INSTITUTE of MATHEMATICS

ACADEMY of SCIENCES of the CZECH REPUBLIC

# A convergent mixed numerical method for the Navier-Stokes-Fourier system

*Eduard Feireisl*

*Trygve Karper*

*Antonín Novotný*

Preprint No. 48-2014

PRAHA 2014

# A convergent mixed numerical method for the Navier-Stokes-Fourier system

Eduard Feireisl *  Trygve Karper  Antonín Novotný†

Institute of Mathematics of the Academy of Sciences of the Czech Republic
Žitná 25, CZ-115 67 Praha 1, Czech Republic

Department of Mathematical Sciences, Norwegian University of Science and Technology
Alfred Getz vei 1, 7049 Trondheim, Norway

IMATH, EA 2134, Université de Toulon BP 20132, 83957 La Garde, France

**Abstract**

We propose a numerical method for solving the full Navier-Stokes-Fourier system describing the evolution of a general compressible, viscous, and heat conducting fluid. The use a finite volume method for approximating the continuity equation as well as the thermal energy balance, while the momentum equation is discretized by means of the discontinuous Galerkin scheme based on the Crouzeix-Raviart finite elements. The numerical solutions converge, up to a subsequence, to a suitable weak solution of the problem.

**Key words:** Navier-Stokes-Fourier system, finite element discontinuous Galerkin numerical method, finite volume numerical method, convergence

*The research of E.F. leading to these results has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP7/2007-2013)/ ERC Grant Agreement 320078. The Institute of Mathematics of the Academy of Sciences of the Czech Republic is supported by RVO:67985840.

†The work of A.N. was supported by the MODTERCOM project within the APEX programme of the Provence-Alpes-Côte d'Azur region

# Contents

# 1 Introduction

In continuum mechanics, the state of a compressible, viscous and heat conducting fluid is characterized by the principal macroscopic fields: the mass density $\varrho = \varrho(t,x)$, the absolute temperature $\vartheta = \vartheta(t,x)$, and the velocity $\mathbf{u} = \mathbf{u}(t,x)$, depending on the time $t \in (0,T)$ and the reference spatial position $x$ in the physical domain $\Omega \subset R^3$. The time evolution of these quantities is governed by the *Navier-Stokes-Fourier system*:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{1.1}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}), \tag{1.2}$$

$$c_v \left[\partial_t(\varrho\vartheta) + \mathrm{div}_x(\varrho\vartheta\mathbf{u})\right] + \mathrm{div}_x \mathbf{q}(\vartheta, \nabla_x\vartheta) = \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \vartheta \frac{\partial p(\varrho,\vartheta)}{\partial \vartheta} \mathrm{div}_x \mathbf{u}, \tag{1.3}$$

where $p = p(\varrho, \vartheta)$ denotes the pressure, $\mathbb{S}(\nabla_x \mathbf{u})$ is the viscous stress tensor, $c_v$ the specific heat at constant volume, and $\mathbf{q}(\vartheta, \nabla_x \vartheta)$ the heat flux. Note that the effect of external mechanical and heat forces is omitted for the sake of simplicity.

## 1.1 Constitutive relations

For the system to be thermodynamically consistent, certain constitutive restrictions must be imposed on the thermodynamic functions. Given the complexity of the problem, we focus on the simplest possible situation still tractable by the available theoretical tools. Specifically, we suppose that:

- The specific heat at constant volume $c_v$ is a positive constant.
- The internal energy $e(\varrho, \vartheta)$ can be written in the form

$$e(\varrho, \vartheta) = c_v\vartheta + P(\varrho).$$

  Moreover, we suppose that the pressure takes the specific form

$$p(\varrho, \vartheta) = a\varrho^\gamma + b\varrho + \varrho\vartheta, \ a, b > 0, \ \gamma > 3, \tag{1.4}$$

  therefore the specific internal energy reads

$$e(\varrho, \vartheta) = c_v\vartheta + \frac{a}{\gamma - 1}\varrho^\gamma + b\varrho\log(\varrho), \ c_v > 0. \tag{1.5}$$

- The fluid is linearly viscous, with $\mathbb{S}$ determined by *Newton's rheological law*

$$\mathbb{S}(\nabla_x\mathbf{u}) = \mu\left(\nabla_x\mathbf{u} + \nabla_x^t\mathbf{u} - \frac{2}{3}\mathrm{div}_x\mathbf{u}\mathbb{I}\right) + \eta\mathrm{div}_x\mathbf{u}\mathbb{I}, \ \mu > 0, \ \eta \geq 0, \tag{1.6}$$

  with constant viscosity coefficients $\mu$ and $\eta$. Accordingly, we may write

$$\mathrm{div}_x\mathbb{S}(\nabla_x\mathbf{u}) = \mu\Delta\mathbf{u} + \lambda\nabla_x\mathrm{div}_x\mathbf{u}, \ \lambda = \frac{1}{3}\mu + \eta > 0. \tag{1.7}$$

- The heat flux $\mathbf{q}$ obeys *Fourier's law*

$$\mathbf{q} = -\kappa(\vartheta)\nabla_x\vartheta = -\nabla_x K(\vartheta), \ K(\vartheta) = \int_0^\vartheta \kappa(z) \ \mathrm{d}z, \tag{1.8}$$

  where the heat conductivity coefficient $\kappa$ is a continuously differentiable function of the temperature satisfying

$$\kappa = \kappa(\vartheta), \ \underline{\kappa}(1 + \vartheta^2) \leq \kappa(\vartheta) \leq \overline{\kappa}(1 + \vartheta^2), \ \underline{\kappa} > 0. \tag{1.9}$$

The specific form of the constitutive relations (1.4 - 1.9) is inspired by similar hypotheses introduced in [14]. In particular, the assumption $\gamma > 3$ is optimal in view of the available analytical methods, see [14, Chapter 6].

## 1.2 Boundary conditions

We adopt the standard *no-slip* hypothesis

$$\mathbf{u}|_{\partial\Omega} = 0 \tag{1.10}$$

for the velocity accompanied with the no-flux boundary condition

$$\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \mathbf{n}|_{\partial\Omega} = 0, \tag{1.11}$$

meaning, in particular, that the fluid is both mechanically and energetically autonomous.

## 1.3 Weak formulation

The problem (1.1 - 1.3), (1.10), (1.11) is supplemented by the initial conditions

$$\varrho(0,\cdot) = \varrho_0,\ \vartheta(0,\cdot) = \vartheta_0,\ \mathbf{u}(0,\cdot) = \mathbf{u}_0,\ \varrho_0 > 0,\ \vartheta_0 > 0 \text{ in } \overline{\Omega}. \tag{1.12}$$

We adopt the following *weak formulation* introduced in [14, Chapter 4]:

**Definition 1.1** *We say that a trio of functions* $[\varrho, \vartheta, \mathbf{u}]$ *is a* weak solution *to the problem (1.1 - 1.3), (1.10 - 1.12) in* $(0,T) \times \Omega$ *if:*

$$\varrho \in L^\infty(0,T; L^\gamma(\Omega)),\ \vartheta \in L^2(0,T; L^6(\Omega)),\ \mathbf{u} \in L^2(0,T; W_0^{1,2}(\Omega; R^3)), \tag{1.13}$$

$$\varrho\mathbf{u} \in L^\infty(0,T; L^{\frac{2\gamma}{\gamma+1}}(\Omega; R^3)),\ \varrho\vartheta \in L^\infty(0,T; L^1(\Omega)); \tag{1.14}$$

$$\varrho \geq 0,\ \vartheta > 0 \ a.a. \ in \ (0,T) \times \Omega; \tag{1.15}$$

$$\int_0^T \int_\Omega \Big[ \varrho \partial_t \varphi + \varrho \mathbf{u} \cdot \nabla_x \varphi \Big] \ \mathrm{d}x \ \mathrm{d}t = - \int_\Omega \varrho_0 \varphi(0,\cdot) \ \mathrm{d}x \tag{1.16}$$

*for any* $\varphi \in C_c^\infty([0,T) \times \overline{\Omega})$*;*

$$\int_0^T \int_\Omega \Big[ \varrho\mathbf{u} \cdot \partial_t \varphi + \varrho \mathbf{u} \otimes \mathbf{u} : \nabla_x \varphi + p(\varrho, \vartheta) \mathrm{div}_x \varphi \Big] \ \mathrm{d}x \ \mathrm{d}t \tag{1.17}$$

$$= \int_0^T \int_\Omega \Big[ \mu \nabla_x \mathbf{u} : \nabla_x \varphi + \lambda \mathrm{div}_x \mathbf{u} \ \mathrm{div}_x \varphi \Big] \ \mathrm{d}x \ \mathrm{d}t - \int_\Omega \varrho_0 \mathbf{u}_0 \cdot \varphi(0,\cdot) \ \mathrm{d}x$$

*for any* $\varphi \in C_c^\infty([0,T) \times \Omega; R^3)$*;*

$$\int_0^T \int_\Omega \Big[ c_v \Big( \varrho\vartheta \partial_t \varphi + \varrho\vartheta \mathbf{u} \cdot \nabla_x \varphi \Big) - \overline{K(\vartheta)} \Delta \varphi \Big] \ \mathrm{d}x \ \mathrm{d}t \tag{1.18}$$

$$+\int_0^T \int_\Omega \left[ \mu |\nabla_x \mathbf{u}|^2 + \lambda |\mathrm{div}_x \mathbf{u}|^2 \right] \varphi \ \mathrm{d}x \ \mathrm{d}t - \int_0^T \int_\Omega \varrho \vartheta \mathrm{div}_x \mathbf{u} \varphi \ \mathrm{d}x \ \mathrm{d}t \leq \int_\Omega c_v \varrho_0 \vartheta_0 \varphi(0, \cdot) \ \mathrm{d}x$$

*for any* $\varphi \in C_c^\infty([0,T) \times \overline{\Omega})$, $\varphi \geq 0$, $\nabla_x \varphi \cdot \mathbf{n}|_{\partial\Omega} = 0$, *where*

$$\overline{\varrho K(\vartheta)} = \varrho K(\vartheta); \tag{1.19}$$

$$\int_\Omega \left[ \frac{1}{2} \varrho |\mathbf{u}|^2 + c_v \varrho \vartheta + \frac{a}{\gamma - 1} \varrho^\gamma + b \varrho \log(\varrho) \right] (\tau, \cdot) \ \mathrm{d}x \tag{1.20}$$

$$\leq \int_\Omega \left[ \frac{1}{2} \varrho_0 |\mathbf{u}_0|^2 + c_v \varrho_0 \vartheta_0 + \frac{a}{\gamma - 1} \varrho_0^\gamma + b \varrho_0 \log(\varrho_0) \right] \ \mathrm{d}x$$

*for a.a.* $\tau \in (0,T)$.

The key idea behind the present weak formulation is replacing one *equation* - the thermal energy balance (1.3) - by the *two* inequalities, specifically (1.18) supplemented with the total energy balance (1.20). It can be shown that the resulting concept of weak solution complies with the obligatory *compatibility principle*, namely, any weak solution in the sense of Definition 1.1 that enjoys the necessary smoothness is a classical solution of (1.1 - 1.3). Moreover, conditional regularity as well as the weak-strong uniqueness property were established in [13]. In particular, any weak solution in the sense of Definition 1.1, emanating from regular initial data and belonging to the class

$$\varrho, \vartheta \in L^\infty((0,T) \times \Omega), \ \mathbf{u}, \mathrm{div}_x \mathbf{u} \in L^\infty((0,T) \times \Omega)$$

is necessarily a regular solution of (1.1 - 1.3), see [13, Theorem 2.2]. The interested reader may consult the monograph [14] for the relevant mathematical theory including a global-in-time existence result for the Navier-Stokes-Fourier system in the class of weak solutions specified in Definition 1.1.

## 1.4 Numerical method

Our main goal in this paper is to propose a numerical method for solving the Navier-Stokes-Fourier system (1.1 - 1.3) and to show its stability and convergence towards a weak solution specified in Definition 1.1. To this end, we adapt the discontinuous Galerkin finite element scheme proposed by Karlsen, Karper [19], [20] for the compressible Navier-Stokes system, combined with a finite volume method to solve the thermal energy balance.

The time evolution of the system is approximated by the implicit time discretization scheme, where the resulting stationary problems at each time steps are solved by means of a combination of a finite volume - finite element scheme on a regular tetrahedral mesh. With our choice of the no-slip boundary conditions (1.10), it is convenient to approximate the velocity field by means of the finite elements of Crouzeix-Raviart type, while the convective terms are discretized by the standard upwind scheme. In particular, we use the specific form of the upwind term in the momentum equation proposed by Karlsen, Karper [20] that is compatible with all the necessary steps for showing compactness of the family of approximate solutions, cf. [21]. The thermal energy balance is approximated by a finite volume scheme, similar to those used by Eymard et al. [9], [11] for solving non-linear degenerate reaction diffusion equations. Here, the proof of convergence of the method is based on compactness of embeddings of the associated discrete Sobolev spaces and an adaptation of a renormalization method proposed in [14, Chapter 6].

The key ideas of our approach can be summarized as follows:

- the specific discretization of the upwind term in the momentum method introduced in [20];
- adding artificial viscosity in the continuity method in the spirit of Eymard et al. [10] to enable "by parts integration" in the convergence proof;
- renormalization of the thermal energy method and the "biting limit" passage following [14, Chapter 6].

The paper is organized as follows. In Section 2, we introduce the necessary numerical framework including the basic notation and several useful properties of the underlying function spaces. The numerical scheme is introduced in Section 3, where we also state our main result concerning convergence towards a weak solution of the Navier-Stokes-Fourier system proved in the remaining part of the paper. In Section 4, we derive a renormalized version of the continuity and thermal energy balance as well as the discrete version of the total energy balance. Section 5 is devoted to the stability of the scheme, containing the uniform bounds necessary for the limit passage. In Section 6, we discuss the problem of consistency of the method rewriting finally the numerical scheme in terms of the standard weak formulation based on smooth test functions. Having established consistency, we show convergence of the scheme by adapting the steps of [14, Chapter 7]. Here, similarly to the existence theory, the key idea is the weak continuity property of the effective viscous flux discovered by Lions [22], combined with the renormalization technique of [14] applied to the thermal energy balance.

# 2 Finite elements/volumes preliminaries

In this section, we collect the necessary apparatus of the numerical analysis. We tacitly assume the reader to be fairly familiar with the techniques used in numerical analysis; we refer to standard texts as Brezzi, Fortin [3] for details. The following convention will be used systematically in the text: For two numerical quantities $a$, $b$, we shall write

$$a \lesssim b \text{ if } a \leq cb,\ c > 0 \text{ a constant},\ a \approx b \text{ if } a \lesssim b \text{ and } b \lesssim a.$$

Here, "constant" typically means a generic quantity independent of the size of the mesh and the time step used in the numerical scheme as well as other parameters as the case may be.

## 2.1 Mesh

We suppose that the physical space is a *polyhedral bounded domain* $\Omega \subset R^3$ that admits a *tetrahedral* mesh $E_h$; the individual elements in the mesh will be denoted by $E \in E_h$. Faces in the mesh are denoted as $\Gamma$, whereas $\Gamma_h$ is the set of all faces. Moreover, the set of faces $\Gamma \subset \partial\Omega$ is denoted $\Gamma_{h,\mathrm{ext}}$, while $\Gamma_{h,\mathrm{int}} = \Gamma_h \setminus \Gamma_{h,\mathrm{ext}}$. The size ( diameter $h_E$ of its elements $E$ in the mesh) is proportional to a positive parameter $h$.

In addition, we require the mesh to be admissible in the sense of Eymard et al. [11, Definition 2.1]:

- For $E, F \in E_h$, $E \neq F$, the intersection $E \cap F$ is either a vertex, or an edge, or a face $\Gamma \in \Gamma_h$.
- There is a family of control points $\{x_E \mid x_E \in E,\ E \in E_h\}$ such that the segment $[x_E, x_F]$ for two adjacent elements $E$, $F$ of the mesh is perpendicular to their common face $\Gamma$. We denote $d_\Gamma \equiv |x_E - x_F|$.
- The mesh is *shape regular* in the sense that

$$\inf_{E \in E_h} \inf_{\Gamma \subset \partial E} \mathrm{dist}[x_E, \Gamma] \gtrsim h. \tag{2.1}$$

**Remark 2.1** *1) The above properties are satisfied, for instance, by the well-centered meshes discussed by Vanderzee et al. [23], [24], where the point $x_E$ is simply the circumcenter of the tetrahedron $E$.*

*2) The mesh described above is regular in the sense of the classical finite volume literature, meaning that there exists $\theta_0 > 0$ such that*

$$\inf \Big( \{ \frac{\underline{h}_E}{h_E}, E \in E_h \} \cup \{ \frac{h_E}{h_L}, \frac{h_L}{h_E}, \Gamma = \partial E \cap \partial L \cap \Gamma_{h,\mathrm{int}} \} \Big) > \theta_0, \tag{2.2}$$

*where $\underline{h}_E$ is the diameter of the largest ball included in $E$.*

Each face $\Gamma \in \Gamma_h$ is associated with a fixed normal vector $\mathbf{n}$. On the other hand, we write $\Gamma_E$ whenever the face $\Gamma_E \subset \partial E$ is considered as a part of the boundary of the element $E$. In such a case, the normal vector to $\Gamma_E$ is always the *outer* normal vector with respect to $E$.

For a function $g$, continuous on each element $E$, we denote

$$g^+|_\Gamma = \lim_{\delta \to 0+} g(\cdot + \delta \mathbf{n}),\ g^-|_\Gamma = \lim_{\delta \to 0+} g(\cdot - \delta \mathbf{n}),\ [[g]]_\Gamma = g^+ - g^-, \{g\}_\Gamma = \frac{1}{2} \Big( g^+ + g^- \Big). \tag{2.3}$$

## 2.2 Piecewise constant finite elements

We introduce the space

$$Q_h(\Omega) = \left\{ v \in L^2(\Omega) \ \middle| \ v|_E = a_E \in R \right\}$$

of piecewise constant functions along with the associated orthogonal projection

$$\Pi_h^Q : L^2(\Omega) \to Q_h(\Omega), \ \Pi_h^Q[v]|_E = \frac{1}{|E|} \int_E v \ \mathrm{d}x;$$

we will occasionally denote

$$\Pi_h^Q[v] \equiv \hat{v}.$$

We recall Poincaré's inequality

$$\left\| v - \Pi_h^Q[v] \right\|_{L^q(\Omega)} \stackrel{<}{\sim} h \| \nabla_x v \|_{L^q(\Omega;R^3)}, \ 1 \le q \le \infty \text{ for any } v \in W^{1,q}(\Omega), \tag{2.4}$$

together with Jensen's inequality

$$\| \Pi_h^Q[v] \|_{L^q(\Omega)} \stackrel{<}{\sim} \| v \|_{L^q(\Omega)}, \ 1 \le q \le \infty \text{ for any } v \in L^q(\Omega). \tag{2.5}$$

In addition, we introduce another projection operator with the target space $Q_h$, namely

$$\Pi_h^B : C(\overline{\Omega}) \to Q_h(\Omega), \ \Pi_h^B[v]|_E = v(x_E).$$

It is easy to check that

$$\left\| v - \Pi_h^B[v] \right\|_{L^\infty(\Omega)} \stackrel{<}{\sim} h \| \nabla_x v \|_{L^\infty(\Omega;R^3)} \text{ for any Lipschitz } v. \tag{2.6}$$

## 2.3 Crouzeix-Raviart finite elements

A differential operator $D$ acting on the $x-$variable will be discretized as

$$D_h v|_E = D(v|_E) \text{ for any } v \text{ differentiable on each element } E \in E_h.$$

The *Crouzeix-Raviart finite element spaces* (see Brezzi and Fortin [3], among others) are defined as

$$V_h(\Omega) = \left\{ v \in L^2(\Omega) \ \middle| \ v|_E = \text{affine function}, \ E \in E_h, \ \int_\Gamma [[v]] \ \mathrm{dS}_x = 0 \text{ for any } \Gamma \in \Gamma_{h,\mathrm{int}} \right\}, \tag{2.7}$$

together with

$$V_{h,0}(\Omega) = \left\{ v \in V_h \ \middle| \ \int_\Gamma v \ \mathrm{dS}_x = 0 \text{ for any } \Gamma \in \Gamma_{h,\mathrm{ext}} \right\}. \tag{2.8}$$

Next, we introduce the associated projection

$$\Pi_h^V : W^{1,p}(\Omega) \to V_h(\Omega),$$

$$\int_\Gamma \Pi_h^V[v] \ \mathrm{dS}_x = \int_\Gamma v \ \mathrm{dS}_x \text{ for any } \Gamma \in \Gamma_h,\ p > 1.$$

We have,

$$\int_\Omega \mathrm{div}_h \Pi_h^V[\mathbf{u}] \ w \ \mathrm{d}x = \int_\Omega \mathrm{div}_h \mathbf{u} \ w \ \mathrm{d}x \text{ for any } w \in Q_h(\Omega), \tag{2.9}$$

Poincaré's inequality,

$$\|v - \Pi_h^V[v]\|_{L^2(\Omega)} \lesssim h \|\nabla_h v\|_{L^2(\Omega;R^3)} \text{ for any } v \in V_h(\Omega), \tag{2.10}$$

along with the error estimates

$$\left\| v - \Pi_h^V[v] \right\|_{L^p(\Omega)} + h \left\| \nabla_h \left( v - \Pi_h^V[v] \right) \right\|_{L^p(\Omega;R^3)} \lesssim h^m \|v\|_{W^{m,p}(\Omega)},\ m = 1, 2,\ 1 < p < \infty, \tag{2.11}$$

for any $v \in W^{m,p}(\Omega)$, see Crouzeix and Raviart [6], and [21, Lemma 2.7].

Finally, we recall the well-known property of the Crouzeix-Raviart finite elements,

$$\int_\Omega \nabla_h v \cdot \nabla_h \Pi_h^V[\varphi] \ \mathrm{d}x = \int_\Omega \nabla_h v \cdot \nabla_x \varphi \ \mathrm{d}x \text{ for all } v \in V_{h,0}(\Omega),\ \varphi \in W_0^{1,2}(\Omega), \tag{2.12}$$

see [21, Lemma 2.11] and the estimate for jumps in the Crouzeix-Raviart space,

$$\sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma [[v]]^2 \mathrm{d}S_x \lesssim h \|v\|^2_{H^1_{V_h}(\Omega)} \text{ for all } v \in V_{h,0}(\Omega), \tag{2.13}$$

see Gallouët et al. [17, Lemma 2.2].

## 2.4 Convective terms, upwinds

The *upwind* operator $\mathrm{Up}[r, \mathbf{u}]$ on a face $\Gamma$ is defined as

$$\mathrm{Up}[r, \mathbf{u}] = r^- [\tilde{\mathbf{u}} \cdot \mathbf{n}]^+ + r^+ [\tilde{\mathbf{u}} \cdot \mathbf{n}]^-, \tag{2.14}$$

where we have denoted

$$[c]^+ = \max\{c, 0\},\ [c]^- = \min\{c, 0\},\ \tilde{v} := \tilde{v}_\Gamma = \frac{1}{|\Gamma|} \int_\Gamma v \ \mathrm{dS}_x.$$

For $r, F \in Q_h(\Omega)$, $\mathbf{u} \in V_h(\Omega)$, $\phi \in C^1(\overline{\Omega})$ arbitrary functions, we may use Green's theorem to compute

$$\int_\Omega r\mathbf{u} \cdot \nabla_x \phi \ \mathrm{d}x = \sum_{E \in E_h} \int_E r\mathbf{u} \cdot \nabla_x (\phi - F) \ \mathrm{d}x = \sum_{E \in E_h} \int_{\partial E} (\phi - F) r\mathbf{u} \cdot \mathbf{n} \ \mathrm{dS}_x + \int_\Omega (F - \phi) r \mathrm{div}_h \mathbf{u} \ \mathrm{d}x. \tag{2.15}$$

**Remark 2.2** *Recalling our convention that when integrating over the element boundary $\partial E$, the symbol* $\mathbf{n}$ *denotes always the* outer *normal vector, we have*

$$g^-|_{\Gamma_E} = g,$$

*while $g^+|_{\Gamma_E}$ is the value of a function $g$ on the adjacent element. In particular,*

$$\sum_{\Gamma\in\Gamma_h} \int_\Gamma \mathrm{Up}[r,\mathbf{u}]\ [[g]]\ \mathrm{dS}_x = -\sum_{E\in E_h}\sum_{\Gamma_E\subset\partial E}\int_{\Gamma_E} \mathrm{Up}[r,\mathbf{u}] g\ \mathrm{dS}_x \tag{2.16}$$

$$= -\sum_{E\in E_h}\sum_{\Gamma_E\subset\partial E}\int_{\Gamma_E} g\left(r[\tilde{\mathbf{u}}\cdot\mathbf{n}]^+ + r^+[\tilde{\mathbf{u}}\cdot\mathbf{n}]^-\right)\ \mathrm{dS}_x$$

*for any $r, g \in Q_h$, $\mathbf{u} \in V_{h,0}$.*

Using formula (2.16), we may compute the first integral on the right-hand side of (2.15):

$$\sum_{E\in E_h}\int_{\partial E} (\phi - F) r\mathbf{u}\cdot\mathbf{n}\ \mathrm{dS}_x$$

$$= \sum_{E\in E_h}\int_{\partial E}\phi r\mathbf{u}\cdot\mathbf{n}\ \ \mathrm{dS}_x - \sum_{E\in E_h}\sum_{\Gamma_E\subset\partial E}\int_{\Gamma_E} Fr\Big([\tilde{\mathbf{u}}\cdot\mathbf{n}]^+ + [\tilde{\mathbf{u}}\cdot\mathbf{n}]^-\Big)\ \mathrm{dS}_x$$

$$= \sum_{\Gamma\in\Gamma_h}\int_\Gamma \mathrm{Up}[r,\mathbf{u}][[F]]\ \mathrm{dS}_x + \sum_{E\in E_h}\int_{\partial E}\phi r\mathbf{u}\cdot\mathbf{n}\ \ \mathrm{dS}_x - \sum_{E\in E_h}\sum_{\Gamma_E\subset\partial E}\int_{\Gamma_E} F(r - r^+)[\tilde{\mathbf{u}}\cdot\mathbf{n}]^-\ \mathrm{dS}_x.$$

Plugging the resulting expression in (2.15) we obtain a universal formula which is the key ingredient of the consistency proof for convective terms, namely

$$\int_\Omega r\mathbf{u}\cdot\nabla_x\phi\ \mathrm{d}x = \sum_{\Gamma\in\Gamma_h}\int_\Gamma \mathrm{Up}[r,\mathbf{u}][[F]]\ \mathrm{dS}_x \tag{2.17}$$

$$+\sum_{E\in E_h}\sum_{\Gamma_E\subset\partial E}\int_{\Gamma_E}(F-\phi)\ [[r]]\ [\tilde{\mathbf{u}}\cdot\mathbf{n}]^-\ \mathrm{dS}_x + \sum_{E\in E_h}\int_{\partial E}\phi r(\mathbf{u}-\tilde{\mathbf{u}})\cdot\mathbf{n} + \int_\Omega (F-\phi) r\mathrm{div}_h\mathbf{u}\ \mathrm{d}x$$

for any $r, F \in Q_h(\Omega)$, $\mathbf{u} \in V_{h,0}(\Omega)$, $\phi \in C^1(\overline{\Omega})$.

