{u})\mathbb{Q} - \mathbb{Q}\omega(\mathbf{u})] : \mathbb{A} = 2(\mathbb{A}\mathbb{Q}) : \omega(\mathbf{u}) = 0, \quad (4.3)$$

which holds provided that \mathbb{A} and $\mathbb{A}\mathbb{Q}$ are symmetric. The relation (4.2) may be seen as a kind of *renormalized* energy balance for \mathbb{Q} . It is worth noting that, thanks to our hypothesis $\xi = 0$, this relation is independent of the velocity \mathbf{u} .

4.1 A maximum principle

Our first goal is to show that

$$\|\mathbb{Q}(t, \cdot)\|_{L^\infty(\mathbb{R}^3; \mathbb{R}^{3 \times 3})} \leq r_2 \text{ for all } t \geq 0 \quad (4.4)$$

provided that the initial datum \mathbb{Q}_0 satisfies (3.2). To this end, it is enough to take G in (4.2) such that

$$G(z) = 0 \text{ for } z \in [0, r_2^2], \quad G' \geq 0, G'' \geq 0, \quad G(z) > 0 \text{ for } z > r_2^2.$$

In view of the hypotheses (1.7), (3.2) we have

$$\int_{\mathbb{R}^3} G(|\mathbb{Q}|^2)(t, \cdot) \, dx \leq \int_{\mathbb{R}^3} G(|\mathbb{Q}_0|^2) \, dx = 0 \text{ for all } t \geq 0$$

yielding the desired conclusion (4.4).

In what follows, in view of (4.4), we may assume, by virtue of (1.6) and (1.7), that

$$0 \leq F(\mathbb{Q}) \leq \alpha |\mathbb{Q}|^2, \quad \alpha > 0. \quad (4.5)$$

4.2 Asymptotic smallness of \mathbb{Q}

As a consequence of (4.4) and the energy inequality (3.1), we deduce that

$$\sup_{t \geq 0} [\|\mathbb{Q}(t, \cdot)\|_{L^\infty(\mathbb{R}^3; \mathbb{R}^{3 \times 3})} + \|F(\mathbb{Q}(t, \cdot))\|_{L^1(\mathbb{R}^3)} + \|\nabla_x \mathbb{Q}(t, \cdot)\|_{L^2(\mathbb{R}^3; \mathbb{R}^{3 \times 3 \times 3})}] \leq c. \quad (4.6)$$

In addition, there exists a sequence $t_n \rightarrow \infty$ such that

$$\Delta_x \mathbb{Q}(t_n, \cdot) - \mathcal{L}[\partial F(\mathbb{Q})](t_n, \cdot) = g_n \rightarrow 0 \text{ in } L^2(\mathbb{R}^3; \mathbb{R}^{3 \times 3}) \text{ as } n \rightarrow \infty. \quad (4.7)$$

Our goal is to show that (4.6), (4.7) imply that $\mathbb{Q}(t_n, \cdot)$ tends uniformly to zero, at least for a suitable subsequence of times. To this end, we need the following result that may be of independent interest.

Lemma 4.1. *Let $F \in C^2(B_{r_2})$, $F(0) = 0$. Suppose that \mathbb{Q} is a solution of the stationary problem*

$$-\Delta \mathbb{Q} + \mathcal{L}[\partial F(\mathbb{Q})] = 0 \text{ in } \mathbb{R}^3 \quad (4.8)$$

satisfying

$$|\mathbb{Q}| \leq r_2, \quad |\nabla_x \mathbb{Q}|^2, \quad F(\mathbb{Q}) \in L^1(\mathbb{R}^3). \quad (4.9)$$

Then \mathbb{Q} satisfies Pochozhaev's identity

$$\int_{\mathbb{R}^3} \left(\frac{1}{2} |\nabla_x \mathbb{Q}|^2 + 3F(\mathbb{Q}) \right) dx = 0. \quad (4.10)$$

In particular, $\mathbb{Q} \equiv 0$ provided that $F \geq 0$ in B_{r_2} .

Proof. We use the standard Pochožaev type argument. To begin we claim that any solution of (4.8) is smooth (at least C^2) because of the standard elliptic theory.

We multiply the equation on $\mathbf{x} \cdot \nabla_x \mathbb{Q}$ which is a symmetric traceless tensor. Accordingly

$$\begin{aligned} 0 &= -\Delta \mathbb{Q} : [\mathbf{x} \cdot \nabla_x \mathbb{Q}] + \partial F(\mathbb{Q}) : [\mathbf{x} \cdot \nabla_x \mathbb{Q}] \\ &= -\operatorname{div}_x ((\mathbf{x} \cdot \nabla_x \mathbb{Q}) : \nabla_x \mathbb{Q}) + |\nabla_x \mathbb{Q}|^2 + \frac{1}{2} \mathbf{x} \cdot \nabla_x |\nabla_x \mathbb{Q}|^2 + \nabla_x F(\mathbb{Q}) \cdot \mathbf{x}. \end{aligned}$$

Integrating the expression on the right-hand side over a ball $B_R \subset \mathbb{R}^3$ of the radius R , we obtain

$$\begin{aligned} \int_{\partial B_R} (\mathbf{x} \cdot \nabla_x \mathbb{Q}) : (\nabla_x \mathbb{Q} \cdot \mathbf{n}) \, dS_x - \int_{\partial B_R} \frac{1}{2} |\nabla_x \mathbb{Q}|^2 \mathbf{x} \cdot \mathbf{n} \, dS_x - \int_{\partial B_R} F(\mathbb{Q}) \mathbf{x} \cdot \mathbf{n} \, dS_x \\ + \frac{1}{2} \int_{B_R} |\nabla_x \mathbb{Q}|^2 \, dx + 3 \int_{B_R} F(\mathbb{Q}) \, dx = 0. \end{aligned} \quad (4.11)$$

