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Abstract

We discuss the impact of certain recent results concerning stability of weak solutions to the complete Navier-Stokes-Fourier system on the problem of convergence of the associated numerical schemes. In particular, we show that solutions of certain numerical schemes converge unconditionally to the exact solution as long as the numerical solutions remain bounded.

Key words: Navier-Stokes-Fourier system, weak solution, mixed finite-volume finite-element numerical scheme, convergence

1 Introduction

The problem of stability and convergence of numerical methods used for simulation of fluids in continuum mechanics is of great theoretical and, obviously, also practical interest. In [9], we proposed a numerical scheme for solving the *Navier-Stokes-Fourier system* describing the motion of a general viscous, compressible, and heat conducting fluid in terms of the three state variables:

the mass density .... $\varrho = \varrho(t,x)$,
the absolute temperature .... $\vartheta = \vartheta(t,x)$,
the velocity field .... $\mathbf{u} = \mathbf{u}(t,x)$,

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evaluated at the time $t \in [0, T)$ and the Eulerian reference position $x \in \Omega \subset R^3$. Given the initial state

$$\varrho(0, \cdot) = \varrho_0, \ \vartheta(0, \cdot) = \vartheta_0, \ \mathbf{u}(0, \cdot) = \mathbf{u}_0, \tag{1.1}$$

the time evolution of the fluid is described by means of the field equations

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{1.2}$$

$$\partial_(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}), \tag{1.3}$$

$$\partial_t(\varrho e(\varrho, \vartheta)) + \mathrm{div}_x(\varrho e(\varrho, \vartheta)\mathbf{u}) + \nabla_x \mathbf{q} = \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - p(\varrho, \vartheta)\mathrm{div}_x \mathbf{u}, \tag{1.4}$$

where $p = p(\varrho, \vartheta)$ is the pressure, $e = e(\varrho, \vartheta)$ the specific internal energy, $\mathbb{S} = \mathbb{S}(\nabla_x \mathbf{u})$ the viscous stress tensor, here determined by *Newton's rheological law*

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \right) + \eta \mathrm{div}_x \mathbf{u} \mathbb{I}, \tag{1.5}$$

and, finally, $\mathbf{q}$ denotes the heat flux given by *Fourier's law*

$$\mathbf{q} = -\kappa \nabla_x \vartheta. \tag{1.6}$$

In agreement with the principles of classical thermodynamics, the internal energy $e = e(\varrho, \vartheta)$ and the pressure $p(\varrho, \vartheta)$ comply with *Maxwell's relation*

$$\frac{\partial e(\varrho, \vartheta)}{\partial \varrho} = \frac{1}{\varrho^2} \left(p(\varrho, \vartheta) - \vartheta \frac{\partial p(\varrho, \vartheta)}{\partial \vartheta} \right) \tag{1.7}$$

and satisfy the *thermodynamic stability condition*

$$\frac{\partial p(\varrho, \vartheta)}{\partial \varrho} > 0, \ \frac{\partial e(\varrho, \vartheta)}{\partial \vartheta} > 0, \tag{1.8}$$

see e.g. Bechtel, Rooney, and Forest [1], Gallavotti [11].

To avoid problems connected with the boundary behavior of the fluid, we impose the periodic boundary conditions, meaning

$$\Omega = \left([-1, 1]|_{\{-1,1\}} \right)^3$$

is the 3-dimensional flat torus. We refer to Ebin [5] for the physical relevance of these conditions, in particular the connection with the complete slip on flat boundaries.

For the sake of simplicity, we have omitted the influence of external forces and heat sources in (1.3), (1.4). Accordingly, the resulting system is conservative, the total mass and energy are constants of motion:

$$\int_\Omega \varrho(t, \cdot) \ \mathrm{d}x = \int_\Omega \varrho_0 \ \mathrm{d}x, \ \int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) \right) (t, \cdot) \ \mathrm{d}x = \int_\Omega \left(\frac{1}{2} \varrho_0 |\mathbf{u}_0|^2 + \varrho_0 e(\varrho_0, \vartheta_0) \right) \ \mathrm{d}x \tag{1.9}$$

for any $t \in (0,T)$.

In this paper, we review some recent results on the existence and stability of the weak solutions to the *Navier-Stokes-Fourier system* (1.2–1.4) and their relevance in the study of convergence of the numerical scheme proposed in [9]. More specifically, we plan to address the following topics:

- *Weak and strong solvability*

  We identify a class of admissible constitutive relations under which the problem admits a global weak or a local strong solution for any finite energy initial data, see Section 2.