Finally, we recall the Poincaré type inequality

$$\|v - \tilde{v}_\Gamma\|_{L^q(E)} \stackrel{<}{\sim} ch\|\nabla v\|_{L^q(E)},\ 1 \le q \le \infty \text{ for any } v \in C^1(\overline{E}),\ \Gamma \subset \partial E, \tag{2.18}$$

and report Jensen's inequality

$$\int_{\partial E}|\tilde{v}|\mathrm{d}S \equiv \sum_{\Gamma\subset\partial E}\int_\Gamma |\tilde{v}|^q\ \mathrm{dS}_x \stackrel{<}{\sim} \sum_{\Gamma\subset\partial E}\int_\Gamma |v|^q\ \mathrm{dS}_x \equiv \int_{\partial E}|v|^q \mathrm{dS}_x,\ 1 \le q < \infty, \tag{2.19}$$

for any $v \in C(\overline{E})$, $E \in E_h$.

## 2.5 $L^p - L^q$ and trace estimates for finite elements

The following estimates are easy to obtain by means of scaling arguments. To begin, we claim that

$$\|v\|_{L^q(\partial E)} \stackrel{<}{\sim} \frac{1}{h^{1/q}} \Big( \|v\|_{L^q(E)} + h\|\nabla_x v\|_{L^q(E;R^3)} \Big),\ 1 \le q \le \infty \text{ for any } v \in C^1(\overline{E}), \tag{2.20}$$

from which we readily deduce that

$$\|w\|_{L^q(\partial E)} \stackrel{<}{\sim} \frac{1}{h^{1/q}} \|w\|_{L^q(E)} \text{ for any } 1 \le q \le \infty,\ w \in P_m, \tag{2.21}$$

where $P_m$ denotes the space of polynomials of order $m$.

In a similar way, we obtain

$$\|w\|_{L^p(E)} \stackrel{<}{\sim} h^{3\left(\frac{1}{p} - \frac{1}{q}\right)} \|w\|_{L^q(E)}\ 1 \le q < p \le \infty,\ w \in P_m, \tag{2.22}$$

and, making use of the inequality

$$\left( \sum a_i^p \right)^{1/p} \le \left( \sum a_i^q \right)^{1/q} \text{ whenever } p \ge q, \tag{2.23}$$

we deduce the global version

$$\|w\|_{L^p(\Omega)} \le c h^{3\left(\frac{1}{p} - \frac{1}{q}\right)} \|w\|_{L^q(\Omega)}\ 1 \le q < p \le \infty, \text{ for any } w|_E \in P_m(E),\ E \in E_h. \tag{2.24}$$

**Remark 2.3** *For future use, we record a version of (2.22) and (2.24) for the functions of the time variable $t \in (0,T)$, where the discretization is of order $\Delta t$, specifically,*

$$\|w\|_{L^p(\Delta t)} \stackrel{<}{\sim} (\Delta t)^{\left(\frac{1}{p} - \frac{1}{q}\right)} \|w\|_{L^q(\Delta t)}\ 1 \le q < p \le \infty, \tag{2.25}$$

*and*

$$\|w\|_{L^p(0,T)} \stackrel{<}{\sim} (\Delta t)^{\left(\frac{1}{p} - \frac{1}{q}\right)} \|w\|_{L^q(0,T)}\ 1 \le q < p \le \infty. \tag{2.26}$$

## 2.6 Discrete Sobolev spaces

We finish this introductory part by a short excursion in the theory of discrete analogues of the classical Sobolev spaces. We introduce a discrete $H^1$-(semi)norm

$$\|v\|^2_{H^1_{Q_h}(\Omega)} = \sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma \frac{[[v]]^2}{h} \, \mathrm{dS}_x$$

for $v \in Q_h(\Omega)$. We report the following estimates

$$\|v\|_{L^6(\Omega)} \stackrel{<}{\sim} \|v\|_{H^1_{Q_h}(\Omega)} + \|v\|_{L^2(\Omega)}, \tag{2.27}$$

see Chainais-Hillairet, Droniou [4, Lemma 6.1],

$$\int_{x \in K, \mathrm{dist}[x, \partial\Omega] > |\xi|} |v(x) - v(x - \xi)|^2 \, \mathrm{d}x \stackrel{<}{\sim} \left( |\xi|^2 + h|\xi| \right) \|v\|^2_{H^1_{Q_h}(\Omega; R^3)}, \tag{2.28}$$

for any compact $K \subset \Omega$ and any $v \in Q_h(\Omega)$, see Eymard, Gallouët, Herbin [8, Section5].

**Remark 2.4** *As a matter of fact, validity of (2.28) can be extended to $\Omega$ provided the latter complies with certain geometric restrictions, for the case of convex domain see Christiansen, Munthe-Kaas and Owren [5].*

Next, we may define a discrete $H^1-$norm on space $V_{h,0}(\Omega)$ setting

$$\|v\|^2_{H^1_{V_h}(\Omega)} = \int_\Omega \left( |\nabla_h v|^2 \right) \, \mathrm{d}x. \tag{2.29}$$

In view of future analysis, it seems convenient to have the functions in $V_{h,0}$ defined on the whole space $R^3$, extending them by zero outside $\Omega$. Keeping this convention in mind, we have

$$\|v\|_{L^6(\Omega)} \stackrel{<}{\sim} \|v\|_{H^1_{V_h}(\Omega)}, \tag{2.30}$$

and

$$\int_{R^3} |v(x) - v(x - \xi)|^2 \, \mathrm{d}x \stackrel{<}{\sim} \left( |\xi|^2 + h|\xi| \right) \|v\|^2_{H^1_{V_h}(\Omega)} \tag{2.31}$$

for any $v \in V_{h,0}(\Omega)$, see Gallouët et al. [17, Lemma 3.2].

The next assertion follows from (2.28), (2.31) and can be seen as a special case of the results in [5, Proposition 5.67]:

**Lemma 2.1** *For any function $v \in V_{h,0}$ there exists $R^V_h[v] \in C^\infty_c(\Omega)$ such that*

$$\|\nabla_x R^V_h[v]\|_{L^2(\Omega; R^3)} \stackrel{<}{\sim} \|v\|_{H^1_{V_h}(\Omega)}, \ \|v - R^V_h[v]\|_{L^2(\Omega; R^3)} \stackrel{<}{\sim} h\|v\|_{H^1_{V_h}(\Omega)}.$$

*Similarly, for any $g \in Q_h(\Omega)$ there is $R^Q_h[g] \in C^\infty(\overline{\Omega})$ such that*

$$\|\nabla_x R^Q_h[g]\|_{L^2(K; R^3)} \stackrel{<}{\sim} \|g\|_{H^1_{Q_h}(\Omega)}, \ \|g - R^Q_h[g]\|_{L^2(K; R^3)} \stackrel{<}{\sim} h\|g\|_{H^1_{Q_h}(\Omega)}$$

*for any compact $K \subset \Omega$.*

**Remark 2.5** *The regularizing operators $R_h^V[v]$, $R_h^Q[v]$ can be constructed with the help of the spatial convolution with a regularizing kernel applied to a suitable extension of the function $v$, see [5] for details. The compact set $K$ at the left hand side of the second inequality in Lemma 2.1 can be replaced by $\Omega$ under the same geometric conditions on $\Omega$ as evoked in Remark 2.4, see again [5].*

Finally, we claim the following result that can be proved exactly as [15, Theorem 10.7]:

**Lemma 2.2** *Let $r \geq 0$ be such that*

$$0 < \int_\Omega r \ \mathrm{d}x = M, \ \int_\Omega r^\gamma \ \mathrm{d}x \leq K.$$

*Then there exists a constant $C$ depending only on $M$ and $K$ but not on $h$ such that*

$$\|v\|_{L^2(\Omega)} \leq C(M, K) \left( \|v\|_{H^1_{Q_h}(\Omega)} + \int_\Omega r|v| \ \mathrm{d}x \right).$$

*for any $v \in Q_h(\Omega)$.*

# 3 Numerical scheme, main result

Having collected the necessary preliminary material, we are ready to introduce the numerical scheme to solve the Navier-Stokes-Fourier system.

## 3.1 Numerical scheme

We start by approximating the initial data by their projections onto the space $Q_h(\Omega)$:

$$\varrho_h^0 = \Pi_h^Q[\varrho_0], \ \vartheta_h^0 = \Pi_h^Q[\vartheta_0], \ \mathbf{u}_h^0 = \Pi_h^Q[\mathbf{u}_0]. \tag{3.1}$$

Introducing the discrete time derivative

$$D_t b_h^k = \frac{b_h^k - b_h^{k-1}}{\Delta t}, \ \Delta t \approx h,$$

we define successively the sequence of numerical solutions $[\varrho_h^k, \vartheta_h^k, \mathbf{u}_h^k]_{h>0}$, $k = 1, 2, \ldots$,

$$\varrho_h^k, \ \vartheta_h^k \in Q_h(\Omega), \ \mathbf{u}_h^k \in V_{h,0}(\Omega)$$

satisfying:

Continuity method

$$\int_\Omega D_t \varrho_h^k \phi \ \mathrm{d}x - \sum_{\Gamma\in\Gamma_h} \int_\Gamma \mathrm{Up}[\varrho_h^k, \mathbf{u}_h^k] \ [[\phi]] \ \mathrm{dS}_x + h^{1-\varepsilon} \sum_{\Gamma\in\Gamma_{h,\mathrm{int}}} \int_\Gamma [[\varrho_h^k]][[\phi]] \ \mathrm{dS}_x = 0 \tag{3.2}$$

for all $\phi \in Q_h(\Omega)$, with a parameter $0 < \varepsilon < 1$;

Momentum method

$$\int_\Omega D_t(\varrho_h^k \widehat{\mathbf{u}}_h^k)\cdot \phi \ \mathrm{d}x - \sum_{\Gamma\in\Gamma_h} \int_\Gamma \mathrm{Up}[\varrho_h^k \widehat{\mathbf{u}}_h^k, \mathbf{u}_h^k]\cdot [[\widehat{\phi}]] \ \mathrm{dS}_x \tag{3.3}$$

$$+ \int_\Omega \left[\mu \nabla_h \mathbf{u}_h^k : \nabla_h \phi + \lambda \mathrm{div}_h \mathbf{u}_h^k \mathrm{div}_h \phi\right] \ \mathrm{d}x - \int_\Omega p(\varrho_h^k, \vartheta_h^k) \mathrm{div}_h \phi \ \mathrm{d}x$$

$$+ h^{1-\varepsilon} \sum_{\Gamma\in\Gamma_{h,\mathrm{int}}} \int_\Gamma [[\varrho_h^k]] \left\{ \widehat{u_h^k} \right\} \cdot [[\widehat{\phi}]] \ \mathrm{dS}_x = 0$$

for any $\phi \in V_{h,0}(\Omega)$;

Thermal energy method

$$c_v \int_\Omega D_t(\varrho_h^k \vartheta_h^k)\phi \ \mathrm{d}x - c_v \sum_{\Gamma\in\Gamma_h} \int_\Gamma \mathrm{Up}[\varrho_h^k \vartheta_h^k, \mathbf{u}_h^k] \ [[\phi]] \ \mathrm{dS}_x + \sum_{\Gamma\in\Gamma_{h,\mathrm{int}}} \int_\Gamma \frac{1}{d_\Gamma} [[K(\vartheta_h^k)]] \ [[\phi]] \ \mathrm{dS}_x \tag{3.4}$$

$$= \int_\Omega \left[\mu |\nabla_h \mathbf{u}_h^k|^2 + \lambda |\mathrm{div}_h \mathbf{u}_h^k|^2\right] \phi \ \mathrm{d}x - \int_\Omega \varrho_h^k \vartheta_h^k \mathrm{div}_h \mathbf{u}_h^k \phi \ \mathrm{d}x$$

for any $\phi \in Q_h(\Omega)$.

**Remark 3.1** *The terms involving $h^{1-\varepsilon}$, are needed for technical reasons explained in detail in Section 7.2.1. They are numerical counterparts of the artificial viscosity regularization used in [14, Chapter 7] and were introduced by Eymard et al. [10] to prove convergence of the momentum method. The same approach was applied in [21].*

### 3.2 Main result

In order to state our main result, it is convenient to extend the numerical solution to be defined for *any* $t \geq 0$. To this end, we set

$$\varrho_h(t,\ldots) = \varrho_h^0,\ \vartheta_h(t,\ldots) = \vartheta_h^0,\ \mathbf{u}_h(t,\cdot) = \mathbf{u}_h^0 \text{ for } t \leq 0,$$

$$\varrho_h(t,\cdot) = \varrho_h^k,\ \vartheta_h(t,\cdot) = \vartheta_h^k,\ \mathbf{u}_h(t,\cdot) = \mathbf{u}_h^k \text{ for } t \in [k\Delta t, (k+1)\Delta t),\ k = 1, 2, \ldots,$$

and, accordingly, the discrete time derivative of a quantity $v_h$ reads

$$D_t v_h(t,\cdot) = \frac{v_h(t) - v_h(t - \Delta t)}{\Delta t},\ t > 0.$$

Our main result reads as follows:

**Theorem 3.1** *Let $\Omega \subset R^3$ be a bounded polyhedral domain admitting a tetrahedral mesh satisfying the hypotheses specified in Section 2.1 for any $h > 0$. Let the assumptions (1.4–1.7) be verified. Let $[\varrho_h, \vartheta_h, \mathbf{u}_h]_{h>0}$ be a family of numerical solutions constructed by means of the scheme (3.1 - 3.4) such that*

$$\varrho_h > 0,\ \vartheta_h > 0 \text{ for all } h > 0,$$

*with*

$$\Delta t \approx h.$$

*Then, at least for a suitable subsequence,*

$$\varrho_h \to \varrho \text{ weakly-(*) in } L^\infty(0,T; L^\gamma(\Omega)) \text{ and strongly in } L^1((0,T)\times\Omega),$$

$$\vartheta_h \to \vartheta \text{ weakly in } L^2(0,T; L^6(\Omega)),$$

$$\mathbf{u}_h \to \mathbf{u} \text{ weakly in } L^2(0,T; L^6(\Omega; R^3)),\ \nabla_h \mathbf{u}_h \to \nabla_x \mathbf{u} \text{ weakly in } L^2((0,T)\times\Omega; R^{3\times 3}),$$

*where $[\varrho, \vartheta, \mathbf{u}]$ is a weak solution of the problem (1.1 - 1.3), (1.10 - 1.12) in $(0,T)\times\Omega$ in the sense of Definition 1.1.*

The rest of the paper is devoted to the proof of Theorem 3.1. Note that the *existence* of the numerical solutions $[\varrho_h, \vartheta_h, \mathbf{u}_h]$ can be shown by means of a fixed point argument similarly to [21].

# 4 Renormalization

Mimicking the principal steps of the existence theory developed in [14], we introduce renormalized variants of the continuity method (3.2), the temperature method (3.4) as well as the total energy balance (1.20).

## 4.1 Equation of continuity

Take $b'(\varrho_h^k)\phi$ as a test function in the continuity method:

$$
\begin{aligned}
&\int_\Omega \phi \frac{\varrho_h^k - \varrho_h^{k-1}}{\Delta t} b'(\varrho_h^k) \ \mathrm{d}x \\
&= \int_\Omega \phi \left[ \frac{b(\varrho_h^k) - b(\varrho_h^{k-1})}{\Delta t} + \frac{\Delta t}{2} b''(\xi_h^k) \left( \frac{\varrho_h^k - \varrho_h^{k-1}}{\Delta t} \right)^2 \right] \ \mathrm{d}x \\
&= \int_\Omega D_t b(\varrho_h^k)\phi \ \mathrm{d}x + \int_\Omega \frac{\Delta t}{2} b''(\xi_h^k) \left( \frac{\varrho_h^k - \varrho_h^{k-1}}{\Delta t} \right)^2 \phi \ \mathrm{d}x
\end{aligned}
$$

for a certain $\xi_h^k \in \mathrm{co}\{\varrho_h^{k-1}, \varrho_h^k\}$, where we have denoted

$$
\mathrm{co}\{A, B\} = [\min\{A, B\}, \max\{A, B\}].
$$

Similarly, the upwind term can be handled as follows:

$$
\begin{aligned}
&\sum_{\Gamma \in \Gamma_h} \int_\Gamma \mathrm{Up}[\varrho_h^k, \mathbf{u}_h^k] \ [[b'(\varrho_h^k)\phi]] \ \mathrm{dS}_x = - \sum_{E \in E_h} \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} \mathrm{Up}[\varrho_h^k, u_h^k] b'(\varrho_h^k)\phi \ \mathrm{dS}_x \\
&= - \sum_{E \in E_h} \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} \phi b'(\varrho_h^k) \left[ \varrho_h^k [\tilde{\mathbf{u}}_h^k \cdot \mathbf{n}]^+ + \left(\varrho_h^k\right)^+ [\tilde{\mathbf{u}}_h^k \cdot \mathbf{n}]^- \right] \ \mathrm{dS}_x \\
&= - \sum_{E \in E_h} \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} \phi \left[ b(\varrho_h^k) [\tilde{\mathbf{u}}_h^k \cdot \mathbf{n}]^+ + b\left( \left(\varrho_h^k\right)^+ \right) [\tilde{\mathbf{u}}_h^k \cdot \mathbf{n}]^- \right] \ \mathrm{dS}_x \\
&\quad + \sum_{E \in E_h} \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} \phi \left( b(\varrho_h^k) - b'(\varrho_h^k)\varrho_h^h \right) [\tilde{\mathbf{u}}_h^k \cdot \mathbf{n}]^+ \ \mathrm{dS}_x \\
&\quad + \sum_{E \in E_h} \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} \phi \left( b\left( \left(\varrho_h^k\right)^+ \right) - b'(\varrho_h^k) \left(\varrho_h^k\right)^+ \right) [\tilde{\mathbf{u}}_h^k \cdot \mathbf{n}]^- \ \mathrm{dS}_x \\
&= \sum_{\Gamma \in \Gamma_h} \int_\Gamma \mathrm{Up}[b(\varrho_h^k), \mathbf{u}_h^k] \ [[\phi]] \ \mathrm{dS}_x + \sum_{E \in E_h} \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} \phi \left( b(\varrho_h^k) - b'(\varrho_h^k)\varrho_h^k \right) \tilde{\mathbf{u}} \cdot \mathbf{n} \ \mathrm{d}S_x
\end{aligned}
$$

$$
+\sum_{E\in E_h}\sum_{\Gamma_E\subset\partial E}\int_{\Gamma_E}\phi\left[b\left(\left(\varrho_h^k\right)^+\right)-b'(\varrho_h^k)\left(\left(\varrho_h^k\right)^+-\varrho_h^k\right)-b(\varrho_h^k)\right][\tilde{\mathbf{u}}_h^k\cdot\mathbf{n}]^-\ \mathrm{dS}_x
$$

$$
=\sum_{\Gamma\in\Gamma_h}\int_\Gamma \mathrm{Up}[b(\varrho_h^k),\mathbf{u}_h^k]\ [[\phi]]\ \mathrm{dS}_x+\int_\Omega\phi\ \left(b(\varrho_h^k)-b'(\varrho_h^k)\varrho_h^k\right)\mathrm{div}_h\mathbf{u}_h^k\ \mathrm{d}x
$$

$$
+\frac{1}{2}\sum_{E\in E_h}\sum_{\Gamma_E\subset\partial E}\int_{\Gamma_E}\phi\ b''(\eta_h^k)\left(\left(\varrho_h^k\right)^+-\varrho_h^k\right)^2[\tilde{\mathbf{u}}_h^k\cdot\mathbf{n}]^-\ \mathrm{dS}_x.
$$

We thus obtain the *renormalized continuity method*:

$$
\int_\Omega D_t b(\varrho_h^k)\phi\ \mathrm{d}x-\sum_{\Gamma\in\Gamma_h}\int_\Gamma \mathrm{Up}[b(\varrho_h^k),\mathbf{u}_h^k]\ [[\phi]]\ \mathrm{dS}_x+\int_\Omega\phi\ \left(b'(\varrho_h^k)\varrho_h^k-b(\varrho_h^k)\right)\mathrm{div}_h\mathbf{u}_h^k\ \mathrm{d}x \tag{4.1}
$$

$$
=-\int_\Omega\frac{\Delta t}{2}b''(\xi_{\varrho,h}^k)\left(\frac{\varrho_h^k-\varrho_h^{k-1}}{\Delta t}\right)^2\phi\ \mathrm{d}x-h^{1-\varepsilon}\sum_{\Gamma\in\Gamma_{h,\mathrm{int}}}\int_\Gamma\phi\ b''(\overline{\eta}_{\varrho,h}^k)[[\varrho_h^k]]^2\ \mathrm{dS}_x
$$

$$
-\frac{1}{2}\sum_{\Gamma\in\Gamma_{h,\mathrm{int}}}\int_\Gamma\phi\ b''(\eta_{\varrho,h)}^k[[\varrho_h^k]]^2|\tilde{\mathbf{u}}_h^k\cdot\mathbf{n}|\ \mathrm{dS}_x
$$

for any $\phi\in Q_h(\Omega)$, where

$$
\xi_{\varrho,h}^k\in\mathrm{co}\{\varrho_h^{k-1},\varrho_h^k\}\text{ on each element }E\in E_h,\ \eta_{\varrho,h}^k,\overline{\eta}_{\varrho,h}^k\in\mathrm{co}\{\varrho_h^k,(\varrho_h^k)^+\}\text{ on each face }\Gamma\in\Gamma_{h,\mathrm{int}}.
$$

## 4.2 Thermal energy balance

Similarly to the previous section, we use the quantities $\chi'(\vartheta_h^k)\phi$ as test functions in the thermal energy method (3.4). After a bit tedious but straightforward manipulation summarized in Lemma 8.1 in Appendix, we arrive at the *renormalized thermal energy method*:

$$
c_v\int_\Omega D_t\left(\varrho_h^k\chi(\vartheta_h^k)\right)\phi\ \mathrm{d}x-c_v\sum_{\Gamma\in\Gamma_h}\int_\Gamma\mathrm{Up}(\varrho_h^k\chi(\vartheta_h^k),\mathbf{u}_h^k)\ [[\phi]]\ \mathrm{dS}_x+\sum_{\Gamma\in\Gamma_{h,\mathrm{int}}}\int_\Gamma\frac{1}{d_\Gamma}[[K(\vartheta_h^k)]]\ [[\chi'(\vartheta_h^k)\phi]]\ \mathrm{dS}_x \tag{4.2}
$$

$$
=\int_\Omega\left(\mu|\nabla_h\mathbf{u}_h^k|^2+\lambda|\mathrm{div}_h\mathbf{u}_h^k|^2\right)\chi'(\vartheta_h^k)\phi\ \mathrm{d}x-\int_\Omega\chi'(\vartheta_h^k)\varrho_h^k\vartheta_h^k\mathrm{div}_h\mathbf{u}_h^k\phi\ \mathrm{d}x
$$

$$
-c_v\frac{\Delta t}{2}\int_\Omega\chi''(\xi_{\vartheta,h}^k)\varrho_h^{k-1}\left(\frac{\vartheta_h^k-\vartheta_h^{k-1}}{\Delta t}\right)^2\phi\ \mathrm{d}x+\frac{c_v}{2}\sum_{E\in E_h}\sum_{\Gamma_E\subset\partial E}\int_{\Gamma_E}\phi\chi''(\eta_{\vartheta,h}^k)[[\vartheta_h^k]]^2\left(\varrho_h^k\right)^+[\tilde{\mathbf{u}}_h^k\cdot\mathbf{n}]^-\ \mathrm{dS}_x
$$

$$-h^{1-\varepsilon}c_v \sum_{\Gamma\in\Gamma_{h,\mathrm{int}}} \int_\Gamma [[\varrho_h^k]]\ [[\Big(\chi(\vartheta_h^k) - \chi'(\vartheta_h^k)\vartheta_k^k\Big)\,\phi]]\ \mathrm{dS}_x$$

for any $\phi \in Q_h(\Omega)$, with

$$\xi_{\vartheta,h}^k \in \mathrm{co}\{\vartheta_h^{k-1}, \vartheta_h^k\},\ \ \eta_{\vartheta,h}^k \in \mathrm{co}\{\vartheta_h^k, (\vartheta_h^k)^+\}.$$

## 4.3 Total energy balance

The total energy balance is the sum of the kinetic and internal energy equations. In order to derive its discrete counterpart, we take $\phi = \mathbf{u}_h^k$ as test function in the momentum method (3.3), $\phi = -\frac{1}{2}|\hat{\mathbf{u}}_h^k|^2$ in the momentum method (3.2) and $\phi = 1$ in the thermal energy method.

To begin, we claim that

$$\int_\Omega p(\varrho_h^k, \vartheta_h^k)\mathrm{div}_h\mathbf{u}_h^k\ \mathrm{d}x - \int_\Omega \left(\mu\left|\nabla_h\mathbf{u}_h^k\right|^2 + \lambda\left|\mathrm{div}_h\mathbf{u}_h^k\right|^2\right)\ \mathrm{d}x \tag{4.3}$$

$$= \int_\Omega \left[a\left(\varrho_h^k\right)^\gamma + b\varrho_h^k\right)\mathrm{div}_h\mathbf{u}_h^k\ \mathrm{d}x + \int_\Omega \varrho_h^k\vartheta_h^k\mathrm{div}_h\mathbf{u}_h^k\ \mathrm{d}x - \int_\Omega \left(\mu\left|\nabla_h\mathbf{u}_h^k\right|^2 + \lambda\left|\mathrm{div}_h\mathbf{u}_h^k\right|^2\right)\ \mathrm{d}x,$$

where, in accordance with the renormalized continuity method (4.1),

$$\int_\Omega \left[a\left(\varrho_h^k\right)^\gamma + b\varrho_h^k\right]\mathrm{div}_h\mathbf{u}_h^k\ \mathrm{d}x = -\int_\Omega D_t\left[\frac{a}{\gamma-1}\left(\varrho_h^k\right)^\gamma + b\varrho_h^k\log(\varrho_h^k)\right]\ \mathrm{d}x \tag{4.4}$$

$$-\frac{1}{2}\sum_{\Gamma\in\Gamma_{h,\mathrm{int}}}\int_\Gamma b''(\eta_{\varrho,h}^k)[[\varrho_h^k]]^2|\tilde{\mathbf{u}}_h^k\cdot\mathbf{n}|\ \mathrm{dS}_x$$

$$-h^{1-\varepsilon}\sum_{\Gamma\in\Gamma_{h,\mathrm{int}}}\int_\Gamma b''(\overline{\eta}_{\varrho,h}^k)[[\varrho_h^k]]^2\ \mathrm{dS}_x - \int_\Omega \frac{\Delta t}{2}b''(\xi_{\varrho,h}^k)\left(\frac{\varrho_h^k-\varrho_h^{k-1}}{\Delta t}\right)^2\ \mathrm{d}x,$$

with $b(r) = a/(\gamma-1)r^\gamma + br\log(r)$.