Since \mathbb{Q} satisfies (4.9) there exists a sequence $R_n \rightarrow \infty$ such that

$$R_n \int_{\partial B_{R_n}} (|\nabla_x \mathbb{Q}|^2 + F(\mathbb{Q})) \, dx \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Thus we may take $R = R_n$ in (4.11) and let $n \rightarrow \infty$ to conclude that

$$\int_{\mathbb{R}^3} \left(\frac{1}{2} |\nabla_x \mathbb{Q}|^2 + 3F(\mathbb{Q}) \right) \, dx = 0.$$

□

Going back to (4.7) we may assume, shifting $\mathbb{Q}(t_n, \cdot)$ in x as the case may be, that

$$|\mathbb{Q}(t_n, 0)| \geq \frac{1}{2} \sup_{x \in \mathbb{R}^3} |\mathbb{Q}(t_n, x)|. \quad (4.12)$$

Now, the relations (4.6), (4.7) imply that, at least for a suitable subsequence

$$\mathbb{Q}(t_n, 0) \rightarrow \tilde{\mathbb{Q}} \text{ in } C_{\text{loc}}(\mathbb{R}^3; \mathbb{R}^{3 \times 3}),$$

where $\tilde{\mathbb{Q}}$ is a solution of the stationary equation (4.8) belonging to the class (4.9), whence, by Lemma 4.1, $\tilde{\mathbb{Q}} = 0$.

Thus, making use of (4.12), we obtain that

$$\|\mathbb{Q}(t_n, \cdot)\|_{L^\infty(\mathbb{R}^3; \mathbb{R}^{3 \times 3})} \rightarrow 0 \text{ as } t_n \rightarrow \infty, \quad (4.13)$$

at least for a suitable subsequence.

Finally, we may use the same arguments as in Section 4.1 to deduce from (4.13) and the hypothesis (1.6) the property

$$|\mathbb{Q}(t, \cdot)| < r_1 \text{ for all } t \text{ large enough.} \quad (4.14)$$

More specifically, one could take in (4.2) the function

$$G(z) = \left[\left(z - \frac{r_1^2}{4} \right)_+ \right]^2.$$

Then, noting that

$$G'(|\mathbb{Q}|^2) \partial F(\mathbb{Q}) : \mathbb{Q} \geq -cG(|\mathbb{Q}|^2) \text{ for } |\mathbb{Q}| \in [r_1, r_2],$$

in view of (4.13) we can choose \tilde{t} such that $\|\mathbb{Q}(\tilde{t}, \cdot)\|_{L^\infty} \leq r_1/2$ and apply Gronwall's lemma starting from the time \tilde{t} . Hence, in view of (1.7), we may assume that F , in addition to (4.5), satisfies

$$\partial F(\mathbb{Q}) : \mathbb{Q} \geq 0. \quad (4.15)$$

Thus, revisiting (4.13), we may infer that

$$\|\mathbb{Q}(t, \cdot)\|_{L^\infty(\mathbb{R}^3, \mathbb{R}^{3 \times 3})} \rightarrow 0 \text{ as } t \rightarrow \infty. \quad (4.16)$$

4.3 L^2 -decay of \mathbb{Q}

Using an approximation by smooth functions, we can take

$$G(z) = \sqrt{z}$$

in (4.2). It is worth noting that the above function G is not convex; nevertheless, by a direct computation one can check that the sum $2G'(|\mathbb{Q}|^2)|\nabla_x \mathbb{Q}|^2 + G''(|\mathbb{Q}|^2)|\nabla_x |\mathbb{Q}|^2|^2$ is nonnegative anyway. Hence, by virtue of (4.14), (4.15), we can conclude that

$$\|\mathbb{Q}(t, \cdot)\|_{L^1(\mathbb{R}^3; \mathbb{R}^{3 \times 3})} \leq c \text{ for all } t > 0. \quad (4.17)$$

To be more precise, we have to notice that (4.14) has been justified so far only for t greater than some (sufficiently large) time \tilde{T} . Hence, relation (4.17) should be proved first on the time interval $[0, \tilde{T}]$ by integrating (4.2) and using the Gronwall lemma (indeed, on $[0, \tilde{T}]$, the right hand side of (4.2) needs not be negative), and subsequently extended for $t \geq \tilde{T}$ by means of (4.14). Finally, taking $G(z) = z$ in (4.2) we obtain

$$\frac{d}{dt} \int_{\mathbb{R}^3} \frac{1}{2} |\mathbb{Q}|^2 dx + \int_{\mathbb{R}^3} |\nabla_x \mathbb{Q}|^2 dx \leq 0 \text{ for all } t > 0 \text{ large enough.} \quad (4.18)$$