- *Weak = strong*

  We adopt the concept of *relative energy*, introduced for the Navier-Stokes-Fourier system in [10], and show that a weak and strong solution emanating from the same initial data coincide as long as the strong one exists. In particular, we use the recent results [7] on conditional regularity of the strong solutions to show that *bounded* weak solutions are in fact regular on the whole existence interval as soon as they start from regular initial data, see Section 3.

- *Numerical method*

  In Section 4, we introduce and briefly recall the basic properties of the numerical scheme [9].

- *Convergence of numerical solutions*

  Combining the results collected in the preceding part we show that the numerical scheme converges to a unique exact solution of the Navier-Stokes-Fourier system provided the numerical solutions remain bounded uniformly with respect to the time step and mesh refinement proportional to a small parameter $h \to 0$, see Section 5.

2 Weak and strong solutions

To keep the presentation as simple as possible, we suppose that the internal energy $e = e(\varrho, \vartheta)$ can be written in the form

$$e(\varrho, \vartheta) = E_{\rm therm}(\vartheta) + E_{\rm el}(\varrho),$$

where the "thermal" component $E_{\rm therm}$ is a linear function of the absolute temperature. In addition, for technical reasons that will become clear later, the "elastic" part $E_{\rm el}$ must posses certain coercivity properties for large values of ϱ. Consequently, in agreement with (1.7), we set

$$p(\varrho, \vartheta) = p_e(\varrho) + \varrho\vartheta,\ \ e(\varrho, \vartheta) = c_v\vartheta + P(\varrho),\ \ P(\varrho) = \int_1^\varrho \frac{p_e(z)}{z^2}\ {\rm d}z,\ \ c_v > 0, \tag{2.1}$$

where, in accordance with (1.8),

$$p_e'(\varrho) > 0 \text{ for } \varrho \geq 0,\ c_v > 0, \text{ and } \lim_{\varrho \to \infty} \frac{p_e'(\varrho)}{\varrho^\gamma} = p_\infty > 0 \text{ for a certain } \gamma > 3. \tag{2.2}$$

In this new setting, the internal energy balance (1.4) can be written in a concise form

$$c_v\Big(\partial_t(\varrho\vartheta) + {\rm div}_x(\varrho\vartheta\mathbf{u})\Big) + {\rm div}_x\mathbf{q} = \mathbb{S}(\nabla_x\mathbf{u}) : \nabla_x\mathbf{u} - \varrho\vartheta{\rm div}_x\mathbf{u}. \tag{2.3}$$

Moreover, we suppose that

$$\mu > 0,\ \eta \geq 0 \text{ are constant, while } \underline{\kappa}(1+\vartheta^2) \leq \kappa(\vartheta) \leq \overline{\kappa}(1+\vartheta^2) \text{ for } \vartheta \geq 0, \tag{2.4}$$

where $\underline{\kappa} > 0$.

The existence of *strong* local-in-time solutions to the Navier-Stokes-Fourier system (1.1 - 1.3), (2.3) under much less restrictive assumptions than (2.1 - 2.4) was studied by a number of authors under different regularity assumption on the initial data. Here we use the result of Valli [18], [19], where the strong solutions live in the class

$$\varrho,\ \vartheta \in C([0,T];W^{3,2}(\Omega)),\ \mathbf{u} \in C([0,T];W^{3,2}(\Omega;R^3)),\ \varrho,\ \vartheta > 0, \tag{2.5}$$

provided the initial data enjoy the same regularity. Note that no compatibility conditions are necessary in the case of periodic boundary conditions. Related results concerning local as well as global existence of smooth solutions for the initial data close to an equilibrium were obtained by many authors, see e.g. Hoff [12], Matsumura and Nishida [15], [16], Valli and Zajaczkowski [20], Tani [17].

In the future, we will always assume that the initial data are smooth enough to guarantee the existence of local-in-time solutions in the class (2.5), meaning

$$\varrho_0,\ \vartheta_0 \in W^{3,2}(\Omega),\ \mathbf{u}_0 \in W^{3,2}(\Omega;R^3),\ \varrho_0,\ \vartheta_0 > 0. \tag{2.6}$$

As observed in [7], these strong solutions possess all the necessary derivatives appearing in (1.2), (1.3), (2.3), at least in the open time interval $(0,T)$.

