On bounded solutions to the compressible isentropic Euler system

Ondřej Kreml

Joint work with Elisabetta Chiodaroli

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Setting of the problem

We study the compressible isentropic Euler system in the whole 2D space

$$\left\{\begin{array}{rcl} \partial_t \rho + \mathrm{div}_x(\rho v) & = & 0 \\ \partial_t(\rho v) + \mathrm{div}_x\left(\rho v \otimes v\right) + \nabla_x[p(\rho)] & = & 0 \\ \rho(\cdot, 0) & = & \rho^0 \\ v(\cdot, 0) & = & v^0 \, . \end{array}\right. \tag{1}$$

Unknowns:

- $\rho(x, t)$ ... density
- $v(x, t)$ ... velocity

The pressure $p(\rho)$ is given.

Introduction

- It is a hyperbolic system of conservation laws
- The theory of hyperbolic conservation laws is far frome being completely understood
- Solutions develop singularities in finite time even for smooth initial data
- Admissibility comes into play due to the entropy inequality (”selector” of physical solutions in case of existence of many solutions)
- There are satisfactory results in the case of scalar conservation laws (in 1D as well as in multi-D), there is a lot of entropies: $\Rightarrow$ Kruzkov, 1970: Well–posedness theory in BV.
- There are also satisfactory results in the case of systems of conservation laws in 1D: Lax, Glimm, Bianchini, Bressan, ...

Introduction II

Back to our case, the Euler system:

- In more than 1D there is only one (entropy, entropy flux) pair, which is

$$\left(\rho\varepsilon(\rho) + \frac{\rho|v|^2}{2}, \left(\rho\varepsilon(\rho) + \frac{\rho|v|^2}{2} + p(\rho)\right) v\right)$$

  with the internal energy $\varepsilon(\rho)$ given through

$$p(\rho) = \rho^2\varepsilon'(\rho).$$

- Local existence of strong (and therefore admissible) solutions is proved
- On the other hand global existence of weak solutions in general (it is a system in multi D!) is still an open problem, there are only partial results
- The weak–strong uniqueness property holds for this system

Weak solution

Definition 1

By a *weak solution* of Euler system on $\mathbb{R}^2 \times [0,\infty)$ we mean a pair $(\rho, v) \in L^\infty(\mathbb{R}^2 \times [0,\infty))$ such that the following identities hold for every test functions $\psi \in C_c^\infty(\mathbb{R}^2 \times [0,\infty))$, $\phi \in C_c^\infty(\mathbb{R}^2 \times [0,\infty))$:

$$\int_0^\infty \int_{\mathbb{R}^2} [\rho\partial_t\psi + \rho v \cdot \nabla_x\psi]\, dxdt + \int_{\mathbb{R}^2} \rho^0(x)\psi(x,0)dx \; = \; 0$$

$$\int_0^\infty \int_{\mathbb{R}^2} [\rho v \cdot \partial_t\phi + \rho v \otimes v : \nabla_x\phi + p(\rho)\operatorname{div}_x \phi]\, dxdt + \int_{\mathbb{R}^2} \rho^0(x)v^0(x) \cdot \phi(x,0)dx \; = \; 0.$$

Admissible weak solution

Definition 2

A bounded weak solution (ρ, v) of Euler system is *admissible* if it satisfies the following inequality for every nonnegative test function $\varphi \in C_c^\infty(\mathbb{R}^2 \times [0,\infty))$:

$$\int_0^\infty \int_{\mathbb{R}^2} \left[\left(\rho\varepsilon(\rho) + \rho \frac{|v|^2}{2} \right) \partial_t \varphi \right.$$
$$\left. + \left(\rho\varepsilon(\rho) + \rho \frac{|v|^2}{2} + p(\rho) \right) v \cdot \nabla_x \varphi \right] dxdt$$
$$+ \int_{\mathbb{R}^2} \left(\rho^0(x)\varepsilon(\rho^0(x)) + \rho^0(x) \frac{|v^0(x)|^2}{2} \right) \varphi(x,0)\, dx \; \geq \; 0\,.$$

Previous result

Motivated by the result of László Székelyhidi about vortex sheet initial data for incompressible Euler equations we studied similar problem for the compressible isentropic Euler system and proved the following

Theorem 3 (Chiodaroli, De Lellis, K.)

Let $p(\rho) = \rho^2$. There exist Lipschitz initial data (ρ^0, v^0) for which there are infinitely many bounded admissible weak solutions (ρ, v) of Euler system on $\mathbb{R}^2 \times [0, \infty)$ with $\inf \rho > 0$. These solutions are all locally Lipschitz on a finite interval of time where they therefore all coincide with the unique classical solution.

The proof is based on analysis of the Riemann problem and a suitable application of the theory of De Lellis and Székelyhidi for incompressible Euler equations.