Recalling (4.16) and using the standard interpolation inequalities, we get

$$\|\mathbb{Q}\|_{L^2(\mathbb{R}^3; \mathbb{R}^{3 \times 3})} \leq \|\mathbb{Q}\|_{L^1(\mathbb{R}^3; \mathbb{R}^{3 \times 3})}^{2/5} \|\mathbb{Q}\|_{L^6(\mathbb{R}^3; \mathbb{R}^{3 \times 3})}^{3/5} \leq c \|\nabla_x \mathbb{Q}\|_{L^2(\mathbb{R}^3; \mathbb{R}^{27})}^{3/5},$$

whence (4.18) implies

$$\|\mathbb{Q}(t, \cdot)\|_{L^2(\mathbb{R}^3; \mathbb{R}^{3 \times 3})}^2 \leq c(1+t)^{-3/2} \text{ for all } t \geq 0. \quad (4.19)$$

If F satisfies the hypothesis (3.4) the L^2 -decay rate is exponential, specifically

$$\|\mathbb{Q}(t, \cdot)\|_{L^2(\mathbb{R}^3; \mathbb{R}^{3 \times 3})}^2 \leq c \exp(-dt) \text{ for all } t \geq 0 \text{ and some } d > 0. \quad (4.20)$$

Indeed, since F is twice continuously differentiable, (3.4) implies strict positivity of the Hessian of F at $\mathbb{Q} = 0$, in particular, F is strictly convex in a neighborhood of zero. Consequently,

$$\partial F(\mathbb{Q}) : \mathbb{Q} \geq F(\mathbb{Q}) \geq \lambda |\mathbb{Q}|^2$$

and (4.20) follows from (4.2) with $G(z) = z$.

5 Decay for the Navier-Stokes system via the Fourier splitting method

Using elementary inequalities, the (differential version of the) energy inequality (3.1) can be rewritten in the form

$$\frac{d}{dt} \int_{\mathbb{R}^3} \left[\frac{1}{2} |\mathbf{u}|^2 + \frac{1}{2} |\nabla_x \mathbb{Q}|^2 + F(\mathbb{Q}) \right] dx + \int_{\mathbb{R}^3} \left[|\nabla_x \mathbf{u}|^2 + \frac{1}{2} |\Delta_x \mathbb{Q}|^2 - c |\mathcal{L}[\partial F(\mathbb{Q})]|^2 \right] dx \leq 0. \quad (5.1)$$

Moving the last integrand to the right hand side, and noting that, thanks to (4.5),

$$\int_{\mathbb{R}^3} |\mathcal{L}[\partial F(\mathbb{Q})]|^2 dx \leq c \int_{\mathbb{R}^3} |\mathbb{Q}|^2 dx, \quad (5.2)$$

we then obtain

$$\frac{d}{dt} \int_{\mathbb{R}^3} \left[\frac{1}{2} |\mathbf{u}|^2 + \frac{1}{2} |\nabla_x \mathbb{Q}|^2 + F(\mathbb{Q}) \right] (t, \cdot) dx + \int_{\mathbb{R}^3} \left[|\nabla_x \mathbf{u}|^2 + \frac{1}{2} |\Delta_x \mathbb{Q}|^2 \right] dx \leq c(1+t)^{-3/2}, \quad (5.3)$$

thanks also to (4.19).

The extra term on the right-hand side of (5.3) is responsible for the loss of the optimal decay rate in the general case. Actually, if F satisfies (3.4), then the energy inequality reads

$$\frac{d}{dt} \int_{\mathbb{R}^3} \left[\frac{1}{2} |\mathbf{u}|^2 + \frac{1}{2} |\nabla_x \mathbb{Q}|^2 + F(\mathbb{Q}) \right] (t, \cdot) dx + c \int_{\mathbb{R}^3} \left[\frac{1}{2} |\nabla_x \mathbf{u}|^2 + \frac{1}{2} |\nabla_x \mathbb{Q}|^2 + F(\mathbb{Q}) \right] (t, \cdot) dx \leq 0, \quad c > 0. \quad (5.4)$$

Indeed we have

$$\int_{\mathbb{R}^3} |\Delta \mathbb{Q} - \mathcal{L} [\partial F(\mathbb{Q})]|^2 dx \geq c \int_{\mathbb{R}^3} \left[\frac{1}{2} |\nabla_x \mathbb{Q}|^2 + F(\mathbb{Q}) \right] dx, \quad c > 0,$$

for any \mathbb{Q} in an open neighborhood of zero. To see this, denote

$$-\Delta \mathbb{Q} + \mathcal{L} [\partial F(\mathbb{Q})] = \mathbb{G}$$

and observe that

$$\int_{\mathbb{R}^3} \left[\frac{1}{2} |\nabla_x \mathbb{Q}|^2 + F(\mathbb{Q}) \right] dx \leq c \int_{\mathbb{R}^3} [|\nabla_x \mathbb{Q}|^2 + \partial F(\mathbb{Q}) : \mathbb{Q}] dx = c \int_{\mathbb{R}^3} \mathbb{G} : \mathbb{Q} dx;$$

where the term on the right-hand side may be “absorbed” by means of the Cauchy-Schwartz inequality provided that F satisfies (3.4).

