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Abstract

We consider a "semi-relativistic" model of radiative viscous compressible Navier-Stokes-Fourier system coupled to the radiative transfer equation extending the classical model introduced in [10] and we study some of its singular limits (low Mach and diffusion) in the case of well-prepared initial data and Dirichlet boundary condition for the velocity field. In the low Mach number case we prove the convergence toward the incompressible Navier-Stokes system coupled to a system of two stationary transport equations. In the diffusion case we prove the convergence toward the compressible Navier-Stokes with modified state functions (equilibrium case) or toward the compressible Navier-Stokes coupled to a diffusion equation (non equilibrium case).

Key words: Radiation hydrodynamics, Navier-Stokes-Fourier system, weak solution, low Mach number limit, diffusion limits, well-prepared initial data

1 Introduction

In recent works [12] [13] singular limits (low Mach number limit and diffusion limits) for a simplified model of radiation hydrodynamics introduced by Teleaga, Seaïd, Gasser, Klar and Struckmeier in [29] have been presented, incorporating the effects of radiation in a simplified classical setting (special relativity appears only in the persistence of the speed light c in the system) and neglecting the radiative source in the momentum equation. A more realistic model relaxing this last hypothesis was studied in [10] however this more complete model suffers from a non manifestly positive production rate of total entropy, preventing ones from studying these singular limits.

Our idea in the present paper is to introduce in the complete model of [10] a perturbed Planck's function and a suitable (relativistic) velocity cut off (this is the meaning we give to "semi-relativistic" model) allowing to recover this crucial positivity property for the production rate of total entropy. As the perturbation will be small (going formally to zero as $c \to \infty$), one can expect to obtain the correct limit regimes.

The motion of the fluid is still described by standard non-relativistic fluid mechanics giving the evolution of the mass density $\varrho = \varrho(t,x)$, the velocity field $\vec{u} = \vec{u}(t,x)$, and the temperature $\vartheta = \vartheta(t,x)$ as functions of the time t and the spatial coordinate $x \in \Omega \subset \mathbb{R}^3$. The effect of radiation is still incorporated in the radiative intensity $I = I(t,x,\vec{\omega},\nu)$, depending on the direction $\vec{\omega} \in \mathcal{S}^2$, where $\mathcal{S}^2 \subset \mathbb{R}^3$ denotes the unit sphere, and the frequency $\nu \geq 0$, but we take into account their relativistic corrections. The evolution of I is described by a transport equation with a source term and the fluid-radiation coupling is expressed through radiative sources in the momentum and energy equations. More precisely, the system of equations to be studied reads as follows:

$$\partial_t \varrho + \mathrm{div}_x(\varrho\vec{u}) = 0 \quad \text{in } (0,T)\times\Omega, \tag{1.1}$$

$$\partial_t(\varrho\vec{u}) + \mathrm{div}_x(\varrho\vec{u}\otimes\vec{u}) + \nabla_x p(\varrho,\vartheta) = \mathrm{div}_x\mathbb{S} - \vec{S}_F \quad \text{in } (0,T)\times\Omega, \tag{1.2}$$

$$\partial_t\left(\frac{1}{2}\varrho|\vec{u}|^2 + \varrho e(\varrho,\vartheta)\right) + \mathrm{div}_x\left(\left(\frac{1}{2}\varrho|\vec{u}|^2 + \varrho e(\varrho,\vartheta) + p\right)\vec{u} + \vec{q} - \mathbb{S}\vec{u}\right) = -S_E \quad \text{in } (0,T)\times\Omega, \tag{1.3}$$

$$\frac{1}{c}\partial_t I + \vec{\omega}\cdot\nabla_x I = S \quad \text{ in } (0,T)\times\Omega\times(0,\infty)\times\mathcal{S}^2. \tag{1.4}$$

The symbol $p = p(\varrho,\vartheta)$ denotes the thermodynamic pressure and $e = e(\varrho,\vartheta)$ is the specific internal energy, related through Maxwell's equation

$$\frac{\partial e}{\partial \varrho} = \frac{1}{\varrho^2}\left(p(\varrho,\vartheta) - \vartheta\frac{\partial p}{\partial \vartheta}\right). \tag{1.5}$$

In (1.2) $\mathbb{S}$ is the viscous stress tensor given by $\mathbb{S} = \mu\left(\nabla_x\vec{u} + \nabla_x^t\vec{u} - \frac{2}{3}\mathrm{div}_x\vec{u}\right) + \eta\,\mathrm{div}_x\vec{u}\,\mathbb{I}$, where the viscosity coefficients $\mu = \mu(\vartheta) > 0$ and $\eta = \eta(\vartheta) \geq 0$ are effective functions of the temperature. Similarly in (1.3) $\vec{q}$ is the heat flux given by Fourier's law $\vec{q} = -\kappa\nabla_x\vartheta$, with the heat conductivity coefficient $\kappa = \kappa(\vartheta) > 0$.

We suppose that the radiative source S is given by

$$S = \sigma_a\Big[B(\nu,\vec{\omega},\vec{u},\vartheta) - I(t,x,\nu,\vec{\omega})\Big] + \sigma_s\left(\frac{1}{4\pi}\int_{\mathcal{S}^2} I(t,x,\nu,\vec{\omega}')\,d\vec{\omega}' - I(t,x,\nu,\vec{\omega})\right) =: S_{a,e} + S_s. \tag{1.6}$$

In the right-hand side the first term is the emission-absorption contribution where $\sigma_a > 0$ is the absorption coefficient and B is a perturbation of the equilibrium Planck's function given by

$$B(\nu,\vec{\omega},\vec{u},\vartheta) = \frac{2h}{c^2}\frac{\nu^3}{e^{\frac{h\nu}{k\vartheta}\left(1-\alpha\frac{\vec{\omega}\cdot\vec{u}}{c}\right)} - 1}, \tag{1.7}$$

where h is the Planck's constant, k is the Boltzmann's constant and $0 \leq \alpha(\vartheta) \leq 1$ is a smooth function, to be determined below. One observes that for $\frac{|\vec{u}|}{c} << 1$ one recovers the standard equilibrium Planck's function $B(\nu,\vartheta) = \frac{2h}{c^2}\frac{\nu^3}{e^{\frac{h\nu}{k\vartheta}}-1}$.

Note that the idea of this kind of perturbation is not new and has been extensively used in recent works on radiative transfer [4],[6],[9],[8], for exemple in the $M1$ Levermore model [18],[19].

The second term in S is the scattering contribution where $\sigma_s > 0$ is the scattering coefficient and in the right-hand sides of (1.2) and (1.3) appear the coupling sources.

$$\vec{S}_F(t,x) = \frac{1}{c}\int_0^\infty\int_{\mathcal{S}^2}\vec{\omega}S\,\mathrm{d}\vec{\omega}\,\mathrm{d}\nu, \tag{1.8}$$

and

$$S_E(t,x) = \int_0^\infty\int_{\mathcal{S}^2} S\,\mathrm{d}\vec{\omega}\,\mathrm{d}\nu. \tag{1.9}$$

We first suppose that the transport coefficients are smooth functions satisfying $\sigma_a(\vartheta,\vec{u}) = \chi(|\vec{u}|)\tilde{\sigma}_a(\vartheta) \geq 0$ and $\sigma_s(\vartheta) \geq 0$ and that both depend neither on angular variable (1.1 - 1.4) (isotropy of radiation), nor on frequency (the so called "grey" hypothesis).

The function χ appearing in the emission-absorption coefficient is a C^∞ cut-off satisfying

$$\chi(s) = \begin{cases} 1 & \text{if } \; s \leq c, \\ 0 & \text{if } \; s \geq c+\beta, \end{cases}$$

for an arbitrary $\beta > 0$. The role of this cut-off is to deal with the singularity of B and its meaning is the following: in the "over-relativistic" regime ($|\vec{u}| \geq c$) where special relativity would be violated, we decide to decouple matter and radiation. Of course this is an arbitrary choice but only a meaningless region with respect to physics is concerned (recall that in the relativistic setting [6], Lorentz factors of the type $\left(1 - \frac{\vec{u}^2}{c^2}\right)^{1/2}$ become singular for $|\vec{u}| = c$).

More restrictions on properties of these constitutive quantities will be imposed in Section 2 below.

Finally system (1.1 - 1.4) is supplemented with the boundary conditions:

$$\vec{u}|_{\partial\Omega} = 0, \;\; \vec{q}\cdot\vec{n}|_{\partial\Omega} = 0, \tag{1.10}$$

$$I(t,x,\nu,\vec{\omega})|_{\Gamma_-} = 0,$$
$$\Gamma_- \equiv \{\{x,\omega\} \in \partial\Omega \times S^2, \vec{\omega}\cdot\vec{n} \leq 0\}, \tag{1.11}$$

where $\vec{n}$ denotes the outer normal vector to $\partial\Omega$, and initial conditions

$$(\varrho(t,x),\ \vec{u}(t,x),\ \vartheta(t,x),\ I(t,x,\omega,\nu))|_{t=0} = \left(\varrho^0(x),\ \vec{u}^0(x),\ \vartheta^0(x),\ I^0(x,\vec{\omega},\nu)\right), \tag{1.12}$$

for any $x \in \Omega$, $\vec{\omega} \in \mathcal{S}^2$,$\nu \in \mathbb{R}_+$.

The relativistic version of system (1.1 - 1.11) has been introduced by Pomraning [25] and Mihalas and Weibel-Mihalas [24] and investigated more recently in astrophysics and laser applications (in the inviscid case) by Lowrie, Morel and Hittinger [22] and Buet and Desprès [6], with a special attention to asymptotic regimes, and this last paper was a deep source of inspiration for the present work. Let us mention that a simplified version of the system (non relativistic non conducting fluid at rest) has been investigated by Golse and Perthame in [16] where global existence was proved under very mild hypotheses (transport coefficients may be singular). A global existence result was also proved in [10] for the simplified model (without relativistic corrections), under some cut-off hypotheses on transport coefficients. As previously aforementioned it is not clear how to justify singular limits on this system because of the lack of (manifest) positivity of the entropy production rate however it was shown recently in [12] [13] that this difficulty disappears in the model of Teleaga, Seaïd, Gasser, Klar and Struckmeier [29], where the radiative momentum source is absent in the right hand side of (1.11).

Our goal in the present work is then to show that provided we use the aforementioned "semi relativistic" framework, the positivity of the entropy production rate is restored and singular limits can actually be performed for the problem (1.1 -1.11) by using the ideas of [14].

The paper is organized as follows. In Section 2, we list the principal hypotheses imposed on constitutive relations, introduce the concept of weak solution to problem (1.1 - 1.11), and state the existence result for our model. In Section 3 we investigate the low Mach number limit and in Section 4 we study both the equilibrium and non-equilibrium diffusion limits.

2 Hypotheses and existence result

We consider the pressure in the form

$$p(\varrho,\vartheta) = \vartheta^{5/2} P\left(\frac{\varrho}{\vartheta^{3/2}}\right) + \frac{a}{3}\vartheta^4,\ a > 0, \tag{2.1}$$

where $P : [0,\infty) \to [0,\infty)$ is a given function with the following properties:

$$P \in C^1[0,\infty),\ P(0) = 0,\ P'(Z) > 0,\ \text{for all } Z \geq 0, \tag{2.2}$$

$$0 < \frac{\frac{5}{3}P(Z) - P'(Z)Z}{Z} < c \text{ for all } Z \geq 0, \tag{2.3}$$

$$\lim_{Z\to\infty} \frac{P(Z)}{Z^{5/3}} = p_\infty > 0. \tag{2.4}$$

After Maxwell's equation (1.5), the specific internal energy e is

$$e(\varrho,\vartheta) = \frac{3}{2}\left(\frac{\vartheta^{5/2}}{\varrho}\right) P\left(\frac{\varrho}{\vartheta^{3/2}}\right) + a\frac{\vartheta^4}{\varrho}, \tag{2.5}$$

and the associated specific entropy reads

$$s(\varrho,\vartheta) = M\left(\frac{\varrho}{\vartheta^{3/2}}\right) + \frac{4a}{3}\frac{\vartheta^3}{\varrho} \quad \text{with } M'(Z) = -\frac{3}{2}\frac{\frac{5}{3}P(Z) - P'(Z)Z}{Z^2} < 0. \tag{2.6}$$

The transport coefficients μ, η, and κ are continuously differentiable functions of the absolute temperature such that

$$0 < c_1(1+\vartheta) \le \mu(\vartheta),\ \ \mu'(\vartheta) < c_2,\ \ 0 \le \eta(\vartheta) \le c(1+\vartheta), \tag{2.7}$$

$$0 < c_1(1+\vartheta^3) \le \kappa(\vartheta) \le c_2(1+\vartheta^3) \tag{2.8}$$

for any $\vartheta \ge 0$. Moreover we assume that σ_a and σ_s are smooth functions such that

$$0 \le \sigma_a(\vartheta,\vec{u}),\ \ \sigma_s(\vartheta) \le c_1,\ \ \sigma_a(\vartheta,\vec{u})B(\nu,\vec{\omega},\vec{u},\vartheta) \le c_2, \tag{2.9}$$

$$\sigma_a(\vartheta,\vec{u})B(\nu,\vec{\omega},\vec{u},\vartheta) \le h(\nu),\ \ h \in L^1(0,\infty), \tag{2.10}$$

where $c_{1,2,3}$ are positive constants. Relations (2.9 - 2.10) represent "cut-off" hypotheses neglecting the effect of radiation at large temperature and ultra relativistic velocities (see [26] for physical motivations).

