Qualitative properties of solutions to the complete fluid systems

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Navier-Stokes-Fourier system

Mass conservation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum balance

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u})$$

$$\mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}) \equiv \mu \Delta \mathbf{u} + \lambda \nabla_x \mathrm{div}_x \mathbf{u}, \ \ \mu > 0, \ \ \lambda \geq 0$$

Internal energy equation

$$c_v \left[\partial_t(\varrho \vartheta) + \mathrm{div}_x(\varrho \vartheta \mathbf{u}) \right] - \mathrm{div}_x \left(\kappa(\vartheta) \nabla_x \vartheta \right)$$

$$= \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \vartheta p_\vartheta(\varrho, \vartheta) \mathrm{div}_x \mathbf{u}, \ \ \kappa(\vartheta) > 0$$

Total energy balance

Pressure

$$p(\varrho,\vartheta) = a\varrho^\gamma + b\varrho + \varrho\vartheta,\ \ \gamma > 3,\ \ a, b > 0$$

Boundary conditions

$$\mathbf{u}|_{\partial\Omega} = 0,\ \ \nabla_x\vartheta \cdot \mathbf{n}|_{\partial\Omega} = 0$$

Total energy balance

$$E(t) = \int \left[\frac{1}{2}\varrho|\mathbf{u}|^2 + c_v\varrho\vartheta + \frac{a}{\gamma - 1}\varrho^\gamma + b\varrho\log(\varrho) \right]$$

$$\frac{\mathrm{d}}{\mathrm{d}t} E(t) = 0$$

Existence of smooth solutions (classical theory)

Local solutions

A. Valli, W.Zajaczkowski [1982] Existence of classical *local-in-time* solutions in the class:

$$\varrho \in C([0, T_{\rm max}); W^{3,2}(\Omega)), \ \vartheta_0 \in C([0, T_{\rm max}); W^{3,2}(\Omega))$$

$$\mathbf{u} \in C([0, T_{\rm max}); W^{3,2}(\Omega; R^3))$$

Global solutions

A.Matsumura, T.Nishida [1980,1983] Existence of classical *global-in-time* solutions in the same class for the initial data sufficiently close to a static state

Navier-Stokes-Fourier system (weak form)

Mass conservation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum balance

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u})$$

Thermal energy balance

$$c_v \partial_t(\varrho\vartheta) + c_v \mathrm{div}_x(\varrho\vartheta\mathbf{u}) - \Delta(K(\vartheta))$$

$$\boxed{\geq} \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - p_\vartheta(\varrho, \vartheta)\vartheta \mathrm{div}_x \mathbf{u}$$

Total energy balance

$$\int \left[\frac{1}{2}\varrho|\mathbf{u}|^2 + \frac{a}{\gamma - 1}\varrho^\gamma + b\varrho \log(\varrho) + c_v \varrho\vartheta \right] (\tau, \cdot) \ \mathrm{d}x \boxed{\leq} E_0$$

Existence of weak solutions

Existence of weak solutions [E.F.2003]

The Navier-Stokes-Fourier system admits a global-in-time weak solution for any finite energy initial data

Compatibility property of weak solutions

Any weak solution that possesses necessary smoothness is a classical one

Problems

- density oscillations
- temperature concentrations

Euler-Fourier system

Mass conservation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum balance

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x(\varrho \vartheta) = 0$$

Internal energy balance

$$\frac{3}{2}\Big[\partial_t(\varrho\vartheta) + \mathrm{div}_x(\varrho\vartheta\mathbf{u})\Big] - \Delta\vartheta = -\varrho\vartheta \mathrm{div}_x \mathbf{u}$$

Existence of weak solutions

Initial data

$$\varrho_0,\ \vartheta_0,\ \mathbf{u}_0 \in C^3,\ \varrho_0 > 0,\ \vartheta_0 > 0$$

Global existence [E.Chiodaroli, E.F., O.Kreml 2013]

For any (smooth) initial data ϱ_0, ϑ_0, $\mathbf{u}_0$ the Euler-Fourier system admits infinitely many weak solutions on a given time interval $(0, T)$

Regularity class

$$\varrho \in C^2,\ \partial_t \vartheta,\ \nabla_x^2 \vartheta \ \in L^p \text{ for any } 1 \leq p < \infty$$

$$\mathbf{u} \in C_{\rm weak}([0, T]; L^2) \cap L^\infty,\ {\rm div}_x \mathbf{u} \in C^1$$

Blow-up criterion

Blow-up of smooth solutions [E.F., Y.Sun 2014]

Suppose that the initial data ϱ_0, ϑ_0, and $\mathbf{u}_0$ are smooth $(W^{2,3})$. Then the Navier-Stokes-Fourier system admits a strong solution defined on a (possibly short) time interval $(0, T)$.
If

$$\sup_{t \in (0,T)} \left[\|\varrho\|_{L^\infty} + \|\vartheta\|_{L^\infty} + \|\mathbf{u}\|_{L^\infty} \right] < \infty,$$

then the solution can be extended beyond T.

Regularity criterion

Regularity for weak solutions [E.F., Y.Sun 2014]

Suppose that the initial data ϱ_0, ϑ_0, and $\mathbf{u}_0$ are smooth ($W^{2,3}$). Let $[\varrho, \vartheta, \mathbf{u}]$ be a weak solution of the Navier-Stokes-Fourier system such that

$$\sup_{t \in (0,T)} \left[\|\varrho\|_{L^\infty} + \|\vartheta\|_{L^\infty} + \|\mathbf{u}\|_{L^\infty} + \|\mathrm{div}_x \mathbf{u}\|_{L^\infty} \right] < \infty.$$

Then $[\varrho, \vartheta, \mathbf{u}]$ is regular.