5.1 Fourier analysis for the Navier-Stokes system

Let

$$\widehat{v}(t, \xi) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \exp(-i\xi \cdot x) v(t, x) dx$$

denote the Fourier transform of a function v with respect to the spatial variable x . Accordingly, the velocity field \mathbf{u} , solving (2.8), can be written as

$$\widehat{u}_i(t, \xi) = \exp(-|\xi|^2 t) \widehat{u}_{0,i}(\xi) \quad (5.5)$$

$$+ \int_0^t \exp(-|\xi|^2(t-s)) \sum_{j,k} \left[\left(\delta_{i,j} - \frac{\xi_i \xi_j}{|\xi|^2} \right) \xi_k \left(-\widehat{\mathbb{Q} \Delta_x \mathbb{Q}} + \widehat{\Delta_x \mathbb{Q} \mathbb{Q}} - \widehat{\nabla_x \mathbb{Q} \odot \nabla_x \mathbb{Q}} - \widehat{\mathbf{u} \otimes \mathbf{u}} \right)_{j,k} (s, \xi) \right] ds.$$

Now, we observe that

$$\|(\nabla_x \mathbb{Q} \odot \nabla_x \mathbb{Q})(t, \cdot)\|_{L^1(\mathbb{R}^3; \mathbb{R}^3)} + \|(\mathbf{u} \otimes \mathbf{u})(t, \cdot)\|_{L^1(\mathbb{R}^3; \mathbb{R}^3)} \leq c\mathcal{E}(t) \text{ for all } t \geq 0,$$

where

$$\mathcal{E}(t) = \int_{\mathbb{R}^3} \left(\frac{1}{2} |\mathbf{u}|^2 + \frac{1}{2} |\nabla_x \mathbb{Q}|^2 + F(\mathbb{Q}) \right) dx.$$

Consequently,

$$\left| \widehat{\nabla_x \mathbb{Q} \odot \nabla_x \mathbb{Q}}(t, \xi) \right| + \left| \widehat{\mathbf{u} \otimes \mathbf{u}}(t, \xi) \right| \leq c \mathcal{E}(t) \text{ for all } t, \xi. \quad (5.6)$$

Next, writing

$$(\mathbb{Q} \Delta_x \mathbb{Q})_{ij} = \mathbb{Q}_{ik} (\partial_l \partial_l \mathbb{Q}_{kj}) = \partial_l (\mathbb{Q}_{ik} \partial_l \mathbb{Q}_{kj}) - \partial_l \mathbb{Q}_{ik} \partial_l \mathbb{Q}_{kj}$$

and, analogously,

$$(\Delta_x \mathbb{Q} \mathbb{Q})_{ij} = (\partial_l \partial_l \mathbb{Q}_{ik}) \mathbb{Q}_{kj} = \partial_l (\partial_l \mathbb{Q}_{ik} \mathbb{Q}_{kj}) - \partial_l \mathbb{Q}_{ik} \partial_l \mathbb{Q}_{kj},$$

we may therefore infer that

$$\begin{aligned} & \left| -\widehat{\mathbb{Q} \Delta_x \mathbb{Q}}(t, \xi) + \widehat{\Delta_x \mathbb{Q} \mathbb{Q}}(t, \xi) \right| = \left| \operatorname{div}_x \widehat{(\nabla_x \mathbb{Q} \mathbb{Q})} - \operatorname{div}_x \widehat{(\mathbb{Q} \nabla_x \mathbb{Q})} \right| \\ & \leq c(1 + |\xi|) \left(\mathcal{E}(t) + \int_{\mathbb{R}^3} |\mathbb{Q}|^2(t, \cdot) dx \right) \leq c(1 + |\xi|) (\mathcal{E}(t) + (1+t)^{-3/2}) \text{ for all } t, \xi. \end{aligned} \quad (5.7)$$

Thus, combining (5.6), (5.7) with (5.5) we conclude

$$\begin{aligned} & |\widehat{u}_i|(t, \xi) \leq \exp(-|\xi|^2 t) |\widehat{u}_{0,i}|(\xi) \\ & + c \int_0^t \exp(-|\xi|^2(t-s)) |\xi| (1 + |\xi|) (\mathcal{E}(s) + (1+s)^{-3/2}) ds, \quad i = 1, 2, 3. \end{aligned} \quad (5.8)$$

Here again, we remark that (5.8) does not contain the extra term $(1+s)^{-3/2}$ if F satisfies (3.4).

5.2 First decay estimate

Having collected all the necessary ingredients, we are ready to finish the proof of Theorem 3.1. We focus on the case of a general nonlinearity F and then shortly comment on how to modify the arguments when F satisfies (3.4). To begin, note that

$$\begin{aligned} & \int_{\mathbb{R}^3} \left[\frac{1}{2} |\nabla_x \mathbb{Q}|^2 + F(\mathbb{Q}) \right] dx \leq \int_{\mathbb{R}^3} \frac{1}{2} |\Delta_x \mathbb{Q}|^2 + c_1 \left[F(\mathbb{Q}) + \frac{1}{2} |\mathbb{Q}|^2 \right] dx \\ & \leq \int_{\mathbb{R}^3} \left[\frac{1}{2} |\Delta_x \mathbb{Q}|^2 + c_2 |\mathbb{Q}|^2 \right] dx \leq \int_{\mathbb{R}^3} \frac{1}{2} |\Delta_x \mathbb{Q}|^2 dx + c_3 (1+t)^{-3/2}. \end{aligned} \quad (5.9)$$

Adding (5.9) to (5.3) and applying Plancherel's Theorem, we obtain

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^3} \left[\frac{1}{2} |\mathbf{u}|^2 + \frac{1}{2} |\nabla_x \mathbb{Q}|^2 + F(\mathbb{Q}) \right] (t, \cdot) dx + \int_{\mathbb{R}^3} |\xi |\widehat{\mathbf{u}}|^2 d\xi \\ + \int_{\mathbb{R}^3} \left[\frac{1}{2} |\nabla_x \mathbb{Q}|^2 + F(\mathbb{Q}) \right] dx \leq c(1+t)^{-3/2}. \end{aligned} \quad (5.10)$$