In the weak formulation of the Navier-Stokes-Fourier system the equation of continuity (1.1) is replaced by its (weak) renormalized version

$$\int_0^T \int_\Omega ((\varrho+b(\varrho))\partial_t\varphi + (\varrho+b(\varrho))\vec{u}\cdot\nabla_x\varphi + (b(\varrho)-b'(\varrho)\varrho)\mathrm{div}_x\vec{u}\varphi)\ \mathrm{d}x\ \mathrm{d}t = -\int_\Omega (\varrho_0+b(\varrho_0))\varphi(0,\cdot)\ \mathrm{d}x \tag{2.11}$$

satisfied for any $\varphi \in C_c^\infty([0,T)\times\overline{\Omega})$, and any $b \in C^\infty[0,\infty)$, $b' \in C_c^\infty[0,\infty)$, where (2.11) implicitly includes the initial condition $\varrho(0,\cdot) = \varrho_0$. Similarly, the momentum equation (1.2) is replaced by

$$\int_0^T \int_\Omega (\varrho\vec{u}\cdot\partial_t\vec{\varphi} + \varrho\vec{u}\otimes\vec{u} : \nabla_x\vec{\varphi} + p\mathrm{div}_x\vec{\varphi})\ \mathrm{d}x\ \mathrm{d}t$$

$$= \int_0^T \int_\Omega \mathbb{S} : \nabla_x\vec{\varphi}\ \mathrm{d}x\ \mathrm{d}t - \int_0^T \int_\Omega \vec{S}_F\vec{\varphi}\ \mathrm{d}x\ \mathrm{d}t - \int_\Omega (\varrho\vec{u})_0\cdot\vec{\varphi}(0,\cdot)\ \mathrm{d}x, \tag{2.12}$$

for any $\vec{\varphi} \in C_c^\infty([0,T)\times\Omega;\mathbb{R}^3)$. As usual, for (2.12) to make sense, we require that $\vec{u} \in L^2(0,T;W_0^{1,2}(\Omega;\mathbb{R}^3))$ which contains the no-slip boundary condition (1.10). As the term $\mathbb{S}\vec{u}$ in the total energy balance (1.3) is not controlled on the (hypothetical) vacuum zones of vanishing density, we will replace (1.3) by the internal energy equation

$$\partial_t(\varrho e) + \mathrm{div}_x(\varrho e\vec{u}) + \mathrm{div}_x\vec{q} = \mathbb{S} : \nabla_x\vec{u} - p\mathrm{div}_x\vec{u} - S_E + \vec{S}_F\cdot\vec{u}, \tag{2.13}$$

moreover, dividing (2.13) on ϑ and using Maxwell's relation (1.5), we may rewrite (2.13) as the entropy equation

$$\partial_t(\varrho s) + \mathrm{div}_x(\varrho s\vec{u}) + \mathrm{div}_x\left(\frac{\vec{q}}{\vartheta}\right) = \frac{1}{\vartheta}\left(\mathbb{S} : \nabla_x\vec{u} - \frac{\vec{q}\cdot\nabla_x\vartheta}{\vartheta}\right) - \frac{S_E}{\vartheta} + \frac{\vec{S}_F\cdot\vec{u}}{\vartheta} =: \varsigma, \tag{2.14}$$

where the first term of the right hand side $\varsigma_m := \frac{1}{\vartheta}\left(\mathbb{S} : \nabla_x\vec{u} - \frac{\vec{q}\cdot\nabla_x\vartheta}{\vartheta}\right)$ is the (positive) matter entropy production. In order to identify the second term in the right hand side of (2.14), let us recall [1] the formula for the entropy of a photon gas

$$s^R = -\frac{2k}{c^3}\int_0^\infty \int_{\mathcal{S}^2} \nu^2\left[n\log n - (n+1)\log(n+1)\right] d\vec{\omega}d\nu, \tag{2.15}$$

where $n = n(I) = \frac{c^2 I}{2h\alpha^3\nu^3}$ is the occupation number. Defining the radiative entropy flux

$$\vec{q}^R = -\frac{2k}{c^2}\int_0^\infty \int_{\mathcal{S}^2} \nu^2\left[n\log n - (n+1)\log(n+1)\right]\vec{\omega}\ d\vec{\omega}d\nu, \tag{2.16}$$

and using the radiative transfer equation, we get the equation

$$\partial_t s^R + \mathrm{div}_x\vec{q}^R = -\frac{k}{h}\int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu}\log\frac{n}{n+1}\ S\ d\vec{\omega}d\nu =: \varsigma^R. \tag{2.17}$$

Checking the identity $\log \frac{n(B)}{n(B)+1} = -\frac{h\nu}{k\vartheta}\left(1-\alpha\frac{\vec\omega\cdot\vec u}{c}\right)$ where B is the Planck's function, using the definition of S and taking into account that $\vec S_F = (\sigma_a+\sigma_s)\int_0^\infty\int_{\mathcal{S}^2}\omega I\; d\vec\omega I\; d\nu$ (the transport coefficients $\sigma_{a,s}$ do not depend on $\vec\omega$), the right-hand side of (2.17) rewrites

$$\varsigma^R = -\frac{k}{h}\int_0^\infty\int_{\mathcal{S}^2}\frac{1}{\nu}\left\{\left[\log\frac{n(I)}{n(I)+1}-\log\frac{n(B)}{n(B)+1}\right]\sigma_a(B-I)+\left[\log\frac{n(I)}{n(I)+1}-\log\frac{n(\tilde I)}{n(\tilde I)+1}\right]\sigma_s(\tilde I-I)\right\}d\vec\omega d\nu,$$

$$+\frac{1}{\vartheta}\int_0^\infty\int_{\mathcal{S}^2}\left(1-\alpha\frac{\vec\omega\cdot\vec u}{c}\right)S\; d\vec\omega d\nu - \frac{\alpha\sigma_s}{\sigma_a+\sigma_s}\frac{\vec S_F\cdot\vec u}{\vartheta}.$$

Choosing now

$$\alpha = \frac{\sigma_a+\sigma_s}{\sigma_a+2\sigma_s}, \tag{2.18}$$

we get

$$\varsigma^R = -\frac{k}{h}\int_0^\infty\int_{\mathcal{S}^2}\frac{1}{\nu}\left[\log\frac{n(I)}{n(I)+1}-\log\frac{n(B)}{n(B)+1}\right]\sigma_a(B-I)\; d\vec\omega d\nu$$

$$-\frac{k}{h}\int_0^\infty\int_{\mathcal{S}^2}\frac{1}{\nu}\left[\log\frac{n(I)}{n(I)+1}-\log\frac{n(\tilde I)}{n(\tilde I)+1}\right]\sigma_s(\tilde I-I)\; d\vec\omega d\nu + \frac{1}{\vartheta}S_E - \frac{\vec S_F\cdot\vec u}{\vartheta}. \tag{2.19}$$

From (2.14),(2.17) and (2.19) we obtain finally

$$\partial_t\left(\varrho s+s^R\right)+\mathrm{div}_x\left(\varrho s\vec u+\vec q^R\right)+\mathrm{div}_x\left(\frac{\vec q}{\vartheta}\right)$$

$$=\frac{1}{\vartheta}\left(\mathbb{S}:\nabla_x\vec u-\frac{\vec q\cdot\nabla_x\vartheta}{\vartheta}\right)-\frac{k}{h}\int_0^\infty\int_{\mathcal{S}^2}\frac{1}{\nu}\left[\log\frac{n(I)}{n(I)+1}-\log\frac{n(B)}{n(B)+1}\right]\sigma_a(B-I)\; d\vec\omega d\nu$$

$$-\frac{k}{h}\int_0^\infty\int_{\mathcal{S}^2}\frac{1}{\nu}\left[\log\frac{n(I)}{n(I)+1}-\log\frac{n(\tilde I)}{n(\tilde I)+1}\right]\sigma_s(\tilde I-I)\; d\vec\omega d\nu, \tag{2.20}$$

and equation (2.14) is replaced in the weak formulation by the inequality

$$\int_0^T\int_\Omega\left([\varrho s+s^R]\partial_t\varphi+\varrho s\vec u\cdot\nabla_x\varphi+[\frac{\vec q}{\vartheta}+\vec q^R]\cdot\nabla_x\varphi\right)\;\mathrm{d}x\;\mathrm{d}t$$

$$\le -\int_\Omega(\varrho s+s^R)_0\varphi(0,\cdot)\;\mathrm{d}x-\int_0^T\int_\Omega\frac{1}{\vartheta}\left(\mathbb{S}:\nabla_x\vec u-\frac{\vec q\cdot\nabla_x\vartheta}{\vartheta}\right)\varphi\;\mathrm{d}x\;\mathrm{d}t$$

$$-\frac{k}{h}\int_0^T\int_\Omega\left[\int_0^\infty\int_{\mathcal{S}^2}\frac{1}{\nu}\left[\log\frac{n(I)}{n(I)+1}-\log\frac{n(B)}{n(B)+1}\right]\sigma_a(B-I)\; d\vec\omega d\nu\right.$$

$$\left.+\int_0^\infty\int_{\mathcal{S}^2}\frac{1}{\nu}\left[\log\frac{n(I)}{n(I)+1}-\log\frac{n(\tilde I)}{n(\tilde I)+1}\right]\sigma_s(\tilde I-I)\; d\vec\omega d\nu\right]\varphi\;\mathrm{d}x\mathrm{d}x\;\mathrm{d}t \tag{2.21}$$

for any $\varphi\in C_c^\infty([0,T)\times\overline{\Omega})$, $\varphi\ge 0$, where the sign of all the terms in the right hand side may be controlled.

Since replacing equation (1.3) by inequality (2.21) would result in a formally under-determined problem, system (2.11), (2.12), (2.21) must be supplemented with the total energy balance

$$\int_\Omega\left(\frac12\varrho|\vec u|^2+\varrho e(\varrho,\vartheta)+E^R\right)(\tau,\cdot)\;\mathrm{d}x+\int_0^\tau\int_{\Gamma_+}\int_0^\infty\vec\omega\cdot\vec n I(t,x,\vec\omega,\nu)\;\mathrm{d}\nu\;\mathrm{d}\vec\omega\;\mathrm{d}S_x\;\mathrm{d}t,$$

$$=\int_\Omega\left(\frac{1}{2\varrho_0}|(\varrho\vec u)_0|^2+(\varrho e)_0+E_{R,0}\right)\;\mathrm{d}x, \tag{2.22}$$

where $\Gamma_+ = \{(x, \vec{\omega}) \in \partial\Omega \times \mathcal{S}^2 \ : \ \vec{\omega} \cdot \vec{n}_x > 0\}$.

The radiative energy E^R is defined by

$$E^R(t,x) = \frac{1}{c} \int_{\mathcal{S}^2} \int_0^\infty I(t,x,\vec{\omega},\nu) \, \mathrm{d}\vec{\omega} \, \mathrm{d}\nu, \tag{2.23}$$

with $E_{R,0} = \int_{\mathcal{S}^2} \int_0^\infty I_0(\cdot,\vec{\omega},\nu) \, \mathrm{d}\vec{\omega} \, \mathrm{d}\nu$, and for later purposes we also define the radiative momentum

$$\vec{F}^R(t,x) = \int_{\mathcal{S}^2} \int_0^\infty \vec{\omega} I(t,x,\vec{\omega},\nu) \, \mathrm{d}\vec{\omega} \, \mathrm{d}\nu, \tag{2.24}$$

and the radiative tensor

$$\mathbb{P}^R(t,x) = \frac{1}{c} \int_{\mathcal{S}^2} \int_0^\infty \vec{\omega} \otimes \vec{\omega} I(t,x,\vec{\omega},\nu) \, \mathrm{d}\vec{\omega} \, \mathrm{d}\nu. \tag{2.25}$$

Definition 2.1 *We say that $\varrho, \vec{u}, \vartheta, I$ is a weak solution of problem (1.1 - 1.11) if*

$$\varrho \geq 0, \ \vartheta > 0 \text{ for a.a. } (t,x) \times \Omega, \ I \geq 0 \text{ a.a. in } (0,T) \times \Omega \times \mathcal{S}^2 \times (0,\infty),$$
$$\varrho \in L^\infty(0,T; L^{5/3}(\Omega)), \ \vartheta \in L^\infty(0,T; L^4(\Omega)),$$
$$\vec{u} \in L^2(0,T; W_0^{1,2}(\Omega; \mathbb{R}^3)), \ \vartheta \in L^2(0,T; W^{1,2}(\Omega)),$$
$$I \in L^\infty((0,T) \times \Omega \times \mathcal{S}^2 \times (0,\infty)), \ I \in L^\infty(0,T; L^1(\Omega \times \mathcal{S}^2 \times (0,\infty)),$$

and if ϱ, $\vec{u}$, ϑ, I satisfy the integral identities (2.11), (2.12), (2.21), (2.22), together with the transport equation (1.4). The existence result reads now

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Assume that the thermodynamic functions p, e, s satisfy hypotheses (2.1-2.6), that B satisfies (1.7) and (2.18) and that the transport coefficients μ, λ, κ, σ_a, and σ_s comply with (2.7 - 2.10). Let $\{\varrho_\varepsilon, \vec{u}_\varepsilon, \vartheta_\varepsilon, I_\varepsilon\}_{\varepsilon>0}$ be a family of weak solutions to problem (1.1 - 1.11) in the sense of Definition 2.1 such that*

$$\varrho_\varepsilon(0,\cdot) \equiv \varrho_{\varepsilon,0} \to \varrho_0 \ \text{in } L^{5/3}(\Omega), \tag{2.26}$$

$$\int_\Omega (\frac{1}{2}\varrho_\varepsilon |\vec{u}_\varepsilon|^2 + \varrho_\varepsilon e(\varrho_\varepsilon, \vartheta_\varepsilon) + E_{R,\varepsilon})(0,\cdot) \, \mathrm{d}x \equiv \int_\Omega \left(\frac{1}{2\varrho_{0,\varepsilon}} |(\varrho\vec{u})_{0,\varepsilon}|^2 + (\varrho e)_{0,\varepsilon} + E_{R,0,\varepsilon} \right) \, \mathrm{d}x \leq E_0, \tag{2.27}$$

$$\int_\Omega [\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon) + s^R(I_\varepsilon)](0,\cdot) \, \mathrm{d}x \equiv \int_\Omega (\varrho s + s^R)_{0,\varepsilon} \, \mathrm{d}x \geq S_0,$$

and

$$0 \leq I_\varepsilon(0,\cdot) \equiv I_{0,\varepsilon}(\cdot) \leq I_0, \ |I_{0,\varepsilon}(\cdot,\nu)| \leq h(\nu) \text{ for a certain } h \in L^1(0,\infty).$$

Then

$$\varrho_\varepsilon \to \varrho \text{ in } C_{\mathrm{weak}}([0,T]; L^{5/3}(\Omega)),$$
$$\vec{u}_\varepsilon \to \vec{u} \text{ weakly in } L^2(0,T; W_0^{1,2}(\Omega; \mathbb{R}^3)),$$
$$\vartheta_\varepsilon \to \vartheta \text{ weakly in } L^2(0,T; W^{1,2}(\Omega)),$$

and

$$I_\varepsilon \to I \text{ weakly-(*) in } L^\infty((0,T) \times \Omega \times \mathcal{S}^2 \times (0,\infty)),$$

at least for suitable subsequences, where $\{\varrho, \vec{u}, \vartheta, I\}$ is a weak solution of problem (1.1 - 1.11).