Next, as a consequence of thermal energy method (3.4),

$$\int_\Omega \varrho_h^k\vartheta_h^k\mathrm{div}_h\mathbf{u}_h^k\ \mathrm{d}x = -c_v\int_\Omega D_t\left(\varrho_h^k\vartheta_h^k\right)\ \mathrm{d}x + \int_\Omega \left(\mu\left|\nabla_h\mathbf{u}_h^k\right|^2 + \lambda\left|\mathrm{div}_h\mathbf{u}_h^k\right|^2\right)\ \mathrm{d}x. \tag{4.5}$$

Moreover, by the same token as in Lemma 8.1 (see also [21, Section 4]),

$$\int_\Omega D_t(\varrho_h^k\hat{\mathbf{u}}_h^k)\cdot\mathbf{u}_h^k\ \mathrm{d}x - \sum_{\Gamma_{h,\mathrm{int}}}\int_\Gamma \mathrm{Up}[\varrho_h^k\hat{\mathbf{u}}_h^k, \mathbf{u}_h^k]\cdot[[\hat{\mathbf{u}}_h^k]]\ \mathrm{dS}_x \tag{4.6}$$

$$-\frac{1}{2}\int_\Omega D_t\varrho_h^k|\hat{\mathbf{u}}_h^k|^2\ \mathrm{d}x + \frac{1}{2}\sum_{\Gamma_{h,\mathrm{int}}}\int_\Gamma \mathrm{Up}[\varrho_h^k, \mathbf{u}_h^k]\ [[\ |\hat{\mathbf{u}}_h^k|^2\ ]]\ \mathrm{dS}_x$$

$$= \frac{1}{2}\int_\Omega \varrho_h^k |\hat{\mathbf{u}}_h^k|^2 \ \mathrm{d}x + \frac{\Delta t}{2}\int_\Omega \varrho_h^{k-1}\Big|\frac{\hat{\mathbf{u}}_h^k - \hat{\mathbf{u}}_h^{k-1}}{\Delta t}\Big|^2 \ \mathrm{d}x$$
$$-\frac{1}{2}\sum_{E\in E_h}\sum_{\Gamma_E\in\partial E}\int_{\Gamma_E}\left(\varrho_h^k\right)^+ [[\hat{\mathbf{u}}_h^k]]^2\,[\tilde{\mathbf{u}}_h^k\cdot\mathbf{n}]^- \ \mathrm{dS}_x$$

and

$$h^{1-\varepsilon}\sum_{\Gamma_{h,\mathrm{int}}}\Big[\int_\Gamma [[\varrho_h^k]]\,\{\hat{\mathbf{u}}_h^k\}\cdot[[\hat{\mathbf{u}}_h^k]] \ \mathrm{dS}_x - \frac{1}{2}\int_\Gamma [[\varrho_h^k]]\,[[|\hat{\mathbf{u}}_h^k|^2]] \ \mathrm{dS}_x\Big] = 0 \tag{4.7}$$

Gathering formulas (4.3–4.7) we obtain the total energy balance

$$D_t\int_\Omega\left[\frac{1}{2}\varrho_h^k|\hat{\mathbf{u}}_h^k|^2 + c_v\varrho_h^k\vartheta_h^k + \frac{a}{\gamma-1}\left(\varrho_h^k\right)^\gamma + b\varrho_h^k\log(\varrho_h^k)\right] \ \mathrm{d}x \tag{4.8}$$
$$+\frac{\Delta t}{2}\int_\Omega\left(b''(\xi_{\varrho,h}^k)\left|\frac{\varrho_h^k-\varrho_h^{k-1}}{\Delta t}\right|^2 + \varrho_h^{k-1}\left|\frac{\hat{\mathbf{u}}_h^k-\hat{\mathbf{u}}_h^{k-1}}{\Delta t}\right|^2\right) \ \mathrm{d}x$$
$$-\sum_{E\in E_h}\sum_{\Gamma_E\subset\partial E}\int_{\Gamma_E}\left(\varrho_h^k\right)^+[\tilde{\mathbf{u}}_h^k\cdot\mathbf{n}]^-\frac{[[\hat{\mathbf{u}}_h^k]]^2}{2} \ \mathrm{dS}_x$$
$$+h^{1-\varepsilon}\sum_{\Gamma\in\Gamma_{h,\mathrm{int}}}\int_\Gamma b''(\overline{\eta}_{\varrho,h}^k)\,[[\varrho_h^k]]^2 \ \mathrm{dS}_x - \frac{1}{2}\sum_{\Gamma\in\Gamma_{h,\mathrm{int}}}\int_\Gamma b''(\eta_{\varrho,h}^k)\,[[\varrho_h^k]]^2\,[\tilde{\mathbf{u}}_h^k\cdot\mathbf{n}]^- \ \mathrm{dS}_x = 0,$$

where numbers $\xi_{\varrho,h}^k$, $\eta_{\varrho,h}^k$, $\overline{\eta}_{\varrho,h}^k$ are defined in (4.1). Consequently,

$$D_t\int_\Omega\left[\frac{1}{2}\varrho_h^k|\hat{\mathbf{u}}_h^k|^2 + c_v\varrho_h^k\vartheta_h^k + \frac{a}{\gamma-1}\left(\varrho_h^k\right)^\gamma + b\varrho_h^k\log(\varrho_h^k)\right] \ \mathrm{d}x \tag{4.9}$$
$$+\frac{\Delta t}{2}\int_\Omega\left(A\left|\frac{\varrho_h^k-\varrho_h^{k-1}}{\Delta t}\right|^2 + \varrho_h^{k-1}\left|\frac{\hat{\mathbf{u}}_h^k-\hat{\mathbf{u}}_h^{k-1}}{\Delta t}\right|^2\right) \ \mathrm{d}x$$
$$-\sum_{E\in E_h}\sum_{\Gamma_E\subset\partial E}\int_{\Gamma_E}\left(\varrho_h^k\right)^+[\tilde{\mathbf{u}}_h^k\cdot\mathbf{n}]^-\frac{\left|\hat{\mathbf{u}}_h^k-\left(\hat{\mathbf{u}}_h^k\right)^+\right|^2}{2} \ \mathrm{dS}_x + \frac{A}{2}\sum_{\Gamma\in\Gamma_{h,\mathrm{int}}}\int_\Gamma\left(h^{1-\varepsilon}+|\tilde{\mathbf{u}}_h^k\cdot\mathbf{n}|\right)[[\varrho_h^k]]^2 \ \mathrm{dS}_x \le 0$$
$$\text{with } A = \min_{\varrho>0}\left\{a\gamma\varrho^{\gamma-2}+\frac{b}{\varrho}\right\} > 0.$$

# 5 Stability

In this section, we derive *uniform bounds* for the family $[\varrho_h, \vartheta_h, \mathbf{u}_h]_{h>0}$ independent of the time step $\Delta t \approx h$ and the element size $h$.

## 5.1 Mass conservation, energy bounds

Taking $\phi \equiv 1$ in the continuity method (3.2) we obtain

$$\int_\Omega \varrho_h(t,\cdot)\ \mathrm{d}x = \int_\Omega \varrho_h^0\ \mathrm{d}x = \int_\Omega \varrho_0\ \mathrm{d}x \text{ for any } h > 0, \tag{5.1}$$

meaning the total mass is conserved by the scheme.

Next, the energy balance (4.9) yields

$$\int_\Omega \left[\frac{1}{2}\varrho_h|\widehat{\mathbf{u}}_h|^2 + c_v\varrho_h\vartheta_h + \frac{a}{\gamma-1}\left(\varrho_h\right)^\gamma + b\varrho_h\log(\varrho_h)\right](\tau,\cdot)\ \mathrm{d}x \tag{5.2}$$

$$\leq \int_\Omega \left[\frac{1}{2}\varrho_h^0|\widehat{\mathbf{u}}_h^0|^2 + c_v\varrho_h^0\vartheta_h^0 + \frac{a}{\gamma-1}\left(\varrho_h^0\right)^\gamma + b\varrho_h^0\log(\varrho_h^0)\right]\ \mathrm{d}x \equiv E_{0,h},\ E_{0,h} \lesssim 1;$$

whence, in particular,

$$\sup_{\tau\in(0,T)}\|\sqrt{\varrho_h}\widehat{\mathbf{u}}_h(\tau,\cdot)\|_{L^2(\Omega)} \lesssim 1, \tag{5.3}$$

$$\sup_{\tau\in(0,T)}\|\varrho_h\vartheta_h(\tau,\cdot)\|_{L^1(\Omega)} \lesssim 1, \tag{5.4}$$

$$\sup_{\tau\in(0,T)}\|\varrho_h[\log\vartheta_h]^+(\tau,\cdot)\|_{L^1(\Omega)} \lesssim 1, \tag{5.5}$$

and

$$\sup_{\tau\in(0,T)}\|\varrho_h(\tau,\cdot)\|_{L^\gamma(\Omega)} \lesssim 1, \tag{5.6}$$

where the bounds are uniform for $h \to 0$.

Finally, we record the bounds on the numerical dissipation:

$$\sum_{k\geq 0}\int_\Omega \left[\left|\varrho_h^k - \varrho_h^{k-1}\right|^2 + \varrho_h^{k-1}\left|\widehat{\mathbf{u}}_h^k - \widehat{\mathbf{u}}_h^{k-1}\right|^2\right]\ \mathrm{d}x \lesssim 1, \tag{5.7}$$

$$-\sum_{E\in E_h}\sum_{\Gamma_E\subset\partial E}\int_0^T\int_{\Gamma_E}\left(\varrho_h\right)^+[\tilde{\mathbf{u}}_h\cdot\mathbf{n}]^-\left|\widehat{\mathbf{u}}_h - (\widehat{\mathbf{u}}_h)^+\right|^2\ \mathrm{dS}_x\ \mathrm{d}t \lesssim 1 \tag{5.8}$$

and

$$\sum_{\Gamma\in\Gamma_{h,\mathrm{int}}}\int_0^T\int_\Gamma\left(|\tilde{\mathbf{u}}_h\cdot\mathbf{n}| + h^{1-\varepsilon}\right)[[\varrho_h]]^2\ \mathrm{dS}_x\ \mathrm{d}t \lesssim 1. \tag{5.9}$$

## 5.2 Entropy bounds

The entropy bounds are obtained by taking $\chi = \log$ and $\phi = 1$ in (4.2):

$$c_v \int_\Omega D_t\left(\varrho_h^k \log(\vartheta_h^k)\right) \, \mathrm{d}x \geq - \sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma \frac{1}{d_\Gamma} [[K(\vartheta_h^k)]] \, [[(\vartheta_h^k)^{-1}]] \, \mathrm{dS}_x \tag{5.10}$$

$$+ \int_\Omega \left(\mu |\nabla_h \mathbf{u}_h^k|^2 + \lambda |\mathrm{div}_h \mathbf{u}_h^k|^2\right) \frac{1}{\vartheta_h^k} \, \mathrm{d}x - \int_\Omega \varrho_h^k \mathrm{div}_h \mathbf{u}_h^k \, \mathrm{d}x$$

$$+ \frac{\Delta t}{2} c_v \int_\Omega (\xi_{\vartheta,h}^k)^{-2} \varrho_h^{k-1} \left(\frac{\vartheta_h^k - \vartheta_h^{k-1}}{\Delta t}\right)^2 \mathrm{d}x$$

$$- \frac{1}{2} c_v \sum_{E \in E_h} \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} (\eta_{\vartheta,h}^k)^{-2} \left(\vartheta_h^k - \left(\vartheta_h^k\right)^+\right)^2 \left(\varrho_h^k\right)^+ [\tilde{\mathbf{u}}_h^k \cdot \mathbf{n}]^- \, \mathrm{dS}_x$$

$$- h^{1-\varepsilon} c_v \sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma [[\varrho_h^k]] \, [[\log(\vartheta_h^k)]] \, \mathrm{dS}_x,$$

where, in accordance with the renormalized continuity method (4.1),

$$\int_\Omega \varrho_h^k \mathrm{div}_h \mathbf{u}_h^k \, \mathrm{d}x \leq - \int_\Omega D_t\left(\varrho_h^k \log(\varrho_h^k)\right) \mathrm{d}x. \tag{5.11}$$

In order to reveal the piece of information hidden in (5.10), we need the following technical lemma.

**Lemma 5.1** *Let $F$ and $G$ be two convex functions defined on an open interval $I \subset R$ and such that*

$$F' \geq 0, \ G' \leq 0.$$

*Then*

$$(F(B) - F(A))(G(A) - G(B)) \geq -\frac{1}{4} F'\left(\frac{A+B}{2}\right) G'\left(\frac{A+B}{2}\right) (A-B)^2$$

*for any $A, B \in I$.*

**Proof:**

Without loss of generality, we may suppose $A < B$ and set $C = \frac{A+B}{2}$. Since $F$ is non-decreasing and convex, we have

$$F(B) - F(A) = F(B) - F(C) + F(C) - F(A) \geq F(B) - F(C) \geq F'(C)(B-C) = \frac{1}{2} F'(C)(B-A), \tag{5.12}$$

and, similarly,

$$G(A)-G(B)=G(A)-G(C)+G(C)-G(B)\geq G(A)-G(C)\geq G'(C)(A-C)=-\frac{1}{2}G'(C)(B-A).$$

Q.E.D.

Now, we first use the hypothesis (1.9) to deduce that

$$-\int_\Gamma \frac{1}{h}[[\tilde{K}(\vartheta_h^k)]]\ [[(\vartheta_h^k)^{-1}]]\ \mathrm{dS}_x \leq -\frac{1}{\underline{\kappa}}\int_\Gamma \frac{1}{h}[[K(\vartheta_h^k)]]\ [[(\vartheta_h^k)^{-1}]]\ \mathrm{dS}_x, \tag{5.13}$$

where

$$\tilde{K}(\vartheta)=\vartheta+\frac{1}{3}\vartheta^3-\ \text{a convex function for } \vartheta\geq 0.$$

Consequently, we may apply Lemma 5.1 with $F=\tilde{K}(\vartheta)$, $G=\vartheta^{-1}$ to obtain

$$\sum_{\Gamma\in\Gamma_{h,\mathrm{int}}}\int_0^T\int_\Gamma \frac{1}{h}\left(\frac{1+(\{\vartheta_h\})^2}{(\{\vartheta_h\})^2}\right)[[\vartheta_h]]^2\ \mathrm{dS}_x\ \mathrm{d}t \stackrel{<}{\sim} -\int_0^T\left[\sum_{\Gamma\in\Gamma_{h,\mathrm{int}}}\int_\Gamma\frac{1}{h}[[K(\vartheta_h)]]\ [[(\vartheta_h)^{-1}]]\ \mathrm{dS}_x\right]\ \mathrm{d}t$$

in particular

$$\sum_{\Gamma\in\Gamma_{h,\mathrm{int}}}\int_0^T\int_\Gamma \frac{[[\vartheta_h]]^2}{h}\mathrm{dS}_x\ \mathrm{d}t \stackrel{<}{\sim} -\int_0^T\left[\sum_{\Gamma\in\Gamma_{h,\mathrm{int}}}\int_\Gamma\frac{1}{h}[[K(\vartheta_h)]]\ [[(\vartheta_h)^{-1}]]\ \mathrm{dS}_x\right]\ \mathrm{d}t.$$

Next observe that we have a similar bound for $\log(\vartheta_h)$, specifically,

$$\sum_{\Gamma\in\Gamma_{h,\mathrm{int}}}\int_0^T\int_\Gamma \frac{[[\log(\vartheta_h)]]^2}{h}\mathrm{dS}_x\ \mathrm{d}t \stackrel{<}{\sim} -\int_0^T\left[\sum_{\Gamma\in\Gamma_{h,\mathrm{int}}}\int_\Gamma\frac{1}{h}[[K(\vartheta_h)]]\ [[(\vartheta_h)^{-1}]]\ \mathrm{dS}_x\right]\ \mathrm{d}t. \tag{5.14}$$

Indeed it is enough to observe that

$$\log(A)-\log(B)\stackrel{<}{\sim}\frac{A-B}{\sqrt{AB}}\ \text{for all } 0<B\leq A,$$

or, equivalently,

$$\log(Z)+\frac{1}{\sqrt{Z}}\stackrel{<}{\sim}\sqrt{Z}\ \text{for all } Z\geq 1.$$

Now, the desired estimate follows by taking $\tilde{K}(\vartheta)=\vartheta$ in (5.13).

The estimates (5.9), (5.14) control the $h^{1-\varepsilon}$-term in (5.10). Consequently, writing $\varrho_h \log \vartheta_h = \varrho_h[\log\vartheta_h]^+ + \varrho_h[\log\vartheta_h]^-$, using (5.11) and the energy bounds (5.3 - 5.6), we deduce from (5.10),

$$\sup_{\tau\in(0,T)} \|\varrho_h \log(\vartheta_h)(\tau,\cdot)\|_{L^1(\Omega)} \lesssim 1, \tag{5.15}$$

$$\int_0^T \int_\Omega \frac{1}{\vartheta_h} |\nabla_h \mathbf{u}_h|^2 \ \mathrm{d}x \ \mathrm{d}t \lesssim 1, \tag{5.16}$$

$$\sum_{\Gamma\in\Gamma_{h,\mathrm{int}}} \int_0^T \int_\Gamma \frac{[[\vartheta_h]]^2}{h} \mathrm{dS}_x \ \mathrm{d}t \lesssim 1, \tag{5.17}$$

and

$$\sum_{\Gamma\in\Gamma_{h,\mathrm{int}}} \int_0^T \int_\Gamma \frac{[[\log(\vartheta_h)]]^2}{h} \mathrm{dS}_x \ \mathrm{d}t \lesssim 1. \tag{5.18}$$

Formulas (5.17–5.18) together with the evident inequality $A^\alpha - B^\alpha \lesssim A - B + \log A - \log B$, $0 < B < A$, $\alpha \in (0,1)$ imply

$$\sum_{\Gamma\in\Gamma_{h,\mathrm{int}}} \int_0^T \int_\Gamma \frac{[[\vartheta_h^\alpha]]^2}{h} \mathrm{dS}_x \ \mathrm{d}t \lesssim 1, \ 0 \le \alpha \le 1. \tag{5.19}$$

We also collect the bounds resulting from numerical dissipation:

$$\sum_{k\ge 0} \int_\Omega (\xi^k_{\vartheta,h})^{-2} \varrho_h^{k-1} \left(\vartheta_h^k - \vartheta_h^{k-1}\right)^2 \ \mathrm{d}x \lesssim 1, \ \xi^k_{\vartheta,h} \in \mathrm{co}\{\vartheta_h^{k-1}, \vartheta_h^k\}, \tag{5.20}$$

and

$$-\sum_{E\in E_h} \sum_{\Gamma_E \subset \partial E} \int_0^T \int_{\Gamma_E} (\eta_{\vartheta,h})^{-2} [[\vartheta_h]]^2 \left(\varrho_h\right)^+ [\tilde{\mathbf{u}}_h \cdot \mathbf{n}]^- \ \mathrm{dS}_x \ \mathrm{d}t \lesssim 1, \ \eta_{\vartheta,h} \in \mathrm{co}\{\vartheta_h, \vartheta_h^+\}. \tag{5.21}$$

Finally, we apply Lemma 2.2 to deduce

$$\|\vartheta_h\|_{L^2(0,T;L^6(\Omega))} + \|\log(\vartheta_h)\|_{L^2(0,T;L^6(\Omega))} \lesssim 1 \tag{5.22}$$

by virtue of (2.27) and (5.1), (5.6), (5.15), (5.17), and (5.18).

### 5.3 Temperature estimates

Our ultimate goal in this sections is to derive refined estimates for the family of approximate temperatures. To this end, take $\chi = (\vartheta_h^k)^\alpha$, $0 < \alpha < 1$, together with the test function $\phi = 1$, in the renormalized thermal energy balance (4.2) to obtain

$$
\begin{aligned}
&-\alpha \sum_{\Gamma \in \Gamma_{h,\rm int}} \int_\Gamma \frac{1}{h} [[K(\vartheta_h)]]\ [[(\vartheta_h)^{\alpha-1}]]\ {\rm dS}_x + \alpha\mu \int_\Omega \vartheta_h^{\alpha-1} |\nabla_h \mathbf{u}_h|^2\ {\rm d}x\ {\rm d}t \qquad (5.23)\\
&+ c_v \alpha(1-\alpha) \frac{\Delta t}{2} \sum_{k=1} \int_\Omega (\xi_{\vartheta,h}^k)^{\alpha-2} \varrho_h^{k-1} \left( \frac{\vartheta_h^k - \vartheta_h^{k-1}}{\Delta t} \right)^2\ {\rm d}x \\
&+ \frac{c_v}{2} \alpha(1-\alpha) \sum_{E \in E_h} \sum_{\Gamma_E \subset \partial E} \int_\Gamma (\eta_{\vartheta,h}^k)^{\alpha-2} [[\vartheta_h^k]]^2 \left( \varrho_h^k \right)^+ [\tilde{\mathbf{u}}_h^k \cdot \mathbf{n}]^-\ {\rm dS}_x \\
&\stackrel{<}{\sim} c_v \int_\Omega D_t (\varrho_h^k (\vartheta_h^k)^\alpha)\ {\rm d}x + \alpha \int_\Omega \varrho_h^k (\vartheta_h^k)^\alpha {\rm div}_h \mathbf{u}_h^k\ {\rm d}x + h^{1-\varepsilon} c_v (1-\alpha) \sum_{\Gamma \in \Gamma_{h,\rm int}} \int_\Gamma [[\varrho_h^k]]\ [[(\vartheta_h^k)^\alpha]]\ {\rm dS}_x
\end{aligned}
$$

Note that, as $\alpha < 1$, the extra $h^{1-\varepsilon}$-term is small in view of (5.9), (5.19).

It follows from (5.6), (5.22) that

$$
\varrho_h \vartheta_h^{\frac{\alpha+1}{2}} \text{ is bounded in } L^2(0,T;L^2(\Omega));
$$

whence

$$
\Big| \int_\Omega \varrho_h (\vartheta_h)^\alpha {\rm div}_h \mathbf{u}_h\ {\rm d}x \Big| \leq c + \frac{1}{2} \alpha\mu \int_\Omega \vartheta_h^{\alpha-1} |\nabla_h \mathbf{u}_h|^2\ {\rm d}x \text{ with some } c > 0.
$$

Also, by virtue of (5.4–5.6),

$$
\Big| \int_0^T \int_\Omega D_t (\varrho_h^k (\vartheta_h^k)^\alpha)\ {\rm d}x\ {\rm d}t \Big| \text{ is bounded.}
$$

Consequently, (5.23) implies

$$
- \sum_{\Gamma \in \Gamma_{h,\rm int}} \int_\Gamma \int_0^T \frac{1}{h} [[K(\vartheta_h)]]\ [[(\vartheta_h)^{\alpha-1}]]\ {\rm dS}_x \stackrel{<}{\sim} 1 \text{ for all } 0 < \alpha < 1. \qquad (5.24)
$$

Note however that (5.23) "blows up" when $\alpha$ approaches the extremal value 1.

Now another application of Lemma 5.1 (with $F(\vartheta) = \vartheta + \frac{1}{3}\vartheta^3$, $G(\vartheta) = \vartheta^{\alpha-1}$) gives rise to

$$
\sum_{\Gamma \in \Gamma_{h,\rm int}} \int_0^T \int_\Gamma \frac{1}{h} \left\{ \vartheta_h \right\}^\alpha [[\vartheta_h]]^2\ {\rm dS}_x\ {\rm d}t \stackrel{<}{\sim} 1 \text{ for all } 0 < \alpha < 1
$$

yielding by (5.12) with $F(\vartheta) = \vartheta^{1+\frac{\alpha}{2}}$,

$$\sum_{\Gamma \in \Gamma_h} \int_0^T \int_\Gamma \frac{[[\vartheta_h^{1+\frac{\alpha}{2}}]]^2}{h} \, \mathrm{dS}_x \stackrel{<}{\sim} 1 \text{ for all } 0 \leq \alpha < 1; \tag{5.25}$$

whence, by virtue of (2.27) and Lemma 2.2,

$$\|\vartheta_h\|_{L^p(0,T;L^q(\Omega))} \stackrel{<}{\sim} 1 \text{ for any } 1 \leq p < 3, \ 1 \leq q < 9. \tag{5.26}$$

Finally, revisiting the thermal energy method (3.4) with $\phi = 1$, in the light of the previous estimates, we conclude

$$\int_0^T \int_\Omega |\nabla_h \mathbf{u}_h|^2 \, \mathrm{d}x \, \mathrm{d}t \stackrel{<}{\sim} 1, \tag{5.27}$$

and, by virtue of (2.30),

$$\|\mathbf{u}_h\|^2_{L^2(0,T;L^6(\Omega;R^3))} \stackrel{<}{\sim} 1. \tag{5.28}$$

# 6 Consistency

Having collected all the available uniform bounds, our next task is to verify that our numerical method is *consistent* with the variational formulation of the original problem.

## 6.1 Continuity method

For $\phi \in C^1(\overline{\Omega})$, take $\Pi_h^Q[\phi]$ as a test function in the continuity method (3.1). Using the formula (2.17) for $r = \varrho_h^k$, $\mathbf{u} = \mathbf{u}_h^k$, $F = \Pi_h^Q[\phi]$ we check without difficulty that

$$\int_\Omega \varrho_h^k \mathbf{u}_h^k \cdot \nabla_x \phi \, \mathrm{d}x$$

$$= \sum_{\Gamma \in \Gamma_h} \int_\Gamma \mathrm{Up}[\varrho_h^k, \mathbf{u}_h^k] \, [[ \, \Pi_h^Q[\phi] \, ]] \, \mathrm{dS}_x - \sum_{E \in E_h} \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} \left( \phi - \Pi_h^Q[\phi] \right) [[\varrho_h^k]] \, [\tilde{\mathbf{u}}_h^k \cdot \mathbf{n}]^- \, \mathrm{dS}_x$$

$$+ \sum_{E \in E_h} \int_{\partial E} \phi \varrho_h^k (\mathbf{u}_h^k - \tilde{\mathbf{u}}_h^k) \cdot \mathbf{n} \, \mathrm{dS}_x.$$

Note that here

$$\int_\Omega (\Pi_h^Q[\phi] - \phi) \varrho_h^k \mathrm{div}_h \mathbf{u}_h^k \, \mathrm{d}x = \sum_{E \in E_h} \int_E (\Pi_h^Q[\phi] - \phi) \varrho_h^k \mathrm{div}_h \mathbf{u}_h^k \, \mathrm{d}x = 0$$

as $\mathrm{div}_h \mathbf{u}_h^k$ is constant on each element $E$.