Next, we have

$$\int_{\mathbb{R}^3} |\xi |\widehat{\mathbf{u}}|^2 d\xi = \int_{|\xi| < R(t)} |\xi |\widehat{\mathbf{u}}|^2 d\xi + \int_{|\xi| \geq R(t)} |\xi |\widehat{\mathbf{u}}|^2 d\xi \geq R^2(t) \int_{|\xi| \geq R(t)} |\widehat{\mathbf{u}}|^2 d\xi. \quad (5.11)$$

Replacing (5.11) into (5.10), we infer

$$\frac{d}{dt} \mathcal{E}(t) + R^2(t) \mathcal{E}(t) \leq R^2(t) \int_{|\xi| < R(t)} |\widehat{\mathbf{u}}|^2 d\xi + c(1+t)^{-3/2} \text{ for any } 0 \leq R(t) \leq 1. \quad (5.12)$$

In order to evaluate the integral on the right hand side, we notice that, in agreement with (5.8),

$$\begin{aligned} \int_{|\xi| < R(t)} |\widehat{\mathbf{u}}|^2 d\xi \\ \leq c \|\mathbf{u}_0\|_{L^1(\mathbb{R}^3; \mathbb{R}^3)}^2 \int_{|\xi| < R(t)} \exp(-2|\xi|^2 t) d\xi \\ + c \int_{|\xi| < R(t)} \left(\int_0^t \exp(-|\xi|^2(t-s)) |\xi|(1+|\xi|) (\mathcal{E}(s) + (1+s)^{-3/2}) ds \right)^2 d\xi. \end{aligned} \quad (5.13)$$

Now, let us evaluate the first integral on the right hand side: passing to polar coordinates and then substituting $r := \rho^3(t+1)^{3/2}$ we get

$$\begin{aligned} \int_{|\xi| < R(t)} \exp(-2|\xi|^2 t) d\xi &\leq c \int_0^{R(t)} \exp(-2\rho^2 t) \rho^2 d\rho \leq c \int_0^{R(t)} \exp(-2\rho^2(t+1)) \rho^2 d\rho \\ &= c(t+1)^{-3/2} \int_0^{R^3(t)(t+1)^{3/2}} e^{-2r^{2/3}} dr \leq c(t+1)^{-3/2} \int_0^{+\infty} e^{-2r^{2/3}} dr \leq c(t+1)^{-3/2}, \end{aligned} \quad (5.14)$$

where we also assumed (and used) the fact that $R(t)$ will be chosen to be smaller than 1.

Collecting (5.12 - 5.14), we then conclude that

$$\frac{d}{dt} \mathcal{E}(t) + R^2(t) \mathcal{E}(t) \quad (5.15)$$

$$\begin{aligned}
&\leq c \left[(t+1)^{-3/2} + R^2(t) \int_{|\xi| < R(t)} \left(\int_0^t \exp(-|\xi|^2(t-s)) |\xi| (\mathcal{E}(s) + (1+s)^{-3/2}) ds \right)^2 d\xi \right] \\
&= c \left[(t+1)^{-3/2} + R^2(t) \int_0^{R(t)} \left(\int_0^t \exp(-r^2(t-s)) r^2 (\mathcal{E}(s) + (1+s)^{-3/2}) ds \right)^2 dr \right]
\end{aligned}$$

for any $0 \leq R(t) \leq 1$.

5.3 A bootstrap argument

The inequality (5.15) is a starting point of a bootstrap procedure to deduce the desired decay estimate (3.3). We start with an auxiliary assertion.

Lemma 5.1. *Let $\gamma \in (0, 1)$, $\mu > 0$ and $\gamma < \mu$. If*

$$\frac{d}{dt} \mathcal{E}(t) + (1+t)^{-\gamma} \mathcal{E}(t) \leq c(1+t)^{-\mu}, \quad (5.16)$$

then

$$\mathcal{E}(t) \leq c(\gamma, \mathcal{E}(0))(1+t)^{-\mu+\gamma}. \quad (5.17)$$

Proof. Let $g(t) = \exp[(1-\gamma)^{-1}(1+t)^{1-\gamma}]$. Then (5.16) yields

$$\frac{d}{dt} (g(t) \mathcal{E}(t)) \leq c g(t) (1+t)^{-\mu}.$$

Hence,

$$\mathcal{E}(t) \leq g(t)^{-1} \mathcal{E}(0) + c g(t)^{-1} \int_0^t (1+s)^{-\mu} g(s) ds =: I_1(t) + I_2(t). \quad (5.18)$$

Clearly it is sufficient to handle I_2 . Noting that $g(t) = (1+t)^\gamma g'(t)$ and then integrating by parts, we obtain

$$I_2(t) = c g(t)^{-1} \int_0^t (1+s)^{\gamma-\mu} g'(s) ds \leq c(1+t)^{\gamma-\mu} + c(\mu-\gamma) g(t)^{-1} \int_0^t g(s) (1+s)^{\gamma-\mu-1} ds.$$