Setting

$$K(\vartheta) = \int_0^\vartheta \kappa(z)\ {\rm d}z,$$

we introduce a *weak* solution to the Navier-Stokes-Fourier system:

Definition 2.1 *We say that* $[\varrho, \vartheta, \mathbf{u}]$ *is a* weak solution *of the Navier-Stokes-Fourier system (1.1 - 1.4) if:*

- *The functions $[\varrho, \vartheta, \mathbf{u}]$ belong to the regularity class:*

$$\varrho \geq 0,\ \ \varrho \in C_{\rm weak}([0,T]; L^\gamma(\Omega)) \ \text{for some}\ \gamma > 1,$$

$$\left\{ \begin{array}{c} \mathbf{u} \in L^2(0,T; W^{1,2}(\Omega; R^3)), \\ \\ (\varrho \mathbf{u}) \in C_{\rm weak}([0,T]; L^{2\gamma/(\gamma+1)}(\Omega; R^3)), \\ \\ \varrho |\mathbf{u}|^2 \in L^\infty(0,T; L^1(\Omega)), \end{array} \right\}$$

$$\vartheta > 0 \ \text{a.a. in}\ (0,T) \times \Omega, K(\vartheta) \in L^1((0,T)\times\Omega),\ \ \varrho\vartheta \in L^2((0,T)\times \Omega).$$

- *The equations (1.2-1.4) are replaced by a family of integral identities:*

$$\int_\Omega \varrho \varphi \ {\rm d}x \Big|_{t=\tau_1}^{t=\tau_2} = \int_{\tau_1}^{\tau_2} \int_\Omega \left[\varrho \partial_t \varphi + \varrho \mathbf{u} \cdot \nabla_x \varphi \right] \ {\rm d}x{\rm d}t,\ \ \varrho(0,\cdot) = \varrho_0, \tag{2.7}$$

for any $0 \leq \tau_1 < \tau_2 \leq T$ and any test function $\varphi \in C^\infty_c([0,T] \times \overline{\Omega})$;

$$\int_\Omega \varrho \mathbf{u} \cdot \varphi \ {\rm d}x \Big|_{t=\tau_1}^{t=\tau_2} = \tag{2.8}$$

$$\int_{\tau_1}^{\tau_2} \int_\Omega \left[\varrho \mathbf{u} \cdot \partial_t \varphi + \varrho \mathbf{u} \otimes \mathbf{u} : \nabla_x \varphi + p(\varrho, \vartheta) {\rm div}_x \varphi - \mathbb{S}(\nabla_x \varphi) : \nabla_x \varphi \right] \ {\rm d}x \ {\rm d}t,$$

$$\varrho \mathbf{u}(0, \cdot) = \varrho_0 \mathbf{u}_0,$$

for any $0 \leq \tau_1 < \tau_2 \leq T$, and any test function $\varphi \in C^\infty_c([0,T] \times \Omega; R^3)$ in the case of the no-slip (1.7), $\varphi \in C^\infty_c([0,T] \times \overline{\Omega}; R^3)$, $\varphi \cdot \mathbf{n}|_{\partial\Omega} = 0$ in the case of the complete slip (1.8);

$$c_v \int_\Omega \varrho \vartheta \varphi \ {\rm d}x \Big|_{t=\tau_1}^{t=\tau_2} \geq \int_{\tau_1}^{\tau_2} \int_\Omega \left[c_v \left(\varrho \vartheta \partial_t \varphi + \varrho \vartheta \mathbf{u} \cdot \nabla_x \varphi \right) - \overline{K(\vartheta)} \Delta \varphi \right] \ {\rm d}x \ {\rm d}t \tag{2.9}$$

$$+ \int_{\tau_1}^{\tau_2} \int_\Omega \left[\mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \varphi - \vartheta \frac{\partial_\vartheta p(\varrho, \vartheta)}{\partial \vartheta} {\rm div}_x \mathbf{u} \varphi \right] \ {\rm d}x \ {\rm d}t$$

for a.a. $0 \leq \tau_1 < \tau_2 \leq T$ including $\tau_1 = 0$, where

$$\varrho \overline{K(\vartheta)} = \varrho K(\vartheta), \tag{2.10}$$

$\varrho\vartheta(0, \cdot) = \varrho_0 \vartheta_0$, and for any test function $\varphi \in C^\infty_c([0,T] \times \overline{\Omega})$, $\varphi \geq 0$;

$$\int_\Omega \left[\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) \right] (\tau, \cdot) \ {\rm d}x \leq \int_\Omega \left[\frac{1}{2} \varrho_0 |\mathbf{u}_0|^2 + \varrho_0 e(\varrho_0, \vartheta_0) \right] \ {\rm d}x \ \text{for a.a.}\ \tau \in [0,T]. \tag{2.11}$$