Now, by Hölder's inequality,

$$\left| \sum_{E \in E_h} \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} \left( \phi - \Pi_h^Q[\phi] \right) [[\varrho_h^k]] \ [\tilde{\mathbf{u}}_h^k \cdot \mathbf{n}]^- \ \mathrm{dS}_x \right| \lesssim \sum_{\Gamma \in \Gamma_h} \int_\Gamma \left| \phi - \Pi_h^Q[\phi] \right| | \ [[\varrho_h^k]] \ | \ \left| \ [\tilde{\mathbf{u}}_h^k \cdot \mathbf{n}]^- \ \right| \ \mathrm{dS}_x$$

$$\lesssim \left( \sum_{\Gamma \in \Gamma_h} \int_\Gamma [[\varrho_h^k]]^2 \left| \ [\tilde{\mathbf{u}}_h^k \cdot \mathbf{n}]^- \ \right| \ \mathrm{dS}_x \right)^{1/2} \left( \sum_{E \in E_h} \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} \left( \phi - \Pi_h^Q[\phi] \right)^2 \left| \ \tilde{\mathbf{u}}_h^k \cdot \mathbf{n} \ \right| \ \mathrm{dS}_x \right)^{1/2},$$

where the first integral on the right-hand side is controlled by (5.9).

Next, another application of Hölder's inequality, combined with Poincaré's inequality (2.4) and the trace estimates (2.20), (2.21), gives rise to

$$\sum_{E \in E_h} \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} \left( \phi - \Pi_h^Q[\phi] \right)^2 \left| \ \tilde{\mathbf{u}}_h^k \cdot \mathbf{n} \ \right| \ \mathrm{dS}_x$$

$$\lesssim \sum_{E \in E_h} \left( \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} \left( \phi - \Pi_h^Q[\phi] \right)^{\frac{6\gamma}{\gamma+6}} \ \mathrm{dS}_x \right)^{\frac{\gamma+6}{3\gamma}} \left( \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} \left| \tilde{\mathbf{u}}_h^k \right|^{\frac{3\gamma}{2\gamma-6}} \ \mathrm{dS}_x \right)^{\frac{2\gamma-6}{3\gamma}}$$

$$\lesssim h \sum_{E \in E_h} \left\| \mathbf{u}_h^k \right\|_{L^{\frac{3\gamma}{2\gamma-6}}(E;R^3)} \left\| \nabla_x \phi \right\|^2_{L^{\frac{6\gamma}{\gamma+6}}(E;R^3)} \lesssim h \left\| \mathbf{u}_h^k \right\|_{L^{\frac{3\gamma}{2\gamma-6}}(\Omega;R^3)} \left\| \nabla_x \phi \right\|^2_{L^{\frac{6\gamma}{\gamma+6}}(\Omega;R^3)}.$$

Now, we use the interpolation $L^p - L^q$ estimates (2.24), (2.26) and (5.28) to conclude

$$h \left\| \mathbf{u}_h^k \right\|_{L^{\frac{3\gamma}{2\gamma-6}}(\Omega;R^3)} \left\| \nabla_x \phi \right\|^2_{L^{\frac{6\gamma}{\gamma+6}}(\Omega;R^3)} \lesssim h^{\min\{1, \frac{5\gamma-12}{2\gamma}\}} \left\| \mathbf{u}_h^k \right\|_{L^6(\Omega;R^3)} \left\| \nabla_x \phi \right\|^2_{L^{\frac{6\gamma}{\gamma+6}}(\Omega;R^3)}$$

$$= h^{\min\{1, \frac{5\gamma-12}{2\gamma}\}} (\Delta t)^{-1/2} (\Delta t)^{1/2} \left\| \mathbf{u}_h^k \right\|_{L^6(\Omega;R^3)} \left\| \nabla_x \phi \right\|^2_{L^{\frac{6\gamma}{\gamma+6}}(\Omega;R^3)} \lesssim h^{\min\{1, \frac{5\gamma-12}{2\gamma}\}} (\Delta t)^{-1/2} \left\| \nabla_x \phi \right\|^2_{L^{\frac{6\gamma}{\gamma+6}}(\Omega;R^3)}.$$

The next step is to estimate

$$\sum_{\Gamma \in \Gamma_h} \int_\Gamma \phi \varrho_h^k (\mathbf{u}_h^k - \tilde{\mathbf{u}}_h^k) \cdot \mathbf{n} \ \mathrm{dS}_x = \sum_{\Gamma \in \Gamma_h} \int_\Gamma (\phi - \tilde{\phi}) \varrho_h^k (\mathbf{u}_h^k - \tilde{\mathbf{u}}_h^k) \cdot \mathbf{n} \ \mathrm{dS}_x,$$

where, by Hölder's inequality, (2.18–2.21),

$$\left| \sum_{\Gamma \in \Gamma_h} \int_\Gamma (\phi - \tilde{\phi}) \varrho_h^k (\mathbf{u}_h^k - \tilde{\mathbf{u}}_h^k) \cdot \mathbf{n} \ \mathrm{dS}_x \right|$$

$$\leq \sum_{\Gamma \in \Gamma_h} \left\| \phi - \tilde{\phi} \right\|_{L^2(\Gamma)} \left\| \varrho_h^k \right\|_{L^\gamma(\Gamma)} \left\| \mathbf{u}_h^k - \tilde{\mathbf{u}}_h^k \right\|_{L^{\frac{2\gamma}{\gamma-2}}(\Gamma;R^3)}$$

$$\lesssim \frac{1}{h}\sum_{E\in E_h}\left(\left\|\phi-\tilde{\phi}\right\|_{L^2(E)}+h\left\|\nabla\phi\right\|_{L^2(\Omega;\mathbb{R}^3)}\right)\left\|\varrho_h^k\right\|_{L^\gamma(E)}\left\|\mathbf{u}_h^k-\tilde{\mathbf{u}}_h^k\right\|_{L^{\frac{2\gamma}{\gamma-2}}(E;R^3)}$$

$$\lesssim h^{\frac{\gamma-3}{\gamma}}\|\varrho_h^k\|_{L^\gamma(\Omega)}\|\nabla_h\mathbf{u}_h^k\|_{L^2(\Omega;R^{3\times 3})}\|\nabla_x\phi\|_{L^2(\Omega;R^3)}.$$

Finally, observing

$$|\sum_{\Gamma\in\Gamma_{h,\mathrm{int}}}\int_\Gamma[[\varrho_h^k]]\ [[\Pi_h^Q[\phi]]]\ \mathrm{dS}_x|\lesssim\sum_{E\in E_h}\sum_{\Gamma_E\subset\partial E\cap\Gamma_{h,\mathrm{int}}}\int_{\Gamma_E}|\,[[\varrho_h^k]]\,|\ |\Pi_h^Q[\phi]-\phi|\ \mathrm{dS}_x$$

we find by using (2.4), (5.9) and (2.20) that the artificial viscosity $h^{1-\varepsilon}$-term is controlled as follows

$$h^{1-\varepsilon}\left|\sum_{\Gamma\in\Gamma_{h,\mathrm{int}}}\int_\Gamma[[\varrho_h^k]]\ [[\Pi_h^Q[\phi]]]\ \mathrm{dS}_x\right|\lesssim h^{1-\frac{\varepsilon}{2}}\omega_h(t)\|\nabla_x\phi\|_{L^2(\Omega;R^3)},\ \ \|\omega_h\|_{L^2(0,T)}\lesssim 1.$$

Since $\Delta t\approx h$, we conclude that there exists $\alpha>0$ such that

$$\int_\Omega[D_t\varrho_h\phi-\varrho_h\mathbf{u}_h\cdot\nabla_x\phi]\ \mathrm{d}x=h^\alpha\left\langle R_h^1,\phi\right\rangle\ \text{in }(0,T)\text{ for all }\phi\in C^1(\overline{\Omega}),\tag{6.1}$$

with

$$\left|\left\langle R_h^1(t),\phi\right\rangle\right|\lesssim r_h^1(t)\|\nabla_x\phi\|_{L^{\frac{6\gamma}{\gamma+6}}(\Omega)},\ \ \|r_h^1\|_{L^2(0,T)}\lesssim 1.\tag{6.2}$$

## 6.2 Momentum method

The next step is to take

$$\Pi_h^V[\phi],\ \ \phi\in C_c^1(\Omega;R^3),$$

as a test function in the momentum method (3.3). Seeing that, in accordance with (2.9), (2.12),

$$\int_\Omega\left[\mu\nabla_h\mathbf{u}_h^k:\nabla_h\Pi_h^V[\phi]+\lambda\mathrm{div}_h\mathbf{u}_h^k\mathrm{div}_h\Pi_h^V[\phi]\right]\ \mathrm{d}x=\int_\Omega\left[\mu\nabla_h\mathbf{u}_h^k:\nabla_x\phi+\lambda\mathrm{div}_h\mathbf{u}_h^k\mathrm{div}_x\phi\right]\ \mathrm{d}x,$$

and

$$\int_\Omega p(\varrho_h,\vartheta_h)\mathrm{div}_h\Pi_h^V[\phi]\ \mathrm{d}x=\int_\Omega p(\varrho_h,\vartheta_h)\mathrm{div}_x\phi\ \mathrm{d}x,$$

we may rewrite (3.3) in the form

$$\int_\Omega D_t\varrho_h^k\hat{\mathbf{u}}_h^k\cdot\phi\ \mathrm{d}x-\int_\Omega\varrho_h^k\mathbf{u}_h^k\otimes\widehat{\mathbf{u}_h^k}:\nabla_x\phi\ \mathrm{d}x\tag{6.3}$$

$$+\int_\Omega \left[\mu \nabla_h \mathbf{u}_h^k \cdot \nabla_x \phi + \lambda \mathrm{div}_h \mathbf{u}_h^k \mathrm{div}_x \phi\right]\ \mathrm{d}x - \int_\Omega p(\varrho_h^k, \vartheta_h^k) \mathrm{div}_x \phi\ \mathrm{d}x$$

$$= \int_\Omega D_t \varrho_h^k \widehat{\mathbf{u}}_h^k \cdot \left(\phi - \Pi_h^V[\phi]\right)\ \mathrm{d}x + \sum_{\Gamma \in \Gamma_h} \int_\Gamma \mathrm{Up}[\varrho_h^k \widehat{\mathbf{u}}_h^k, \mathbf{u}_h^k] \cdot [[\widehat{\Pi_h^V[\phi]}]]\ \mathrm{dS}_x - \int_\Omega \varrho_h^k \mathbf{u}_h^k \otimes \widehat{\mathbf{u}_h^k} : \nabla_x \phi\ \mathrm{d}x$$

$$-h^{1-\varepsilon} \sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma [[\varrho_h^k]] \left\{\widehat{\mathbf{u}_h^k}\right\} \cdot [[\widehat{\Pi_h^V[\phi]}]]\ \mathrm{dS}_x$$

Our goal is to estimate the four integrals on the right-hand side of (6.3). We proceed in several steps.

#### 6.2.1 Error in the discretized time derivative

We have

$$\int_\Omega D_t \varrho_h^k \widehat{\mathbf{u}}_h^k \cdot \left(\phi - \Pi_h^V[\phi]\right)\ \mathrm{d}x$$

$$\int_\Omega \sqrt{\varrho_h^{k-1}} \sqrt{\varrho_h^{k-1}} \frac{\mathbf{u}_h^k - \mathbf{u}_h^{k-1}}{\Delta t} \cdot \left(\phi - \Pi_h^V[\phi]\right)\ \mathrm{d}x + \int_\Omega \frac{\varrho_h^k - \varrho_h^{k-1}}{\Delta t} \mathbf{u}_h^k \cdot \left(\phi - \Pi_h^V[\phi]\right)\ \mathrm{d}x,$$

where, by virtue of Hölder's inequality and the error estimate (2.11),

$$\left| \int_\Omega \sqrt{\varrho_h^{k-1}} \sqrt{\varrho_h^{k-1}} \frac{\mathbf{u}_h^{k-1} - \mathbf{u}_h^{k-1}}{\Delta t} \cdot \left(\phi - \Pi_h^V[\phi]\right)\ \mathrm{d}x \right|$$

$$\leq \|\varrho_h^{k-1}\|_{L^\gamma(\Omega)}^{1/2} \left( \int_\Omega \varrho_h^{k-1} \left( \frac{\mathbf{u}_h^{k-1} - \mathbf{u}_h^{k-1}}{\Delta t} \right)^2\ \mathrm{d}x \right)^{1/2} \left\| \phi - \Pi_h^V[\phi] \right\|_{L^{\frac{2\gamma}{\gamma-1}}(\Omega)}$$

$$\lesssim \|\varrho_h^{k-1}\|_{L^\gamma(\Omega)}^{1/2} \left( \Delta t \int_\Omega \varrho_h^{k-1} \left( \frac{\mathbf{u}_h^{k-1} - \mathbf{u}_h^{k-1}}{\Delta t} \right)^2\ \mathrm{d}x \right)^{1/2} (\Delta t)^{-1/2} h \left\| \nabla_x \phi \right\|_{L^{\frac{2\gamma}{\gamma-1}}(\Omega;R^3)},$$

where, in accordance with the energy estimates (5.7),

$$\sum_{k \geq 0} \Delta t \left( \Delta t \int_\Omega \varrho_h^{k-1} \left( \frac{\mathbf{u}_h^{k-1} - \mathbf{u}_h^{k-1}}{\Delta t} \right)^2\ \mathrm{d}x \right) \lesssim 1. \tag{6.4}$$

Applying a similar treatment to the second integral we get

$$\left| \int_\Omega \frac{\varrho_h^k - \varrho_h^{k-1}}{\Delta t} \mathbf{u}_h^k \cdot \left(\phi - \Pi_h^V[\phi]\right)\ \mathrm{d}x \right|$$

$$\leq \left( \Delta t \int_\Omega \left( \frac{\varrho_h^k - \varrho_h^{k-1}}{\Delta t} \right)^2 \ \mathrm{d}x \right)^{1/2} \|\mathbf{u}_h^k\|_{L^6(\Omega;R^3)} (\Delta t)^{-1/2} h \|\nabla_x \phi\|_{L^3(\Omega)},$$

where the first integral on the right-hand side is controlled by means of (5.7).

Thus we may infer that

$$\left| \int_\Omega D_t(\varrho_h \hat{\mathbf{u}}_h) \cdot \left( \phi - \Pi_h^V[\phi] \right) \ \mathrm{d}x \right| \lesssim \sqrt{h} \ r_h^2(t) \|\nabla_x \phi\|_{L^\gamma(\Omega)}, \ \|r_h^2\|_{L^2(0,T)} \lesssim 1. \tag{6.5}$$

### 6.2.2 Error in the upwind term

Take $F = (\widehat{\Pi_h^V[\phi]})_i = (\Pi_h^Q \Pi_h^V[\phi])_i$, $r = \varrho_h \hat{\mathbf{u}}_{h,i}$, $i = 1,2,3$ in (2.17) to obtain

$$\sum_{\Gamma \in \Gamma_h} \int_\Gamma \mathrm{Up}[\varrho_h^k \hat{\mathbf{u}}_h^k, \mathbf{u}_h^k] \cdot [[\widehat{\Pi_h^V[\phi]}]] \ \mathrm{dS}_x - \int_\Omega \varrho_h^k \mathbf{u}_h^k \otimes \widehat{\mathbf{u}_h^k} : \nabla_x \phi \ \mathrm{d}x$$

$$= \sum_{E \in E_h} \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} \left( \phi - \Pi_h^Q \Pi_h^V[\phi] \right) \cdot [[\varrho_h^k \widehat{\mathbf{u}_h^k}]] [\tilde{\mathbf{u}}_h^k \cdot \mathbf{n}]^- \ \mathrm{dS}_x + \sum_{E \in E_h} \int_E \varrho_h^k \hat{\mathbf{u}}_h^k (\phi - \Pi_h^Q \Pi_h^V[\phi]) \mathrm{div}_h \mathbf{u}_h^k \ \mathrm{d}x$$

$$+ \sum_{E \in E_h} \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} \phi \cdot \widehat{\mathbf{u}_h^k} \varrho_h^k (\tilde{\mathbf{u}}_h^k - \mathbf{u}_h^k) \cdot \mathbf{n} \ \mathrm{dS}_x$$

$$= \sum_{E \in E_h} \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} (\varrho_h^k)^+ \left( \phi - \Pi_h^Q \Pi_h^V[\phi] \right) [\tilde{\mathbf{u}}_h^k \cdot \mathbf{n}]^- [[\widehat{\mathbf{u}_h^k}]] \ \mathrm{dS}_x$$

$$+ \sum_{E \in E_h} \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} \hat{\mathbf{u}}_h^k \cdot \left( \phi - \Pi_h^Q \Pi_h^V[\phi] \right) [\tilde{\mathbf{u}}_h^k \cdot \mathbf{n}]^- [[\varrho_h^k]] \ \mathrm{dS}_x$$

$$+ \sum_{E \in E_h} \int_E \varrho_h^k \hat{\mathbf{u}}_h^k (\phi - \Pi_h^Q \Pi_h^V[\phi]) \mathrm{div}_h \mathbf{u}_h^k \ \mathrm{d}x$$

$$+ \sum_{E \in E_h} \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} \phi \cdot \widehat{\mathbf{u}_h^k} \varrho_h^k (\tilde{\mathbf{u}}_h^k - \mathbf{u}_h^k) \cdot \mathbf{n} \ \mathrm{dS}_x \equiv I_1 + I_2 + I_3 + I_4.$$

**Step 1:**
Applying Hölder's inequality to $I_1$ we obtain

$$|I_1| = \left| \sum_{E \in E_h} \int_{\partial E} (\varrho_h^k)^+ \left( \Pi_h^Q \Pi_h^V[\phi] - \phi \right) [\tilde{\mathbf{u}}_h^k \cdot \mathbf{n}]^- [[\widehat{\mathbf{u}_h^k}]] \ \mathrm{dS}_x \right|$$

$$\lesssim \left( \sum_{E \in E_h} \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} (\varrho_h^k)^+ \left| \tilde{\mathbf{u}}_h^k \cdot \mathbf{n} \right| [[\widehat{\mathbf{u}_h^k}]]^2 \ \mathrm{dS}_x \right)^{1/2}$$

$$\times \left( \sum_{E \in E_h} \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} (\varrho_h^k)^+ \left| \tilde{\mathbf{u}}_h^k \cdot \mathbf{n} \right| \left( \Pi_h^Q \Pi_h^V [\phi] - \phi \right)^2 \ \mathrm{dS}_x \right)^{1/2},$$

where the first term is bounded in $L^2(0,T)$ in view of the energy estimates (5.8).

Next, as $\mathbf{u}_h^k$ are continuous on each element, we have

$$\sum_{E \in E_h} \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} (\varrho_h^k)^+ \left| \tilde{\mathbf{u}}_h^k \cdot \mathbf{n} \right| \left( \Pi_h^Q \Pi_h^V [\phi] - \phi \right)^2 \ \mathrm{dS}_x$$

$$\leq \sum_{E \in E_h} \|\varrho_h^k\|_{L^q(\partial E)} \|\mathbf{u}_h^k\|_{L^\infty(E,R^3)} \left\| \Pi_h^Q \Pi_h^V [\phi] - \phi \right\|^2_{L^\gamma(\partial E;R^3)}, \ \frac{1}{q} + \frac{2}{\gamma} = 1,$$

where, in accordance with the trace estimates (2.20), (2.21), and the $L^p - L^q$ estimates (2.24),

$$\sum_{E \in E_h} \|\varrho_h^k\|_{L^q(\partial E)} \|\mathbf{u}_h^k\|_{L^\infty(E,R^3)} \left\| \Pi_h^Q \Pi_h^V [\phi] - \phi \right\|^2_{L^\gamma(\partial E;R^3)}$$

$$\leq \frac{1}{h} \|\mathbf{u}_h^k\|_{L^\infty(\Omega,R^3)} \sum_{E \in E_h} \|\varrho_h^k\|_{L^q(E)} \left( \left\| \Pi_h^Q \Pi_h^V [\phi] - \phi \right\|^2_{L^\gamma(E;R^3)} + h^2 \|\nabla_x \phi\|^2_{L^\gamma(E;R^3)} \right)$$

$$\frac{1}{h^{3/2}} \|\mathbf{u}_h^k\|_{L^6(\Omega,R^3)} \sum_{E \in E_h} \|\varrho_h^k\|_{L^q(E)} \left( \left\| \Pi_h^Q \Pi_h^V [\phi] - \phi \right\|^2_{L^\gamma(E;R^3)} + h^2 \|\nabla_x \phi\|^2_{L^\gamma(E;R^3)} \right)$$

$$\leq \frac{1}{h^{3/2}} \|\mathbf{u}_h^k\|_{L^6(\Omega,R^3)} \|\varrho_h^k\|_{L^q(\Omega)} \left( \left\| \Pi_h^Q \Pi_h^V [\phi] - \phi \right\|^2_{L^\gamma(\Omega;R^3)} + h^2 \|\nabla_x \phi\|^2_{L^\gamma(\Omega;R^3)} \right).$$

Finally, by virtue of (2.5), (2.4), (2.11),

$$\left\| \Pi_h^Q \Pi_h^V [\phi] - \phi \right\|_{L^\gamma(\Omega;R^3)} \leq \left\| \Pi_h^Q \left[ \Pi_h^V [\phi] - \phi \right] \right\|_{L^\gamma(\Omega;R^3)} + \left\| \Pi_h^Q [\phi] - \phi \right\|_{L^\gamma(\Omega;R^3)} \tag{6.6}$$

$$\leq \left\| \Pi_h^V [\phi] - \phi \right\|_{L^\gamma(\Omega;R^3)} + \left\| \Pi_h^Q [\phi] - \phi \right\|_{L^\gamma(\Omega;R^3)} \leq ch \|\nabla_x \phi\|_{L^\gamma(\Omega;R^3)}.$$

Thus,

$$|I_1(t)| \leq h^{1/4} z_h(t) \|\nabla \phi\|_{L^\infty(\Omega)} \text{ with } \|z_h\|_{L^1(0,T)} \leq T^\delta (\Delta t)^{-\delta} \|z_h\|_{L^1(0,T)}, \ \delta \in (0,1), \tag{6.7}$$

according to (2.26), since $z_h$ is a piecewise constant function on $(0,T)$. As $\gamma > 3$ and $\Delta t \approx h$, we conclude that

$$|I_1| = \left| \sum_{E \in E_h} \int_{\partial E} \left( \Pi_h^Q \Pi_h^V [\phi] - \phi \right) [\tilde{\mathbf{u}}_h^k \cdot \mathbf{n}]^- [[\widehat{\varrho_h^k \mathbf{u}_h^k}]] \ \mathrm{dS}_x \right| \stackrel{<}{\sim} h^\alpha r_h^3(t) \|\nabla_x \phi\|_{L^\gamma(\Omega)}, \ \|r_h^3\|_{L^\beta(0,T)} \stackrel{<}{\sim} 1, \tag{6.8}$$

where $\alpha > 0$ and $\beta > 1$.

**Step 2:**

Continuing we have

$$|I_2| = \left| \sum_{E \in E_h} \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} \widehat{\mathbf{u}}_h^k \cdot \left( \Pi_h^Q \Pi_h^V [\phi] - \phi \right) [\tilde{\mathbf{u}}_h^k \cdot \mathbf{n}]^- [[\varrho_h^k]] \ \mathrm{dS}_x \right|$$

$$\lesssim \left( - \sum_{\Gamma \in \Gamma_h} \int_\Gamma [\tilde{\mathbf{u}}_h^k \cdot \mathbf{n}]^- [[\varrho_h^k]]^2 \ \mathrm{dS}_x \right)^{1/2} \left( \sum_{E \in E_h} \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} |\widehat{\mathbf{u}}_h^k|^2 |\tilde{\mathbf{u}}_h^k \cdot \mathbf{n}| \left| \Pi_h^Q \Pi_h^V [\phi] - \phi \right|^2 \ \mathrm{dS}_x \right)^{1/2}$$

where, in accordance with (5.9), the first integral is uniformly bounded in $L^2(0,T)$.