Riemann problem

Denote $x = (x_1, x_2) \in \mathbb{R}^2$ and consider the special initial data

$$(\rho^0(x), v^0(x)) := \begin{cases} (\rho_-, v_-) & \text{if } x_2 < 0 \\ (\rho_+, v_+) & \text{if } x_2 > 0, \end{cases} \quad (2)$$

where $\rho_\pm, v_\pm$ are constants.
In particular the initial data are "1D" and there is a classical theory about self-similar solutions to the Riemann problem in 1D (they are unique in the class of BV functions).
In the case of system (1), the initial singularity can resolve to at most 3 structures (rarefaction wave, admissible shock or contact discontinuity) connected by constant states.

First observation

If $v_{-1} = v_{+1}$, then any self-similar solution to (1), (2) has to satisfy $v_1(t,x) = v_{-1} = v_{+1}$ and in particular there is no contact discontinuity in the self-similar solution.
The initial singularity then resolves into at most 2 structures (rarefaction waves or admissible shocks) connected by constant states.

Classification of self-similar solutions I

1) If

$$v_{+2} - v_{-2} \geq \int_0^{\rho_-} \frac{\sqrt{p'(\tau)}}{\tau} \mathrm{d}\tau + \int_0^{\rho_+} \frac{\sqrt{p'(\tau)}}{\tau} \mathrm{d}\tau,$$

then the self-similar solution consists of a $1-$rarefaction wave and a $3-$rarefaction wave. The intermediate state is vacuum, i.e. $\rho_m = 0$.

2) If

$$\left| \int_{\rho_-}^{\rho_+} \frac{\sqrt{p'(\tau)}}{\tau} \mathrm{d}\tau \right| < v_{+2} - v_{-2} < \int_0^{\rho_-} \frac{\sqrt{p'(\tau)}}{\tau} \mathrm{d}\tau + \int_0^{\rho_+} \frac{\sqrt{p'(\tau)}}{\tau} \mathrm{d}\tau,$$

then the self-similar solution consists of a $1-$rarefaction wave and a $3-$rarefaction wave. The intermediate state has $\rho_m > 0$.

3) If $\rho_- > \rho_+$ and

$$-\sqrt{\frac{(\rho_- - \rho_+)(p(\rho_-) - p(\rho_+))}{\rho_-\rho_+}} < v_{+2} - v_{-2} < \int_{\rho_+}^{\rho_-} \frac{\sqrt{p'(\tau)}}{\tau} \mathrm{d}\tau,$$

then the self-similar solution consists of a 1−rarefaction wave and an admissible 3−shock.

4) If $\rho_- < \rho_+$ and

$$-\sqrt{\frac{(\rho_+ - \rho_-)(p(\rho_+) - p(\rho_-))}{\rho_+\rho_-}} < v_{+2} - v_{-2} < \int_{\rho_-}^{\rho_+} \frac{\sqrt{p'(\tau)}}{\tau} \mathrm{d}\tau,$$

then the self-similar solution consists of an admissible 1−shock and a 3−rarefaction wave.

5) If

$$v_{+2} - v_{-2} < -\sqrt{\frac{(\rho_+ - \rho_-)(p(\rho_+) - p(\rho_-))}{\rho_+\rho_-}} \qquad (3)$$

then the self-similar solution consists of an admissible 1−shock and an admissible 3−shock.

Main Theorem

Theorem 4 (Chiodaroli, K.)

Let $p(\rho) = \rho^\gamma$ *with* $\gamma \geq 1$. *For every Riemann data* (2) *such that the self-similar solution to the Riemann problem* (1), (2) *consists of an admissible* $1-$*shock and an admissible* $3-$*shock, i.e.* $v_{-1} = v_{+1}$ *and*

$$v_{+2} - v_{-2} < -\sqrt{\frac{(\rho_- - \rho_+)(p(\rho_-) - p(\rho_+))}{\rho_-\rho_+}}, \tag{4}$$

there exist infinitely many admissible solutions to (1), (2).

Some remarks

- Compared to Theorem 3, the new Theorem 4 widely extends the set of initial data for which there exist infinitely many admissible solutions to the Riemann problem.
- Moreover Theorem 4 gives this result for any pressure law $p(\rho) = \rho^\gamma$, instead of the specific case $\gamma = 2$ in Theorem 3