Remark 2.1 *In the weak formulation, the internal energy equation (1.4) was replaced by* two inequalities *(2.9), (2.11). Although this may seem to lead to a substantial enlargement of the set of possible weak solutions, such a definition still preserves the following compatibility principle: A weak solution is a strong one as soon as it is smooth, see [8, Chapter 4].*

Remark 2.2 *The reader will have noticed the subtle difference between (2.10) and the expected relation*

$$\overline{K(\vartheta)} = K(\vartheta). \tag{2.12}$$

What (2.10) says is that (2.12) holds with the possible exception of the (hypothetical) vacuum zones where $\varrho = 0$*. Unfortunately, the absence of vacuums in the weak solutions even if* $\varrho_0 > 0$ *is an outstanding open problem.*

The *existence* of global-in-time weak solutions to the Navier-Stokes-Fourier system under the restrictions specified through (2.1–2.4) was established in [8, Chapter 7, Theorem 7.1]. Although the weak solutions in the sense of Definition 2.1 suffer several deficiencies, including the inequality signs in (2.9), (2.11), they represent strong solutions of the same problem as soon as they are smooth enough, cf. [8, Chapter 4].

3 Relative energy and applications

Our goal is to show that *bounded* weak solutions to the Navier-Stokes-Fourier system emanating from smooth initial data are, in fact, regular. To this end, we revoke the approach based on the concept of *relative* energy (entropy) proposed by Dafermos [4] in the context of hyperbolic conservation laws and adapted to the present setting in [10]. The main stumbling block is that we need the entropy balance equation rather than (2.9) in order to apply the method proposed in [10]. Recall that, in view of (1.7), there exists a function $s = s(\varrho, \vartheta)$ – the specific entropy – determined, up to an additive constant, through *Gibbs' equation*:

$$\vartheta D s(\varrho, \vartheta) = D e(\varrho, \vartheta) + p(\varrho, \vartheta) D \left(\frac{1}{\varrho} \right). \tag{3.1}$$

For *smooth* solutions, the internal energy equation (1.4) can be equivalently replaced by the entropy balance equation

$$\partial_t (\varrho s(\varrho, \vartheta)) + \mathrm{div}_x (\varrho s(\varrho, \vartheta) \mathbf{u}) + \mathrm{div}_x \left(\frac{\mathbf{q}}{\vartheta} \right) = \frac{1}{\vartheta} \left(\mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right),$$

or, in the context of weak solutions satisfying (2.9), by

$$\int_\Omega \varrho s(\varrho,\vartheta)\varphi \ {\rm d}x\Big|_{t=\tau_1}^{t=\tau_2} \geq \int_{\tau_1}^{\tau_2}\int_\Omega \left[\Big(\varrho s(\varrho,\vartheta)\partial_t\varphi + \varrho s(\varrho,\vartheta)\mathbf{u}\cdot\nabla_x\varphi\Big) + \frac{\kappa(\vartheta)}{\vartheta}\nabla_x\vartheta\cdot\nabla_x\varphi \right] \ {\rm d}x \ {\rm d}t \tag{3.2}$$

$$+\int_{\tau_1}^{\tau_2}\int_\Omega \frac{\varphi}{\vartheta}\left[\mathbb{S}(\nabla_x\mathbf{u}) : \nabla_x\mathbf{u} + \frac{\kappa(\vartheta)|\nabla_x\vartheta|^2}{\vartheta} \right] \ {\rm d}x \ {\rm d}t$$

for a.a. $0 \leq \tau_1 < \tau_2 \leq T$ including $\tau_1 = 0$, and any test function $\varphi \in C^\infty_c([0,T]\times\overline{\Omega})$, $\varphi \geq 0$.

As is well known, the passage from (2.9) to (3.2) in the framework of *weak* solutions is less obvious. Fortunately, we have the following result proved in [7, Lemmas 2.3, 2.4].

Proposition 3.1 *Under the hypotheses (2.2–2.4), let* $[\varrho,\vartheta,\mathbf{u}]$ *be a weak solution of the Navier-Stokes-Fourier system, emanating from the initial data (2.6) and enjoying the extra regularity*

$$\varrho, \ \vartheta, \ {\rm div}_x\mathbf{u} \in L^\infty((0,T)\times\Omega), \ \mathbf{u}\in L^\infty((0,T)\times\Omega;R^3).$$

Then

$$\varrho > 0, \ \vartheta > 0 \ \textit{a.a. in} \ (0,T)\times\Omega$$

and the entropy inequality (3.2) holds.