As for the second integral, we use Hölder's inequality to deduce

$$\sum_{E \in E_h} \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} |\widehat{\mathbf{u}}_h^k|^2 |\tilde{\mathbf{u}}_h^k \cdot \mathbf{n}| \left| \Pi_h^Q \Pi_h^V [\phi] - \phi \right|^2 \ \mathrm{dS}_x$$

$$\leq \sum_{E \in E_h} \| \Pi_h^Q \Pi_h^V [\phi] - \phi \|^2_{L^\gamma(\partial E)} \left( \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} |\widehat{\mathbf{u}}_h^k|^{\frac{2\gamma}{\gamma-2}} |\tilde{\mathbf{u}}_h^k|^{\frac{\gamma}{\gamma-2}} \ \mathrm{dS}_x \right)^{\frac{\gamma-2}{\gamma}}.$$

Next, by virtue of the trace estimate (2.20) and Hölder's inequality,

$$\sum_{E \in E_h} \| \Pi_h^Q \Pi_h^V [\phi] - \phi \|^2_{L^\gamma(\partial E)} \left( \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} |\widehat{\mathbf{u}}_h^k|^{\frac{2\gamma}{\gamma-2}} |\tilde{\mathbf{u}}_h^k|^{\frac{\gamma}{\gamma-2}} \ \mathrm{dS}_x \right)^{\frac{\gamma-2}{\gamma}}$$

$$\lesssim \sum_{E \in E_h} \left( h^{-\frac{2}{\gamma}} \| \Pi_h^Q \Pi_h^V [\phi] - \phi \|^2_{L^\gamma(E)} + h^{2-\frac{2}{\gamma}} \| \nabla_x \phi \|^2_{L^\gamma(E)} \right) \left( \int_{\partial E} |\widehat{\mathbf{u}}_h^k|^{\frac{3\gamma}{\gamma-2}} \ \mathrm{dS}_x \right)^{2\frac{\gamma-2}{3\gamma}} \times$$

$$\times \left( \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} |\tilde{\mathbf{u}}_h^k|^{\frac{3\gamma}{\gamma-2}} \ \mathrm{dS}_x \right)^{\frac{\gamma-2}{3\gamma}}.$$

Furthermore, by (2.20–2.21), (2.5), (2.19), Hölder's inequality and (6.6),

$$\sum_{E \in E_h} \left( h^{-\frac{2}{\gamma}} \| \Pi_h^Q \Pi_h^V [\phi] - \phi \|^2_{L^\gamma(E)} + h^{2-\frac{2}{\gamma}} \| \nabla_x \phi \|^2_{L^\gamma(E)} \right) \left( \int_{\partial E} |\widehat{\mathbf{u}}_h^k|^{\frac{3\gamma}{\gamma-2}} \ \mathrm{dS}_x \right)^{2\frac{\gamma-2}{3\gamma}} \times$$

$$\times \left( \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} |\tilde{\mathbf{u}}_h^k|^{\frac{3\gamma}{\gamma-2}} \ \mathrm{dS}_x \right)^{\frac{\gamma-2}{3\gamma}}$$

$$\lesssim \sum_{E\in E_h} \left( \frac{1}{h} \|\Pi_h^Q \Pi_h^V[\phi] - \phi\|^2_{L^\gamma(E)} + h\|\nabla_x\phi\|^2_{L^\gamma(E)} \right) \left( \int_E |\widehat{\mathbf{u}}_h^k|^{\frac{3\gamma}{\gamma-2}} \ \mathrm{d}x \right)^{2\frac{\gamma-2}{3\gamma}} \left( \int_E |\mathbf{u}_h^k|^{\frac{3\gamma}{\gamma-2}} \ \mathrm{d}x \right)^{\frac{\gamma-2}{3\gamma}}$$

$$\lesssim \frac{1}{h}\|\Pi_h^Q \Pi_h^V[\phi] - \phi\|^2_{L^\gamma(\Omega)} \left\|\widehat{\mathbf{u}}_h^k\right\|^2_{L^{\frac{3\gamma}{\gamma-2}}(\Omega)} \left\|\mathbf{u}_h^k\right\|_{L^{\frac{3\gamma}{\gamma-2}}(\Omega)} + h\|\nabla_x\phi\|^2_{L^\gamma(\Omega)} \left\|\widehat{\mathbf{u}}_h^k\right\|^2_{L^{\frac{3\gamma}{\gamma-2}}(\Omega)} \left\|\mathbf{u}_h^k\right\|_{L^{\frac{3\gamma}{\gamma-2}}(\Omega)}$$

$$\lesssim h\|\nabla_x\phi\|^2_{L^\gamma(\Omega)} \left\|\mathbf{u}_h^k\right\|^3_{L^{\frac{3\gamma}{\gamma-2}}(\Omega)} \lesssim h^{\frac{5}{2}-\max\{\frac{6}{\gamma};\frac{3}{2}\}} \|\nabla_x\phi\|^2_{L^\gamma(\Omega)} \left\|\mathbf{u}_h^k\right\|^3_{L^6(\Omega)} \ \text{provided } \gamma > 3,$$

where we have also used (2.22).

Finally, using the time estimates (2.26) we infer that

$$h^{\frac{5}{2}-\max\{\frac{6}{\gamma};\frac{3}{2}\}} \|\nabla_x\phi\|^2_{L^\gamma(\Omega)} \left\|\mathbf{u}_h^k\right\|^3_{L^6(\Omega)} \leq (\Delta t)^{-\frac{1}{2}} h^{\frac{5}{2}-\max\{\frac{6}{\gamma};\frac{3}{2}\}} \|\nabla_x\phi\|^2_{L^\gamma(\Omega)} (\Delta t)^{\frac{1}{2}} \left\|\mathbf{u}_h^k\right\|^3_{L^6(\Omega)}.$$

Summarizing we conclude

$$|I_2| = \left| \sum_{E\in E_h} \sum_{\Gamma_E\subset\partial E} \int_{\Gamma_E} \widehat{\mathbf{u}}_h^k \cdot \left( \Pi_h^Q \Pi_h^V[\phi] - \phi \right) [\mathbf{u}_h^k \cdot \mathbf{n}]^- [[\varrho_h^k]] \ \mathrm{dS}_x \right| \tag{6.9}$$

$$\lesssim h^\alpha r_h^4(t) \|\nabla_x\phi\|_{L^\gamma(\Omega)}, \ \|r_h^4\|_{L^\beta(0,T)} \lesssim 1 \text{ with some } \alpha > 0, \ \beta > 1,$$

where we have used the same reasoning as in (6.7).

**Step 3:**

Finally, another application of Hölder's inequality gives rise to

$$|I_3| = \left| \sum_{E\in E_h} \int_E \varrho_h^k \widehat{\mathbf{u}}_h^k (\phi - \Pi_h^Q \Pi_h^V[\phi]) \mathrm{div}_h \mathbf{u}_h^k \ \mathrm{d}x \right|$$

$$\leq \|\varrho_h^k\|_{L^\infty(\Omega;R^3)} \sum_{E\in E_h} \|\mathrm{div}_h \mathbf{u}_h^k\|_{L^2(E)} \|\mathbf{u}_h^k\|_{L^6(E)} \left\|\phi - \Pi_h^Q \Pi_h^V[\phi]\right\|_{L^3(E;R^3)}.$$

Proceeding as in (6.6) and employing moreover(2.22),

$$\|\varrho_h^k\|_{L^\infty(\Omega)} \sum_{E\in E_h} \|\mathrm{div}_h \mathbf{u}_h^k\|_{L^2(E)} \|\mathbf{u}_h^k\|_{L^6(E;R^3)} \left\|\phi - \Pi_h^Q \Pi_h^V[\phi]\right\|_{L^3(E;R^3)}$$

$$\lesssim h^{1-\frac{3}{\gamma}} \|\varrho_h^k\|_{L^\gamma(\Omega)} \|\mathrm{div}_h \mathbf{u}_h^k\|_{L^2(\Omega)} \|\mathbf{u}_h^k\|_{L^6(\Omega;R^3)} \|\nabla_x\phi\|_{L^3(\Omega;R^3)}$$

yielding as in the previous step the desired conclusion

$$|I_3| = \left| \sum_{E\in E_h} \int_E \varrho_h^k \widehat{\mathbf{u}}_h^k (\phi - \Pi_h^Q \Pi_h^V[\phi]) \mathrm{div}_h \mathbf{u}_h^k \ \mathrm{d}x \right| \lesssim h^\alpha r_h^5(t) \|\nabla_x\phi\|_{L^\gamma(\Omega)}, \ \|r_h^5\|_{L^\beta(0,T)} \lesssim 1 \tag{6.10}$$

with some $\alpha > 0$, $\beta > 1$.

**Step 4:**

The last integral

$$I_4 = \sum_{E\in E_h} \sum_{\Gamma_E\subset\partial E} \int_{\Gamma_E} \phi\cdot \widehat{\mathbf{u}_h^k}\varrho_h^k(\tilde{\mathbf{u}}_h^k - \mathbf{u}_h^k)\cdot \mathbf{n}\ \mathrm{dS}_x = \sum_{E\in E_h} \sum_{\Gamma_E\subset\partial E} \int_{\Gamma_E} (\phi-\tilde{\phi})\cdot \widehat{\mathbf{u}_h^k}\varrho_h^k(\tilde{\mathbf{u}}_n^k - \mathbf{u}_h^k)\cdot \mathbf{n}\ \mathrm{dS}_x$$

can be handled in the same way as its counter-part in the continuity method, namely

$$|I_4| \le \sum_{\Gamma\in\Gamma_h} \left\|\phi-\tilde{\phi}\right\|_{L^\gamma(\Gamma)} \left\|\varrho_h^k\right\|_{L^\gamma(\Gamma)} \left\|\hat{\mathbf{u}}_h^k\right\|_{L^6(\Gamma;\mathbb{R}^3)} \left\|\mathbf{u}_h^k - \tilde{\mathbf{u}}_h^k\right\|_{L^{\frac{6\gamma}{5\gamma-12}}(\Gamma;R^3)}$$

$$\lesssim \frac{1}{h}\sum_{E\in E_h}\Big(\left\|\phi-\tilde{\phi}\right\|_{L^\gamma(E)} + h\left\|\nabla\phi\right\|_{L^\gamma(E;\mathbb{R}^3)}\Big)\left\|\varrho_h^k\right\|_{L^\gamma(E)} \left\|\hat{\mathbf{u}}_h^k\right\|_{L^6(E;\mathbb{R}^3)} \left\|\mathbf{u}_h^k - \tilde{\mathbf{u}}_h^k\right\|_{L^{\frac{6\gamma}{5\gamma-12}}(E;R^3)}$$

$$\lesssim h^{1-\max\{\frac{6-\gamma}{\gamma},0\}}\|\varrho_h^k\|_{L^\gamma(\Omega)}\|\nabla_h\mathbf{u}_h^k\|^2_{L^2(\Omega;R^{3\times3})}\|\nabla_x\phi\|_{L^\gamma(\Omega;R^3)}. \tag{6.11}$$

Similarly the artificial viscosity term is estimated again as its counter-part in the continuity method, namely

$$h^{1-\varepsilon}\Big|\sum_{\Gamma\in\Gamma_{h,\mathrm{int}}}\int_\Gamma [[\varrho_h^k]]\ \left\{\widehat{\mathbf{u}_h^k}\right\}\cdot[[\widehat{\Pi_h^V[\phi]}]]\ \mathrm{dS}_x\Big| \lesssim h^{1-\varepsilon}\sum_{E\in E_h}\sum_{\Gamma_E\subset\partial E\cap\Gamma_{h,\mathrm{int}}}\int_{\Gamma_E}|\,[[\varrho_h^k]]|\ \left|\left\{\widehat{\mathbf{u}_h^k}\right\}\right|\ |\widehat{\Pi_h^V[\phi]}-\phi|\ \mathrm{dS}_x$$

$$\lesssim h^{1-\varepsilon}\sum_{E\in E_h}\sum_{\Gamma\subset\partial E\cap\Gamma_{h,\mathrm{int}}}\left\|[[\varrho_h^k]]\right\|_{L^2(\Gamma)}\left\|\left\{\widehat{\mathbf{u}_h^k}\right\}\right\|_{L^{\frac{2\gamma}{\gamma-2}}(\Gamma;\mathbb{R}^3)}\Big(\left\|\widehat{\Pi_h^V[\phi]}-\Pi_h^V[\phi]\right\|_{L^\gamma(\Gamma;\mathbb{R}^3)}+\left\|\Pi_h^V[\phi]-\phi\right\|_{L^\gamma(\Gamma;\mathbb{R}^3)}\Big) \tag{6.12}$$

$$\lesssim h^{1/2-\varepsilon/2}\Big(\sum_{\Gamma\in\Gamma_{h,\mathrm{int}}}\left\|\sqrt{h^{1-\varepsilon}}[[\varrho_h^k]]\right\|^2_{L^2(\Gamma)}\Big)^{1/2}\left\|\mathbf{u}_h^k\right\|_{L^6(\Omega;\mathbb{R}^3)}\left\|\nabla_x\phi\right\|_{L^\gamma(\Omega;\mathbb{R}^3)}.$$

Summing up (6.8 - 6.12) we obtain the consistency formulation of the momentum method

$$\int_\Omega D_t(\varrho_h\widehat{\mathbf{u}}_h)\cdot\phi\ \mathrm{d}x - \int_\Omega(\varrho_h\widehat{\mathbf{u}}_h\otimes\mathbf{u}_h):\nabla_x\phi\ \mathrm{d}x \tag{6.13}$$

$$+\int_\Omega\left[\mu\nabla_h\mathbf{u}_h:\nabla_x\phi+\lambda\mathrm{div}_h\mathbf{u}_h\mathrm{div}_x\phi\right]\ \mathrm{d}x - \int_\Omega p(\varrho_h,\vartheta_h)\mathrm{div}_x\phi\ \mathrm{d}x = h^\alpha\left\langle R_h^6,\phi\right\rangle,$$

for a certain $\alpha > 0$, $\phi\in C_c^1(\Omega;R^3)$, where

$$\left|\left\langle R_h^6(t),\phi\right\rangle\right| \le r_h^6(t)\|\nabla_x\phi\|_{L^\gamma(\Omega)},\ \|r_h^6\|_{L^\beta(0,T)}\lesssim 1 \text{ with some } \beta>1. \tag{6.14}$$

## 6.3 Renormalized thermal energy method

Our ultimate goal in this section is to derive a consistency formulation of the *renormalized* thermal energy method (4.2). As a matter of fact, it is (4.2) rather than (3.4) that is used in the proof of convergence of the method. We use a special ansatz for $\chi$ in (4.2), namely

$$\chi \in W^{2,\infty}[0,\infty),\ \chi'(\vartheta) \geq 0,\ \chi''(\vartheta) \leq 0,\quad \chi(\vartheta) = \text{const for all } \vartheta > \vartheta_\chi. \tag{6.15}$$

We start by rewriting

$$\sum_{\Gamma\in\Gamma_{h,\text{int}}} \int_\Gamma \frac{1}{d_\Gamma}[[K(\vartheta_h^k)]]\ [[\chi'(\vartheta_h^k)\phi]]\ \mathrm{dS}_x$$

$$= \sum_{\Gamma\in\Gamma_{h,\text{int}}} \int_\Gamma \frac{1}{d_\Gamma} \{\phi\}\ [[K(\vartheta_h^k)]]\ [[\chi'(\vartheta_h^k)]]\mathrm{dS}_x + \sum_{\Gamma\in\Gamma_{h,\text{int}}} \int_\Gamma \frac{1}{d_\Gamma} \left\{\chi'\left(\vartheta_h^k\right)\right\} [[K(\vartheta_h^k)]]\ [[\phi]]\mathrm{dS}_x$$

for any $\phi \in Q_h(\Omega)$.

The next step is to use $\Pi_h^B[\phi]$ as a test function in (4.2), where $\phi$ is a smooth function satisfying $\nabla_x\phi\cdot\mathbf{n}|_{\partial\Omega} = 0$. Using the error estimate (2.6), together with the uniform bounds established in (5.6), (5.26), and (5.27), we obtain

$$c_v \int_\Omega D_t\left(\varrho_h^k\chi(\vartheta_h^k)\right)\Pi_h^B[\phi]\ \mathrm{d}x - c_v \sum_{\Gamma\in\Gamma_h}\int_\Gamma \mathrm{Up}(\varrho_h^k\chi(\vartheta_h^k),\mathbf{u}_h^k)\ [[\Pi_h^B[\phi]\ ]]\ \mathrm{dS}_x \tag{6.16}$$

$$+ \sum_{\Gamma\in\Gamma_{h,\text{int}}} \int_\Gamma \frac{1}{d_\Gamma}\left\{\chi'\left(\vartheta_h^k\right)\right\}[[K(\vartheta_h^k)]]\ [[\Pi_h^B[\phi]\ ]]\mathrm{dS}_x$$

$$= \int_\Omega \left(\mu|\nabla_h\mathbf{u}_h^k|^2 + \lambda|\mathrm{div}_h\mathbf{u}_h^k|^2\right)\chi'(\vartheta_h^k)\Pi_h^B[\phi])\ \mathrm{d}x - \int_\Omega \chi'(\vartheta_h^k)\vartheta_h^k\varrho_h^k\mathrm{div}_h\mathbf{u}_h^k\Pi_h^B[\phi]\ \mathrm{d}x$$

$$-h^{1-\varepsilon}c_v \sum_{\Gamma\in\Gamma_{h,\text{int}}}\int_\Gamma [[\varrho_h^k]]\ [[\left(\chi(\vartheta_h^k) - \chi'(\vartheta_h^k)\vartheta_k^k\right)\Pi_h^B[\phi]\ ]]\ \mathrm{dS}_x + \langle D_h,\phi\rangle\,,$$

where

$$\langle D_h(t),\phi\rangle = -\sum_{\Gamma\in\Gamma_{h,\text{int}}}\int_\Gamma \frac{1}{d_\Gamma}\left\{\Pi_h^B[\phi]\right\}[[K(\vartheta_h^k)]]\ [[\chi'(\vartheta_h^k)]]\mathrm{dS}_x$$

$$-c_v\frac{\Delta t}{2}\int_\Omega \chi''(\xi_{\vartheta,h}^k)\varrho_h^{k-1}\left(\frac{\vartheta_h^k - \vartheta_h^{k-1}}{\Delta t}\right)^2\Pi_h^B[\phi]\ \mathrm{d}x + \frac{c_v}{2}\sum_{E\in E_h}\sum_{\Gamma_E\subset\partial E}\int_\Gamma \Pi_h^B[\phi]\chi''(\eta_{\vartheta,h}^k)[[\vartheta_h^k]]^2\left(\varrho_h^k\right)^+[\tilde{\mathbf{u}}_h^k\cdot\mathbf{n}]^-\ \mathrm{dS}_x.$$

Consequently,

$$|\langle D_h(t),\phi\rangle| \leq r_h^7(t)\|\phi\|_{L^\infty(\Omega)},\ \langle D_h(t),\phi\rangle \geq 0 \text{ if } \phi \geq 0,\ \|r^7\|_{L^1(0,T)} \lesssim 1. \tag{6.17}$$

We point out that the above bound is readily deduced from (6.16), (5.17), (5.6) and depends on the specific properties of the function $\chi$ stated in (6.15). The sign of $< D_h(t), \phi >$ can be deduced from (6.15) in view of the definition of the projection $\Pi_h^B$.

In the remaining part of this Section, we replace at the left-hand side of equation (6.16) the quantity $\Pi_h^B[\phi]$ by $\phi$ and evaluate the error arising from this replacement.

### 6.3.1 Error in the discretized time derivative

We write

$$\int_\Omega D_t \left( \varrho_h^k \chi(\vartheta_h^k) \right) \Pi_h^B[\phi] \ \mathrm{d}x = \int_\Omega D_t \left( \varrho_h^k \chi(\vartheta_h^k) \right) \phi \ \mathrm{d}x + \int_\Omega D_t \left( \varrho_h^k \chi(\vartheta_h^k) \right) \left( \Pi_h^B[\phi] - \phi \right) \ \mathrm{d}x,$$

where

$$\int_\Omega D_t \left( \varrho_h^k \chi(\vartheta_h^k) \right) \left( \Pi_h^B[\phi] - \phi \right) \ \mathrm{d}x =$$

$$\int_\Omega \frac{\varrho_h^k - \varrho_h^{k-1}}{\Delta t} \chi(\vartheta_h^k) \left( \Pi_h^B[\phi] - \phi \right) \ \mathrm{d}x + \int_\Omega \varrho_h^{k-1} \frac{\chi(\vartheta_h^k) - \chi(\vartheta_h^{k-1})}{\Delta t} \left( \Pi_h^B[\phi] - \phi \right) \ \mathrm{d}x.$$

Since $\chi$ is bounded, we get, similarly to Section 6.2.1,

$$\left| \int_\Omega \frac{\varrho_h^k - \varrho_h^{k-1}}{\Delta t} \chi(\vartheta_h^k) \left( \Pi_h^B[\phi] - \phi \right) \ \mathrm{d}x \right|$$

$$\lesssim \left( \Delta t \int_\Omega \left( \frac{\varrho_h^k - \varrho_h^{k-1}}{\Delta t} \right)^2 \ \mathrm{d}x \right)^{1/2} \sqrt{h} \|\nabla_x \phi\|_{L^\infty(\Omega)},$$

provided $\Delta \approx h$, where the first integral is controlled by (5.7). Similarly,

$$\left| \int_\Omega \varrho_h^{k-1} \frac{\chi(\vartheta_h^k) - \chi(\vartheta_h^{k-1})}{\Delta t} \left( \Pi_h^B[\phi] - \phi \right) \ \mathrm{d}x \right|$$

$$\left| \int_\Omega \sqrt{\varrho_h^{k-1}} \sqrt{\varrho_h^{k-1}} \frac{\chi(\vartheta_h^k) - \chi(\vartheta_h^{k-1})}{\Delta t} \left( \Pi_h^B[\phi] - \phi \right) \ \mathrm{d}x \right|$$

$$\lesssim \left( \Delta t \int_\Omega \varrho_h^{k-1} \left( \frac{\chi(\vartheta_h^k) - \chi(\vartheta_h^{k-1})}{\Delta t} \right)^2 \ \mathrm{d}x \right)^{1/2} \sqrt{h} \|\nabla_x \phi\|_{L^\infty(\Omega;R^3)} \|\varrho_h^{k-1}\|_{L^\gamma(\Omega)}^{1/2},$$

where the first integral can be controlled by means of (5.20). Indeed it is enough to check that

$$\chi(A) - \chi(B) \lesssim \frac{A - B}{A} \text{ whenever } A > B \geq 0 \tag{6.18}$$

as long as $\chi$ belongs to the class (6.15).

Thus we get

$$\left| \int_\Omega D_t \left( \varrho_h \chi(\vartheta_h) \right) \left( \Pi_h^B[\phi] - \phi \right) \ \mathrm{d}x \right| \lesssim \sqrt{h} \ r_h^8(t) \|\nabla_x \phi\|_{L^\infty(\Omega)}, \ \|r_h^8\|_{L^2(0,T)} \lesssim 1. \tag{6.19}$$

### 6.3.2 Error in the upwind term

To handle the upwind term, we use the relation (2.17) obtaining

$$\sum_{\Gamma \in \Gamma_h} \int_\Gamma \mathrm{Up}[\varrho_h^k \chi(\vartheta_h^k), \mathbf{u}_h^k] \ [[\Pi_h^B[\phi] \ ]] \ \mathrm{dS}_x \tag{6.20}$$

$$= \int_\Omega \varrho_h^k \chi(\vartheta_h^k) \mathbf{u}_h^k \cdot \nabla_x \phi \ \mathrm{d}x - \sum_{E \in E_h} \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} \left( \Pi_h^B[\phi] - \phi \right) [[\varrho_h^k \chi(\vartheta_h^k)]] [\tilde{\mathbf{u}}_h^k \cdot \mathbf{n}]^- \ \mathrm{dS}_x$$

$$+ \sum_{E \in E_h} \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} \varrho_h^k \chi(\vartheta_h^k) \phi (\tilde{\mathbf{u}} - \mathbf{u}) \cdot \mathbf{n} \ \mathrm{dS}_x + \sum_{E \in E_h} \int_{E_h} \varrho_h^k \chi(\vartheta_h^k) \mathrm{div}_h \mathbf{u}_h^k \left( \phi - \Pi_h^B \phi \right) \ \mathrm{d}x.$$

Next, we decompose

$$\sum_{E \in E_h} \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} \left( \Pi_h^B[\phi] - \phi \right) [[\varrho_h^k \chi(\vartheta_h^k)]] [\tilde{\mathbf{u}}_h^k \cdot \mathbf{n}]^- \ \mathrm{dS}_x =$$

$$\sum_{E \in E_h} \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} \left( \Pi_h^B[\phi] - \phi \right) \varrho_h^k [[\chi(\vartheta_h^k)]] [\tilde{\mathbf{u}}_h^k \cdot \mathbf{n}]^- \ \mathrm{dS}_x$$

$$+ \sum_{E \in E_h} \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} \left( \Pi_h^B[\phi] - \phi \right) [[\varrho_h^k]] \chi((\vartheta_h^k)^+) [\tilde{\mathbf{u}}_h^k \cdot \mathbf{n}]^- \ \mathrm{dS}_x,$$

where, by means of Hölder's and Jensen's inequalities, the error estimates (2.6), and the trace estimates (2.21),

$$\left| \sum_{E \in E_h} \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} \left( \Pi_h^B[\phi] - \phi \right) (\varrho_h^k) [[\chi(\vartheta_h^k)]] [\tilde{\mathbf{u}}_h^k \cdot \mathbf{n}]^- \ \mathrm{dS}_x \right|$$

$$\lesssim h^{3/2} \|\nabla_x \phi\|_{L^\infty(\Omega)} \left( \sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma \frac{[[\chi(\vartheta_h^k)]]^2}{h} \ \mathrm{dS}_x \right)^{1/2} \left( \sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma |\varrho_h^k|^2 |\tilde{\mathbf{u}}_h^k \cdot \mathbf{n}|^2 \ \mathrm{dS}_x \right)^{1/2}$$

$$\lesssim h^{3/2} \|\nabla_x \phi\|_{L^\infty(\Omega)} \left( \sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma \frac{[[\chi(\vartheta_h^k)]]^2}{h} \ \mathrm{dS}_x \right)^{1/2} \left( \sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma |\varrho_h^k|^2 |\mathbf{u}_h^k|^2 \ \mathrm{dS}_x \right)^{1/2}$$

$$\lesssim h\|\nabla_x\phi\|_{L^\infty(\Omega)}\left(\sum_{\Gamma\in\Gamma_{h,\rm int}}\int_\Gamma\frac{[[\chi(\vartheta_h^k)]]^2}{h}\ {\rm dS}_x\right)^{1/2}\left(\sum_{E\in E_h}\int_E|\varrho_h^k|^2|\mathbf{u}_h^k|^2{\rm d}x\right)^{1/2},$$

where we may use (5.6), (5.28), and (5.17) to control both integrals on the right-hand side in $L^2(0,T)$.

Moreover, seeing that $\chi$ is bounded, the integral

$$\sum_{E\in E_h}\sum_{\Gamma_E\subset\partial E}\int_{\Gamma_E}\left(\Pi_h^B[\phi]-\phi\right)[[\varrho_h^k]]\chi((\vartheta_h^k)^+)[\tilde{\mathbf{u}}_h^k\cdot\mathbf{n}]^-\ {\rm dS}_x$$

can be handled with the help of the energy estimate (5.9), (5.28) and the error estimate (2.6).

Finally, the remaining two integrals on the right-hand side of (6.20) can be handled by means of (2.18) and the available energy bounds (5.6), (5.27).

Thus we have obtained

$$\left|\sum_{\Gamma\in\Gamma_h}\int_\Gamma {\rm Up}[\varrho_h^k\chi(\vartheta_h^k),\mathbf{u}_h^k]\ [[\Pi_h^B[\phi]\ ]]\ {\rm dS}_x-\int_\Omega\varrho_h^k\chi(\vartheta_h^k)\mathbf{u}_h^k\cdot\nabla_x\phi\ {\rm d}x\right|$$

$$\lesssim h^{\frac{\gamma-2}{\gamma}}z(t)\|\phi\|_{W^{1,\infty}(\Omega)},\ \|z\|_{L^1(0,T)}\lesssim 1.$$

Consequently, by the reasoning as in (6.7) we conclude

$$\left|\sum_{\Gamma\in\Gamma_h}\int_\Gamma {\rm Up}[\varrho_h^k\chi(\vartheta_h^k),\mathbf{u}_h^k]\ [[\Pi_h^B[\phi]\ ]]\ {\rm dS}_x-\int_\Omega\varrho_h^k\chi(\vartheta_h^k)\mathbf{u}_h^k\cdot\nabla_x\phi\ {\rm d}x\right| \tag{6.21}$$

$$\lesssim h^\alpha r^9(t)\|\phi\|_{W^{1,\infty}(\Omega)},\ \|r^9\|_{L^\beta(0,T)}\lesssim 1,\ \text{ with some } \alpha>0,\ \beta>1.$$

### 6.3.3 Error in the thermal diffusion

We need the following auxiliary result.