Entropy rate admissibility I

Define the *total energy* of the solutions (ρ, v) to (1) as

$$E[\rho, v](t) = \int_{\mathbb{R}^2} \left(\rho\varepsilon(\rho) + \rho \frac{|v|^2}{2} \right) \mathrm{d}x \tag{5}$$

and the *energy dissipation rate* of (ρ, v) at time t:

$$D[\rho, v](t) = \frac{\mathrm{d}_+ E[\rho, v](t)}{\mathrm{d}t}. \tag{6}$$

In our case the energy is always infinite, so we restrict the integrals to a finite box:

$$E_L[\rho, v](t) = \int_{(-L,L)^2} \left(\rho\varepsilon(\rho) + \rho \frac{|v|^2}{2} \right) \mathrm{d}x \tag{7}$$

$$D_L[\rho, v](t) = \frac{\mathrm{d}_+ E_L[\rho, v](t)}{\mathrm{d}t}. \tag{8}$$

Entropy rate admissibility II

Definition 5 (Entropy rate admissible solution)

A weak solution (ρ, v) of (1) is called *entropy rate admissible* if there exists $L^* > 0$ such that there is no other weak solution $(\overline{\rho}, \overline{v})$ with the property that for some $\tau \geq 0$, $(\overline{\rho}, \overline{v})(x,t) = (\rho, v)(x,t)$ on $\mathbb{R}^2 \times [0, \tau]$ and $D_L[\overline{\rho}, \overline{v}](\tau) < D_L[\rho, v](\tau)$ for all $L \geq L^*$.

This definition is motivated by Dafermos. He proved that for a single equation the entropy rate criterion is equivalent to the viscosity criterion in the class of piecewise smooth solutions. Following the approach of Dafermos, Hsiao proved, in the class of piecewise smooth solutions, the equivalence of the entropy rate criterion and the viscosity criterion for the one-dimensional system of equations of nonisentropic gas dynamics in lagrangian formulation with pressure laws $p(\rho) = \rho^\gamma$ for $\gamma \geq 5/3$ while the same equivalence is disproved for $\gamma < 5/3$.

Main Theorem II

Theorem 6 (Chiodaroli, K.)

Let $p(\rho) = \rho^\gamma$, $1 \leq \gamma < 3$. *There exist Riemann data* (2) *for which the self-similar solution to* (1) *emanating from these data is not entropy rate admissible.*

Theorem 6 ensures that for $1 \leq \gamma < 3$ there exist initial Riemann data (2) for which some of the infinitely many nonstandard solutions constructed as in Theorem 4 dissipate more total energy than the self-similar solution, suggesting in particular that the Dafermos entropy rate admissibility criterion would not pick the self-similar solution as the admissible one.

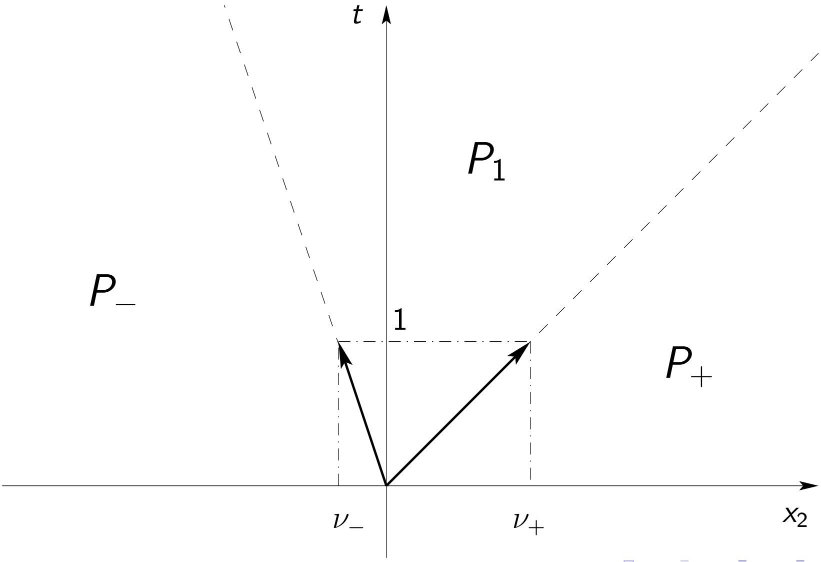
Definition 7 (Fan partition)

A *fan partition* of $\mathbb{R}^2 \times (0, \infty)$ consists of three open sets P_-, P_1, P_+ of the following form

$$\begin{aligned} P_- &= \{(x,t) : t > 0 \quad \text{and} \quad x_2 < \nu_- t\} \\ P_+ &= \{(x,t) : t > 0 \quad \text{and} \quad x_2 > \nu_+ t\} \\ P_1 &= \{(x,t) : t > 0 \quad \text{and} \quad \nu_- t < x_2 < \nu_+ t\} \end{aligned}$$

where $\nu_- < \nu_+$ is an arbitrary couple of real numbers.

Picture of fan partition



Fan subsolution

Definition 8 (Fan subsolution I)

A *fan subsolution* to the compressible Euler equations with initial data (2) is a triple $(\overline{\rho}, \overline{v}, \overline{u}) : \mathbb{R}^2 \times (0, \infty) \to (\mathbb{R}^+, \mathbb{R}^2, \mathcal{S}_0^{2\times 2})$ of bounded measurable functions satisfying the following requirements.