Sketch of the proof: Using bounds (2.1-2.6), the expression (1.7) of B and (2.18) in the radiative transfer equation one gets a uniform bound for I_ε for any $T \geq 0$

$$0 \leq I_\varepsilon(t,x,\nu,\vec{\omega}) \leq C(T)\Big(1 + \|I_0\|_{L^\infty(\Omega \times \mathcal{S}^2 \times (0,\infty))}\Big),$$

which implies that

$$\|S_E\|_{L^\infty([0,T]\times\Omega)} \leq C(T), \quad \|\vec{S}_F\|_{L^\infty([0,T]\times\Omega;\mathbb{R}^3)} \leq C(T),$$

then it is clear by inspection that the rest of the proof of Theorem 2.1 in [10] applies verbatim to Theorem 2.1. Comparing with [10], just note that positivity of total entropy rate allows us now to relax the low-temperature cut-off used in [10] in the transport coefficients (see Condition 2.11 in [10]) □

3 Low Mach number limit

In order to identify the limit regime we perform a scaling, denoting by

$$L_{ref},\ T_{ref},\ U_{ref}, \rho_{ref},\ \vartheta_{ref},\ p_{ref},\ e_{ref},\ \mu_{ref},\ \kappa_{ref},$$

the reference hydrodynamical quantities (length, time, velocity, density, temperature, pressure, energy, viscosity, conductivity) and by $I_{ref},\ \nu_{ref},\ \sigma_{a,ref},\ \sigma_{s,ref}$, the reference radiative quantities (radiative intensity, frequency, absorption and scattering coefficients). We also assume the compatibility conditions $p_{ref} \equiv \rho_{ref} e_{ref},\ \ \nu_{ref} = \frac{k_B \vartheta_{ref}}{h},\ \ I_{ref} = \frac{2h\nu^3_{ref}}{c^2}$ and we denote by $Sr := \frac{L_{ref}}{T_{ref}U_{ref}},\ Ma := \frac{U_{ref}}{\sqrt{\rho_{ref}p_{ref}}},\ Re := \frac{U_{ref}\rho_{ref}L_{ref}}{\mu_{ref}},\ Pe := \frac{U_{ref}p_{ref}L_{ref}}{\vartheta_{ref}\kappa_{ref}},\ \mathcal{C} := \frac{c}{U_{ref}}$, the Strouhal, Mach, Reynolds, Péclet (dimensionless) and "infrarelativistic" numbers corresponding to hydrodynamics, and by $\mathcal{L} := L_{ref}\sigma_{a,ref},\ \mathcal{L}_s := \frac{\sigma_{s,ref}}{\sigma_{a,ref}},\ \mathcal{P} := \frac{2k_B^4\vartheta^4_{ref}}{h^3c^3\ \rho_{ref}e_{ref}}$, various dimensionless numbers corresponding to radiation. Using these scalings and using carets to symbolize renormalized variables we get

$$\hat{S} = \mathcal{L}\hat{\sigma}_a\left(B(\hat{\nu},\vec{\omega},\hat{\vec{u}},\hat{\vartheta}) - \hat{I}\right) + \mathcal{L}\mathcal{L}_s\hat{\sigma}_s\left(\frac{1}{4\pi}\int_{\mathcal{S}^2}\hat{I}(\cdot,\vec{\omega})\ \mathrm{d}\vec{\omega} - \hat{I}\right).$$

We have also for the non dimensional α

$$\alpha = \frac{\widehat{\sigma}_a + \mathcal{L}_s\widehat{\sigma}_s}{\widehat{\sigma}_a + 2\mathcal{L}_s\widehat{\sigma}_s}. \tag{3.1}$$

Omitting the carets in the following, we get first the scaled equation for I, in the region $(0,T)\times\Omega\times(0,\infty)\times\mathcal{S}^2$

$$\frac{Sr}{\mathcal{C}}\ \partial_t I + \vec{\omega}\cdot\nabla_x I = s = \mathcal{L}\sigma_a\left(B - I\right) + \mathcal{L}\mathcal{L}_s\sigma_s\left(\frac{1}{4\pi}\int_{\mathcal{S}^2} I\ \mathrm{d}\vec{\omega} - I\right), \tag{3.2}$$

where we used the same notation B for the dimensionless Planck function $B(\nu,\vec{\omega},\vec{u},\vartheta) = \dfrac{\nu^3}{e^{\frac{\nu}{\vartheta}(1-\alpha\frac{\vec{\omega}\cdot\vec{u}}{\mathcal{C}})} - 1}$. We denote also by $E^R = \int_{\mathcal{S}^2}\int_0^\infty I\ \mathrm{d}\nu\ \mathrm{d}\vec{\omega}$, the renormalized radiative energy, by $\vec{F}^R = \int_{\mathcal{S}^2}\int_0^\infty \vec{\omega}I\ \mathrm{d}\nu\ \mathrm{d}\vec{\omega}$, the renormalized radiative momentum, by $\mathbb{P}^R = \int_{\mathcal{S}^2}\int_0^\infty \vec{\omega}\otimes\vec{\omega}I\ \mathrm{d}\nu\ \mathrm{d}\vec{\omega}$, the renormalized radiative tensor, by $s_E = \int_{\mathcal{S}^2}\int_0^\infty s\ \mathrm{d}\nu\ \mathrm{d}\vec{\omega}$, the renormalized radiative energy source, by $\vec{s}_F = \int_{\mathcal{S}^2}\int_0^\infty \vec{\omega}I\ \mathrm{d}\nu\ \mathrm{d}\vec{\omega}s$, the renormalized radiative momentum source, by $\vec{s}^R = -\int_0^\infty\int_{\mathcal{S}^2}\nu^2\left[n\log n - (n+1)\log(n+1)\right]\ d\vec{\omega}d\nu$, the renormalized radiative entropy with $n = n(I) = \frac{I}{\alpha^3\nu^3}$, by $\vec{q}^R = -\int_0^\infty\int_{\mathcal{S}^2}\nu^2\left[n\log n - (n+1)\log(n+1)\right]\vec{\omega}\ d\vec{\omega}d\nu$, the renormalized radiative entropy flux.

In order to analyze the low Mach number regime we put $Ma = \varepsilon$, where $\varepsilon > 0$ is small. We also suppose that the flow is strongly under-relativistic so $\mathcal{C} = O(\varepsilon^{-1})$ and that a small amount of radiation is present so $\mathcal{P} = \varepsilon$. Finally we put $Sr = 1$, $Pe = 1$, $Re = 1$, $\mathcal{L} = \mathcal{L}_s = 1$ in the previous system.

Taking the first moment of (3.2) with respect to $\vec{\omega}$, we get first equations for E^R and $\vec{S}_F$

$$\varepsilon\partial_t E^R + \mathrm{div}_x\vec{F}^R = s_E, \tag{3.3}$$

$$\varepsilon\partial_t\vec{F}_R + \mathrm{div}_x\mathbb{P}^R = \vec{s}_F, \tag{3.4}$$

then the scaled system reads

$$\varepsilon\partial_t I + \vec{\omega}\cdot\nabla_x I = \sigma_a\left(B - I\right) + \sigma_s\left(\frac{1}{4\pi}\int_{\mathcal{S}^2} I\ \mathrm{d}\vec{\omega} - I\right), \tag{3.5}$$

$$\partial_t\varrho + \mathrm{div}_x(\varrho\vec{u}) = 0, \tag{3.6}$$

$$\partial_t\left(\varrho\vec{u} + \varepsilon^2\vec{F}^R\right) + \mathrm{div}_x\left(\varrho\vec{u}\otimes\vec{u} + \varepsilon\mathbb{P}^R\right) + \frac{1}{\varepsilon^2}\ \nabla_x p(\varrho,\vartheta) - \ \mathrm{div}_x\mathbb{S} = 0. \tag{3.7}$$

$$\partial_t \left(\frac{\varepsilon^2}{2}\ \varrho|\vec{u}|^2 + \varrho e + \varepsilon E^R \right) + \mathrm{div}_x \left(\left(\frac{\varepsilon^2}{2}\ \varrho|\vec{u}|^2 + \varrho e + p \right) \vec{u} + \vec{F}^R + \vec{q} - \varepsilon^2 \mathbb{S}\vec{u} \right) = 0, \tag{3.8}$$

$$\partial_t \left(\varrho s + \varepsilon s^R \right) + \mathrm{div}_x \left(\varrho s \vec{u} + \vec{q}^R \right) + \ \mathrm{div}_x \left(\frac{\vec{q}}{\vartheta} \right) \geq \frac{1}{\vartheta} \left(\varepsilon^2 \mathbb{S} : \nabla_x \vec{u} - \frac{\vec{q} \cdot \nabla_x \vartheta}{\vartheta} \right)$$
$$+ \int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu} \left[\log \frac{n(I)}{n(I)+1} - \log \frac{n(B)}{n(B)+1} \right] \sigma_a (I - B)\ d\vec{\omega} d\nu$$
$$+ \int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu} \left[\log \frac{n(I)}{n(I)+1} - \log \frac{n(\tilde{I})}{n(\tilde{I})+1} \right] \sigma_s (I - \tilde{I})\ d\vec{\omega} d\nu =: \varsigma_\varepsilon, \tag{3.9}$$

with the conservation of total energy

$$\frac{d}{dt} \int_\Omega \left(\frac{\varepsilon^2}{2}\ \varrho|\vec{u}|^2 + \varrho e + \varepsilon E^R \right)\ dx + \int_0^\infty \int_{\Gamma_+} \vec{\omega} \cdot \vec{n} I\ d\Gamma_+ d\nu = 0, \tag{3.10}$$

where

$$\Gamma_+ \equiv \{\{x, \omega\} \in \partial\Omega \times S^2,\ \vec{\omega} \cdot \vec{n} \geq 0\}. \tag{3.11}$$

In order to compute the limit system, we consider now the formal expansions

$$\begin{cases} I = I_0 + \varepsilon I_1 + O(\varepsilon^2), \\ \varrho = \varrho_0 + \varepsilon \varrho_1 + O(\varepsilon^2), \\ \vec{u} = \vec{u}_0 + \varepsilon \vec{u}_1 + O(\varepsilon^2), \\ \vartheta = \vartheta_0 + \varepsilon \vartheta_1 + O(\varepsilon^2). \end{cases} \tag{3.12}$$

We first note from (3.7) that $\nabla p = O(\varepsilon^2)$ which leads, using the arguments of [14], to $\nabla p_0 = \nabla p_1 = 0$ and we get

$$\varrho_0 = Cte, \quad \vartheta_0 = Cte \quad \text{and} \quad \partial_\vartheta p(\varrho_0, \vartheta_0)\vartheta_1 + \partial_\varrho p(\varrho_0, \vartheta_0)\varrho_1 = Cte. \tag{3.13}$$

From (3.6) follows the incompressibility condition

$$\mathrm{div}_x \vec{u}_0 = 0, \tag{3.14}$$

and

$$\partial_t \varrho_1 + \mathrm{div}_x \left(\varrho_0 \vec{u}_1 + \varrho_1 \vec{u}_0 \right) = 0. \tag{3.15}$$

Now observing that $\alpha = \frac{\sigma_a(\vartheta_0)+\sigma_s(\vartheta_0)}{\sigma_a(\vartheta_0)+2\sigma_s(\vartheta_0)} + 0(\varepsilon)$ we have

$$B(\nu, \vec{\omega}, \vec{u}, \vartheta) = B_0(\nu, \vartheta_0) + \left(\alpha \vec{\omega} \cdot \vec{u}_0 \vartheta_0 + \vartheta_1 \right) \partial_\vartheta B_0(\nu, \vartheta_0) \varepsilon + O(\varepsilon^2),$$

where $B_0(\nu, \vartheta_0) = B(\nu, \vec{\omega}, \vec{u}_0, \vartheta_0)|_{\alpha=0}$ so we get from (3.5) two linear steady-state transport equations for the first two moments I_0 and I_1

$$\vec{\omega} \cdot \nabla_x I_0 = \sigma_a(\vartheta_0) \left(B_0(\nu, \vartheta_0) - I_0 \right) + \sigma_s(\vartheta_0) \left(\frac{1}{4\pi} \int_{\mathcal{S}^2} I_0\ \mathrm{d}\vec{\omega} - I_0 \right), \tag{3.16}$$

and

$$\vec{\omega} \cdot \nabla_x I_1 = \sigma_a(\vartheta_0) \Big[\left(\alpha_0 \vec{\omega} \cdot \vec{u}_0 \vartheta_0 + \vartheta_1 \right) \partial_\vartheta B_0(\nu, \vartheta_0) - I_1 \Big] + \partial_\vartheta \sigma_a(\vartheta_0) \left(B_0(\nu, \vartheta_0) - I_0 \right) \vartheta_1$$
$$+ \partial_\vartheta \sigma_s(\vartheta_0) \left(\frac{1}{4\pi} \int_{\mathcal{S}^2} I_0\ \mathrm{d}\vec{\omega} - I_0 \right) \vartheta_1 + \sigma_s(\vartheta_0) \left(\frac{1}{4\pi} \int_{\mathcal{S}^2} I_1\ \mathrm{d}\vec{\omega} - I_1 \right), \tag{3.17}$$

and the limit momentum equation is

$$\varrho_0 \Big(\partial_t \vec{u}_0 + \mathrm{div}_x (\vec{u}_0 \otimes \vec{u}_0) \Big) + \nabla_x \Pi - \ \mathrm{div}_x \Big(\mu_0 \left(\nabla_x \vec{u}_0 + \nabla_x^t \vec{u}_0 \right) \Big) = 0, \tag{3.18}$$

where $\mu_0 = \mu(\vartheta_0)$ and the gradient term contains higher correctors.