**Lemma 6.1** *Let $\phi\in C^2(\overline{\Omega})$ such that $\nabla_x\phi\cdot\mathbf{n}|_{\partial\Omega}=0$.*

*Then*

$$\left|\sum_{\Gamma\in\Gamma_{h,\rm int}}\int_{\Gamma_h}\frac{1}{d_\Gamma}[[v]]\ [[\Pi_h^B[\phi]]]{\rm dS}_x+\int_\Omega v\Delta\phi\ {\rm d}x\right|\le h\|v\|_{H^1_{Q_h}(\Omega)}\|\phi\|_{C^2(\overline{\Omega})}$$

*for any $v\in Q^h(\Omega)$.*

**Proof:**
We start by writing

$$\int_\Omega v\Delta\phi \ \mathrm{d}x = \sum_{E\in E_h}\int_E v\Delta\phi \ \mathrm{dx} = \sum_{E\in E_h}\int_{\partial E} v\nabla_x\phi\cdot\mathbf{n}\ \mathrm{dS}_x = -\sum_{\Gamma\in\Gamma_{h,\mathrm{int}}}[[v]]\nabla_x\phi\cdot\mathbf{n}\ \mathrm{dS}_x.$$

Recalling the definition of the projection $\Pi_h^B$ we observe that

$$\left|\nabla_x\phi\cdot\mathbf{n} - \frac{[[\Pi_h^B\phi]]}{d_\Gamma}\right| \lesssim h\|\phi\|_{C^2(\overline{\Omega})} \text{ on any face } \Gamma;$$

whence it remains to estimate

$$\left|\sum_{\Gamma\in\Gamma_{h,\mathrm{int}}}\int_\Gamma [[v]]\ \mathrm{dS}_x\right| \leq \left(\sum_{\Gamma\in\Gamma_{h,\mathrm{int}}}\int_\Gamma \frac{[[v]]^2}{d_\Gamma}\ \mathrm{dS}_x\right)^{1/2}\left(\sum_{\Gamma\in\Gamma_{h,\mathrm{int}}}\int_\Gamma h\mathrm{dS}_x\right)^{1/2} \lesssim \|v\|_{H^1_{Q_h}(\Omega)}|\Omega|^{1/2}.$$

Q.E.D.

Now, we are ready to deal with the diffusion term

$$\sum_{\Gamma\in\Gamma_{h,\mathrm{int}}}\int_\Gamma \frac{1}{d_\Gamma}\left\{\chi'\left(\vartheta_h^k\right)\right\}[[K(\vartheta_h^k)]]\ [[\Pi_h^B[\phi]]]\mathrm{dS}_x.$$

Introducing a new function $K_\chi$,

$$K'_\chi(\vartheta) = \chi'(\vartheta)K'(\vartheta),$$

we may write the diffusive term with the help of the mean-value theorem as

$$\left\{\chi'\left(\vartheta_h^k\right)\right\}[[K(\vartheta_h^k)]] = [[K_\chi(\vartheta_h^k)]] + c_h^k(x)[[\vartheta_n^k]]^2,$$

where $c_h^k$ is uniformly bounded. Consequently, we get

$$\sum_{\Gamma\in\Gamma_{h,\mathrm{int}}}\int_\Gamma \frac{1}{d_\Gamma}\left\{\chi'\left(\vartheta_h^k\right)\right\}[[K(\vartheta_h^k)]]\ [[\Pi_h^B[\phi]]]\mathrm{dS}_x$$

$$= \sum_{\Gamma\in\Gamma_{h,\mathrm{int}}}\int_\Gamma \frac{1}{d_\Gamma}[[K_\chi(\vartheta_h^k)]]\ [[\Pi_h^B[\phi]]]\mathrm{dS}_x + \sum_{\Gamma\in\Gamma_{h,\mathrm{int}}}\int_\Gamma c_h^k\frac{[[\vartheta_h^k]]^2}{d_\Gamma}[[\Pi_h^B[\phi]]]\mathrm{dS}_x.$$

Seeing that

$$|[[\Pi_h^B\phi]]| \leq h\|\nabla_x\phi\|_{L^\infty(\Omega)}$$

we can estimate the last integral using the entropy bounds (5.17), (5.18). Combining the above results with Lemma 6.1, we get

$$\sum_{\Gamma\in\Gamma_{h,\rm int}} \int_\Gamma \frac{1}{d_\Gamma} \left\{\chi'\left(\vartheta_h^k\right)\right\} [[K(\vartheta_h^k)]]\ [[\Pi_h^B[\phi]]] {\rm dS}_x \tag{6.22}$$

$$= - \int_\Omega K_\chi(\vartheta_h^k)\Delta\phi \ {\rm d}x + h^\alpha r^{10}\|\phi\|_{C^2(\overline{\Omega})},\ \|r^{10}\|_{L^\beta(0,T)} \lesssim 1,\ \alpha>0, \beta>1,$$

where we have used (2.26) in the same way as in (6.7).

### 6.3.4 Error estimates in the remaining terms

By virtue of (5.6), (5.28), (5.26), (6.15), (2.6), (2.26),

$$\Big|\int_\Omega \Big(\mu|\nabla_h \mathbf{u}_h^k|^2 + \lambda|{\rm div}_h\mathbf{u}_h^k|^2\Big)\chi'(\vartheta_h^k)\Big(\Pi_h^B[\phi]-\phi\Big)\ {\rm d}x\Big| \tag{6.23}$$

$$+\Big|\int_\Omega \chi'(\vartheta_h^k)\vartheta_h^k\varrho_h^k{\rm div}_h\mathbf{u}_h^k\Big(\Pi_h^B[\phi]-\phi\Big)\ {\rm d}x\Big| \lesssim h^\alpha r^{11}(t)\|\nabla\phi\|_{L^\infty(\Omega)},\ \|r^{11}\|_{L^\beta(0,T)} \lesssim 1,\ \alpha>0,\beta>1.$$

Finally, we note that the similar treatment, in combination with the estimate (5.9), can be applied to eliminate the $h^{1-\varepsilon}$-term on the right-hand side of (6.16), namely

$$\Big|h^{1-\varepsilon}\sum_{\Gamma\in\Gamma_{h,\rm int}}\int_\Gamma [[\varrho_h^k]]\ [[\Big(\chi(\vartheta_h^k)-\chi'(\vartheta_h^k)\vartheta_k^k\Big)\ \Pi_h^B[\phi]\ ]]\ {\rm dS}_x\Big| \tag{6.24}$$

$$\lesssim h^\alpha\ r^{12}(t)\ \|\phi\|_{W^{1,\infty}(\Omega)},\ \|r^{12}\|_{L^\beta(0,T)} \lesssim 1,\ \alpha>0,\beta>1.$$

Thus, going back to formula (6.16) while summing up the estimates (6.17), (6.19), (6.21), (6.22–6.24), we obtain the consistency formulation of the renormalized thermal energy method:

$$\int_\Omega D_t\left(\varrho_h^k\chi(\vartheta_h^k)\right)\phi\ {\rm d}x - \int_\Omega \varrho_h^k\chi(\vartheta_h^k)\mathbf{u}_h^k\cdot\nabla_x\phi\ {\rm d}x - \int_\Omega K_\chi(\vartheta_h^k)\Delta\phi\ {\rm d}x \tag{6.25}$$

$$= \int_\Omega \Big(\mu|\nabla_h\mathbf{u}_h^k|^2+\lambda|{\rm div}_h\mathbf{u}_h^k|^2\Big)\chi'(\vartheta_h^k)\phi\ {\rm d}x - \int_\Omega \chi'(\vartheta_h^k)\vartheta_h^k\varrho_h^k{\rm div}_h\mathbf{u}_h^k\phi\ {\rm d}x + \langle D_h,\phi\rangle + h^\alpha\left\langle R_h^{13},\phi\right\rangle,$$

with some $\alpha>0$, for any test function $\phi\in C^2(\overline{\Omega})$, $\nabla_x\phi\cdot\mathbf{n}|_{\partial\Omega}=0$, and any $\chi$ satisfying (6.15), where $D_h$ satisfies (6.17), and

$$\left|\left\langle R_h^{13}(t),\phi\right\rangle\right| \lesssim r_h^{13}(t)\|\phi\|_{C^2(\overline{\Omega})},\ \|r_h^{13}\|_{L^\beta(0,T)} \lesssim 1 \text{ with some } \beta>1. \tag{6.26}$$

**Remark 6.1** *The required regularity of the test functions $\phi$ in (6.25) can be relaxed to*

$$\phi \in W^{2,\infty}(\Omega),\ \nabla_x \phi \cdot \mathbf{n}|_{\Gamma} = 0 \text{ on any face } \Gamma \subset \partial\Omega.$$

# 7 Convergence

We are ready to establish convergence of our numerical scheme to a weak solution of the problem. We take advantage of the consistency formulation derived in the preceding section that converts the problem to the framework of the mathematical theory developed in [14]. The reader may want to check the material of [14, Chapters 6,7] for technical details omitted in the present text. We also refer to [1] for a complete existence proof based on the technique of time discretization.

## 7.1 Pressure estimates

The uniform bound (5.6) is not sufficient for passing to limit in the "elastic" pressure component $\varrho^\gamma$. To get better integrability of the pressure, we introduce an inverse of the divergence commonly referred to as Bogovskii's operator $\mathcal{B}$, see Bogovskii [2]. The operator acts on integrable functions of zero mean in $\Omega$ and enjoys the following properties:

$$\mathcal{B}[r] \in W_0^{1,p}(\Omega; R^3),\ \mathrm{div}_x \mathcal{B}[r] = r \text{ for any } r \in L^p(\Omega),\ \int_\Omega r \ \mathrm{d}x = 0,\ 1 < p < \infty, \tag{7.1}$$

$$\|\mathcal{B}[r]\|_{W_0^{1,p}(\Omega;R^3)} \stackrel{<}{\sim} \|r\|_{L^p(\Omega)},\ 1 < p < \infty, \tag{7.2}$$

$$\|\mathcal{B}[r]\|_{L^q(\Omega;R^3)} \stackrel{<}{\sim} \|r\|_{W_0^{-1,q}(\Omega)},\ 1 < q < \infty, \tag{7.3}$$

where the symbol $W_0^{-1,q}$ denotes the dual to the Sobolev space $W^{1,q'}$ (not to be confused with $W^{-1,q}$ - the dual to $W_0^{1,q'}$). We refer to Galdi [16, Chapter 3] for a detailed proof of the properties (7.1), (7.2), and to Geissert, Heck and Hieber [18] for (7.3). It is important that the afore-mentioned estimates are valid for any *Lipschitz* domains, in particular for $\Omega$.

The pressure estimates are obtained by taking

$$\phi = \mathcal{B}\left[\varrho_h - \frac{1}{|\Omega|}\int_\Omega \varrho_h \ \mathrm{d}x\right]$$

as a test function in the consistency formulation (6.13) of the momentum method:

$$\int_0^T \int_\Omega \varrho_h^2 (b + \vartheta_h) \ \mathrm{d}x\mathrm{d}t + \int_0^T \int_\Omega \varrho_h^{\gamma+1} \ \mathrm{d}x\mathrm{d}t = \frac{1}{|\Omega|}\int_\Omega \varrho_h \ \mathrm{d}x \int_0^T \int_\Omega p(\varrho_h, \vartheta_h) \ \mathrm{d}x\mathrm{d}t \tag{7.4}$$

$$+\int_0^T \int_\Omega D_t(\varrho_h \widehat{\mathbf{u}}_h) \cdot \mathcal{B}\left[\varrho_h - \frac{1}{|\Omega|}\int_\Omega \varrho_h \ \mathrm{d}x\right] \ \mathrm{d}x \ \mathrm{d}t - h^\alpha \int_0^T \left\langle R_h^6, \mathcal{B}\left[\varrho_h - \frac{1}{|\Omega|}\int_\Omega \varrho_h \ \mathrm{d}x\right]\right\rangle \ \mathrm{d}t$$

$$-\int_0^T \int_\Omega (\varrho_h \widehat{\mathbf{u}}_h \otimes \mathbf{u}_h) : \nabla_x \mathcal{B}\left[\varrho_h - \frac{1}{|\Omega|}\int_\Omega \varrho_h \ \mathrm{d}x\right] \ \mathrm{d}x \ \mathrm{d}t$$

$$+\int_0^T \int_\Omega \left(\mu \nabla_h \mathbf{u}_h \cdot \nabla_x \ \mathcal{B}\left[\varrho_h - \frac{1}{|\Omega|}\int_\Omega \varrho_h \ \mathrm{d}x\right] + \lambda \mathrm{div}_h \mathbf{u}_h \left[\varrho_h - \frac{1}{|\Omega|}\int_\Omega \varrho_h \ \mathrm{d}x\right]\right) \ \mathrm{d}x \ \mathrm{d}t$$

Combining the stability estimate (5.6) with (7.1) we observe that

$$\mathcal{B}\left[\varrho_h - \frac{1}{|\Omega|}\int_\Omega \varrho_h \ \mathrm{d}x\right] \in L^\infty(0,T; W_0^{1,\gamma}(\Omega));$$

whence admissible for (6.13). In particular, thanks to (6.14), (5.6), (7.2) we get

$$h^\alpha \int_0^T \left\langle R_h^6, \mathcal{B}\left[\varrho_h - \frac{1}{|\Omega|}\int_\Omega \varrho_h \ \mathrm{d}x\right]\right\rangle \ \mathrm{d}t \to 0 \text{ as } h \to 0.$$

Next, we apply by part integration to the time "derivative" to deduce

$$\int_0^T \int_\Omega D_t(\varrho_h \widehat{\mathbf{u}}_h) \cdot \mathcal{B}\left[\varrho_h - \frac{1}{|\Omega|}\int_\Omega \varrho_h \ \mathrm{d}x\right] \ \mathrm{d}x \ \mathrm{d}t \tag{7.5}$$

$$= \sum_{k\geq 1} \int_\Omega \left(\varrho_h^k \widehat{\mathbf{u}}_h^k - \varrho_h^{k-1} \widehat{\mathbf{u}}_h^{k-1}\right) \cdot \mathcal{B}\left[\varrho_h^k - \frac{1}{|\Omega|}\int_\Omega \varrho_h^k \ \mathrm{d}x\right] \ \mathrm{d}x$$

$$= \int_\Omega \varrho_h \widehat{\mathbf{u}}_h \cdot \mathcal{B}\left[\varrho_h - \frac{1}{|\Omega|}\int_\Omega \varrho_h \ \mathrm{d}x\right](T,\cdot) \ \mathrm{d}x - \int_\Omega \varrho_h^0 \widehat{\mathbf{u}}_h^0 \cdot \mathcal{B}\left[\varrho_h^1 - \frac{1}{|\Omega|}\int_\Omega \varrho_h^1 \ \mathrm{d}x\right] \ \mathrm{d}x$$

$$-\sum_{k\geq 1} \int_\Omega \left[\varrho_h^{k-1} \widehat{\mathbf{u}}_h^{k-1} \cdot \left(\mathcal{B}\left[\varrho_h^k - \frac{1}{|\Omega|}\int_\Omega \varrho_h^k \ \mathrm{d}x\right] - \mathcal{B}\left[\varrho_h^{k-1} - \frac{1}{|\Omega|}\int_\Omega \varrho_h^{k-1} \ \mathrm{d}x\right]\right)\right] \ \mathrm{d}x$$

$$= \int_\Omega \varrho_h \widehat{\mathbf{u}}_h \cdot \mathcal{B}\left[\varrho_h - \frac{1}{|\Omega|}\int_\Omega \varrho_h \ \mathrm{d}x\right](T,\cdot) \ \mathrm{d}x - \int_\Omega \varrho_h^0 \widehat{\mathbf{u}}_h^0 \cdot \mathcal{B}\left[\varrho_h^1 - \frac{1}{|\Omega|}\int_\Omega \varrho_h^1 \ \mathrm{d}x\right] \ \mathrm{d}x$$

$$-\int_0^T \int_\Omega \varrho_h(t-\Delta t) \widehat{\mathbf{u}}_h(t-\Delta t) \cdot \mathcal{B}[D_t \varrho_h] \ \mathrm{d}x \ \mathrm{d}t.$$

We observe that the expression on the right-hand side of (7.4) is bounded uniformly for $h \to 0$. Indeed combining the estimates (2.5), (5.3), (5.6), and (5.28) we have

$$\sup_{\tau \in (0,T)} \|\varrho_h \widehat{\mathbf{u}}_h(\tau,\cdot)\|_{L^q(\Omega,R^3)} \stackrel{<}{\sim} 1, \ q = \frac{2\gamma}{\gamma+1}, \tag{7.6}$$

$$\|\varrho_h \mathbf{u}_h\|_{L^2(0,T;L^s(\Omega;R^3))} \stackrel{<}{\sim} 1,\ \|\varrho_h \widehat{\mathbf{u}}_h\|_{L^2(0,T;L^s(\Omega;R^3))} \stackrel{<}{\sim} 1,\ s = \frac{6\gamma}{\gamma+6} > 2 \text{ if } \gamma > 3, \tag{7.7}$$

while, by virtue of (7.3) and the consistency formulation (6.1) of the continuity method

$$\mathcal{B}[D_t \varrho_h] \text{ is bounded in } L^2(0,T;L^{s'}(\Omega,R^3)),\ s' = \frac{6\gamma}{5\gamma-6}$$

uniformly for $h \to 0$.

Finally, by virtue of (7.7) and (5.28)

$$\|\varrho_h \widehat{\mathbf{u}}_h \otimes \mathbf{u}_h\|_{L^2(0,T;L^s(\Omega;R^9))} \stackrel{<}{\sim} 1, \quad s = \frac{3\gamma}{\gamma+3} > 3/2 \text{ if } \gamma > 3; \tag{7.8}$$

consequently, due to (7.1), the remaining integrals on the right-hand side of (7.4) can be estimated in the same way as in [14, Chapter 5] and we may conclude that

$$\|\varrho_h\|_{L^{\gamma+1}((0,T)\times\Omega)} \stackrel{<}{\sim} 1. \tag{7.9}$$

## 7.2 Weak sequential compactness

In accordance with the uniform estimates (5.6), (5.26), and (5.28) we may assume that

$$\varrho_h \to \varrho \text{ weakly-(*) in } L^\infty(0,T;L^\gamma(\Omega)), \tag{7.10}$$

$$\vartheta_h \to \vartheta \text{ weakly in } L^p(0,T;L^q(\Omega)) \text{ for any } 1 \le p < 3,\ 1 \le q < 9, \tag{7.11}$$

and

$$\mathbf{u}_h \to \mathbf{u} \text{ weakly in } L^2(0,T;L^6(\Omega;R^3)) \tag{7.12}$$

at least for a suitable subsequence $h \to 0$. Moreover, we have $\varrho \ge 0$, and, by virtue of (5.1),

$$\int_\Omega \varrho(\tau,\cdot)\ \mathrm{d}x = \int_\Omega \varrho_0\ \mathrm{d}x \text{ for a.a. } t \in (0,T).$$

As a consequence of (5.22), and since $\vartheta \mapsto \log\vartheta$ is a concave function

$$\log(\vartheta) \in L^2(0,T;L^6(\Omega)),\ \text{in particular } \vartheta > 0 \text{ a.a. in } (0,T)\times\Omega.$$

Next, it follows from (2.10) that

$$\|\widehat{\mathbf{u}_h} - \mathbf{u}_h\|_{L^2((0,T)\times\Omega;R^3)} \to 0, \tag{7.13}$$

in particular,

$$\widehat{\mathbf{u}_h} \to \mathbf{u} \text{ weakly in } L^2(0,T;L^6(\Omega;R^3)). \tag{7.14}$$

The final observation is that (5.27) implies

$$\nabla_h \mathbf{u}_h \to \nabla_x \mathbf{u} \text{ weakly in } L^2((0,T)\times\Omega), \tag{7.15}$$

where the limit velocity field satisfies

$$\mathbf{u} \in L^2(0,T;W_0^{1,2}(\Omega;R^3)).$$

**Remark 7.1** *The fact that the weak limit of $\nabla_h \mathbf{u}_h$ coincides with $\nabla_x \mathbf{u}$ follows from the "density" of the spaces $V_{h,0}$ in $W_0^{1,2}$ stated in (2.11) and the estimate (2.13).*

#### 7.2.1 Convergence of convective terms and of the thermal pressure $\varrho\vartheta$

To establish the weak convergence of convective terms, we need the following result which is a variant of [20, Lemma 2.3].

**Lemma 7.1** *Let $\{v_h\}_{h>0}$, $\{w_h\}_{h>0}$ be two sequences of functions in $(0,T)\times\Omega$ such that*

$$v_h, w_h \text{ are constant functions of the time on any interval } [k\Delta t,(k+1)\Delta t),\ k=0,1,\dots,\ \Delta t \approx h,$$

$$v_h \to v \text{ weakly-* in } L^{p_1}(0,T;L^{q_1}(\Omega)),\ w_h \to w \text{ weakly-* in } L^{p_2}(0,T;L^{q_2}(\Omega)),$$

$$\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{q_1}+\frac{1}{q_2}\le 1,\ p_i>1, q_i>1, i=1,2,$$

$$\left|\int_\Omega D_t v_h \phi \ \mathrm{d}x\right| \le r_h(t)\|\phi\|_{W^{k,p}(\Omega)} \text{ for certain } k,p\ge 1,\ \|r^h\|_{L^1(0,T)} \stackrel{<}{\sim} 1, \tag{7.16}$$

$$\|w_h(t,x)-w_h(t,x-\xi)\|_{L^{p_2}(0,T;L^{q_2}(\Omega))} \to 0 \text{ as } |\xi|\to 0 \text{ uniformly in } h. \tag{7.17}$$

*Then*

$$v_h w_h \to vw \text{ in the sense of distributions in } (0,T)\times\Omega.$$

In agreement with the estimates (5.25), (5.27) and the compactness properties of the spaces $H^1_{V_h}$, $H^1_{Q_h}$ stated in (2.31), (2.28) we observe that the sequences

$$\{\mathbf{u}_h\}_{h>0},\ \{\vartheta_h\}_{h>0},\ \text{and}\{\chi(\vartheta_h)\}_{h>0},\ \text{with } \chi \text{ as in (6.15)}$$

satisfy the hypothesis (7.17) with $p_2 = q_2 = 2$, while the hypothesis (7.16) can be checked for $\varrho_h$, $\varrho_h \widehat{\mathbf{u}_h}$, and $\varrho_h \chi(\vartheta_h)$ by means of the consistency formulations (6.1), (6.13), and (6.25), respectively. Thus a successive application of Lemma 7.1 gives rise to the following limits:

$$\varrho_h \mathbf{u}_h \to \varrho \mathbf{u} \text{ weakly in } L^2(0,T; L^{\frac{6\gamma}{\gamma+6}}(\Omega; R^3)) \tag{7.18}$$

$$\varrho_h \widehat{u_h} \otimes \mathbf{u}_h \to \varrho \mathbf{u} \otimes \mathbf{u} \text{ weakly in } L^q((0,T) \times \Omega) \text{ for some } q > 1, \tag{7.19}$$

$$\varrho_h \chi(\vartheta_h) \to \varrho \overline{\chi(\vartheta)} \text{ weakly-(*) in } L^\infty(0,T; L^\gamma(\Omega)), \tag{7.20}$$

and

$$\varrho_h \chi(\vartheta_h) \mathbf{u}_h \to \varrho \overline{\chi(\vartheta)} \mathbf{u} \text{ weakly in } L^2(0,T; L^{\frac{6\gamma}{\gamma+6}}(\Omega). \tag{7.21}$$

Here and hereafter we denote $\overline{f(\varrho, \vartheta, \mathbf{u})}$ a $L^1$-weak limit of the sequence $\overline{f(\varrho_h, \vartheta_h, \mathbf{u}_h)}$, in particular, $\overline{\chi(\vartheta)}$ denotes a weak limit of $\chi(\vartheta_h)$.

Employing Lemma 7.1 on any compact subset $K$ in $\Omega$ to treat the product $\varrho_h \vartheta_h$, we get in view of (2.28), (5.17), (6.1)

$$\varrho_h \vartheta_h \to \varrho \vartheta \text{ weakly in } L^2((0,T) \times K), \tag{7.22}$$

where we have also used (7.10), (7.11).

**Remark 7.2** *As for the exponent $q$ in (7.19), we recall that*

$$\varrho_h \widehat{\mathbf{u}_h} \in L^\infty(0,T; L^{\frac{2\gamma}{\gamma+1}}(\Omega)) \cap L^2(0,T; L^{\frac{6\gamma}{\gamma+6}}(\Omega; R^3)) \hookrightarrow L^r((0,T) \times \Omega) \textit{ for a certain } r > 2$$

*by interpolation.*

### 7.2.2 Limit in the continuity and momentum methods

Letting $h \to 0$ in (6.1), (6.13) we obtain

$$\int_0^T \int_\Omega \Big[ \varrho \partial_t \varphi + \varrho \mathbf{u} \cdot \nabla_x \varphi \Big] \, \mathrm{d}x \, \mathrm{d}t = - \int_\Omega \varrho_0 \varphi(0, \cdot) \, \mathrm{d}x \tag{7.23}$$

for any $\varphi \in C^1_c([0,T) \times \overline{\Omega})$;

$$\int_0^T \int_\Omega \Big[ \varrho \mathbf{u} \cdot \partial_t \varphi + \varrho \mathbf{u} \otimes \mathbf{u} : \nabla_x \varphi + \varrho \vartheta \mathrm{div}_x \varphi + a \overline{\varrho^\gamma} \mathrm{div}_x \varphi + b \varrho \mathrm{div}_x \varphi \Big] \, \mathrm{d}x \, \mathrm{d}t \tag{7.24}$$

$$= \int_0^T \int_\Omega \Big[ \mu \nabla_x \mathbf{u} : \nabla_x \varphi + \lambda \mathrm{div}_x \mathbf{u} \, \mathrm{div}_x \varphi \Big] \, \mathrm{d}x \, \mathrm{d}t - \int_\Omega \varrho_0 \mathbf{u}_0 \cdot \varphi(0, \cdot) \, \mathrm{d}x$$

for any $\varphi \in C^1_c([0,T) \times \Omega; R^3)$.