Main tools

- Relative energy functional
- Entropy formulation
- Weak strong uniqueness

Numerical solution

FV framework

regular tetrahedral mesh, $Q_h = \{ v \mid v = \text{piece-wise constant} \}$

FE framework - Crouzeix - Raviart

$$V_h = \left\{ v \;\middle|\; v = \text{piece-wise affine, } \tilde{v}_\Gamma \text{ continuous on face } \Gamma \right\}$$

$$\tilde{v}_\Gamma \equiv \frac{1}{|\Gamma|} \int_\Gamma v \, \mathrm{dS}_x$$

Upwind discretization of convective terms

$$\langle h\mathbf{u}; \nabla_x \varphi \rangle_E \approx \sum_\Gamma \int_\Gamma h(\cdot - \tilde{\mathbf{u}}_\Gamma \cdot \mathbf{n}) \tilde{\mathbf{u}}_\Gamma \cdot \mathbf{n} [[\varphi]] \, \mathrm{dS}_x$$

$$\equiv \sum_\Gamma \int_\Gamma \mathrm{Up}[h, \mathbf{u}][[\varphi]] \, \mathrm{dS}_x$$

Numerical scheme [Karlsen-Karper], I

Equation of continuity

$$\int_\Omega D_t \varrho_h^k \varphi_h \ \mathrm{d}x \equiv \int_\Omega \frac{\varrho_h^k - \varrho_h^{k-1}}{\Delta t} \varphi_h \ \mathrm{d}x = \sum_{\Gamma \in \Gamma_h} \int_\Gamma \mathrm{Up}[\varrho_h^k, u_h^k] \ [[\varphi_h]] \ \mathrm{dS}_x$$

$$-h^\alpha \boxed{\sum_{\Gamma \in \Gamma_h} \int_\Gamma [[\varrho_h^k]][[\varphi_h]] \ \mathrm{dS}_x} \ \text{for all } \varphi_h \in Q_h$$

Numerical scheme [Karlsen-Karper], II

Momentum equation

$$\int_\Omega D_t\left(\varrho_h^k \widehat{\mathbf{u}}_h^k\right)\cdot\varphi_h \, \mathrm{d}x - \sum_{\Gamma\in\Gamma_h}\int_\Gamma \mathrm{Up}[\varrho_h^k\widehat{\mathbf{u}}_h^k, \mathbf{u}_h^k] \cdot [[\widehat{\varphi}_h]]\mathrm{dS}_x$$

$$-\int_\Omega p(\varrho_h^k,\vartheta_h^k)\mathrm{div}_x\varphi_h \, \mathrm{d}x - h^\alpha \boxed{\sum_{\Gamma\in\Gamma_h}\int_\Gamma [[\varrho_h^k]][[\widehat{\varphi}_h]]\left\{\widehat{\mathbf{u}}_h^k\right\}\mathrm{dS}_x}$$

$$= -\int_\Omega \left(\mu\nabla_h\mathbf{u}_h : \nabla_h\varphi + \lambda\mathrm{div}_h\mathbf{u}_h^k\mathrm{div}_h\varphi_h\right)\, \mathrm{d}x$$

for all $\varphi_h \in V_h(\Omega)$

Numerical scheme [Karlsen-Karper], III

Energy equation

$$\int_\Omega D_t\left(\varrho_h^k \vartheta_h^k\right)\varphi_h \, \mathrm{d}x - \sum_{\Gamma\in\Gamma_h}\int_\Gamma \mathrm{Up}[\varrho_h^k\vartheta_h^k, \mathbf{u}_h^k]\,[[\varphi_h]]\mathrm{dS}_x$$

$$+\sum_{\Gamma\in\Gamma_h}\int_\Gamma \frac{1}{d_h}[[K(\vartheta_h^k)]]\,[[\varphi_h]]\mathrm{dS}_x$$

$$= \int_\Omega \left(\mu|\nabla_h \mathbf{u}_h^k|^2 + \lambda|\mathrm{div}_h \mathbf{u}_h^k|^2\right)\varphi_h \, \mathrm{d}x$$

$$- \int_\Omega \vartheta_h^k \partial_\vartheta p(\varrho_h^k, \vartheta_h^k)\mathrm{div}_x \mathbf{u}_h^k \varphi_h \, \mathrm{d}x$$

for all $\varphi_h \in Q_h(\Omega)$

Convergence of the numerical scheme

Convergence of the numerical scheme [E.F., T.Karper, A.Novotný 2014]

The numerical solutions converge, up to a subsequence, to a weak solution of the Navier-Stokes-Fourier system

Synergy analysis - numerics, assumptions

Numerical solutions with regular initial data

Suppose that $[\varrho_h, \vartheta_h, \mathbf{u}_h]$ is a sequence of numerical solutions for regular initial data

Boundedness

Suppose that

$$\varrho_h^k, \ \vartheta_h^k, \ \mathbf{u}_h^k, \ \mathrm{div}_h \mathbf{u}_h^k$$

are bounded independently of the order of discretization h.

Synergy analysis - numerics, conclusion

Conclusion

The numerical solutions converge to a weak solution with

$$\sup_{t \in (0,T)} \left[\|\varrho\|_{L^\infty} + \|\vartheta\|_{L^\infty} + \|\mathbf{u}\|_{L^\infty} + \|\mathrm{div}_x \mathbf{u}\|_{L^\infty} \right] < \infty.$$

Consequently:

- the limit solution is smooth
- the limit solution is unique
- the numerical scheme converges unconditionally
- error estimates (?)