With the entropy inequality (3.2) at hand, we may use the machinery developed in [10] based on the relative energy functional

$$\mathcal{E}\left(\varrho,\vartheta,\mathbf{u}\ \Big|r,\Theta,\mathbf{U}\right) = \frac{1}{2}\int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u}-\mathbf{U}|^2 + H_\Theta(\varrho,\vartheta) - \frac{\partial H_\Theta(r,\Theta)}{\partial\varrho}(\varrho - r) - H_\Theta(r,\Theta) \right] \ {\rm d}x,$$

where

$$H_\Theta(\varrho,\vartheta) = \varrho\Big(e(\varrho,\vartheta) - \Theta s(\varrho,\vartheta)\Big)$$

is the so-called ballistic free energy.

It was shown in [7] that any weak solution $[\varrho,\vartheta,\mathbf{u}]$ enjoying the extra regularity specified in the hypotheses of Proposition 3.1 satisfies the *relative energy inequality*:

$$\left[\mathcal{E}\left(\varrho,\vartheta,\mathbf{u}\Big|r,\Theta,\mathbf{U}\right)\right]_{t=0}^{t=\tau} + \int_0^\tau\int_\Omega \frac{\Theta}{\vartheta}\left(\mathbb{S}(\nabla_x\mathbf{u}) : \nabla_x\mathbf{u} + \frac{\kappa(\vartheta)|\nabla_x\vartheta|^2}{\vartheta} \right) \ {\rm d}x \ {\rm d}t \tag{3.3}$$

$$\leq \int_0^\tau\int_\Omega \Big(\varrho(\mathbf{U}-\mathbf{u})\cdot\partial_t\mathbf{U} + \varrho(\mathbf{U}-\mathbf{u})\otimes\mathbf{u} : \nabla_x\mathbf{U} - p(\varrho,\vartheta){\rm div}_x\mathbf{U} + \mathbb{S}(\nabla_x\mathbf{u}) : \nabla_x\mathbf{U} \Big) \ {\rm d}x \ {\rm d}t$$

$$-\int_0^\tau\int_\Omega \Big(\varrho\Big(s(\varrho,\vartheta) - s(r,\Theta)\Big)\partial_t\Theta + \varrho\Big(s(\varrho,\vartheta) - s(r,\Theta)\Big)\mathbf{u}\cdot\nabla_x\Theta \Big) \ {\rm d}x \ {\rm d}t$$

$$+\int_0^\tau \int_\Omega \frac{\kappa(\vartheta)\nabla_x \vartheta}{\vartheta} \cdot \nabla_x \Theta \ \mathrm{d}x \ \mathrm{d}t$$

$$+\int_0^\tau \int_\Omega \left(\left(1 - \frac{\varrho}{r}\right) \partial_t p(r, \Theta) - \frac{\varrho}{r} \mathbf{u} \cdot \nabla_x p(r, \Theta) \right) \ \mathrm{d}x \ \mathrm{d}t$$

for any trio of sufficiently regular test functions $[r, \Theta, \mathbf{U}]$ defined in $[0, T] \times \Omega$, $r > 0$, $\Theta > 0$.

Taking $r = \tilde{\varrho}$, $\Theta = \tilde{\vartheta}$, $\mathbf{U} = \tilde{\mathbf{u}}$ - the (hypothetical) strong solution emanating from the same initial data as the weak solution $[\varrho, \vartheta, \mathbf{u}]$ - we may use a Gronwall's type argument to deduce the following result, see [7, Lemma 3.2]:

Lemma 3.1 *Under the hypotheses of Proposition 3.1, let $[\tilde{\varrho}, \tilde{\vartheta}, \tilde{\mathbf{u}}]$ be a strong solution of the Navier-Stokes-Fourier system defined in $(0, T) \times \Omega$, emanating from the initial data*

$$\tilde{\varrho}(0, \cdot) = \varrho(0, \cdot), \ \tilde{\vartheta}(0, \cdot) = \vartheta(0, \cdot), \ \tilde{\mathbf{u}}(0, \cdot) = \mathbf{u}(0, \cdot),$$

and belonging to the regularity class

$$\left\{ \begin{array}{l} \varrho, \vartheta \in C([0, T]; W^{3,2}(\Omega)), \ \mathbf{u} \in C([0, T]; W^{3,2}(\Omega; R^3)), \\ \\ \vartheta \in L^2(0, T; W^{4,2}(\Omega)), \ \mathbf{u} \in L^2(0, T; W^{4,2}(\Omega; R^3)), \\ \\ \partial_t \vartheta \in L^2(0, T; W^{2,2}(\Omega)), \ \partial_t \mathbf{u} \in L^2(0, T; W^{2,2}(\Omega; R^3)). \end{array} \right\}$$