At first order, the energy equation

$$\partial_t \left(\frac{\varepsilon^2}{2}\ \varrho|\vec{u}|^2 + \varrho e \right) + \mathrm{div}_x \left(\left(\frac{\varepsilon^2}{2}\ \varrho|\vec{u}|^2 + \varrho e + p \right) \vec{u} + \vec{q} - \varepsilon^2 \mathbb{S}\vec{u} \right) = S_E,$$

gives

$$\varrho_0 \partial_\vartheta e(\varrho_0, \vartheta_0)\, \partial_t \vartheta_1 + \Big(\varrho_0 \partial_\varrho e(\varrho_0, \vartheta_0) + e_0(\varrho_0, \vartheta_0) \Big) \partial_t \varrho_1 + \mathrm{div}_x \Big([\varrho_0 \partial_\vartheta e(\varrho_0, \vartheta_0)\vartheta_1 + (\varrho_0 \partial_\varrho e(\varrho_0, \vartheta_0) + e_0(\varrho_0, \vartheta_0))\, \varrho_1)\, \vec{u}_0 \Big)$$

$$+ \mathrm{div}_x \Big(\varrho_0 e_0(\varrho_0, \vartheta_0) \vec{u}_1 - \kappa(\varrho_0, \vartheta_0) \nabla_x \vartheta_1 \Big) = S_{E1}. \tag{3.19}$$

After (3.17), the right hand side rewrites

$$S_{E1} = \int_0^\infty \int_{\mathcal{S}^2} \{ \partial_\vartheta \sigma_a(\vartheta_0) \left(B(\nu, \vartheta_0) - I_0 \right) \vartheta_1 + \sigma_a(\vartheta_0) \left((\alpha_0 \vec{\omega} \cdot \vec{u}_0 \vartheta_0 + \vartheta_1)\, \partial_\vartheta B(\nu, \vartheta_0) - I_1 \right) \} \, d\vec{\omega}\ d\nu$$

$$= \int_0^\infty \int_{\mathcal{S}^2} \Big\{ \partial_\vartheta \sigma_a(\vartheta_0) \Big(B(\nu, \vartheta_0) - I_0 \Big) \vartheta_1 + \sigma_a(\vartheta_0) \Big(\partial_\vartheta B(\nu, \vartheta_0) \vartheta_1 - I_1 \Big) \Big\} d\vec{\omega}\ d\nu.$$

From (3.15) we get $p(\varrho_0, \vartheta_0)\vec{u}_1 = -\frac{p(\varrho_0, \vartheta_0)}{\varrho_0} \left(\partial_t \varrho_1 + \mathrm{div}_x \varrho_1 \vec{u}_0 \right)$, and from (3.13) and (3.14) we have $\partial_t \varrho_1 + \mathrm{div}_x \varrho_1 \vec{u}_0 = -\frac{\partial_\vartheta p(\varrho_0, \vartheta_0)}{\partial_\varrho p(\varrho_0, \vartheta_0)} \left(\partial_t \vartheta_1 + \mathrm{div}_x \vartheta_1 \vec{u}_0 \right)$, so plugging these identities into (3.19), we end with

$$\varrho_0 \left(\partial_\vartheta e(\varrho_0, \vartheta_0) - \frac{\partial_\vartheta p(\varrho_0, \vartheta_0)}{\partial_\varrho p(\varrho_0, \vartheta_0)} \left(\partial_\varrho e(\varrho_0, \vartheta_0) - \frac{p_0(\varrho_0, \vartheta_0)}{\varrho_0^2} \right) \right) \Big(\partial_t \vartheta_1 - \mathrm{div}_x(\vartheta_1 \vec{u}_0) \Big)$$

$$-\mathrm{div}_x \Big(\kappa(\varrho_0, \vartheta_0) \nabla_x \vartheta_1 \Big) = \int_0^\infty \int_{\mathcal{S}^2} \Big\{ \partial_\vartheta \sigma_a(\vartheta_0) \Big(B(\nu, \vartheta_0) - I_0 \Big) \vartheta_1 + \sigma_a(\vartheta_0) \Big(\partial_\vartheta B(\nu, \vartheta_0) \vartheta_1 - I_1 \Big) \Big\} d\vec{\omega}\ d\nu.$$

Putting $\vec{U} = \vec{u}_0$, $\Theta = \vartheta_1$, $\overline{\varrho} = \varrho_0$, $\overline{\vartheta} = \vartheta_0$, $\overline{\mu} = \mu(\vartheta_0)$, $\mathbb{D}(\vec{U}) = \frac{1}{2} \left(\nabla \vec{U} + \nabla^T \vec{U} \right)$, $\overline{\kappa} = \kappa(\varrho_0, \vartheta_0)$, $\overline{c_P} = \partial_\vartheta e(\varrho_0, \vartheta_0) - \frac{\partial_\vartheta p(\varrho_0, \vartheta_0)}{\partial_\varrho p(\varrho_0, \vartheta_0)} \left(\partial_\varrho e(\varrho_0, \vartheta_0) - \frac{p_0(\varrho_0, \vartheta_0)}{\varrho_0^2} \right)$ we obtain the limit system in $(0, T) \times \Omega$

$$\mathrm{div}_x \vec{U} = 0, \tag{3.20}$$

$$\overline{\varrho} \left(\partial_t \vec{U} + \mathrm{div}_x (\vec{U} \otimes \vec{U}) \right) + \nabla_x \Pi - 2 \mathrm{div}_x \left(\overline{\mu}\ \mathbb{D}(\vec{U}) \right) = 0, \tag{3.21}$$

$$\overline{\varrho}\ \overline{c_P} \left(\partial_t \Theta + \mathrm{div}_x \left(\Theta \vec{U} \right) \right) - \mathrm{div}_x \left(\overline{\kappa} \nabla \Theta \right)$$

$$= \left\{ \int_0^\infty \partial_\vartheta \left(\sigma_a(\overline{\vartheta}) B(\nu, \overline{\vartheta}) + \sigma_a(\overline{\vartheta}) \partial_\vartheta B(\nu, \overline{\vartheta}) \right)\ d\nu + \int_0^\infty \int_{\mathcal{S}^2} \partial_\vartheta \sigma_a(\overline{\vartheta}) I_0\ d\vec{\omega}\ d\nu \right\} \Theta - \int_0^\infty \int_{\mathcal{S}^2} \sigma_a(\overline{\vartheta}) I_1\ d\vec{\omega}\ d\nu, \tag{3.22}$$

$$\vec{\omega} \cdot \nabla_x I_0 = \sigma_a(\overline{\vartheta}) \left(B(\nu, \overline{\vartheta}) - I_0 \right) + \sigma_s(\overline{\vartheta}) \left(\frac{1}{4\pi} \int_{\mathcal{S}^2} I_0\ \mathrm{d}\vec{\omega} - I_0 \right), \tag{3.23}$$

$$\vec{\omega} \cdot \nabla_x I_1 = \left\{ \sigma_a(\overline{\vartheta}) \partial_\vartheta B(\nu, \overline{\vartheta}) + \partial_\vartheta \sigma_a \overline{\vartheta}) B(\nu, \overline{\vartheta}) + \partial_\vartheta \sigma_s(\overline{\vartheta}) \left(\frac{1}{4\pi} \int_{\mathcal{S}^2} I_0(x, \nu, \vec{\omega}')\ \mathrm{d}\vec{\omega}' - I_0(x, \nu, \vec{\omega}) \right) \right\} \Theta$$

$$-\sigma_a(\overline{\vartheta}) I_1(x, \nu, \vec{\omega}) + \sigma_s(\overline{\vartheta}) \left(\frac{1}{4\pi} \int_{\mathcal{S}^2} I_1(x, \nu, \vec{\omega}')\ \mathrm{d}\vec{\omega}' - I_1(x, \nu, \vec{\omega}) \right). \tag{3.24}$$

We finally consider the boundary conditions

$$\vec{U}|_{\partial\Omega} = 0,\ \nabla \Theta \cdot \vec{n}|_{\partial\Omega} = 0, \tag{3.25}$$

for (3.20)-(3.22) and

$$I_0(x,\nu,\vec{\omega}) = 0 \text{ for } x \in \partial\Omega,\ \vec{\omega}\cdot\vec{n} \le 0 \tag{3.26}$$

$$I_1(x,\nu,\vec{\omega}) = 0 \text{ for } x \in \partial\Omega,\ \vec{\omega}\cdot\vec{n} \le 0 \tag{3.27}$$

for (3.23) and (3.24), and the initial conditions

$$\vec{U}|_{t=0} = \vec{U}_0, \quad \Theta|_{t=0} = \Theta_0, \tag{3.28}$$

and one has the following result (see [12] for a short proof)

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain.*

For any $T > 0$ the initial-bounday value problem (3.20) - (3.28) has at least a weak solution $(\vec{U}, \Theta, I_0, I_1)$ such that

$$\vec{U} \in L^\infty(0,T;\mathcal{H}(\Omega)) \cap L^2(0,T;\mathcal{V}(\Omega)),$$

with $\mathcal{H}(\Omega) = \{\vec{U} \in L^2(\Omega;\mathbb{R}^3),\ \mathrm{div}_x\vec{U} = 0 \text{ in } \Omega,\ \vec{U}\Big|_{\partial\Omega} = 0\}$, *and* $\mathcal{V}(\Omega) = \mathcal{H}(\Omega) \cap W_0^{1,2}(\Omega;\mathbb{R}^3))$,

$$\Theta \in V_2^{1,1/2}((0,T)\times\Omega),$$

where $V_2^{1,1/2}$ is the energy space defined in [20] p.6,

$$I_0, I_1 \in L^\infty((0,T)\times\Omega)\times\mathcal{S}^2\times\mathbb{R}_+),$$

with

$$\vec{\omega}\cdot\nabla_x I_0,\ \vec{\omega}\cdot\nabla_x I_0 \in L^p((0,T)\times\Omega)\times\mathcal{S}^2\times\mathbb{R}_+),$$

for any $p > 1$.

In the following we consider the convergence from the radiative Navier-Stokes-Fourier system (1.1)-(1.11) to the incompressible limit system (3.20)-(3.28).

3.1 Global existence for the primitive system and uniform estimates

Let us choose initial data such that

$$\begin{cases} \varrho(0,\cdot) = \varrho_{0,\varepsilon} = \overline{\varrho} + \varepsilon\varrho^{(1)}_{0,\varepsilon}, \\ \vec{u}(0,\cdot) = \vec{u}_{0,\varepsilon}, \\ \vartheta(0,\cdot) = \vartheta_{0,\varepsilon} = \overline{\vartheta} + \varepsilon\vartheta^{(1)}_{0,\varepsilon}, \\ I(0,\cdot,\cdot,\cdot) = I_{0,\varepsilon} = \overline{I} + \varepsilon I^{(1)}_{0,\varepsilon}, \end{cases} \tag{3.29}$$

where $\overline{\varrho} > 0$, $\overline{\vartheta} > 0$, $\overline{I} > 0$ and $\int_\Omega \varrho^{(1)}_{0,\varepsilon}\ dx = 0$ for any $\varepsilon > 0$. Recall that after [14] for any locally compact Hausdorff metric space X, $\mathcal{M}(X)$ is the set of signed Borel measures on X and $\mathcal{M}^+(X)$ is the cone of non-negative elements of $\mathcal{M}(X)$. Then we rephraze Theorem 2.1 in the scaled framework as follows

Theorem 3.2. *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Assume that the thermodynamic functions p, e, s satisfy hypotheses (2.1-2.6), and that the transport coefficients μ, η, κ, σ_a, σ_s and the equilibrium function B comply with (2.7 - 2.10). Let the initial data $(\varrho_{0,\varepsilon}, \vec{u}_{0,\varepsilon}, \vartheta_{0,\varepsilon}, I_{0,\varepsilon})$ be given by (3.29), where $(\varrho^{(1)}_{0,\varepsilon}, \vec{u}^{(1)}_{0,\varepsilon}, \vartheta^{(1)}_{0,\varepsilon}, I^{(1)}_{0,\varepsilon})$ are bounded measurable functions.*