To control the last term we simply split it as

$$\begin{aligned}
c(\mu-\gamma) g(t)^{-1} \int_0^t g(s) (1+s)^{\gamma-\mu-1} ds &= c(\mu-\gamma) g(t)^{-1} \int_0^{t/2} g(s) (1+s)^{\gamma-\mu-1} ds \\
&\quad + c(\mu-\gamma) g(t)^{-1} \int_{t/2}^t g(s) (1+s)^{\gamma-\mu-1} ds =: I_3 + I_4
\end{aligned}$$

where

$$I_3 \leq c(\mu - \gamma) \frac{g(t/2)}{g(t)} \int_0^{t/2} (1+s)^{\gamma-\mu-1} ds \leq c \frac{g(t/2)}{g(t)}$$

which decays exponentially fast, and

$$I_4 \leq c(\mu - \gamma) \left(1 + \frac{t}{2}\right)^{\gamma-\mu-1} \int_{t/2}^t \frac{g(s)}{g(t)} ds \leq c(\mu - \gamma)(1+t)^{\gamma-\mu}.$$

The lemma is proved. \square

Now, we are ready to start bootstrapping (5.15). Suppose we have already shown

$$\mathcal{E}(t) \leq c(1+t)^{-\alpha}, \quad 0 \leq \alpha. \quad (5.19)$$

Accordingly, the inequality (5.15) gives rise to

$$\begin{aligned} & \frac{d}{dt} \mathcal{E}(t) + R^2(t) \mathcal{E}(t) \\ & \leq c \left[(t+1)^{-3/2} + R^2(t) \int_0^{R(t)} \left(\int_0^t \exp(-r^2(t-s)) r^2 \frac{1}{(1+s)^\alpha} ds \right)^2 dr \right], \end{aligned} \quad (5.20)$$

where

$$\begin{aligned} \int_0^t \exp(-r^2(t-s)) r^2 \frac{1}{(1+s)^\alpha} ds &= \exp(-r^2 t) r^{2\alpha} \int_0^t \exp(r^2 s) r^2 \frac{1}{(r^2 + r^2 s)^\alpha} ds \\ &= r^{2\alpha} \exp(-r^2 t) \int_0^{r^2 t} \frac{\exp z}{(r^2 + z)^\alpha} dz. \end{aligned}$$

Let us now observe that

$$\exp(-r^2 t) \int_0^{r^2 t} \frac{\exp z}{(r^2 + z)^\alpha} dz \leq \begin{cases} 1, & \text{if } \alpha = 0, \\ \frac{1}{1-\alpha} [(r^2(1+t))^{1-\alpha} - r^{2(1-\alpha)}] \leq \frac{r^{2(1-\alpha)}}{1-\alpha}, & \text{if } 0 < \alpha < 1, \\ \frac{1}{1-\alpha} [(r^2(1+t))^{1-\alpha} - r^{2(1-\alpha)}] \leq \frac{r^{2(1-\alpha)}}{\alpha-1}, & \text{if } 1 < \alpha. \end{cases} \quad (5.21)$$

Hence

$$r^{2\alpha} \exp(-r^2 t) \int_0^{r^2 t} \frac{\exp z}{(r^2 + z)^\alpha} dz \leq \begin{cases} 1, & \text{if } \alpha = 0, \\ c(\alpha) r^2, & \text{if } \alpha > 0, \text{ and } \alpha \neq 1. \end{cases} \quad (5.22)$$

Thus, (5.20) gives

$$\frac{d}{dt}\mathcal{E}(t) + R^2(t)\mathcal{E}(t) \leq \begin{cases} c(\alpha) [(t+1)^{-3/2} + R^3(t)], & \text{if } \alpha = 0, \\ c(\alpha) [(t+1)^{-3/2} + R^7(t)], & \text{if } \alpha > 0, \text{ and } \alpha \neq 1. \end{cases} \quad (5.23)$$

Now, from the uniform boundedness of the energy, we know that (5.19) holds for $\alpha = 0$. Hence, taking $R(t) = (1+t)^{-\beta}$ with

$$\beta = \frac{1}{2} - \frac{\epsilon}{3},$$

where $\epsilon > 0$ is a small number, we obtain $2\beta = 1 - \frac{2\epsilon}{3} < 1$ and $R^3(t) = (1+t)^{-\frac{3}{2}+\epsilon}$ yielding

$$\frac{d}{dt}\mathcal{E}(t) + (1+t)^{-2\beta}\mathcal{E}(t) \leq c(\alpha)(t+1)^{-3/2+\epsilon}.$$

Since $2\beta < 1$ (this is why we subtracted $\epsilon > 0$), Lemma 5.1 can be applied with $\gamma = 2\beta$ and $\mu = 3/2 - \epsilon$, whence

$$\mathcal{E}(t) \leq c(t+1)^{-\mu+2\beta} = c(t+1)^{-\frac{1}{2}+\frac{\epsilon}{3}}.$$

Now we can take $2\beta = \frac{3}{7}$. Repeating the above argument with $\alpha = \frac{1}{2} - \frac{\epsilon}{3}$, and referring to the second row of formula (5.23) yields

$$\mathcal{E}(t) \leq c(t+1)^{-\frac{3}{2}+\frac{3}{7}} = c(t+1)^{-\frac{15}{14}}. \quad (5.24)$$

Remark 5.1. *The above exponent $-15/14$ is not likely to have an optimal character. Indeed, refining the estimation of the right hand side of (5.20), one may get a sharper estimate, at the price of some additional technicalities. Here we sketch an idea of a possible improvement.*

Our aim is to compute

$$\int_0^{R(t)} \left(\int_0^t \exp(-r^2(t-s)) r^2 \frac{1}{(1+s)^\alpha} ds \right)^2 dr. \quad (5.25)$$