Then $[\tilde{\varrho}, \tilde{\vartheta}, \tilde{\mathbf{u}}]$ coincides with the weak solution $[\varrho, \vartheta, \mathbf{u}]$ in $(0, T) \times \Omega$.

We conclude this part by a conditional regularity criterion for the weak solutions proved in [7, Theorem 2.2]:

Theorem 3.1 *Under the hypotheses (2.2–2.4), let $[\varrho, \vartheta, \mathbf{u}]$ be a weak solution of the Navier-Stokes-Fourier system, emanating from regular initial data satisfying (2.6), and enjoying the extra regularity*

$$\varrho, \ \vartheta, \ \mathrm{div}_x \mathbf{u} \in L^\infty((0, T) \times \Omega), \ \mathbf{u} \in L^\infty((0, T) \times \Omega; R^3).$$

Then $[\varrho, \vartheta, \mathbf{u}]$ is a strong (classical) solution of the problem in $(0, T) \times \Omega$.

4 Numerical solutions

We introduce the numerical scheme for solving the Navier-Stokes-Fourier system proposed in [9]. To this end, we suppose that the (periodic) domain Ω admits a *tetrahedral* mesh E_h, parameterized by the typical radius $h \to 0$, with the following properties (cf. Eymard, Gallouët, and Herbin [6]) :

1. Each tetrahedron $E \in E_h$ can be written in the form

$$E = h\mathcal{L}_E[\tilde{E}] + \mathbf{b},$$

where $\tilde{E}$ is a fixed reference tetrahedron, $\mathbf{b} \in R^3$, and $\mathcal{L}_E : R^3 \to R^3$ is a linear mapping such that

$$0 < \underline{L} \leq |\mathrm{det}\mathcal{L}_E| \leq \overline{L} \text{ uniformly for } h \to 0.$$

2. If $K, L \in E_h$, $K \neq L$, and $K \cap L \neq \emptyset$, then $K \cap L$ is either a common face or a common vertex shared by K and L.

3. There is a family of points $x_E \in \mathrm{int}[E]$, $E \in E_h$ such that the segment $[x_K, x_L]$ for two elements K, L sharing a common face Γ is perpendicular to Γ, and

$$\mathrm{dist}[x_K, x_L] \equiv d_\Gamma \geq ch \text{ for a certain } c > 0 \text{ independent of } K, L, h.$$

Motivated by the work of Karper [14] and Karlsen and Karper [13], we introduce a mixed finite-volume finite-element scheme to solve numerically the Navier-Stokes-Fourier system. We denote by Γ_h the family of all faces in the mesh, where each $\Gamma \in \Gamma_h$ is associated with a normal vector $\mathbf{n}$. Furthermore, for a function g continuous on each element $E \in E_h$, we denote

$$g^+|_\Gamma = \lim_{\delta \to 0+} g(\cdot + \delta\mathbf{n}),\ g^-|_\Gamma = \lim_{\delta \to 0+} g(\cdot - \delta\mathbf{n}),\ [[g]]_\Gamma = g^+ - g^-, \{g\}_\Gamma = \frac{1}{2}\left(g^+ + g^-\right).$$

Next, we will use the space

$$Q_h(\Omega) = \left\{ r \in L^2(\Omega) \ \middle|\ r|_E = \text{ constant for any } E \in E_h \right\},$$

and the space of finite elements of Crouzeix-Raviart type (see e.g. Crouzeix and Raviart [3], Brezzi and Fortin [2]),

$$V_h(\Omega) = \left\{ v \in L^2(\Omega) \ \middle|\ v|_E = \text{ affine function}, E \in E_h,\ \int_\Gamma [[v]]_\Gamma \ \mathrm{dS}_x = 0 \text{ for all } \Gamma \in \Gamma_h \right\}.$$