Then for any $\varepsilon > 0$ small enough there exists a weak solution $(\varrho_\varepsilon, \vec{u}_\varepsilon, \vartheta_\varepsilon, I_\varepsilon)$ to the radiative Navier-Stokes system (1.1 - 1.6) for $(t,x,\vec{\omega},\nu) \in (0,T)\times\Omega\times\mathcal{S}^2\times\mathbb{R}_+$, supplemented with the boundary conditions (1.10 - 1.11) and the initial conditions (3.29) such that

$$\int_0^T \int_\Omega \varrho_\varepsilon b(\varrho_\varepsilon)\left(\partial_t\phi + \vec{u}_\varepsilon\cdot\nabla_x\phi\right)\ dx\ dt = \int_0^T \int_\Omega \beta(\varrho_\varepsilon)\mathrm{div}_x u_\varepsilon\ \phi\ dx\ dt - \int_\Omega \varrho_{0,\varepsilon} b(\varrho_{0,\varepsilon})\ \phi(0,\cdot)\ dx, \tag{3.30}$$

for any β such that $\beta \in L^\infty \cap C[0,\infty)$, $b(\varrho) = b(1) + \int_1^\varrho \frac{\beta(z)}{z^2}\, dz$ and any $\phi \in C_c^\infty([0,T)\times\overline{\Omega})$,

$$\int_0^T \int_\Omega \left(\left(\varrho_\varepsilon \vec{u}_\varepsilon + \varepsilon^2 \vec{F}^R \right) \cdot \partial_t \vec{\phi} + \left(\varrho_\varepsilon \vec{u}_\varepsilon \otimes \vec{u}_\varepsilon + \varepsilon \mathbb{P}^R \right) : \nabla_x \vec{\phi} + \frac{p_\varepsilon}{\varepsilon^2}\, \mathrm{div}_x \vec{\phi} \right)\, dx\, dt$$

$$= \int_0^T \int_\Omega \mathbb{S}_\varepsilon : \nabla_x \vec{\phi}\, dx\, dt - \int_\Omega \varrho_{0,\varepsilon} \vec{u}_{0,\varepsilon} \cdot \vec{\phi}(0,\cdot)\, dx, \tag{3.31}$$

for any $\vec{\phi} \in C_c^\infty([0,T)\times\overline{\Omega};\mathbb{R}^3)$ such that $\vec{\phi}\cdot\vec{n}_x\Big|_{\partial\Omega} = 0$, with $p_\varepsilon = p(\varrho_\varepsilon,\vartheta_\varepsilon)$ and $\mathbb{S}_\varepsilon = \mathbb{S}(\vec{u}_\varepsilon,\vartheta_\varepsilon)$,

$$\int_\Omega \left(\frac{\varepsilon^2}{2}\, \varrho_\varepsilon |\vec{u}_\varepsilon|^2 + \varrho_\varepsilon e_\varepsilon + \varepsilon E_\varepsilon^R \right)\, dx\, dt + \int_0^T \int_0^\infty \int_{\Gamma_+} \vec{\omega}\cdot\vec{n}_x I_\varepsilon(t,x,\vec{\omega},\nu)\, d\Gamma\, d\nu\, dt$$

$$= \int_\Omega \left(\frac{\varepsilon^2}{2}\, \varrho_{0,\varepsilon} |\vec{u}_{0,\varepsilon}|^2 + \varrho_{0,\varepsilon} e_{0,\varepsilon} + \varepsilon E_{0,\varepsilon}^R \right)\, dx, \tag{3.32}$$

for a.a. $t \in (0,T)$ with $\Gamma_+ = \{(x,\vec{\omega}) \in \partial\Omega \times \mathcal{S}^2 \ : \ \vec{\omega}\cdot\vec{n}_x \geq 0\}$ and with $e_\varepsilon = e(\varrho_\varepsilon,\vartheta_\varepsilon)$ and $E_\varepsilon^R(t,x) = \int_0^\infty \int_{\mathcal{S}^2} I_\varepsilon(t,x,\vec{\omega},\nu)\, d\vec{\omega}\, d\nu$

$$\int_0^T \int_\Omega \left(\left(\varrho_\varepsilon s_\varepsilon + \varepsilon s_\varepsilon^R \right) \partial_t \varphi + \left(\varrho_\varepsilon s_\varepsilon \vec{u}_\varepsilon + \vec{q}_\varepsilon^R \right) \cdot \nabla_x \varphi \right)\, dx\, dt + \int_0^T \int_\Omega \frac{\vec{q}_\varepsilon}{\vartheta_\varepsilon} \cdot \nabla_x \varphi\, dx\, dt$$

$$+ \left\langle \varsigma_\varepsilon^m + \varsigma_\varepsilon^R ; \varphi \right\rangle_{[\mathcal{M};C]([0,T)\times\overline{\Omega})} = - \int_\Omega \left((\varrho s_{0,\varepsilon} + s_{0,\varepsilon}^R) \varphi(0,\cdot) \right)\, dx, \tag{3.33}$$

where

$$\varsigma_\varepsilon^m \geq \frac{1}{\vartheta_\varepsilon} \left(\mathbb{S}_\varepsilon : \nabla_x \vec{u}_\varepsilon - \frac{\vec{q}_\varepsilon \cdot \nabla_x \vartheta_\varepsilon}{\vartheta_\varepsilon} \right),$$

and

$$\varsigma_\varepsilon^R \geq \frac{k}{h} \int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu} \left(\log \frac{n(I_\varepsilon)}{n(I_\varepsilon)+1} - \log \frac{n(B_\varepsilon)}{n(B_\varepsilon)+1} \right) \sigma_{a\varepsilon} (B_\varepsilon - I_\varepsilon) d\vec{\omega} d\nu$$
$$+ \frac{k}{h} \int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu} \left(\log \frac{n(I_\varepsilon)}{n(I_\varepsilon)+1} - \log \frac{n(\tilde{I}_\varepsilon)}{n(\tilde{I}_\varepsilon)+1} \right) \sigma_{s\varepsilon} (\tilde{I}_\varepsilon - I_\varepsilon) d\vec{\omega} d\nu,$$

for any $\varphi \in C_c^\infty([0,T)\times\overline{\Omega})$ with $\varsigma_\varepsilon^m \in \mathcal{M}^+([0,T)\times\overline{\Omega})$ and $\varsigma_\varepsilon^R \in \mathcal{M}^+([0,T)\times\overline{\Omega})$, and with $\sigma_{a\varepsilon} = \sigma_a(\nu,\vartheta_\varepsilon)$, $\sigma_{s\varepsilon} = \sigma_s(\nu,\vartheta_\varepsilon)$, $B_\varepsilon = B(\nu,\vartheta_\varepsilon)$, $\vec{q}_\varepsilon = \kappa(\varrho_\varepsilon,\vartheta_\varepsilon)\nabla_x\vartheta_\varepsilon$, , $s_\varepsilon = s(\varrho_\varepsilon,\vartheta_\varepsilon)$, $s_\varepsilon^R = s^R(I_\varepsilon)$, $\vec{q}_\varepsilon^R = \vec{q}^R(I_\varepsilon)$ and $\tilde{I}_\varepsilon := \frac{1}{4\pi}\int_{\mathcal{S}^2} I_\varepsilon(t,x,\nu,\vec{\omega})\, \mathrm{d}\vec{\omega}$,

$$\int_0^T \int_\Omega \int_0^\infty \int_{\mathcal{S}^2} \left(\varepsilon \partial_t \psi + \vec{\omega}\cdot\nabla_x \psi \right) I_\varepsilon\, d\vec{\omega}\, d\nu\, dx\, dt + \int_0^T \int_\Omega \int_0^\infty \int_{\mathcal{S}^2} \left[\sigma_{a\varepsilon} \left(B_\varepsilon - I_\varepsilon \right) + \sigma_{s\varepsilon} \left(\tilde{I}_\varepsilon - I_\varepsilon \right) \right] \psi\, d\vec{\omega}\, d\nu\, dx\, dt,$$

$$= \int_\Omega \int_0^\infty \int_{\mathcal{S}^2} \varepsilon I_{0,\varepsilon} \psi(0,x,\vec{\omega},\nu)\, d\vec{\omega}\, d\nu\, dx + \int_0^T \int_{\Gamma_+} \int_0^\infty \vec{\omega}\cdot\vec{n}_x I_\varepsilon \psi\, d\Gamma\, d\nu\, dt, \tag{3.34}$$

for any $\psi \in C_c^\infty([0,T)\times\overline{\Omega}\times\mathcal{S}^2\times\mathbb{R}_+)$.

3.2 Uniform estimates

We adapt from [14] the necessary definitions to the formalism of essential and residual sets. Given three numbers $\overline{\varrho} \in \mathbb{R}_+$, $\overline{\vartheta} \in \mathbb{R}_+$ and $\overline{E} \in \mathbb{R}_+$ we define $\mathcal{O}_{ess}^H$ the set of hydrodynamical essential values

$$\mathcal{O}_{ess}^H := \left\{ (\varrho,\vartheta) \in \mathbb{R}^2 \ : \ \frac{\overline{\varrho}}{2} < \varrho < 2\overline{\varrho},\ \frac{\overline{\vartheta}}{2} < \vartheta < 2\overline{\vartheta} \right\}, \tag{3.35}$$

and $\mathcal{O}^R_{ess}$ the set of radiative essential values

$$\mathcal{O}^R_{ess} := \left\{ E^R \in \mathbb{R} \ : \ \frac{\overline{E}}{2} < E^R < 2\overline{E} \right\}, \tag{3.36}$$

with $\mathcal{O}_{ess} := \mathcal{O}^H_{ess} \cup \mathcal{O}^R_{ess}$, and their residual counterparts

$$\mathcal{O}^H_{res} := (\mathbb{R}_+)^2 \backslash \mathcal{O}^H_{ess}, \quad \mathcal{O}^R_{res} := \mathbb{R}_+ \backslash \mathcal{O}^R_{ess}, \quad \mathcal{O}_{res} := (\mathbb{R}_+)^3 \backslash \mathcal{O}_{ess}. \tag{3.37}$$

Let $\{\varrho_\varepsilon, \vec{u}_\varepsilon, \vartheta_\varepsilon, I_\varepsilon\}_{\varepsilon>0})$ be a family of solutions of the scaled radiative Navier-Stokes system given in Theorem 3.2. We call $\mathcal{M}^\varepsilon_{ess} \subset (0,T) \times \Omega$ the set

$$\mathcal{M}^\varepsilon_{ess} = \left\{ (t,x) \in (0,T) \times \Omega \ : \ (\varrho_\varepsilon(t,x), \vartheta_\varepsilon(t,x), E^R_\varepsilon(t,x)) \in \mathcal{O}_{ess} \right\},$$

and $\mathcal{M}^\varepsilon_{res} = (0,T) \times \Omega \backslash \mathcal{M}^\varepsilon_{ess}$ the corresponding residual set.

To any measurable function h we decompose it into essential and residual parts $h = [h]_{ess} + [h]_{res}$ where $[h]_{ess} = h \cdot \mathbb{I}_{\mathcal{M}^\varepsilon_{ess}}$ and $[h]_{res} = h \cdot \mathbb{I}_{\mathcal{M}^\varepsilon_{res}}$. As in the low Mach number we get, after Lemma 3.1)

Defining now $H_{\overline{\vartheta}} = \varrho e - \overline{\vartheta}\ \varrho s$ as the Helmholtz function for matter and $H^R{}_{\overline{\vartheta}}(I) = E^R - \overline{\vartheta}\ s^R$, as the corresponding radiative function and using (3.33) we rewrite (3.32) as

$$\int_\Omega \left(\frac{\varepsilon^2}{2}\ \varrho_\varepsilon |\vec{u}_\varepsilon|^2 + H_{\overline{\vartheta}}(\varrho_\varepsilon, \vartheta_\varepsilon) + \varepsilon H^R{}_{\overline{\vartheta}}(I_\varepsilon) \right)\ dx + \int_0^T \int_{\Gamma_+} \vec{\omega} \cdot \vec{n}_x I_\varepsilon(t,x,\vec{\omega},\nu)\, d\Gamma\, d\nu\, dt + \overline{\vartheta} \left(\varsigma^m_\varepsilon + \varsigma^R_\varepsilon \right) \left[[0,t] \times \overline{\Omega} \right]$$

$$= \int_\Omega \left(\frac{\varepsilon^2}{2}\ \varrho_{0,\varepsilon} |\vec{u}_{0,\varepsilon}|^2 + \varrho_{0,\varepsilon} e_{0,\varepsilon} + \varepsilon E^R_{0,\varepsilon} \right)\ dx.$$

Observing that the total mass is a constant of motion $M = \int_\Omega \varrho_\varepsilon\ dx = \overline{\varrho}|\Omega|$, we deduce finally that

$$\int_\Omega \left(\frac{1}{2}\ \varrho_\varepsilon |\vec{u}_\varepsilon|^2 + \frac{1}{\varepsilon^2} \left(H_{\overline{\vartheta}}(\varrho_\varepsilon, \vartheta_\varepsilon) - (\varrho_\varepsilon - \overline{\varrho}) \partial_\varrho H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) - H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) \right) + \frac{1}{\varepsilon}\ H^R{}_{\overline{\vartheta}}(I_\varepsilon) \right)\ dx$$

$$+ \frac{1}{\varepsilon^2} \int_0^T \int_{\Gamma_+} \vec{\omega} \cdot \vec{n}_x I_\varepsilon(t,x,\vec{\omega},\nu)\ d\Gamma\ d\nu\ dt + \frac{1}{\varepsilon^2}\ \overline{\vartheta} \left(\varsigma^m_\varepsilon + \varsigma^R_\varepsilon \right) \left[[0,t] \times \overline{\Omega} \right]$$

$$= \int_\Omega \left(\frac{1}{2}\ \varrho_{0,\varepsilon} |\vec{u}_{0,\varepsilon}|^2 + \frac{1}{\varepsilon^2} \left(H_{\overline{\vartheta}}(\varrho_{0,\varepsilon}, \vartheta_{0,\varepsilon}) - (\varrho_{0,\varepsilon} - \overline{\varrho}) \partial_\varrho H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) - H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) \right) \right.$$

$$\left. + \frac{1}{\varepsilon}\ H^R{}_{\overline{\vartheta}}(I_{0,\varepsilon}) \right)\ dx. \tag{3.38}$$

Now, after [14], we have the following properties for matter and radiative Helmholtz functions

Lemma 3.1. *Let $\overline{\varrho} > 0$ and $\overline{\vartheta} > 0$ be two given constants and let $H_{\overline{\vartheta}}(\varrho, \vartheta) = \varrho e - \overline{\vartheta}\ \varrho s$, and $H^R{}_{\overline{\vartheta}}(I) = E^R - \overline{\vartheta}\ s^R$.*