## 7.3 Strong convergence of the density

To show strong convergence of the densities, we use the method of Lions [22] based on the "weak continuity" of the effective viscous flux. To this end, we extend $\varrho_h$ by zero outside $\Omega$, and use the quantities

$$\phi = \varphi \nabla_x \Delta^{-1}[\varrho_h], \tag{7.25}$$

where

$$\varphi(t,x) = \psi(t)\eta(x), \psi \in C_c^\infty(0,T),\ \eta \in C_c^\infty(\Omega),\ -\Delta^{-1}[v] \equiv \mathcal{F}^{-1}_{\xi\to x}\left[\frac{1}{|\xi|^2}\mathcal{F}_{x\to\xi}[1_\Omega v]\right],$$

and $\mathcal{F}$ denotes the standard Fourier transform, as test functions in the consistency formulation (6.13) of the momentum method:

$$\int_0^T \int_\Omega \varphi\Big[p(\varrho_h,\vartheta_h)\varrho_h - \lambda\varrho_h \mathrm{div}_x \mathbf{u}_h\Big] \,\mathrm{d}x\,\mathrm{d}t = \tag{7.26}$$

$$\int_0^T \int_\Omega \Big[\lambda \mathrm{div}_x \mathbf{u}_h - p(\varrho_h,\vartheta_h)\Big]\nabla_x\varphi\cdot\nabla_x(\Delta^{-1}[\varrho_h]) \,\mathrm{d}x\,\mathrm{d}t - h^\alpha \int_0^T \Big\langle R_h^6, \varphi\nabla_x(\Delta^{-1}[\varrho_h])\Big\rangle \,\mathrm{d}t$$

$$+\int_0^T\int_\Omega \mu\nabla_h\mathbf{u}_h : \nabla_x\Big[\varphi\nabla_x(\Delta^{-1}[\varrho_h])\Big] \,\mathrm{d}x\,\mathrm{d}t$$

$$-\int_0^T\int_\Omega (\varrho_h\widehat{\mathbf{u}}_h\otimes\mathbf{u}_h) : \nabla_x\Big(\varphi\nabla_x(\Delta^{-1}[\varrho_h])\Big) \,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega D_t(\varrho_h\widehat{\mathbf{u}}_h)\cdot\varphi\nabla_x(\Delta^{-1}[\varrho_h]) \,\mathrm{d}x\,\mathrm{d}t.$$

We notice that, by virtue of the well known properties of the operator $\nabla\Delta^{-1}$ on $L^p(R^3)$, the function $\phi \in L^\infty(0,T;W_0^{1,\gamma}(\Omega;R^3))$, and it is therefore and admissible test function for the momentum method.

Furthermore, using the by-part integration formula, we get

$$\int_0^T\int_\Omega D_t(\varrho_h\widehat{\mathbf{u}}_h)\cdot\varphi\nabla_x(\Delta^{-1}[\varrho_h]) \,\mathrm{d}x\,\mathrm{d}t = -\int_0^T\int_\Omega \frac{\varphi(t+\Delta t)-\varphi(t)}{\Delta t}\varrho_h\widehat{\mathbf{u}}_h\cdot\nabla_x(\Delta^{-1}[\varrho_h]) \,\mathrm{d}x\,\mathrm{d}t$$

$$-\int_0^T\int_\Omega \varphi\varrho_h(t-\Delta t)\widehat{\mathbf{u}}_h(t-\Delta t)\cdot\nabla_x\Delta^{-1}[D_t\varrho_h] \,\mathrm{d}x\,\mathrm{d}t$$

$$= -\int_0^T\int_\Omega \frac{\varphi(t+\Delta t)-\varphi(t)}{\Delta t}\varrho_h\widehat{\mathbf{u}}_h\cdot\nabla_x(\Delta^{-1}[\varrho_h]) \,\mathrm{d}x\,\mathrm{d}t$$

$$+\int_0^T\int_\Omega \varphi\varrho_h(t-\Delta t)\widehat{\mathbf{u}}_h(t-\Delta t)\cdot\nabla_x\Delta^{-1}\mathrm{div}_x(\varrho_h\mathbf{u}_h) \,\mathrm{d}x\,\mathrm{d}t$$

$$+h^\alpha\int_0^T\int_\Omega \varphi\varrho_h(t-\Delta t)\widehat{\mathbf{u}}_h(t-\Delta t)\cdot\nabla_x\Delta^{-1}[R_h^1] \,\mathrm{d}x\,\mathrm{d}t,$$

where we have used the consistency formulation (6.1).

**Remark 7.3** *Note that (6.1) holds on the whole space, meaning for test functions in $C_c^\infty(R^3)$ provided $\varrho_h$ has been extended to be zero outside $\Omega$. In particular, the error operator $R^1$ can be viewed as a quantity bounded in the space $L^2(0,T;W^{-1,q}(R^3))$, $q = \frac{6\gamma}{5\gamma-6}$.*

Finally, we focus on the term

$$\int_0^T \int_\Omega \nabla_h \mathbf{u}_h : \nabla_x \left[ \varphi \nabla_x (\Delta^{-1}[\varrho_h]) \right] \, \mathrm{d}x \, \mathrm{d}t. \tag{7.27}$$

In order to eliminate the gradient term, we need to perform a "by parts of integration" step, namely

$$\begin{aligned} \int_\Omega \nabla_h \mathbf{u}_h : \nabla_x \phi \, \mathrm{d}x &= \int_\Omega \nabla_h \mathbf{u}_h : \left( \nabla_x - \nabla_x^T \right) \phi \, \mathrm{d}x + \int_\Omega \nabla_h \mathbf{u}_h : \nabla_x^T \phi \, \mathrm{d}x \\ &= \int_\Omega \mathbf{curl}_h \mathbf{u}_h : \mathbf{curl}_x \phi \, \mathrm{d}x + \int_\Omega \nabla_h \mathbf{u}_h : \nabla_x^T \phi \, \mathrm{d}x \\ &= \int_\Omega \mathbf{curl}_h \mathbf{u}_h : \mathbf{curl}_x \phi \, \mathrm{d}x + \int_\Omega \mathrm{div}_h \mathbf{u}_h : \mathrm{div}_x \phi \, \mathrm{d}x + < \mathcal{E}; \phi > \end{aligned} \tag{7.28}$$

where

$$< \mathcal{E}; \phi >= \int_\Omega \left( \nabla_h \mathbf{u}_h : \nabla_x^T \phi - \mathrm{div}_h \mathbf{u}_h \mathrm{div}_x \phi \right) \mathrm{d}x = \sum_{E \in E_h} \int_{\partial E} \left( \mathbf{u}_h \cdot \nabla \phi \cdot \mathbf{n} - \mathbf{u}_h \cdot \mathbf{n} \mathrm{div}_x \phi \right) \mathrm{d}S_x.$$

Noticing that both the mean value of $\mathbf{u}_h$ over each face and the function $\nabla \phi$ are continuous over each face, we write following [21, Lemma 8.2],

$$< \mathcal{E}; \phi >= \sum_{E \in E_h} \sum_{\Gamma \subset \partial E} \int_\Gamma \left( (\mathbf{u}_h - \tilde{\mathbf{u}}_{h,\Gamma}) \cdot (\nabla_x \phi - [\widetilde{\nabla_x \phi}]_\Gamma) \cdot \mathbf{n} - (\mathbf{u}_h - \tilde{\mathbf{u}}_{h,\Gamma}) \cdot \mathbf{n} (\mathrm{div}_x \phi - [\widetilde{\mathrm{div}_x \phi}]_\Gamma \right) \mathrm{d}S_x.$$

Consequently, by virtue of (2.18) and (2.20–2.21) we finally get,

$$\begin{aligned} | < \mathcal{E}; \phi > | &\lesssim \sum_{E \in E_h} \sum_{\Gamma \subset \partial E} \| \mathbf{u}_h - \tilde{\mathbf{u}}_{h,\Gamma} \|_{L^2(\Gamma, R^9)} \| \nabla_x \phi - [\widetilde{\nabla_x \phi}]_\Gamma \|_{L^2(\Gamma; R^9)} \\ &\lesssim h^{-1} \sum_{E \in E_h} \sum_{\Gamma \subset \partial E} \| \mathbf{u}_h - \tilde{\mathbf{u}}_{h,\Gamma} \|_{L^2(E, R^9)} \left( \| \nabla_x \phi - [\widetilde{\nabla_x \phi}]_\Gamma \|_{L^2(E; R^9)} + h \| \nabla_x^2 \phi \|_{L^2(E; R^9)} \right) \\ &\lesssim h \| \nabla_h \mathbf{u}_h \|_{L^2(\Omega; R^9)} \| \nabla^2 \phi \|_{L^2(\Omega; R^{27})}, \end{aligned} \tag{7.29}$$

for any $\phi \in W^{2,2}(\Omega; R^3) \cap W_0^{1,2}(\Omega)$.

Now, coming back to the original test function (7.25), we may write

$$< \mathcal{E};\phi >= \int_0^T \int_\Omega \Big\{ \nabla_h \mathbf{u}_h : \nabla_x^T \left[ \varphi \nabla_x \left( \Delta^{-1} \left[ R_h^Q[\varrho_h] \right] \right) \right] - \mathrm{div}_h \mathbf{u}_h \mathrm{div}_x \left[ \varphi \nabla_x \left( \Delta^{-1} \left[ R_h^Q[\varrho_h] \right] \right) \right] \Big\} \ \mathrm{d}x \ \mathrm{d}t$$

$$+ \int_0^T \int_\Omega \Big\{ \nabla_h \mathbf{u}_h : \nabla_x^T \left[ \varphi \nabla_x \left( \Delta^{-1} \left[ \varrho_h - R_h^Q[\varrho_h] \right] \right) \right] - \mathrm{div}_h \mathbf{u}_h \mathrm{div}_x \left[ \varphi \nabla_x \left( \Delta^{-1} \left[ \varrho_h - R_h^Q[\varrho_h] \right] \right) \right] \Big\} \ \mathrm{d}x \ \mathrm{d}t$$

where $R_h^Q$ are the regularizing operators introduced in Lemma 2.1. Revoking the bounds (5.9) and applying Lemma 2.1, we obtain

$$\int_0^T \| \varrho_h - R_h^Q[\varrho_h] \|^2_{L^2(\Omega)} \ \mathrm{d}t \stackrel{<}{\sim} h^2 \int_0^T \| \varrho_h \|^2_{H^1_{Q_h}(\Omega)} \ \mathrm{d}t \stackrel{<}{\sim} h^\varepsilon$$

and

$$\| \nabla_x R_h^Q[\varrho_h] \|_{L^2(0,T;L^2(\Omega;R^3))} \approx \left( \int_0^T \| \varrho_h \|^2_{H^1_{Q_h}(\Omega)} \ \mathrm{d}t \right)^{1/2} \stackrel{<}{\sim} h^{\frac{\varepsilon}{2}-1};$$

whence, in accordance with (7.29),

$$| < \mathcal{E};\phi > | \stackrel{<}{\sim} h^{\varepsilon/2} \text{ with } \phi = \nabla_x \left[ \varphi \nabla_x (\Delta^{-1}[\varrho_h]) \right], \tag{7.30}$$

where we have used the fact that $\nabla_x^2 \Delta^{-1}$ is a continuous operator from $L^p(R^3)$ to $L^p(R^3)$, $1 < p < \infty$.

Summing up the previous estimates and regrouping terms in (7.26) we obtain

$$\int_0^T \int_\Omega \varphi \Big[ p(\varrho_h, \vartheta_h) \varrho_h - (\lambda + \mu) \varrho_h \mathrm{div}_x \mathbf{u}_h \Big] \ \mathrm{d}x \ \mathrm{d}t = I_h + \tag{7.31}$$

$$\int_0^T \int_\Omega \Big[ (\lambda + \mu) \mathrm{div}_x \mathbf{u}_h - p(\varrho_h, \vartheta_h) \Big] \nabla_x \varphi \cdot \nabla_x (\Delta^{-1}[\varrho_h]) \ \mathrm{d}x \ \mathrm{d}t$$

$$- \int_0^T \int_\Omega \frac{\varphi(t + \Delta t) - \varphi(t)}{\Delta t} \varrho_h \widehat{\mathbf{u}}_h \cdot \nabla_x (\Delta^{-1}[\varrho_h]) \ \mathrm{d}x \ \mathrm{d}t$$

$$+ \int_0^T \int_\Omega \mu \mathbf{curl}_h \mathbf{u}_h \cdot \mathbf{curl}_x \left[ \varphi \nabla_x (\Delta^{-1}[\varrho_h]) \right] \ \mathrm{d}x \ \mathrm{d}t - \int_0^T \int_\Omega (\varrho_h \widehat{\mathbf{u}}_h \otimes \mathbf{u}_h) : \Big( \nabla_x \varphi \otimes \nabla_x (\Delta^{-1}[\varrho_h]) \Big) \ \mathrm{d}x \ \mathrm{d}t$$

$$- \int_0^T \int_\Omega \varphi (\varrho_h \widehat{\mathbf{u}}_h \otimes \mathbf{u}_h) : (\nabla_x \otimes \nabla_x)(\Delta^{-1}[\varrho_h]) \ \mathrm{d}x \ \mathrm{d}t + \int_0^T \int_\Omega \varphi (\varrho_h \widehat{\mathbf{u}}_h)(t - \Delta t) \cdot \nabla_x \Delta^{-1} \mathrm{div}_x (\varrho_h \mathbf{u}_h) \ \mathrm{d}x \ \mathrm{d}t,$$

where

$$I_h = h^\alpha \left( \int_0^T \int_\Omega \varphi \varrho_h (t - \Delta t) \widehat{\mathbf{u}}_h (t - \Delta t) \cdot \nabla_x \Delta^{-1}[R_h^1] \ \mathrm{d}x \ \mathrm{d}t - \int_0^T \Big\langle R_h^6, \varphi \nabla_x (\Delta^{-1}[\varrho_h]) \Big\rangle \ \mathrm{d}t + \int_0^T r_h \right)$$

with some $\alpha > 0$ and $r_h$ such that $\| r_h \|_{L^1((0,T)\times\Omega)} \stackrel{<}{\sim} 1$.

**Remark 7.4** *It is worth-noting that this is the only step in the proof, where we have used the artificial regularization term added to the continuity method.*

Now we apply a similar treatment to the limit equation (7.24), specifically, we extend $\varrho$ by 0 outside $\Omega$, and use the test function

$$\phi = \varphi \nabla_x \Delta^{-1}[\varrho].$$

After a straightforward manipulation (cf. [14, Chapter 6]) we arrive at

$$\int_0^T \int_\Omega \varphi \Big[ \overline{p(\varrho,\vartheta)}\varrho - (\lambda+\mu)\varrho \mathrm{div}_x \mathbf{u} \Big] \ \mathrm{d}x \ \mathrm{d}t = \tag{7.32}$$

$$\int_0^T \int_\Omega \Big[ \lambda \mathrm{div}_x \mathbf{u} - \overline{p(\varrho,\vartheta)} \Big] \nabla_x \varphi \cdot \nabla_x (\Delta^{-1}[\varrho]) \ \mathrm{d}x \ \mathrm{d}t - \int_0^T \int_\Omega \partial_t \varphi \varrho \mathbf{u} \cdot \nabla_x (\Delta^{-1}[\varrho]) \ \mathrm{d}x \ \mathrm{d}t$$

$$+ \int_0^T \int_\Omega \mu \mathbf{curl}_x \mathbf{u} \cdot \mathbf{curl}_x \Big[ \varphi \nabla_x (\Delta^{-1}[\varrho]) \Big] \ \mathrm{d}x \ \mathrm{d}t - \int_0^T \int_\Omega (\varrho \mathbf{u} \otimes \mathbf{u}) : \Big( \nabla_x \varphi \otimes \nabla_x (\Delta^{-1}[\varrho]) \Big) \ \mathrm{d}x \ \mathrm{d}t$$

$$- \int_0^T \int_\Omega \varphi (\varrho \mathbf{u} \otimes \mathbf{u}) : (\nabla_x \otimes \nabla_x)(\Delta^{-1}[\varrho]) \ \mathrm{d}x \ \mathrm{d}t + \int_0^T \int_\Omega \varphi \varrho \mathbf{u} \cdot \nabla_x \Delta^{-1} \mathrm{div}_x (\varrho \mathbf{u}) \ \mathrm{d}x \ \mathrm{d}t.$$

The principal idea due to Lions [22] is that all terms on the right-hand side of (7.31) converge to their counterparts in (7.32). The has been proved in the continuous case in [22] and for the time discretization problem in [1, Section 3.3], Lions [22]. The same result at the level of numerical discretization was obtained by Karlsen and Karper [20], Karper [21].

Seeing that the error term $I_h$ in (7.31) vanishes for $h \to 0$, the most difficult task is to show that

$$- \int_0^T \int_\Omega \varphi (\varrho_h \widehat{\mathbf{u}}_h \otimes \mathbf{u}_h) : (\nabla_x \otimes \nabla_x)(\Delta^{-1}[\varrho_h]) \ \mathrm{d}x \ \mathrm{d}t + \int_0^T \int_\Omega \varphi (\varrho_h \widehat{\mathbf{u}}_h)(t - \Delta t) \cdot \nabla_x \Delta^{-1} \mathrm{div}_x (\varrho_h \mathbf{u}_h) \ \mathrm{d}x \ \mathrm{d}t \tag{7.33}$$

$$\to$$

$$- \int_0^T \int_\Omega \varphi (\varrho \mathbf{u} \otimes \mathbf{u}) : (\nabla_x \otimes \nabla_x)(\Delta^{-1}[\varrho]) \ \mathrm{d}x \ \mathrm{d}t + \int_0^T \int_\Omega \varphi \varrho \mathbf{u} \cdot \nabla_x \Delta^{-1} \mathrm{div}_x (\varrho \mathbf{u}) \ \mathrm{d}x \ \mathrm{d}t.$$

Here, we observe that the velocity field $\mathbf{u}_h$ can be approximated by its spatial regularization in the spirit of Lemma 2.1,

$$\| \mathbf{u}_h - R_h^V[\mathbf{u}_h] \|_{L^2(0,T;L^q(\Omega;R^3))} \stackrel{<}{\sim} h^\beta, \ \beta = \beta(q) > 0 \text{ for any } 2 \leq q < 6,$$

where we have used (2.30), (2.11) and interpolation. In particular, we may "replace" $\mathbf{u}_h$ by $R_h^V[\mathbf{u}_h]$ in (7.33). Now, the limit (7.33) can be verified exactly as in [1, Section 3.3] or Karper [21, Lemma 9.3].

Thus we get the desired conclusion - the effective viscous flux identity due to Lions [22]:

$$\int_0^T \int_\Omega \varphi \Big[ p(\varrho_h, \vartheta_h) \varrho_h - (\lambda + \mu) \varrho_h \mathrm{div}_x \mathbf{u}_h \Big] \ \mathrm{d}x \ \mathrm{d}t \to \int_0^T \int_\Omega \varphi \Big[ \overline{p(\varrho, \vartheta)} \varrho - (\lambda + \mu) \varrho \mathrm{div}_x \mathbf{u} \Big] \ \mathrm{d}x \ \mathrm{d}t \quad (7.34)$$

as $h \to 0$ for any $\varphi \in C_c^\infty((0,T) \times \Omega)$, which, in view of (7.22) and the monotonicity of the density dependent part of the pressure, namely

$$a\overline{\varrho^{\gamma+1}} + b\overline{\varrho^2} \geq a\overline{\varrho^\gamma}\varrho + b\varrho^2,$$

yields the crucial relation

$$\overline{\varrho \mathrm{div}_x \mathbf{u}} \geq \varrho \mathrm{div}_x \mathbf{u}. \quad (7.35)$$

The relation (7.35) implies strong convergence $\varrho_h \to \varrho$ a.a. in $(0,T) \times \Omega$. Indeed the regularization procedure of DiPerna and Lions [7] can be applied to show that $\varrho$ is a renormalized solution of the continuity equation, in particular,

$$\int_\Omega \varrho \log(\varrho)(\tau, \cdot) \ \mathrm{d}x + \int_0^\tau \int_\Omega \varrho \mathrm{div}_x \mathbf{u} \ \mathrm{d}x \ \mathrm{d}t \leq \int_\Omega \varrho_0 \log(\varrho_0) \ \mathrm{d}x \text{ for any } \tau \in [0,T], \quad (7.36)$$

cf. [14, Chapter 6]. On the other hand, passing to the limit in the renormalized continuity method (4.1) for $b(\varrho) = \varrho \log(\varrho)$ for a spatially homogeneous test function $\phi = \phi(t)$ yields

$$\int_\Omega \overline{\varrho \log(\varrho)}(\tau, \cdot) \ \mathrm{d}x + \int_0^\tau \int_\Omega \overline{\varrho \mathrm{div}_x \mathbf{u}} \ \mathrm{d}x \ \mathrm{d}t \leq \int_\Omega \varrho_0 \log(\varrho_0) \ \mathrm{d}x \text{ for a.a } \tau \in (0,T). \quad (7.37)$$

Combining (7.35 - 7.37) we get

$$\overline{\varrho \log(\varrho)} = \varrho \log(\varrho)$$

yielding

$$\varrho_h \to \varrho \text{ in } L^1((0,T) \times \Omega), \quad (7.38)$$

by virtue of the strict convexity of the function $\varrho \mapsto \varrho \log \varrho$. Moreover, coming with this information back to (7.22), we get

$$\varrho_h \vartheta_h \to \varrho \vartheta \text{ weakly in } L^2((0,T) \times \Omega). \quad (7.39)$$

Having established the strong convergence of the density, we may remove the bar in the momentum equation (7.24). We may also pass to the limit in (4.9) to obtain the energy inequality (1.20). Thus our ultimate goal is to perform the "renormalized" limit letting $\alpha \to 0$ in (7.51) to recover the thermal energy balance (1.18). This will be done in the next section.

## 7.4 Convergence of the temperature

Our goal is to establish strong (a.a.) pointwise convergence of the approximate temperature away from the (hypothetical) vacuum region. To this end, we first apply once more Lemma 7.1 verifying its hypotheses by means of (5.16), (2.28), (6.25) to obtain at the end

$$\varrho_h \chi(\vartheta_h)\vartheta_h \to \varrho \overline{\chi(\vartheta)}\vartheta \text{ weakly in } L^2(0,T; L^{\frac{6\gamma}{\gamma+6}}(\Omega))$$

for any $\chi$ as in (6.15), from which we deduce, taking (5.26) into account, that

$$\varrho_h \vartheta_h^2 \to \varrho \vartheta^2 \text{ weakly in } L^q((0,T)\times\Omega) \text{ for a certain } q > 1. \tag{7.40}$$

However, relation (7.40) and (7.38) together with (5.6) implies

$$\vartheta_h \to \vartheta \text{ (strongly) in } L^2\left(\left\{(t,x) \in (0,T)\times\Omega \;\middle|\; \varrho(t,x) > 0\right\}\right), \tag{7.41}$$

see [14, Chapter 6] for details. As a consequence, we may replace $\overline{\chi(\vartheta)}$ by $\chi(\vartheta)$ whenever this expression is multiplied by $\varrho$.

### 7.4.1 Renormalized temperature method

The limit passage in (6.25) is slightly more complicated. Taking a *nonnegative* test function $\varphi$ our aim is to convert to an inequality and accordingly to get rid of the error term $D_h$. To this end, however, we need the function

$$[\vartheta, z] \mapsto \chi'(\vartheta) z^2 \text{ to be convex in } [0,\infty)\times R$$

for the integral

$$\int_0^T \int_\Omega \Big[\mu |\nabla_h \mathbf{u}_h|^2 + \lambda |\mathrm{div}_h \mathbf{u}_h|^2\Big] \chi'(\vartheta_h)\varphi \, \mathrm{d}x$$

to be weakly lower semi-continuous. As observed in [14, Chapter 4, Lemma 4.8.] this equivalent, at least if $\chi$ is smooth, to a structural condition

$$\chi'(0) > 0,\ \chi''(\vartheta) \le 0,\ \lim_{\vartheta\to\infty} \chi'(\vartheta) = 0,\ \chi'''(\vartheta)\chi'(\vartheta) \ge 2(\chi''(\vartheta))^2 \text{ for all } \vartheta \ge 0. \tag{7.42}$$

A typical example of such a function is $\chi = T_\alpha$,

$$T_\alpha(0) = 0,\ T'_\alpha(\vartheta) = \frac{1}{(1+\vartheta)^\alpha},\ 0 < \alpha < 1.$$

Unfortunately, the condition (7.42) is not compatible with the hypothesis (6.15) in the sense that functions satisfying (7.42) cannot have compactly supported first derivative as required in (6.15) and frequently used in Section 6.3.

In view of these difficulties, we introduce a family of functions $T_{\alpha,m}$,

$$T_{\alpha,m}(\vartheta) = T_\alpha(\vartheta) \text{ for } \vartheta < m,\ T_{\alpha,m}(m) = T_\alpha(m),\ T'_{\alpha,m}(\vartheta) = \max\left\{T'_\alpha(m) + T''_\alpha(m)(\vartheta - m); 0\right\} \text{ for } \vartheta \geq m. \tag{7.43}$$

Clearly, $\chi = T_{\alpha,m}$ complies with (6.15) as soon as $m > 0$ is finite.