Finally, we introduce the standard upwind operator $\mathrm{Up}[r, \mathbf{v}]$ defined on each face $\Gamma \in \Gamma_h$ as

$$\mathrm{Up}[r, \mathbf{v}] = r^-[\tilde{\mathbf{v}} \cdot \mathbf{n}]^+ + r^+[\tilde{\mathbf{v}} \cdot \mathbf{n}]^-,\ r \in Q_h(\Omega),\ \mathbf{v} \in V_h(\Omega; R^3),$$

where we set

$$[c]^+ = \max\{c, 0\},\ [c]^- = \min\{c, 0\},\ \tilde{v} := \tilde{v}_\Gamma = \frac{1}{|\Gamma|}\int_\Gamma v \ \mathrm{dS}_x.$$

and

$$\widehat{v}|_E = \frac{1}{|E|}\int_E v \ \mathrm{d}x \text{ for } E \in E_h.$$

With the previous notation, the numerical method to approximate the Navier-Stokes-Fourier system can be formulated as follows, see [9]:

1. Set
$$\varrho_h^0 = \widehat{\varrho_0},\ \vartheta_h^0 = \widehat{\vartheta_0},\ \mathbf{u}_h^0 = \widehat{\mathbf{u}_0}.$$

2. For $k = 1, 2, \ldots$, the approximate solutions $[\varrho_h^k, \vartheta_h^k, \mathbf{u}_h^k]$, $\varrho_h^k, \vartheta_h^k \in Q_h(\Omega)$, $\mathbf{u}_h^k \in V_h(\Omega; R^3)$ are defined recursively by the scheme
$$\int_\Omega \left[\frac{\varrho_h^k - \varrho^{k-1}}{\Delta t}\right] \phi \ \mathrm{d}x - \sum_{\Gamma \in \Gamma_h} \int_\Gamma \mathrm{Up}[\varrho_h^k, \mathbf{u}_h^k] \ [[\phi]] \ \mathrm{dS}_x + h^\alpha \sum_{\Gamma \in \Gamma_h} \int_\Gamma [[\varrho_h^k]] \ [[\phi]] \ \mathrm{dS}_x = 0, \tag{4.1}$$
for any $\phi \in Q_h(\Omega)$;
$$\int_\Omega \left[\frac{(\varrho_h^k \widehat{\mathbf{u}}_h^k) - (\varrho_h^{k-1} \widehat{\mathbf{u}}_h^{k-1})}{\Delta t}\right] \cdot \phi \ \mathrm{d}x - \sum_{\Gamma \in \Gamma_h} \int_\Gamma \mathrm{Up}[\varrho_h^k \widehat{\mathbf{u}}_h^k, \mathbf{u}_h^k] \cdot [[\widehat{\phi}]] \ \mathrm{dS}_x \tag{4.2}$$
$$+ \int_\Omega \left[\mu \nabla_h \mathbf{u}_h^k : \nabla_h \phi + \lambda \mathrm{div}_h \mathbf{u}_h^k \mathrm{div}_h \phi\right] \ \mathrm{d}x - \int_\Omega p(\varrho_h^k, \vartheta_h^k) \mathrm{div}_h \phi \ \mathrm{d}x$$
$$+ h^\alpha \sum_{\Gamma \in \Gamma_h} \int_\Gamma [[\varrho_h^k]] \left\{\widehat{u_h^k}\right\} \cdot [[\widehat{\phi}]] \mathrm{dS}_x = 0$$
for any $\phi \in V_h(\Omega; R^3)$, where $\nabla_h v|_E = \nabla_x v|_E$;
$$c_v \int_\Omega \left[\frac{\varrho_h^k \vartheta_h^k - \varrho_h^{k-1} \vartheta_h^{k-1}}{\Delta t}\right] \phi \ \mathrm{d}x - c_v \sum_{\Gamma \in \Gamma_h} \int_\Gamma \mathrm{Up}[\varrho_h^k \vartheta_h^k, \mathbf{u}_h^k] \ \ [[\phi]] \ \ \mathrm{dS}_x \tag{4.3}$$
$$+ \sum_{\Gamma \in \Gamma_h} \int_\Gamma \frac{1}{d_\Gamma} \left[\left[K(\vartheta_h^k)\right]\right] \ \ [[\phi]] \ \ \mathrm{dS}_x$$
$$= \int_\Omega \left[\mu |\nabla_h \mathbf{u}_h^k|^2 + \lambda |\mathrm{div}_h \mathbf{u}_h^k|^2\right] \phi \ \mathrm{d}x - \int_\Omega \varrho_h^k \vartheta_h^k \mathrm{div}_h \mathbf{u}_h^k \phi \ \mathrm{d}x$$
for any $\phi \in Q_h(\Omega)$.