Let us assume $t \geq 1$. Then, for $\gamma \in (0, 1)$ (to be chosen later on), we have

$$r^2 e^{-r^2 t} \int_0^t e^{r^2 s} \frac{1}{(1+s)^\alpha} ds = r^2 e^{-r^2 t} \int_0^{t^\gamma} e^{r^2 s} \frac{1}{(1+s)^\alpha} ds + r^2 e^{-r^2 t} \int_{t^\gamma}^t e^{r^2 s} \frac{1}{(1+s)^\alpha} ds =: J_1 + J_2.$$

Now,

$$J_2 \leq cr^2 e^{-r^2 t} \frac{1}{(1+t^\gamma)^\alpha} \int_{t^\gamma}^t e^{r^2 s} ds \leq cr^2 e^{-r^2 t} \left(\frac{e^{r^2 t} - e^{-r^2 t^\gamma}}{r^2} \right) \frac{1}{(1+t)^\alpha} \leq \frac{c}{(1+t)^{\alpha\gamma}}.$$

On the other hand,

$$J_1 \leq r^2 e^{-r^2 t} \int_0^{t^\gamma} e^{r^2 s} ds = r^2 e^{-r^2 t} r^{-2} (e^{r^2 t^\gamma} - 1) \leq e^{-r^2 t + r^2 t^\gamma} r^2 t^\gamma,$$

where we used the fact that $1 - e^{-x} \leq x$ for $x \geq 0$.

Hence, going back to (5.25),

$$R^2(t) \int_0^{R(t)} J_2^2(r) \, dr \leq cR^3(t)(1+t)^{-2\alpha\gamma}, \quad (5.26)$$

whereas

$$R^2(t) \int_0^{R(t)} J_1^2(r) \, dr \leq cR^2(t) \int_0^{R(t)} e^{-2r^2t+2r^2t^\gamma} r^4 t^{2\gamma} \, dr \leq cR^2(t) \int_0^{R(t)} e^{-r^2t} r^4 t^{2\gamma} \, dr, \quad (5.27)$$

where the last inequality holds provided we restrict ourselves to $t \geq 2^{\frac{1}{1-\gamma}}$. To estimate the last integrand, we apply the elementary inequality $s^2 e^{-s} \leq 1$ for all $s \geq 0$. Hence, we immediately get

$$R^2(t) \int_0^{R(t)} J_1^2(r) \, dr \leq cR^2(t) \int_0^{R(t)} t^{2\gamma-2} \, dr \leq cR^3(t)t^{2\gamma-2}, \quad (5.28)$$

holding at least for

$$R(t) = (1+t)^{-\beta}, \quad 0 \leq \beta < 1/2, \quad 2^{\frac{1}{1-\gamma}} \leq t < \infty. \quad (5.29)$$

Hence, collecting (5.26) and (5.29), (5.20) yields the inequality

$$\frac{d}{dt} \mathcal{E}(t) + R^2(t) \mathcal{E}(t) \leq c \left[(t+1)^{-3/2} + R^3(t)(1+t)^{-2\alpha\gamma} + R^3(t)(1+t)^{2\gamma-2} \right]. \quad (5.30)$$

In principle, this holds only as far as t satisfies the latter (5.29). On the other hand, it is readily seen that (5.30) holds trivially when t is small. Hence, playing on the choices of β and γ , and performing a bootstrap argument, one may provide an improvement of the exponent $-15/14$ in (5.24). Due to the technical character of such an argument (in particular, it is necessary to take care of the various constants and to keep $\gamma \leq 1 - \delta$ for some $\delta > 0$ independent of the bootstrap step), we prefer to omit the details.

6 Examples

We consider the class of polynomial potentials investigated by Paicu and Zarnescu [7], specifically,

$$F(\mathbb{Q}) = \frac{a}{2} |\mathbb{Q}|^2 + \frac{b}{3} \text{trace}[\mathbb{Q}^3] + \frac{c}{4} |\mathbb{Q}|^4, \quad (6.1)$$

observing that

$$\mathcal{L}[\partial F(\mathbb{Q})] = a\mathbb{Q} + b \left(\mathbb{Q}^2 - \frac{1}{3} \text{trace}[\mathbb{Q}^2] \mathbb{I} \right) + c|\mathbb{Q}|^2 \mathbb{Q}.$$

Here and hereafter, a, b, c are real parameters.

6.1 The case $a > 0$

If $a > 0$, the isotropic state is at least locally stable. It is easy to check that there exists an open neighborhood \mathcal{O} of 0 such that F satisfies the hypotheses of Theorem 3.1 including (3.4). Accordingly, we get

$$\|\mathbf{u}(t, \cdot)\|_{L^2(\mathbb{R}^3; \mathbb{R}^3)} + \|\mathbb{Q}(t, \cdot)\|_{W^{1,2}(\mathbb{R}^3; \mathbb{R}^{3 \times 3})} \leq c(1+t)^{-3/4} \quad (6.2)$$

provided that there exists t_0 such that $\mathbb{Q}(t_0, \cdot) \in \mathcal{O}$ a.e. in \mathbb{R}^3 . The decay rate is global (unconditional) if $c > 0$ and $|b| \leq \bar{b}(a, c)$.

6.2 The case $a \leq 0, c > 0$

We consider solutions belonging to the class

$$\mathbb{Q}(t, \cdot) \in D^{1,2}(\mathbb{R}^3; \mathbb{R}^{3 \times 3}), \quad F(\mathbb{Q}) \in L^1(\mathbb{R}^3), \quad (6.3)$$

where $D^{1,2}$ denotes the completion of C_c^∞ under the $\|\nabla_x \mathbb{Q}\|_{L^2}$ norm.