Passing to the limit in the consistency formulation (6.25) of the renormalized temperature method we get

$$\int_0^T \int_\Omega \left[\varrho T_{\alpha,m}(\vartheta)\partial_t\varphi + \varrho T_{\alpha,m}(\vartheta)\mathbf{u}\cdot\nabla_x\varphi + \overline{K_{\alpha,m}(\vartheta)}\Delta\varphi\right]\ \mathrm{d}x\ \mathrm{d}t + \int_\Omega \varrho_0 T_{\alpha,m}(\vartheta_0)\varphi(0,\cdot)\ \mathrm{d}x \tag{7.44}$$

$$+ \int_0^T \int_\Omega \left[\mu|\nabla_x\mathbf{u}|^2 + \lambda|\mathrm{div}_x\mathbf{u}|^2\right]T'_\alpha(\vartheta)\varphi\ \mathrm{d}x \leq \int_0^T \int_\Omega T'_{\alpha,m}(\vartheta)\vartheta\overline{\varrho\mathrm{div}_x\mathbf{u}}\varphi\ \mathrm{d}x\ \mathrm{d}t + \langle\mathcal{M}_{\alpha,m},\varphi\rangle\,,$$

$$\varrho\overline{K_{\alpha,m}(\vartheta)} = \varrho K_{\alpha,m}(\vartheta),\ \overline{\varrho\mathrm{div}_x\mathbf{u}} = \varrho\mathrm{div}_x\mathbf{u} \tag{7.45}$$

for any test function

$$\varphi \in C^1_c([0,T)\times\overline{\Omega})),\ \varphi \geq 0,\ \nabla^2_x\varphi \in L^\infty((0,T)\times\Omega, R^{3\times 3}),\ \nabla_x\varphi\cdot\mathbf{n}|_{\partial\Omega} = 0, \tag{7.46}$$

where we have denoted

$$K_{\alpha,m} \equiv K_{T_{\alpha,m}}.$$

The extra term $\mathcal{M}_{\alpha,m}$ - the weak-(*) limit of $(\mu|\nabla_h\mathbf{u}_h|^2 + \lambda|\mathrm{div}_h\mathbf{u}_h|^2)\left|T'_\alpha(\vartheta_h) - T'_{\alpha,m}(\vartheta_h)\right|$ - is a kind of defect measure satisfying

$$|\langle\mathcal{M}_{\alpha,m},\varphi\rangle| \stackrel{<}{\sim} T'_\alpha(m)\|\varphi\|_{L^\infty(\Omega)}. \tag{7.47}$$

As $\vartheta_h$ satisfy (5.25), we get

$$\nabla_x\vartheta,\ \nabla_x\overline{K_{\alpha,m}(\vartheta)} \in L^2((0,T)\times\Omega; R^3),$$

in particular, we may integrate by parts in the convective term

$$\int_0^T \int_\Omega \overline{K_{\alpha,m}(\vartheta)}\Delta\varphi\ \mathrm{d}x\ \mathrm{d}t = -\int_0^T \int_\Omega \nabla_x\overline{K_{\alpha,m}(\vartheta)}\cdot\nabla_x\varphi\ \mathrm{d}x\ \mathrm{d}t. \tag{7.48}$$

With this convention, the integral formula (7.44) makes sense for test functions satisfying

$$\varphi \geq 0, \varphi \in L^\infty((0,T)\times\Omega)\cap L^2(0,T;W^{1,2}(\Omega)),\ \partial_t\varphi \in L^2((0,T)\times\Omega). \tag{7.49}$$

Next, we observe that the class (7.46) is in fact dense in (7.49). This follows from the statement proved in Appendix (Lemma 8.2): For any $\phi \in W^{1,2}(\Omega)\cap L^\infty(\Omega)$, $\phi \geq 0$, there exists a sequence $\{\phi_n\}$,

$$\phi_n \geq 0,\ \phi_n \in W^{2,2}(\Omega),\ \nabla_x\phi_n\cdot\mathbf{n}|_{\partial\Omega} = 0,\ |\phi_n| \stackrel{<}{\sim} 1,\ \nabla_x\phi_n \to \nabla_x\phi \text{ in } L^2((0,T)\times\Omega).$$

Thus we may assume that (7.44), with the convention (7.48), holds for any test function belonging to (7.49).

Now consider a test function $\varphi$,

$$\varphi \geq 0,\ \varphi \in W^{1,3}((0,T)\times\Omega)\cap L^\infty((0,T)\times\Omega),\ \Delta\varphi \in L^\infty((0,T)\times\Omega),\ \nabla_x\varphi\cdot\mathbf{n}|_{\partial\Omega} = 0. \tag{7.50}$$

Keeping $\alpha > 0$ fixed, we may let $m \to \infty$ in (7.44) to conclude that

$$\int_0^T \int_\Omega \Big[\varrho T_\alpha(\vartheta)\partial_t\varphi + \varrho T_\alpha(\vartheta)\mathbf{u}\cdot\nabla_x\varphi + \overline{K_\alpha(\vartheta)}\Delta\varphi\Big]\ \mathrm{d}x\ \mathrm{d}t + \int_\Omega \varrho_0 T_\alpha(\vartheta_0)\varphi(0,\cdot)\ \mathrm{d}x \tag{7.51}$$

$$+\int_0^T \int_\Omega \Big[\mu|\nabla_x\mathbf{u}|^2 + \lambda|\mathrm{div}_x\mathbf{u}|^2\Big]T'_\alpha(\vartheta)\varphi\ \mathrm{d}x \leq \int_0^T \int_\Omega T'_\alpha(\vartheta)\vartheta\overline{\varrho\mathrm{div}_x\mathbf{u}}\varphi\ \mathrm{d}x\ \mathrm{d}t,$$

$$\varrho\overline{K_\alpha(\vartheta)} = \varrho K_\alpha(\vartheta), \tag{7.52}$$

for any test function $\varphi$ as in (7.50), where

$$K_\alpha \equiv K_{T_\alpha}.$$

Note that, by virtue of the hypothesis (1.9) and the uniform bounds (5.26),

$$\overline{K_{\alpha,m}(\vartheta)} \to \overline{K_\alpha(\vartheta)} \text{ in } L^1((0,T)\times\Omega) \text{ as } m\to\infty,$$

where $\overline{K_\alpha(\vartheta)}$ denotes a weak limit of $K_\alpha(\vartheta_h)$ for $h \to 0$.

### 7.4.2 Biting limit in the renormalized thermal energy balance

Our last task in the proof of Theorem 3.1 is to perform the limit $\alpha \to 0$ in (7.51) to obtain (1.18). As observed in [14, Chapter 7], the only problem are estimates of the terms $\overline{K_\alpha(\vartheta)}$ on the vacuum set

$$\Big\{(t,x)\in(0,T)\times\Omega\ \Big|\ \varrho(t,x) = 0\Big\}.$$

We note that the estimates (5.4) together with (5.26) are strong enough to justify the limit

$$\overline{K_\alpha(\vartheta)} \to \overline{K(\vartheta)} = K(\vartheta) \text{ for } \alpha\to 0 \text{ on the set } \Big\{(t,x)\in(0,T)\times\Omega\ \Big|\ \varrho(t,x) > 0\Big\}. \tag{7.53}$$

In order to get the desired bounds on the vacuum set, we follow the procedure elaborated in [14, Chapter 7, Section 7.5.2]. To begin, we observe that the estimates (5.4), (5.26) yield

$$\int_0^T \int_{\{\varrho>\omega\}} |\overline{K_\alpha(\vartheta)}|^r \mathrm{d}x = \int_0^T \int_{\{\varrho>\omega\}} |K_\alpha(\vartheta)|^r \mathrm{d}x\ \mathrm{d}t \stackrel{<}{\sim} c(\omega),\ \omega > 0,\ \text{for a certain } r > 1, \tag{7.54}$$

uniformly for $\alpha \to 0$.

In order to derive a bound for $\overline{K_\alpha(\vartheta)}$ on the hypothetical vacuum zone, we follow the approach of [14, Chapter 7, Section 7.5.2]. As

$$\int_\Omega \varrho(\tau,\cdot)\ \mathrm{d}x = \int_\Omega \varrho_0\ \mathrm{d}x = M_0 > 0 \text{ for any } \tau \in [0,T],$$

we have

$$\int_{\varrho > \omega} \varrho(\tau,\cdot)\ \mathrm{d}x \geq M - \omega|\Omega|.$$

On the other hand, by Hölder's inequality,

$$\int_{\varrho>\omega} \varrho\ \mathrm{d}x \leq |\{\varrho > \omega\}|^{\frac{\gamma-1}{\gamma}} \|\varrho\|_{L^\gamma(\Omega)}.$$

Combining these two inequalities we conclude that there is a function $d(\omega)$ independent of $\tau \in [0,T]$ such that

$$|\{\varrho(\tau,\cdot) > 2\omega\}| \geq d(\omega) > 0 \text{ whenever } 0 \leq 2\omega|\Omega| < M. \tag{7.55}$$

For $0 < \omega < M/2|\Omega|$ fixed, take a function $B \in C^\infty(R)$ such that

$$B \text{ non-increasing, } B(z) = 0 \text{ for } z \leq \omega, \ B(z) = -1 \text{ for } z \geq 2\omega.$$

Now we take $\eta = \eta(\tau,\cdot)$ to be the unique solution of the Neumann problem

$$\Delta\eta = B(\varrho) - \frac{1}{|\Omega|}\int_\Omega B(\varrho)\ \mathrm{d}x \text{ in } \Omega, \ \int_\Omega \eta\ \mathrm{d}x = 0, \ \nabla_x\eta\cdot\mathbf{n}|_{\partial\Omega} = 0.$$

Since $\Omega$ is Lipschitz, we deduce from the elliptic theory that

$$\|\nabla_x\eta(\tau,\cdot)\|_{L^q(\Omega)} \lesssim 1 \text{ for a certain } q > 3,$$

cf. [12], in particular,

$$\eta(\tau,\cdot) \geq \underline{\eta} \text{ for all } \tau \in [0,T].$$

Seeing that $\eta - \underline{\eta}$ belongs to the class (7.50) we can take it as a test function in the renormalized temperature equation (7.51) to obtain

$$\int_0^T \int_\Omega \overline{K_\alpha(\vartheta)} \left( B(\varrho) - \frac{1}{|\Omega|}\int_\Omega B(\varrho)\ \mathrm{d}x \right)\ \mathrm{d}x\ \mathrm{d}t$$

$$\lesssim \int_0^T \int_\Omega \varrho\vartheta|\mathbf{u}||\nabla_x\eta|\ \mathrm{d}x\ \mathrm{d}t + \sup_{\tau\in[0,T]} \int_\Omega \varrho\vartheta|\eta|\ \mathrm{d}x + \int_0^T \int_\Omega \vartheta\varrho|\mathrm{div}_x\mathbf{u}||\eta|\ \mathrm{d}x\ \mathrm{d}t$$

$$-\int_0^T \int_\Omega \partial_t \eta \varrho T_\alpha(\vartheta)\ \mathrm{d}x\ \mathrm{d}t.$$

Furthermore, as $\varrho$ satisfies the renormalized continuity equation, we may compute

$$\partial_t \eta = \Delta_N^{-1}\left[\partial_t B(\varrho) - \frac{1}{|\Omega|}\int_\Omega \partial_t B(\varrho)\ \mathrm{d}x\right] = -\Delta_N^{-1}[\mathrm{div}_x(B(\varrho)\mathbf{u})]$$

$$+\Delta_N^{-1}\left[(B(\varrho) - B'(\varrho)\varrho)\mathrm{div}_x\mathbf{u} - \frac{1}{|\Omega|}\int_\Omega (B(\varrho) - B'(\varrho)\varrho)\mathrm{div}_x\mathbf{u}\ \mathrm{d}x\right].$$

In view of the elliptic estimates available for the Neumann Laplacean on *Lipschitz* domains (see for instance Fabes, Mendez, and Mitrea [12]) and the integrability properties of $\varrho$, $\vartheta$, $\mathbf{u}$ established in the preceding section, we may infer that

$$\int_0^T \int_\Omega \overline{K_\alpha(\vartheta)}\left(B(\varrho) - \frac{1}{|\Omega|}\int_\Omega B(\varrho)\ \mathrm{d}x\right)\ \mathrm{d}x\ \mathrm{d}t \stackrel{<}{\sim} 1 \tag{7.56}$$

uniformly for $\alpha \to 0$. Combining the relations (7.54 - 7.56) we deduce the desired conclusion

$$\int_0^T \int_\Omega \overline{K_\alpha(\vartheta)}\ \mathrm{d}x\ \mathrm{d}t \stackrel{<}{\sim} 1 \text{ uniformly for } \alpha > 0, \tag{7.57}$$

cf. [14, Chapter 7, Section 7.5.2]. Thus, letting $\alpha \to 0$ we get

$$\overline{K_\alpha(\vartheta)} \nearrow \overline{K(\vartheta)} \in L^1((0,T)\times\Omega),\ \varrho\overline{K(\vartheta)} = \varrho K(\vartheta).$$

Passing to the limit for $\alpha \to 0$ in the renormalized temperature equation (7.51) we obtain (1.18). We have proved Theorem 3.1

# 8 Appendix

This section collects the proofs omitted in the text.

**Lemma 8.1** *Suppose that* $g_h^k \in Q_h(\Omega)$, $\chi \in C^2(R)$.

*Then*

$$\int_\Omega \chi'(g_h^k) D_t(\varrho_h^k g_h^k)\Phi\ \mathrm{d}x - \sum_{\Gamma\in\Gamma_h}\int_\Gamma \mathrm{Up}\left[\varrho_h^k g_h^k, \mathbf{u}_h^k\right]\ [[\chi'(g_h^k)\Phi]]\ \mathrm{dS}_x$$

$$= \int_\Omega D_t(\varrho_h^k \chi(g_h^k))\Phi\ \mathrm{d}x - \sum_{\Gamma\in\Gamma_h}\int_\Gamma \mathrm{Up}\left[\varrho_h^k\chi(g_h^k), \mathbf{u}_h^k\right]\ [[\Phi]]\ \mathrm{dS}_x$$

$$+\frac{\Delta t}{2}\int_\Omega \chi''(\xi_h^k)\varrho_h^{k-1}\left(\frac{g_h^k-g_h^{k-1}}{\Delta t}\right)^2 \Phi \ \mathrm{d}x - \frac{1}{2}\sum_{E\in E_h}\sum_{\Gamma_E\subset\partial E}\int_{\Gamma_E}\chi''(\eta_h^k)[[g_h^k]]^2\left(\varrho_h^k\right)^+\left[\tilde{\mathbf{u}}_h^k\cdot\mathbf{n}\right]^- \ \mathrm{dS}_x$$

$$+h^{1-\varepsilon}\sum_{\Gamma\in\Gamma_{h,\mathrm{int}}}\int_\Gamma [[\varrho_h^k]]\ [[\Big(\chi(g_h^k)-\chi'(g_h^k)g_h^k\Big)\ \Phi]]\ \mathrm{dS}_x$$

*for any* $\Phi\in Q_h(\Omega)$, *where*

$$\xi_h^k\in\mathrm{co}\left\{g_h^{k-1},g_h^k\right\},\ \ \eta_k^k\in\mathrm{co}\left\{g_h^k,\left(g_h^k\right)^+\right\}.$$

**Proof:**

We have

$$\int_\Omega \chi'(g_h^k)D_t(\varrho_h^k g_h^k)\Phi\ \mathrm{d}x = \int_\Omega\chi'(g_h^k)\frac{\varrho_h^k g_h^k-\varrho_h^{k-1}g_h^{k-1}}{\Delta t}\Phi\ \mathrm{d}x \tag{8.1}$$

$$=\int_\Omega\left[g_h^k\chi'(g_h^k)\frac{\varrho_h^k-\varrho_h^{k-1}}{\Delta t}+\chi'(g_h^k)\varrho_h^{k-1}\frac{g_h^k-g_h^{k-1}}{\Delta t}\right]\Phi\ \mathrm{d}x$$

$$=\int_\Omega\left[g_h^k\chi'(g_h^k)D_t\varrho_h^k+\varrho_h^{k-1}D_t\chi(g_h^k)\right]\Phi\ \mathrm{d}x+\int_\Omega\frac{\Delta t}{2}\varrho_h^{k-1}\chi''(\xi_h^k)\left(\frac{g_h^k-g_h^{k-1}}{\Delta t}\right)^2\Phi\ \mathrm{d}x$$

with $\xi_h^k\in\mathrm{co}\{g_h^{k-1},g_h^k\}$.

The upwind term reads

$$\sum_{\Gamma\in\Gamma_h}\int_\Gamma \mathrm{Up}[\varrho_h^k g_h^k,\mathbf{u}_h^k]\ [[\chi'(g_h^k)\Phi]]\mathrm{dS}_x \tag{8.2}$$

$$=-\sum_{E\in E_h}\sum_{\Gamma_E\subset\partial E}\int_{\Gamma_E}\chi'(g_h^k)\left(\varrho_h^k g_h^k\ [\tilde{\mathbf{u}}_h^k\cdot\mathbf{n}]^+ + \left(\varrho_h^k g_h^k\right)^+[\tilde{\mathbf{u}}_h^k\cdot\mathbf{n}]^-\right)\Phi\ \mathrm{dS}_x$$

$$=-\sum_{E\in E_h}\sum_{\Gamma_E\subset\partial E}\int_{\Gamma_E}\chi'(g_h^k)g_h^k\left(\varrho_h^k\ [\tilde{\mathbf{u}}_h^k\cdot\mathbf{n}]^+ + \left(\varrho_h^k\right)^+[\tilde{\mathbf{u}}_h^k\cdot\mathbf{n}]^-\right)\Phi\ \mathrm{dS}_x$$

$$+\sum_{E\in E_h}\sum_{\Gamma_E\subset\partial E}\int_{\Gamma_E}\chi'(g_h^k)\left(\varrho_h^k\right)^+[\tilde{\mathbf{u}}_h^k\cdot\mathbf{n}]^-\left(g_h^k-\left(g_h^k\right)^+\right)\Phi\ \mathrm{dS}_x$$

$$=\sum_{\Gamma\in\Gamma_h}\int_\Gamma\mathrm{Up}(\varrho_h^k,\mathbf{u}_h^k)\ [[\chi'(g_h^k)g_h^k\Phi]]\ \mathrm{dS}_x$$

$$+\sum_{E\in E_h}\sum_{\Gamma_E\subset\partial E}\int_{\Gamma_E}\left(\varrho_h^k\right)^+[\tilde{\mathbf{u}}_h^k\cdot\mathbf{n}]^-\left(\chi(g_h^k)-\chi\left(\left(g_h^k\right)^+\right)\right)\Phi\ \mathrm{dS}_x$$

$$+\frac{1}{2}\sum_{E\in E_h}\sum_{\Gamma_E\subset\partial E}\int_{\Gamma_E}\chi''(\eta_h^k)\Big(g_h^k-\left(g_h^k\right)^+\Big)^2\left(\varrho_h^k\right)^+[\tilde{\mathbf{u}}_h^k\cdot\mathbf{n}]^-\Phi\ \mathrm{dS}_x$$

$$= \sum_{\Gamma \in \Gamma_h} \int_\Gamma \mathrm{Up}(\varrho_h^k, \mathbf{u}_h^k)\ [[\chi'(g_h^k) g_h^k \Phi]]\ \mathrm{dS}_x - \sum_{\Gamma \in \Gamma_h} \int_\Gamma \mathrm{Up}(\varrho_h^k, \mathbf{u}_h^k)\ [[\chi(g_h^k)\Phi]]\ \mathrm{dS}_x$$

$$+ \sum_{\Gamma \in \Gamma_h} \int_\Gamma \mathrm{Up}(\varrho_h^k \chi(g_h^k), \mathbf{u}_h^k)\ [[\Phi]]\ \mathrm{dS}_x$$

$$+ \frac{1}{2} \sum_{E \in E_h} \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} \chi''(\eta_h^k) \Big( g_h^k - \left(g_h^k\right)^+ \Big)^2 \left(\varrho_h^k\right)^+ [\tilde{\mathbf{u}}_h^k \cdot \mathbf{n}]^- \Phi\ \mathrm{dS}_x.$$

Combining (8.1), (8.2) with the continuity method (3.2) we obtain

$$\int_\Omega D_t \left(\varrho_h^k g_h^k\right) \chi'(g_h^k) \Phi\ \mathrm{d}x - \sum_{\Gamma \in \Gamma_h} \int_\Gamma \mathrm{Up}[\varrho_h^k g_h^k, \mathbf{u}_h^k]\ [[\chi'(g_h^k)\Phi]] \mathrm{dS}_x$$

$$= \int_\Omega \varrho_h^{k-1} D_t \chi(g_h^k) \Phi\ \mathrm{d}x + \sum_{\Gamma \in \Gamma_h} \int_\Gamma \mathrm{Up}(\varrho_h^k, \mathbf{u}_h^k)\ [[\chi(g_h^k)\phi]]\ \mathrm{dS}_x + \frac{\Delta t}{2} \int_\Omega \chi''(\xi_h^k) \varrho_h^{k-1} \left( \frac{g_h^k - g_h^{k-1}}{\Delta t} \right)^2 \Phi\ \mathrm{d}x$$

$$- \sum_{\Gamma \in \Gamma_h} \int_\Gamma \mathrm{Up}(\varrho_h^k \chi(g_h^k), \mathbf{u}_h^k)\ [[\Phi]]\ \mathrm{dS}_x$$

$$- \frac{1}{2} \sum_{E \in E_h} \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} \chi''(\eta_h^k) \Big( g_h^k - \left(g_h^k\right)^+ \Big)^2 \left(\varrho_h^k\right)^+ [\tilde{\mathbf{u}}_h^k \cdot \mathbf{n}]^- \Phi\ \mathrm{dS}_x$$

$$- h^{1-\varepsilon} \sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma [[\varrho_h^k]]\ [[\chi'(g_h^k) g_h^k \Phi]]\ \mathrm{dS}_x.$$

Finally, we use $\chi(g_h^k)\Phi$ as a test function in the continuity method (3.2) to deduce the desired conclusion.

Q.E.D.

**Lemma 8.2** *For any $\phi \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$, $\phi \geq 0$, there exists a sequence $\{\phi_n\}$,*

$$\phi_n \geq 0,\ \phi_n \in W^{2,2}(\Omega),\ \nabla_x \phi_n \cdot \mathbf{n}|_{\partial\Omega} = 0,\ |\phi_n| \lesssim 1,\ \nabla_x \phi_n \to \nabla_x \phi \ \textit{in}\ L^2((0,T) \times \Omega).$$

**Proof:** Let $\phi \geq 0$, $\phi \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$. Since $\Omega$ is Lipschitz, we may assume that $\phi$ is defined on the whole space $R^3$ with the same properties. The set of edges $\mathcal{E} \subset \partial\Omega$, meaning the all intersections of the boundary faces, is of zero $W^{1,2}-$capacity in $R^3$. Therefore we may construct a sequence $\{\phi_n\}_{n \geq 0}$ such that

$$\phi_n \in C_c(R^3) \cap W^{1,2}(R^3), 0 \leq \phi_n \lesssim 1,\ \phi_n \to \phi \text{ in } W^{1,2}(R^3), \tag{8.3}$$

and

$$\phi_n \equiv 0 \text{ in an open neighborhood of } \mathcal{E}. \tag{8.4}$$

Thus it is enough to show the conclusion of the Lemma for functions $\phi$ on $\overline{\Omega}$ belonging to the class (8.3), (8.4). Since these vanish on the edges, we may extend them on an open neighborhood of $\overline{\Omega}$ as *even* functions with respect to the normal vector on all faces. Now we use the standard regularizing kernels to construct the sequence $\{\phi_n\}_{n>0}$ with the desired properties. In particular, all $\phi_n$ satisfy $\nabla_x \phi_n \cdot \mathbf{n}|_{\partial\Omega} = 0$.

Q.E.D.

# References


[1] H. Abels and E. Feireisl. On a diffuse interface model for a two-phase flow of compressible viscous fluids. *Indiana Univ. Math. J.*, 57(2):659–698, 2008.

[2] M. E. Bogovskii. Solution of some vector analysis problems connected with operators div and grad (in Russian). *Trudy Sem. S.L. Sobolev*, **80**(1):5–40, 1980.

[3] F. Brezzi and M. Fortin. *Mixed and hybrid finite element methods*, volume 15 of *Springer Series in Computational Mathematics*. Springer-Verlag, New York, 1991.

[4] C. Chainais-Hillairet and J. Droniou Finite volume schemes for non-coercive elliptic problems with Neumann boundary conditions. *IMA J. Numer. Anal.*, **31**(1):61–85, 2011.

[5] S. H. Christiansen, H. Z. Munthe-Kaas, and B. Owren. Topics in structure-preserving discretization. *Acta Numer.*, **20**:1–119, 2011.

[6] M. Crouzeix and P.A. Raviart. Conforming and nonconforming finite element methods for solving the stationary Stokes equations. *Rev. Française Automat. Informat. Recherche Opérationnelle Sér. Rouge*, **7**:33–75, 1973.

[7] R.J. DiPerna and P.-L. Lions. Ordinary differential equations, transport theory and Sobolev spaces. *Invent. Math.*, **98**:511–547, 1989.

[8] R. Eymard, T. Gallouët, and R. Herbin. Convergence of finite volume schemes for semilinear convection diffusion equations. *Numer. Math.*, **82**(1):91–116, 1999.

[9] R. Eymard, T. Gallouët, and R. Herbin. Finite volume methods. In *Handbook of numerical analysis, Vol. VII*, Handb. Numer. Anal., VII, pages 713–1020. North-Holland, Amsterdam, 2000.

[10] R. Eymard, T. Gallouët, R. Herbin, and J. C. Latché. A convergent finite element-finite volume scheme for the compressible Stokes problem. II. The isentropic case. *Math. Comp.*, **79**(270):649–675, 2010.

[11] R. Eymard, T. Gallouët, R. Herbin, and A. Michel. Convergence of a finite volume scheme for nonlinear degenerate parabolic equations. *Numer. Math.*, **92**(1):41–82, 2002.

[12] E. Fabes, O. Mendez, and M. Mitrea. Boundary layers on Sobolev-Besov spaces and Poisson's equation for the Laplacian in Lipschitz domains. *J. Funct. Anal.*, **159**(2):323–368, 1998.

[13] Feireisl and Y. Sun. Conditional regularity of very weak solutions to the Navier-Stokes-Fourier system. 2014. Preprint IM Prague.

[14] E. Feireisl. *Dynamics of viscous compressible fluids.* Oxford University Press, Oxford, 2004.

[15] E. Feireisl and A. Novotný. *Singular limits in thermodynamics of viscous fluids.* Birkhäuser-Verlag, Basel, 2009.

[16] G. P. Galdi. *An introduction to the mathematical theory of the Navier - Stokes equations, I.* Springer-Verlag, New York, 1994.

[17] T. Gallouet, E. Herbin, and J.-C. Latché. A convergent finite element-finite colume scheme for the compressible Stokes problem. I. The isothermal case. *Math. Comput.*, **78**(267):1333–1352, 2009.

[18] M. Geißert, H. Heck, and M. Hieber. On the equation $\operatorname{div} u = g$ and Bogovskiĭ's operator in Sobolev spaces of negative order. In *Partial differential equations and functional analysis*, volume 168 of *Oper. Theory Adv. Appl.*, pages 113–121. Birkhäuser, Basel, 2006.

[19] K. H. Karlsen and T. K. Karper. Convergence of a mixed method for a semi-stationary compressible Stokes system. *Math. Comp.*, **80**(275):1459–1498, 2011.

[20] K. H. Karlsen and T. K. Karper. A convergent mixed method for the Stokes approximation of viscous compressible flow. *IMA J. Numer. Anal.*, 32(**3**):725–764, 2012.

[21] T. K. Karper. A convergent FEM-DG method for the compressible Navier-Stokes equations. *Numer. Math.*, **125**(3):441–510, 2013.

[22] P.-L. Lions. *Mathematical topics in fluid dynamics, Vol.2, Compressible models*. Oxford Science Publication, Oxford, 1998.

[23] E. Vanderzee, A. N. Hirani, D. Guoy, and E. A. Ramos. Well-centered triangulation. *SIAM J. Sci. Comput.*, **31**(6):4497–4523, 2009/10.

[24] E. VanderZee, A. N. Hirani, D. Guoy, V. Zharnitsky, and E. A. Ramos. Geometric and combinatorial properties of well-centered triangulations in three and higher dimensions. *Comput. Geom.*, **46**(6):700–724, 2013.