In (4.1–4.3), the time step Δt is taken proportional to the diameter h, and $0 < \alpha < 1$, where the $\alpha-$dependent terms represent a kind of artificial viscosity used also in the purely analytical context in [8].

As shown in [9, Theorem 3.1], the numerical method (4.1–4.3) is *convergent*. In order to state the result, we extend the numerical solutions to be defined at *any* time as
$$v_h(t, x) = v_h^k(x) \text{ for } t \in [k\Delta t, (k+1)\Delta t).$$

Theorem 4.1 *Let*

$$0 < \alpha < 1, \ \Delta t = h > 0,$$

and let the hypotheses (2.2–2.4), (2.6) be satisfied. Let $[\varrho_h, \vartheta_h, \mathbf{u}_h]_{h>0}$ *be a family of numerical solutions constructed by means of the scheme (4.1 – 4.3) such that*

$$\varrho_h > 0, \ \vartheta_h > 0 \text{ for all } h > 0.$$

Then, at least for a suitable subsequence,

$$\varrho_h \to \varrho \text{ weakly-(*) in } L^\infty(0,T;L^\gamma(\Omega)) \text{ and strongly in } L^1((0,T)\times\Omega),$$

$$\vartheta_h \to \vartheta \text{ weakly in } L^2(0,T;L^6(\Omega)),$$

$$\mathbf{u}_h \to \mathbf{u} \text{ weakly in } L^2(0,T;L^6(\Omega;R^3)), \ \nabla_h \mathbf{u}_h \to \nabla_x \mathbf{u} \text{ weakly in } L^2((0,T)\times\Omega;R^{3\times 3}),$$

where $[\varrho, \vartheta, \mathbf{u}]$ *is a weak solution of the problem of the Navier-Stokes-Fourier system in the sense specified in Definition 2.1.*

Remark 4.1 *Strictly speaking, this result was obtained in [9] for* Ω *- a regular bounded domain in* R^3 *- supplemented with suitable boundary conditions for* ϑ *and* $\mathbf{u}$*. However, the proof can be easily adapted to the periodic setting used in this paper.*

5 Unconditional convergence

From the practical point of view, the convergence results established in Theorem 4.1 is not very satisfactory, in particular, it holds up to a suitable subsequence. As the weak solutions to the Navier-Stokes-Fourier system are not (known to be) unique, it is therefore not a priori excluded that there is another subsequence converging to a different solution of the same problem. Combining Theorem 3.1 with Theorem 4.1 we may deduce the following *unconditional* convergence statement that can be seen as an example of "synergy" between analysis and numerics:

Theorem 5.1 *Let*

$$0 < \alpha < 1, \ \Delta t = h > 0,$$

and let the hypotheses (2.2–2.4), (2.6) be satisfied. Let $[\varrho_h, \vartheta_h, \mathbf{u}_h]_{h>0}$ *be a family of numerical solutions constructed by means of the scheme (4.1 – 4.3) such that*

$$\varrho_h > 0, \ \vartheta_h > 0,$$

and, in addition,

$$\varrho_h, \ \vartheta_h, \ |\mathbf{u}_h|, \ |\mathrm{div}_h \mathbf{u}_h| \le M \tag{5.1}$$

a.a. in $(0,T)\times\Omega$ *for a certain* M *independent of* h.

Then

$$\varrho_h \to \varrho \text{ weakly-(*) in } L^\infty(0,T;L^\gamma(\Omega)) \text{ and strongly in } L^1((0,T)\times\Omega),$$

$$\vartheta_h \to \vartheta \text{ weakly in } L^2(0,T;L^6(\Omega)),$$

$$\mathbf{u}_h \to \mathbf{u} \text{ weakly in } L^2(0,T;L^6(\Omega;R^3)),\ \nabla_h\mathbf{u}_h \to \nabla_x\mathbf{u} \text{ weakly in } L^2((0,T)\times\Omega;R^{3\times3}),$$

where $[\varrho,\vartheta,\mathbf{u}]$ *is the (strong) solution of the problem of the Navier-Stokes-Fourier system in* $(0,T)\times\Omega$ *emanating from the initial data (2.6).*

Proof:

From Theorem 4.1 we know that there is a subsequence of numerical solutions that converges to a weak solution of the Navier-Stokes-Fourier system. However, by virtue of the assumption (5.1), the latter belongs to the class specified in Theorem 3.1, in particular, the limit solution is strong, unique, and the whole sequence of numerical solutions converges to it.

Q.E.D.

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