There exist positive constants $C_j = C_j(\overline{\varrho}, \overline{\vartheta})$ for $j = 1, \cdots, 8$ such that

$$C_1 \left(|\varrho - \overline{\varrho}|^2 + |\vartheta - \overline{\vartheta}|^2 \right) \leq H_{\overline{\vartheta}}(\varrho, \vartheta) - (\varrho - \overline{\varrho}) \partial_\varrho H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) - H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta})$$

$$\leq C_2 \left(|\varrho - \overline{\varrho}|^2 + |\vartheta - \overline{\vartheta}|^2 \right), \tag{3.39}$$

for all $(\varrho, \vartheta) \in \mathcal{O}^H_{ess}$,

$$H_{\overline{\vartheta}}(\varrho, \vartheta) - (\varrho - \overline{\varrho}) \partial_\varrho H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) - H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta})$$

$$\geq \inf_{\tilde{\varrho}, \tilde{\vartheta} \in \mathcal{O}_{res}} \left\{ H_{\overline{\vartheta}}(\tilde{\varrho}, \tilde{\vartheta}) - (\tilde{\varrho} - \overline{\varrho}) \partial_\varrho H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) - H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) \right\} = C_3, \tag{3.40}$$

for all $(\varrho, \vartheta) \in \mathcal{O}^H_{res}$,

$$H_{\overline{\vartheta}}(\varrho, \vartheta) - (\varrho - \overline{\varrho})\partial_\varrho H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) - H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) \geq C_4 \left(\varrho e(\varrho, \vartheta) + \varrho |s(\varrho, \vartheta)|\right), \tag{3.41}$$

for all $(\varrho, \vartheta) \in \mathcal{O}^H_{res}$,

$$C_5 \int_0^\infty \int_{\mathcal{S}^2} |I_\varepsilon - B(\nu, \vartheta)|^2 d\vec{\omega}\ d\nu \leq H^R(I_\varepsilon) \leq C_6 \int_0^\infty \int_{\mathcal{S}^2} |I_\varepsilon - B(\nu, \vartheta)|^2 d\vec{\omega}\ d\nu, \tag{3.42}$$

for all $I_\varepsilon \in \mathcal{O}^R_{ess}$,

$$E^R(I) - \overline{\vartheta}\ s^R(I) \geq \inf_{\tilde{I} \in \mathcal{O}_{res}} E^R(\tilde{I}) - \overline{\vartheta}\ s^R(\tilde{I}) = C_7, \tag{3.43}$$

for all $E \in \mathcal{O}^R_{res}$,

$$E^R(I) - \overline{\vartheta}\ s^R(I) \geq C_8 \left(E^R(I) + |s^R(I)|\right) \tag{3.44}$$

for all $E \in \mathcal{O}^R_{res}$

Proof: The points 1 to 3 are proved in [14] and we have $E^R(I) - \overline{\vartheta}\ s^R(I) = \int_0^\infty \int_{\mathcal{S}^2} \psi\ d\vec{\omega}\ d\nu$, where $\psi(t, x, \vec{\omega}, \nu; I) = I + \nu^2 \overline{\vartheta} \left(n(I) \log n(I) - (n(I)+1)\log(n(I)+1)\right)$, with $n(I) = \frac{I}{\nu^3}$. Computing $\partial_I \psi = 1 + \frac{\overline{\vartheta}}{\nu}\ \log \frac{n(I)}{n(I)+1}$ and $\partial^2_{II}\psi = \frac{\overline{\vartheta}}{\nu^4}\ \frac{I}{n(n+1)}$, observing that $\partial_I \psi|_{I = B(\nu, \overline{\vartheta})} = 0$ and that $\partial^2_{II}\psi\big|_{I = B(\nu, \overline{\vartheta})} > 0$ and applying Taylor formula with $\overline{I} = B(\nu, \overline{\vartheta})$, we get (3.42). As $I \to \psi(t, x, \vec{\omega}, \nu; I)$ is decreasing for $I < \overline{I}$ and is increasing for $I > \overline{I}$, we get (3.43). Moreover the convexity of ψ implies (3.44) $\square$

From Definition (3.38), we obtain the following energy estimates

Lemma 3.2. *Suppose that the initial data satisfy*

$$\|[\varrho_{0,\varepsilon} - \overline{\varrho}]\|^2_{L^2(\Omega)} \leq C\varepsilon^2,\ \|[\vartheta_{0,\varepsilon} - \overline{\vartheta}]\|^2_{L^2(\Omega)} \leq C\varepsilon^2,\ \|E^R_{0,\varepsilon} - \overline{E}\|^2_{L^2(\Omega)} \leq C\varepsilon,\ \|\sqrt{\varrho_{0,\varepsilon}}\ \vec{u}_{0,\varepsilon}\|_{L^2(\Omega;\mathbb{R}^3)} \leq C,$$

the following estimates hold

$$\operatorname*{ess\ sup}_{t \in (0,T)} |\mathcal{M}^\varepsilon_{res}(t)| \leq C\varepsilon^2. \tag{3.45}$$

$$\operatorname*{ess\ sup}_{t \in (0,T)} \|[\varrho_\varepsilon - \overline{\varrho}]_{ess}(t)\|^2_{L^2(\Omega)} \leq C\varepsilon^2, \tag{3.46}$$

$$\operatorname*{ess\ sup}_{t \in (0,T)} \|[\vartheta_\varepsilon - \overline{\vartheta}]_{ess}(t)\|^2_{L^2(\Omega)} \leq C\varepsilon^2, \tag{3.47}$$

$$\operatorname*{ess\ sup}_{t \in (0,T)} \|[E^R_\varepsilon - \overline{E}]_{ess}(t)\|^2_{L^2(\Omega)} \leq C\varepsilon, \tag{3.48}$$

$$\operatorname*{ess\ sup}_{t \in (0,T)} \|[\varrho_\varepsilon e(\varrho_\varepsilon, \vartheta_\varepsilon)]_{res}(t)\|_{L^1(\Omega)} \leq C\varepsilon^2, \tag{3.49}$$

$$\operatorname*{ess\ sup}_{t \in (0,T)} \|[\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon)]_{res}(t)\|_{L^1(\Omega)} \leq C\varepsilon^2, \tag{3.50}$$

$$\operatorname*{ess\ sup}_{t \in (0,T)} \|[E^R(I_\varepsilon)]_{res}(t)\|_{L^1(\Omega)} \leq C\varepsilon, \tag{3.51}$$

$$\operatorname*{ess\ sup}_{t \in (0,T)} \|[s^R(I_\varepsilon)]_{res}(t)\|_{L^1(\Omega)} \leq C\varepsilon. \tag{3.52}$$

Moreover

$$\left(\varsigma^m_\varepsilon + \varsigma^R_\varepsilon\right)\left[[0,t] \times \overline{\Omega}\right] \leq C\varepsilon^2, \tag{3.53}$$

$$\operatorname*{ess\ sup}_{t \in (0,T)} \|\sqrt{\varrho_\varepsilon}\ \vec{u}_\varepsilon(t)\|_{L^2(\Omega;\mathbb{R}^3)} \leq C. \tag{3.54}$$

Proof: Estimate (4.61) follow after (3.40). Bounds (4.62),(4.63) and (4.68) follow after (3.39) and (3.42). Bounds (4.66) and (4.67) follow after (3.41) and finally (4.68) and (4.69) follow after (3.44) $\square$

From the properties of thermodynamical functions and the hypotheses on transport coefficients, we finally quote from [14] the supplementary estimates

Proposition 3.1. *Suppose that the quantities $e = e(\varrho, \vartheta)$, $s = s(\varrho, \vartheta)$ and $p = p(\varrho, \vartheta)$ satisfy hypotheses (2.1)-(2.6) and that the transport coefficients $\mu = \mu(\vartheta)$, $\eta = \eta(\vartheta)$, $\kappa = \kappa(\vartheta)$, $\sigma_a = \sigma_a(\vartheta)$, $\sigma_a = \sigma_a(\vartheta)$ satisfy the growth conditions (2.7)-(2.10). The following estimates hold*

$$\operatorname*{ess\ sup}_{t \in (0,T)} \int_\Omega \left([\varrho_\varepsilon]^{\frac{5}{3}}]_{res} + [\vartheta_\varepsilon]^4]_{res} \right)(t)\ dx \leq C\varepsilon^2, \tag{3.55}$$

$$\int_0^T \|\vec{u}_\varepsilon(t)\|^2_{W^{1,2}(\Omega;\mathbb{R}^3)}\ dt \leq C, \tag{3.56}$$

$$\int_0^T \left\| \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon}(t) \right\|^2_{W^{1,2}(\Omega)}\ dt \leq C, \tag{3.57}$$

$$\int_0^T \left\| \frac{\log(\vartheta_\varepsilon) - \log(\overline{\vartheta})}{\varepsilon}(t) \right\|^2_{W^{1,2}(\Omega)}\ dt \leq C, \tag{3.58}$$

where the generic constant C does not depend on ε.

3.3 Convergence toward the target system

Our goal is now to prove that the incompressible system (3.20)-(3.28) is the limit, in a suitable sense of the primitive system (3.30)-(3.34). Our result is as follows

Theorem 3.3. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class $C^{2,\nu}$. Assume that the thermodynamic functions p, e, s satisfy hypotheses (2.1 - 2.6) with $P \in C^1[0,\infty) \cap C^2(0,\infty)$, and that the transport coefficients μ, λ, κ, σ_a, σ_s and the equilibrium function B comply with (2.7 - 2.10).*

Let $(\varrho_\varepsilon, \vec{u}_\varepsilon, \vartheta_\varepsilon, I_\varepsilon)$ be a weak solution to the scaled radiative Navier-Stokes system (1.1 - 1.6) for $(t, x, \vec{\omega}, \nu) \in [0,T] \times \Omega \times \mathcal{S}^2 \times \mathbb{R}_+$, supplemented with the boundary conditions (1.10 - 1.11) and the initial conditions $(\varrho_{0,\varepsilon}, \vec{u}_{0,\varepsilon}, \vartheta_{0,\varepsilon}, I_{0,\varepsilon})$ be given by

$$\varrho_\varepsilon(0,\cdot) = \overline{\varrho} + \varepsilon\varrho^{(1)}_{0,\varepsilon}, \vec{u}_\varepsilon(0,\cdot) = \vec{u}_{0,\varepsilon}, \vartheta_\varepsilon(0,\cdot) = \overline{\vartheta} + \varepsilon\vartheta^{(1)}_{0,\varepsilon}, \mathbb{I}_\varepsilon(0,\cdot) = \overline{I} + \varepsilon I^{(1)}_{0,\varepsilon},$$

where

$$\overline{\varrho} > 0,\ \overline{\vartheta} > 0,\ \overline{I} > 0,$$

are constants and

$$\int_\Omega \varrho^{(1)}_{0,\varepsilon}\ dx = 0,\ \int_\Omega \vartheta^{(1)}_{0,\varepsilon}\ dx = 0,\ \int_\Omega I^{(1)}_{0,\varepsilon}\ dx = 0 \ \textit{for all}\ \varepsilon > 0.$$

Assume that

$$\begin{cases} \varrho^{(1)}_{0,\varepsilon} \to \varrho^{(1)}_0 \quad weakly-(*)\ in\ L^\infty(\Omega), \\ \vec{u}^{(1)}_{0,\varepsilon} \to \vec{U}_0 \quad weakly-(*)\ in\ L^\infty(\Omega;\mathbb{R}^3), \\ \vartheta^{(1)}_{0,\varepsilon} \to \vartheta^{(1)}_0 \quad weakly-(*)\ in\ L^\infty(\Omega), \\ I^{(1)}_{0,\varepsilon} \to I^{(1)}_0 \quad weakly-(*)\ in\ L^\infty(\Omega \times \mathcal{S}^2 \times \mathbb{R}_+). \end{cases}$$

Then

$$\operatorname*{ess\ sup}_{t \in (0,T)} \|\varrho_\varepsilon(t) - \overline{\varrho}\|_{L^{\frac{4}{3}}(\Omega)} \leq C\varepsilon,$$

and up to subsequences

$$\vec{u}_\varepsilon \to \vec{U}\ weakly-(*)\ in\ L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3)),$$

$$\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} = \vartheta^{(1)} \to \Theta \ \ weakly - (*) \ in \ L^2(0,T;W^{1,2}(\Omega)),$$

$$I_\varepsilon \to I_0 \ \ weakly - (*) \ in \ L^2(0,T;L^2(\Omega \times \mathcal{S}^2 \times \mathbb{R}_+)),$$

and

$$\frac{I_\varepsilon - \overline{I}}{\varepsilon} = I^{(1)} \to I_1 \ \ weakly - (*) \ in \ L^2(0,T;L^2(\Omega \times \mathcal{S}^2 \times \mathbb{R}_+)),$$

where $(\vec{U}, \Theta, I_0, I_1)$ *solve the radiative Oberbeck-Boussinesq system (3.20)-(3.24).*

Before showing in the last part of this Section that we can pass to the limit $\varepsilon \to 0$ into the various equations of the primitive scaled system and that the limit actually satisfies the target system, let us first quote the following result which is a straightforward extension of Proposition 5.2 of [14] (the proof is omitted)

Proposition 3.2. *Let* $\{\varrho_\varepsilon\}_{\varepsilon>0}, \{\vartheta_\varepsilon\}_{\varepsilon>0}, \{I_\varepsilon\}_{\varepsilon>0}$ *be three sequences of non-negative measurable functions such that*

$$\left[\varrho_\varepsilon^{(1)}\right]_{ess} \to \varrho^{(1)} \ weakly - (*) \ in \ L^\infty(0,T;L^2(\Omega)),$$

$$\left[\vartheta_\varepsilon^{(1)}\right]_{ess} \to \vartheta^{(1)} \ weakly - (*) \ in \ L^\infty(0,T;L^2(\Omega)),$$

$$\left[I_\varepsilon^{(1)}\right]_{ess} \to I^{(1)} \ weakly - (*) \ in \ L^\infty(0,T;L^2(\Omega)), \ a.e. \ in \ \mathcal{S}^2 \times \mathbb{R}_+,$$

where

$$\varrho_\varepsilon^{(1)} = \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}, \ \vartheta_\varepsilon^{(1)} = \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon}, \ I_\varepsilon^{(1)} = \frac{I_\varepsilon - \overline{I}}{\varepsilon}.$$

Suppose that

$$\operatorname{ess} \sup_{t \in (0,T)} |\mathcal{M}^\varepsilon_{res}(t)| \le C\varepsilon^2, \tag{3.59}$$

and let $G, G^R \in C^1(\overline{\mathcal{O}_{ess}})$ *be given functions. Then*

$$\frac{[G(\varrho_\varepsilon, \vartheta_\varepsilon)]_{ess} - G(\overline{\varrho}, \overline{\vartheta})}{\varepsilon} \to \frac{\partial G(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} \ \varrho^{(1)} + \frac{\partial G(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta} \ \vartheta^{(1)},$$

weakly $- (*)$ *in* $L^\infty(0,T;L^2(\Omega))$*, and if we denote*

$$\left[G^R(I_\varepsilon)\right]_{ess} := \left[G^R(I_\varepsilon(\cdot,\cdot,\vec{\omega},\nu))\right]_{ess} = G^R(I_\varepsilon) \cdot \mathbb{I}_{\mathcal{M}^\varepsilon_{ess}}, \ for \ a.a. \ (\vec{\omega},\nu) \in \mathcal{S}^2 \times \mathbb{R}_+,$$

we have

$$\frac{\left[G^R(I_\varepsilon)\right]_{ess} - G^R(\overline{I})}{\varepsilon} \to \frac{\partial G(\overline{I})}{\partial I} \ I^{(1)},$$

weakly $- (*)$ *in* $L^\infty(0,T;L^2(\Omega))$*, a.e. in* $\mathcal{S}^2 \times \mathbb{R}_+$.