(i) There is a fan partition P_-, P_1, P_+ of $\mathbb{R}^2 \times (0, \infty)$ such that

$$(\overline{\rho}, \overline{v}, \overline{u}) = (\rho_-, v_-, u_-)\mathbf{1}_{P_-} + (\rho_1, v_1, u_1)\mathbf{1}_{P_1} + (\rho_+, v_+, u_+)\mathbf{1}_{P_+}$$

where $\rho_1 > 0, v_1, u_1$ are constants and

$$u_\pm = v_\pm \otimes v_\pm - \frac{1}{2}|v_\pm|^2 \mathrm{Id}$$

Fan subsolution II

(ii) There exists a positive constant C such that

$$v_1 \otimes v_1 - u_1 < \frac{C}{2}\mathrm{Id} \text{ a.e.}$$

(iii) The triple $(\overline{\rho}, \overline{v}, \overline{u})$ solves the following system in the sense of distributions:

$$\begin{aligned}
\partial_t \overline{\rho} + \mathrm{div}_x(\overline{\rho}\, \overline{v}) &= 0 \\
\partial_t(\overline{\rho}\, \overline{v}) + \mathrm{div}_x\, (\overline{\rho}\, \overline{u}) & \\
+ \nabla_x \left(p(\overline{\rho}) + \frac{1}{2} \left(C\rho_1 \mathbf{1}_{P_1} + \overline{\rho} |\overline{v}|^2 \mathbf{1}_{P_+ \cup P_-} \right) \right) &= 0
\end{aligned}$$

Admissible fan subsolution

Definition 9 (Admissible fan subsolution)

A fan subsolution $(\overline{\rho}, \overline{v}, \overline{u})$ is said to be *admissible* if it satisfies the following inequality in the sense of distributions

$$
\begin{aligned}
&\partial_t \left(\overline{\rho}\varepsilon(\overline{\rho})\right) + \mathrm{div}_x \left[\left(\overline{\rho}\varepsilon(\overline{\rho}) + p(\overline{\rho})\right)\overline{v}\right] \\
&+ \partial_t \left(\overline{\rho}\frac{|\overline{v}|^2}{2}\mathbf{1}_{P_+\cup P_-}\right) + \mathrm{div}_x \left(\overline{\rho}\frac{|\overline{v}|^2}{2}\overline{v}\mathbf{1}_{P_+\cup P_-}\right) \\
&+ \partial_t \left(\rho_1 \frac{C}{2}\mathbf{1}_{P_1}\right) + \mathrm{div}_x \left(\rho_1 \overline{v}\frac{C}{2}\mathbf{1}_{P_1}\right) \le 0\,.
\end{aligned}
$$

Key Lemma

The true core of our construction is the following Lemma.

Lemma 10

Let $(\tilde{v}, \tilde{u}) \in \mathbb{R}^2 \times \mathcal{S}_0^{2\times 2}$ *and* $C > 0$ *be such that* $\tilde{v} \otimes \tilde{v} - \tilde{u} < \frac{C}{2}\mathrm{Id}$. *For any open set* $\Omega \subset \mathbb{R}^2 \times \mathbb{R}$ *there are infinitely many maps* $(\underline{v}, \underline{u}) \in L^\infty(\Omega, \mathbb{R}^2 \times \mathcal{S}_0^{2\times 2})$ *with the following property*

(i) $\underline{v}$ *and* $\underline{u}$ *vanish identically outside* Ω

(ii)

$$\begin{cases} \mathrm{div}_x\, \underline{v} = 0 \\ \partial_t \underline{v} + \mathrm{div}_x\, \underline{u} = 0 \end{cases}$$

(iii) $(\tilde{v} + \underline{v}) \otimes (\tilde{v} + \underline{v}) - (\tilde{u} + \underline{u}) = \frac{C}{2}\mathrm{Id}$ *a.e. on* Ω.

Proof of this statement is (up to minor modifications) in the first two papers of De Lellis and Székelyhidi on incompressible Euler equations.

Using this Lemma we easily prove the following

Proposition 11

Let p be any C^1 function and $(\rho_\pm, v_\pm)$ be such that there exists at least one admissible fan subsolution $(\overline{\rho}, \overline{v}, \overline{u})$ of the Euler equations with initial data (2). Then there are infinitely many bounded admissible solutions (ρ, v) to (1),(2) such that $\rho = \overline{\rho}$.

To prove Theorem 4 we therefore need to find an admissible fan subsolution.

To prove Theorem 6 we need to prove that the admissible fan subsolution has moreover a special property concerning the energy dissipation which transfers to solutions constructed from this subsolution.

References

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Thank you

Thank you for your attention.