To begin, we observe that $\tilde{\mathbb{Q}} \equiv 0$ is the only stationary solution belonging to the class (6.3). Indeed Pochožaev's identity (4.10) reads

$$\int_{\mathbb{R}^3} \left(\frac{1}{2} |\nabla_x \tilde{\mathbb{Q}}|^2 + \frac{3a}{2} |\tilde{\mathbb{Q}}|^2 + b \operatorname{trace}[\tilde{\mathbb{Q}}^3] + \frac{3c}{4} |\tilde{\mathbb{Q}}|^4 \right) dx = 0,$$

while, as $\tilde{\mathbb{Q}}$ solves the stationary problem,

$$\int_{\mathbb{R}^3} \left(|\nabla_x \tilde{\mathbb{Q}}|^2 + a |\tilde{\mathbb{Q}}|^2 + b \operatorname{trace}[\tilde{\mathbb{Q}}^3] + c |\tilde{\mathbb{Q}}|^4 \right) dx = 0.$$

Consequently,

$$\int_{\mathbb{R}^3} \left(-\frac{1}{2} |\nabla_x \tilde{\mathbb{Q}}|^2 + \frac{a}{2} |\tilde{\mathbb{Q}}|^2 - \frac{c}{4} |\tilde{\mathbb{Q}}|^4 \right) dx = 0,$$

yielding the desired conclusion $\tilde{\mathbb{Q}} = 0$.

We claim the following:

Suppose that

$$\|\mathbb{Q}(t, \cdot)\|_{L^2(\mathbb{R}^3; \mathbb{R}^{3 \times 3})} \leq M \text{ for a.a. } t > 0. \quad (6.4)$$

Then

$$\mathbb{Q}(t, \cdot) \rightarrow 0 \text{ in } L^\infty(\mathbb{R}^3; \mathbb{R}^{3 \times 3}) \text{ as } t \rightarrow \infty. \quad (6.5)$$

In order to see (6.5) we first observe that (6.4) implies that the energy

$$\mathcal{E}(t) = \int_{\mathbb{R}^3} \left[\frac{1}{2} |\mathbf{u}|^2 + \frac{1}{2} |\nabla_x \mathbb{Q}|^2 + F(\mathbb{Q}) \right] (t, \cdot) \, dx$$

remains bounded, more specifically, \mathcal{E} tends to a finite limit as $t \rightarrow \infty$.

This in turn implies the existence of a sequence $t_n \in [n, n+1]$ such that

$$-\Delta \mathbb{Q}(t_n, \cdot) + \mathcal{L}[\partial F(\mathbb{Q}(t_n, \cdot))] \rightarrow 0 \text{ in } L^2(\mathbb{R}^3; \mathbb{R}^{3 \times 3}) \text{ as } t_n \rightarrow \infty.$$

However, (6.4), together with the standard elliptic regularity estimates, implies that $\mathbb{Q}(t_n, \cdot)$ converges uniformly to a stationary solution, meaning to zero. We may therefore infer that

$$\|\mathbb{Q}(t_n, \cdot)\|_{L^\infty(\mathbb{R}^3; \mathbb{R}^{3 \times 3})} = q_n \rightarrow 0 \quad n \rightarrow \infty.$$

Going back to (4.2) and taking $G(z) = z^{p/2}$ we deduce

$$\|\mathbb{Q}(t, \cdot)\|_{L^p(\mathbb{R}^3; \mathbb{R}^{3 \times 3})} \leq \exp K(t-s) \|\mathbb{Q}(s, \cdot)\|_{L^p(\mathbb{R}^3; \mathbb{R}^{3 \times 3})}, \quad p > 2, \quad t \geq s,$$

where K does not depend on p . In particular, by interpolation, we get

$$\begin{aligned} \|\mathbb{Q}(t, \cdot)\|_{L^\infty(\mathbb{R}^3; \mathbb{R}^{3 \times 3})} &= \sup_{x \in \mathbb{R}^3} \|\mathbb{Q}(t, \cdot)\|_{L^\infty(B(x,1); \mathbb{R}^{3 \times 3})} = \sup_{x \in \mathbb{R}^3} \limsup_{p \rightarrow \infty} \|\mathbb{Q}(t, \cdot)\|_{L^p(B(x,1); \mathbb{R}^{3 \times 3})} \\ &\leq \limsup_{p \rightarrow \infty} \|\mathbb{Q}(t, \cdot)\|_{L^p(\mathbb{R}^3; \mathbb{R}^{3 \times 3})} \\ &\leq \exp K(t-t_n) \limsup_{p \rightarrow \infty} \|\mathbb{Q}(t_n, \cdot)\|_{L^p(\mathbb{R}^3; \mathbb{R}^{3 \times 3})} \\ &\leq \exp K(t-t_n) \limsup_{p \rightarrow \infty} \|\mathbb{Q}(t_n, \cdot)\|_{L^2(\mathbb{R}^3; \mathbb{R}^{3 \times 3})}^{2/p} \|\mathbb{Q}(t_n, \cdot)\|_{L^\infty(\mathbb{R}^3; \mathbb{R}^{3 \times 3})}^{1-2/p} \\ &\leq \exp K(t-t_n) \limsup_{p \rightarrow \infty} M^{2/p} q_n^{1-2/p}, \end{aligned}$$

which yields the claim.

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