Moreover if $G, G^R \in C^2(\overline{\mathcal{O}_{ess}})$ *then*

$$\left\| \frac{[G(\varrho_\varepsilon, \vartheta_\varepsilon)]_{ess} - G(\overline{\varrho}, \overline{\vartheta})}{\varepsilon} - \frac{\partial G(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} \ \left[\varrho^{(1)}\right]_{ess} - \frac{\partial G(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta} \ \left[\vartheta^{(1)}\right]_{ess} \right\|_{L^\infty(0,T;L^1(\Omega))} \le C\varepsilon,$$

and

$$\left\| \frac{\left[G^R(I_\varepsilon)\right]_{ess} - G^R(\overline{I})}{\varepsilon} - \frac{\partial G(\overline{I})}{\partial I} \ \left[I^{(1)}\right]_{ess} \right\|_{L^\infty(0,T;L^1(\Omega))} \le C\varepsilon,$$

for a.a. $(\vec{\omega},\nu) \in \mathcal{S}^2 \times \mathbb{R}_+$.

3.4 Proof of Theorem 3.3

1. For the continuity equation, after Proposition 3.1, we know that $\int_0^T \|\vec{u}_\varepsilon(t)\|^2_{W^{1,2}(\Omega;\mathbb{R}^3)}\,dt \le C$ so passing to the limit after possible extraction of a subsequence, we see that $\vec{u}_\varepsilon \to \vec{U}$ weakly in $L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3))$. In the same stroke $\varrho_\varepsilon \to \overline{\varrho}$ weakly in $L^\infty(0,T;L^{5/3}(\Omega;\mathbb{R}^3))$. So we can pass to the limit in the weak continuity equation (3.30) which gives $\int_0^T\int_\Omega \vec{U}\ \nabla_x\phi\ dx\ dt = 0$ for all $\phi \in \mathcal{D}((0,T)\times\overline{\Omega})$, which rewrites

$$\mathrm{div}_x\vec{U} = 0, \quad \text{a.e. in } (0,T)\times\Omega,$$

with $\vec{U}\Big|_{\partial\Omega} = 0$, provided $\partial\Omega$ is regular.

2. For the momentum equation one has only [14] $\varrho_\varepsilon\vec{u}_\varepsilon\otimes\vec{u}_\varepsilon \to \overline{\varrho\vec{U}\otimes\vec{U}}$, weakly in $L^2(0,T;L^{\frac{30}{29}}(\Omega;\mathbb{R}^3))$, however one can show that one can pass to the limit in the convective term and obtain

$$\int_0^T\int_\Omega \overline{\varrho\,\vec{U}\otimes\vec{U}} : \nabla_x\phi\ dx\ dt \to \int_0^T\int_\Omega \overline{\varrho}\ \vec{U}\otimes\vec{U} : \nabla_x\phi\ dx\ dt.$$

Moreover after the hypotheses on the pressure law, the temperature ϑ_ε is bounded in $L^\infty((0,T);L^4(\Omega))\cap L^2(0,T;L^6(\Omega))$, which implies that $\mathbb{S}_\varepsilon \to \mu(\overline{\vartheta})(\nabla_x\vec{U}+\nabla_x^t\vec{U})$, weakly in $L^q(0,T;L^q(\Omega;\mathbb{R}^3))$ for a $q>1$.

Finally after Lemma 3.2 $\|\vec{F}^R\|_{L^2(0,T;L^2(\Omega))} \le C$ and $\|\mathbb{P}\|_{L^2(0,T;L^2(\Omega))} \le C$ so $\varepsilon^2\vec{F}^R \to 0$ and $\varepsilon\mathbb{P}\to 0$ in $L^2(0,T;L^2(\Omega))$. So passing to the limit in momentum equation for a divergence-free test function $\vec{\phi}$, we get

$$\int_0^T\int_\Omega\left(\overline{\varrho}\vec{U}\cdot\partial_t\vec{\phi}+\overline{\varrho}\vec{U}\otimes\vec{U}:\nabla_x\vec{\phi}\right)\ dx\ dt = \int_0^T\int_\Omega\left(\mu(\overline{\vartheta})(\nabla_x\vec{U}+\nabla_x^t\vec{U}):\nabla_x\vec{\phi}\right)\ dx\ dt - \int_\Omega \overline{\varrho}\vec{U}_0\cdot\vec{\phi}\ dx,$$

provided that $\vec{u}_{0,\varepsilon}\to\vec{U}_0$ weakly $*$ in $L^\infty(\Omega;\mathbb{R}^3)$.

As expected in the incompressible limit, pressure no more appears in the limit momentum equation and as in [14], the formal relation between $\varrho^{(1)}$ and $\vartheta^{(1)}$ is recovered by multiplying the momentum equation by ε. One gets

$$\int_0^T\int_\Omega \frac{p_\varepsilon}{\varepsilon}\ \mathrm{div}_x\phi\ dx\ dt = \frac{p_\varepsilon - p_0}{\varepsilon}\ \mathrm{div}_x\phi\ dx\ dt = 0.$$

Using Proposition 3.2 and passing to the limit, we have

$$\int_0^T\int_\Omega\left(\partial_\varrho p(\overline{\varrho},\overline{\vartheta})\varrho^{(1)}+\partial_\vartheta p(\overline{\varrho},\overline{\vartheta})\vartheta^{(1)}\right)\ \mathrm{div}_x\phi\ dx\ dt = 0,$$

which is the weak formulation of

$$\partial_\varrho p(\overline{\varrho},\overline{\varrho})\varrho^{(1)}+\partial_\vartheta p(\overline{\varrho},\overline{\varrho})\vartheta^{(1)} = \text{Const.} \tag{3.60}$$

3. For the radiative transfer equation, using the L^∞ bound shown in the previous sections for I_ε, it is clear that $I_\varepsilon \to I_0$, weakly in $L^2((0,T)\times\Omega\times\mathcal{S}^2\times\mathbb{R}_+)$, and also after Proposition 3.1 $\vartheta_\varepsilon\to\overline{\vartheta}$, weakly in $L^2(0,T;W^{1,2}(\Omega))$.

Using the cut-off hypotheses (2.9)(2.10), we can pass to the limit which gives

$$\int_\Omega\int_0^\infty\int_{\mathcal{S}^2}\vec{\omega}\cdot\nabla_x\psi\ I_0\ d\vec{\omega}\ d\nu\ dx + \int_\Omega\int_0^\infty\int_{\mathcal{S}^2}\left[\sigma_a(\overline{\vartheta})\left(B(\overline{\vartheta})-I_0\right)+\sigma_s(\overline{\vartheta})\left(\tilde{I}_0-I_0\right)\right]\psi\ d\vec{\omega}\ d\nu\ dx,$$

$$= \int_{\Gamma_+}\int_0^\infty \vec{\omega}\cdot\vec{n}_x I_0\ \psi\ d\Gamma\ d\nu,$$

using the same notation for any time-independent test function $\psi\in C_c^\infty(\overline{\Omega}\times\mathcal{S}^2\times\mathbb{R}_+)$, which is the weak formulation of the stationary problem

$$\vec{\omega}\cdot\nabla_x I_0 = S_0, \tag{3.61}$$

with the boundary condition

$$I_0 = 0 \text{ on } \Gamma_+, \tag{3.62}$$

where $S_0 = \sigma_a(\nu, \overline{\vartheta}) \left(B(\nu, \overline{\vartheta}) - I_0\right) + \sigma_s(\overline{\vartheta}) \left(\tilde{I}_0 - I_0\right)$. Now from (3.34)

$$\int_0^T \int_\Omega \int_0^\infty \int_{\mathcal{S}^2} (\varepsilon \partial_t \psi + \vec{\omega} \cdot \nabla_x \psi) \; \frac{I_\varepsilon - I_0}{\varepsilon} \; d\vec{\omega} \; d\nu \; dx \; dt + \int_0^T \int_\Omega \int_0^\infty \int_{\mathcal{S}^2} \left[\frac{S_\varepsilon - S_0}{\varepsilon}\right] \psi \; d\vec{\omega} \; d\nu \; dx \; dt$$

$$= \int_\Omega \int_0^\infty \int_{\mathcal{S}^2} \varepsilon \frac{I_{0,\varepsilon} - I_0}{\varepsilon} \; \psi(0, x, \vec{\omega}, \nu) \; d\vec{\omega} \; d\nu \; dx + \int_0^T \int_{\Gamma_+} \int_0^\infty \vec{\omega} \cdot \vec{n}_x \; \frac{I_\varepsilon - I_0}{\varepsilon} \psi \; d\Gamma \; d\nu \; dt,$$

for any $\psi \in C_c^\infty([0,T] \times \overline{\Omega} \times \mathcal{S}^2 \times \mathbb{R}_+)$, with $S_\varepsilon - S_0 = S(I_\varepsilon) - S(I_0)$. From Proposition 3.2 we get

$$\frac{S_\varepsilon - S_0}{\varepsilon} \to S_1 := \partial_\vartheta (\sigma_a B)(\nu, \overline{\vartheta}) \vartheta^{(1)} - \partial_\vartheta \sigma_a(\overline{\vartheta}) \vartheta^{(1)} I_0 - \sigma_a(\overline{\vartheta}) I_1$$

$$+ \partial_\vartheta \sigma_s(\overline{\vartheta}) \vartheta^{(1)} \tilde{I}_0 + \sigma_s(\overline{\vartheta}) \tilde{I}_1 - \partial_\vartheta \sigma_s(\overline{\vartheta}) \vartheta^{(1)} I_0 - \sigma_s(\overline{\vartheta}) I_1,$$

weakly in $L^\infty((0,T); L^2(\Omega \times \mathcal{S}^2 \times \mathbb{R}_+))$ with $I_1 := I^{(1)}$. Passing to the limit we find the limit equation

$$\int_\Omega \int_0^\infty \int_{\mathcal{S}^2} \vec{\omega} \cdot \nabla_x \psi \; I_1 \; d\vec{\omega} \; d\nu \; dx + \int_\Omega \int_0^\infty \int_{\mathcal{S}^2} S_1 \psi \; d\vec{\omega} \; d\nu \; dx, = \int_{\Gamma_+} \int_0^\infty \vec{\omega} \cdot \vec{n}_x \; I_1 \psi \; d\Gamma \; d\nu, \tag{3.63}$$

using the same notation for any time-independent test function $\psi \in C_c^\infty(\overline{\Omega} \times \mathcal{S}^2 \times \mathbb{R}_+)$, which is the weak formulation of the stationary problem

$$\vec{\omega} \cdot \nabla_x I_1 = S_1, \tag{3.64}$$

with the boundary condition

$$I_1 = 0 \text{ on } \Gamma_+. \tag{3.65}$$

4. For the entropy balance, we rewrite equation (3.33) as

$$\int_0^T \int_\Omega \left\{ \varrho_\varepsilon \; \frac{s_\varepsilon - \overline{s}}{\varepsilon} \; (\partial_t + \vec{u}_\varepsilon \cdot \nabla_x \varphi) + \frac{s_\varepsilon^R - \overline{s^R}}{\varepsilon} \; \varepsilon \partial_t \varphi + \frac{\vec{q}_\varepsilon^R - \overline{\vec{q}^R}}{\varepsilon} \cdot \nabla_x \varphi \right\} \; dx \; dt$$

$$+ \int_0^T \int_\Omega \frac{\kappa(\vartheta_\varepsilon)}{\vartheta_\varepsilon} \nabla_x \left(\frac{\vartheta_\varepsilon}{\varepsilon}\right) \cdot \nabla_x \varphi \; dx \; dt + \frac{1}{\varepsilon} \left\langle \varsigma_\varepsilon^m + \varsigma_\varepsilon^R; \phi \right\rangle_{[\mathcal{M};C]([0,T \times \overline{\Omega})}$$

$$= - \int_\Omega \left\{ \left(\varrho_{0,\varepsilon} \frac{s_{0,\varepsilon} - \overline{s}}{\varepsilon} + \varepsilon \frac{s_\varepsilon^R - \overline{s^R}}{\varepsilon} \right) \varphi(0, \cdot) \right\} \; dx.$$

Similarly to [14], using Proposition 3.2 and energy estimates, we see that $\varrho_\varepsilon \; \frac{s_\varepsilon - \overline{s}}{\varepsilon} \to \overline{\varrho} \left(\partial_\varrho s(\overline{\varrho}, \overline{\vartheta}) \varrho^{(1)} + \partial_\vartheta s(\overline{\varrho}, \overline{\vartheta}) \vartheta^{(1)}\right)$, weakly $*$ in $L^\infty(0,T; L^2(\Omega; \mathbb{R}^3))$, that $\frac{\kappa(\vartheta_\varepsilon)}{\vartheta_\varepsilon} \nabla_x \left(\frac{\vartheta_\varepsilon}{\varepsilon}\right) \to \frac{\kappa(\overline{\vartheta})}{\overline{\vartheta}} \; \nabla_x \vartheta^{(1)}$, weakly $*$ in $L^2(0,T; L^2(\Omega; \mathbb{R}^3))$ and that $\frac{1}{\varepsilon} \left\langle \varsigma_\varepsilon^m + \varsigma_\varepsilon^R; \phi \right\rangle_{[\mathcal{M};C]([0,T \times \overline{\Omega})} \to 0$. Moreover $\varrho_\varepsilon \; \frac{s_\varepsilon - \overline{s}}{\varepsilon} \cdot \vec{u}_\varepsilon \to \overline{\varrho} \left(\partial_\varrho s(\overline{\varrho}, \overline{\vartheta}) \varrho^{(1)} + \partial_\vartheta s(\overline{\varrho}, \overline{\vartheta}) \vartheta^{(1)}\right) \cdot \vec{U}$, weakly $*$ in $L^2(0,T; L^{3/2}(\Omega; \mathbb{R}^3))$. Now applying Proposition 3.2 in the same stroke, we get that $\varepsilon \frac{s_\varepsilon^R - \overline{s^R}}{\varepsilon} \to 0$, weakly $*$ in $L^\infty(0,T; L^2(\Omega; \mathbb{R}^3))$.

Let us compute the limit of $\frac{\vec{q}_\varepsilon^R - \overline{\vec{q}^R}}{\varepsilon}$. We have

$$\vec{q}_\varepsilon^R = \vec{q}^R(I_\varepsilon) = - \int_0^\infty \int_{\mathcal{S}^2} \nu^2 \left\{ n_\varepsilon \log n_\varepsilon - (n_\varepsilon + 1) \log(n_\varepsilon + 1) \right\} \; d\vec{\Omega} \; d\nu,$$

with $n_\varepsilon = n(I_\varepsilon) = \frac{I_\varepsilon}{\nu^3}$. Applying once more Proposition 3.2 with $G^R(I) = n(I)\log n(I) - (n(I)+1)\log(n(I)+1)$ and integrating on $\mathcal{S}^2 \times \mathbb{R}_+$, we find

$$\frac{\vec{q}_\varepsilon^R - \overline{\vec{q}^R}}{\varepsilon} \to \int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu} \log\left(\frac{n(\overline{I})+1}{n(\overline{I})}\right) \vec{\omega}\, I^{(1)}\, d\vec{\omega}\, d\nu,$$

and as $\frac{n(\overline{I})+1}{n(\overline{I})} = \frac{\nu}{\overline{\vartheta}}$, we have

$$\frac{\vec{q}_\varepsilon^R - \overline{\vec{q}^R}}{\varepsilon} \to \frac{1}{\overline{\vartheta}}\, \vec{F}^R(I^{(1)}),$$

with the radiative momentum $\vec{F}^R(I^{(1)}) = \int_0^\infty \int_{\mathcal{S}^2} \vec{\omega}\, I^{(1)}\, d\vec{\omega}\, d\nu$. So

$$\int_0^T \int_\Omega \left(\frac{\vec{q}_\varepsilon^R - \overline{\vec{q}^R}}{\varepsilon}\right) \cdot \nabla_x \varphi\, dx\, dt \to -\int_0^T \int_\Omega \frac{\mathrm{div}_x \vec{F}^R(I^{(1)})}{\overline{\vartheta}}\, \phi\, dx\, dt.$$

As we know from (3.64) that

$$\mathrm{div}_x \vec{F}^R = \int_0^\infty \int_{\mathcal{S}^2} \left[\partial_\vartheta \sigma_a(\overline{\vartheta})\left(B(\nu,\overline{\vartheta}) - I_0\right)\vartheta^{(1)} + \sigma_a(\overline{\vartheta})\left(\partial_\vartheta B(\nu,\overline{\vartheta})\vartheta^{(1)} - I_1\right)\right] d\vec{\omega}\, d\nu,$$

the limit contribution in the right-hand side becomes

$$-\int_0^T \int_\Omega \int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\overline{\vartheta}} \left[\partial_\vartheta \sigma_a(\overline{\vartheta})\left(B(\nu,\overline{\vartheta}) - I_0\right)\vartheta^{(1)} + \sigma_a(\overline{\vartheta})\left(\partial_\vartheta B(\nu,\overline{\vartheta})\vartheta^{(1)} - I_1\right)\right] \phi\;\; d\vec{\omega}\, d\nu\;\; dx\, dt.$$

Gathering all of these terms, we recover the limit equation for entropy

$$-\int_0^T \int_\Omega \overline{\varrho}\left(\partial_\varrho s(\overline{\varrho},\overline{\vartheta})\varrho^{(1)} + \partial_\vartheta s(\overline{\varrho},\overline{\vartheta})\vartheta^{(1)}\right)\left(\partial_t \phi + \vec{U}\cdot\nabla_x \phi(3.64)\right) dx\, dt - \int_0^T \int_\Omega \frac{\kappa(\overline{\varrho},\overline{\vartheta})}{\overline{\vartheta}}\, \nabla_x \vartheta^{(1)}\, \nabla_x \phi\, dx\, dt$$

$$+\int_0^T \int_\Omega \int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\overline{\vartheta}} \left[\partial_\vartheta \sigma_a(\overline{\vartheta})\left(B(\nu,\overline{\vartheta}) - I_0\right)\vartheta^{(1)} + \sigma_a(\overline{\vartheta})\left(\partial_\vartheta B(\nu,\overline{\vartheta})\vartheta^{(1)} - I_1\right)\right] \phi\;\; d\vec{\omega}\, d\nu\;\; dx\, dt$$

$$= -\int_\Omega \overline{\varrho}\left(\partial_\varrho s(\overline{\varrho},\overline{\vartheta})\varrho_0^{(1)} + \partial_\vartheta s(\overline{\varrho},\overline{\vartheta})\vartheta_0^{(1)}\right)\phi(0,\cdot)\, dx,$$

and using (3.60), it is routine to check that we finally obtain the energy equation (3.22).

4 Diffusion limits

Diffusion limits consist in supposing that one of the transport coefficient is small while the other is large. These regimes have been introduced by Lowrie, Morel et Hittinger [22] and also considered by Buet and Desprès [6]. Contrary to the low Mach number limit studied previously, these limits both correspond to a compressible system which introduces a new difficulty in the sense that we need to estimate differences between two variable quantities. We show in this last Section that as in [13] this difficulty may be overcome by using a relative entropy inequality introduced by Feireisl and Novotný [15]. In order to identify the appropriate limit regimes we perform two different scalings.

- The first one corresponds to the *equilibrium diffusion regime* defined in [6] by

$$Ma = Sr = Pe = Re = \mathcal{P} = 1,\ \mathcal{C} = \varepsilon^{-1},\ \mathcal{L}_s = \varepsilon^2 \text{ and } \mathcal{L} = \varepsilon^{-1},$$

leading to the primitive system

$$\varepsilon\, \partial_t I + \vec{\omega}\cdot\nabla_x I = \frac{1}{\varepsilon}\sigma_a\,(B - I) + \varepsilon\sigma_s\left(\frac{1}{4\pi}\int_{\mathcal{S}^2} I\, \mathrm{d}\vec{\omega} - I\right), \tag{4.1}$$

$$\partial_t \varrho + \mathrm{div}_x(\varrho \vec{u}) = 0, \tag{4.2}$$

$$\partial_t \left(\varrho \vec{u} + \varepsilon \vec{F}^R \right) + \mathrm{div}_x \left(\varrho \vec{u} \otimes \vec{u} + \mathbb{P}^R \right) + \nabla_x p - \ \mathrm{div}_x \mathbb{S} = 0. \tag{4.3}$$

$$\partial_t \left(\frac{1}{2} \ \varrho |\vec{u}|^2 + \varrho e + E^R \right) + \mathrm{div}_x \left(\left(\frac{1}{2} \ \varrho |\vec{u}|^2 + \varrho e + p \right) \vec{u} + \frac{\vec{F}^R}{\varepsilon} + \vec{q} - \mathbb{S} \vec{u} \right) = 0, \tag{4.4}$$

$$\partial_t \left(\varrho s + \varepsilon s_R \right) + \mathrm{div}_x \left(\varrho \vec{u} s + \vec{q}_R \right) + \ \mathrm{div}_x \left(\frac{\vec{q}}{\vartheta} \right) \geq \frac{1}{\vartheta} \left(\mathbb{S} : \nabla_x \vec{u} - \frac{\vec{q} \cdot \nabla_x \vartheta}{\vartheta} \right),$$

$$+ \frac{1}{\varepsilon} \int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu} \left[\log \frac{n(I)}{n(I)+1} - \log \frac{n(B)}{n(B)+1} \right] \sigma_a (I - B) \ d\vec{\omega} d\nu$$

$$+ \varepsilon \int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu} \left[\log \frac{n(I)}{n(I)+1} - \log \frac{n(\tilde{I})}{n(\tilde{I})+1} \right] \sigma_s (I - \tilde{I}) \ d\vec{\omega} d\nu. \tag{4.5}$$

with

$$\frac{d}{dt} \int_\Omega \left(\varrho \mathcal{E} + E_R \right) \ dx + \frac{1}{\varepsilon} \int_0^\infty \int_{\Gamma_+} \vec{\omega} \cdot \vec{n} \ I \ d\Gamma_+ d\nu = 0, \tag{4.6}$$

where $\mathcal{E} = \frac{1}{2} \ |\vec{u}|^2 + e$.

- The second one is the “ non-equilibrium diffusion regime” also defined in [6] by

$$Ma = Sr = Pe = Re = \mathcal{P} = 1, \ \mathcal{C} = \varepsilon^{-1}, \ \mathcal{L} = \varepsilon^2 \text{ and } \mathcal{L}_s = \varepsilon^{-1}.$$

One checks that equations (4.2) (4.3) (4.4) and (4.6) still hold in this scaling. The new transport equation is

$$\varepsilon \ \partial_t I + \vec{\omega} \cdot \nabla_x I = \varepsilon \sigma_a \left(B - I \right) + \frac{1}{\varepsilon} \sigma_s \left(\frac{1}{4\pi} \int_{\mathcal{S}^2} I \ \mathrm{d}\vec{\omega} - I \right), \tag{4.7}$$

and the new entropy equation is

$$\partial_t \left(\varrho s + \varepsilon s_R \right) + \mathrm{div}_x \left(\varrho \vec{u} s + \vec{q}_R \right) + \mathrm{div}_x \left(\frac{\vec{q}}{\vartheta} \right) \geq \frac{1}{\vartheta} \left(\mathbb{S} : \nabla_x \vec{u} - \frac{\vec{q} \cdot \nabla_x \vartheta}{\vartheta} \right)$$

$$+ \varepsilon \int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu} \left[\log \frac{n(I)}{n(I)+1} - \log \frac{n(B)}{n(B)+1} \right] \sigma_a (I - B) \ d\vec{\omega} d\nu$$

$$+ \frac{1}{\varepsilon} \int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu} \left[\log \frac{n(I)}{n(I)+1} - \log \frac{n(\tilde{I})}{n(\tilde{I})+1} \right] \sigma_s (I - \tilde{I}) \ d\vec{\omega} d\nu. \tag{4.8}$$

4.1 The equilibrium-diffusion regime

In order to compute the limit system, we use the formal expansions (3.12) and keep the low order terms in (4.1). Introducing the unperturbed Planck's function $B(\nu, \vartheta) = \frac{2h}{c^2} \frac{\nu^3}{e^{\frac{h\nu}{k\vartheta}} - 1}$ and computing in (2.18) the expansion $\alpha = \frac{\sigma_a(\vartheta_0) + \varepsilon^2 \sigma_s}{\sigma_a + 2\varepsilon^2 \sigma_s} = 1 + O(\varepsilon^2)$, we have in (1.7)

$$B(\nu, \vec{\omega}, \vec{u}, \vartheta) = B(\nu, \vartheta_0) + \left(\vec{\omega} \cdot \vec{u}_0 \vartheta_0 + \vartheta_1 \right) \partial_\vartheta B(\nu, \vartheta_0) \varepsilon + O(\varepsilon^2).$$

We get first

$$I_0 = B(\nu, \vartheta_0), \tag{4.9}$$

and

$$I_1 = \left(\vec{\omega} \cdot \vec{u}_0 \vartheta_0 + \vartheta_1 \right) \partial_\vartheta B(\nu, \vartheta_0) - \frac{1}{\sigma_a(\vartheta_0)} \ \vec{\omega} \cdot \nabla_x I_0. \tag{4.10}$$

As the related radiative quantities are

$$\begin{cases} E^R = E_0^R + \varepsilon E_1^R + O(\varepsilon^2), \\ \vec{F}^R = \vec{F}_0^R + \varepsilon \vec{F}_1^R + O(\varepsilon^2), \\ \mathbb{P}^R = \mathbb{P}_0^R + \varepsilon \mathbb{P}_1^R + O(\varepsilon^2), \end{cases} \tag{4.11}$$

using (4.9) and (4.10) we find

$$E_0^R = \int_0^\infty \int_{\mathcal{S}^2} B(\nu, \vartheta_0)\ d\nu = a\vartheta_0^4, \quad \text{with } a = \frac{4\pi^4}{15},$$

$$\vec{F}_0^R = 0.$$

We get also

$$\vec{F}_1^R = \int_0^\infty \int_{\mathcal{S}^2} \Big(\vec{\omega}\,(\vec{\omega}\cdot\vec{u}_0)\,\vartheta_0 \partial_\vartheta B(\nu,\vartheta_0) - \frac{1}{\sigma_a(\vartheta_0)}\ \vec{\omega}\,(\vec{\omega}\cdot\nabla_x B(\nu,\vartheta_0)) \Big)\ d\vec{\omega}\ d\nu,$$

$$= \vartheta_0 \vec{u}_0 \int_0^\infty \int_{\mathcal{S}^2} \vec{\omega}\otimes\vec{\omega}\partial_\vartheta B(\nu,\vartheta_0)\ d\vec{\omega}\ d\nu - \frac{1}{\sigma_a(\vartheta_0)} \nabla_x \int_0^\infty \int_{\mathcal{S}^2} \vec{\omega}\otimes\vec{\omega} B(\nu,\vartheta_0)\ d\vec{\omega}\ d\nu,$$

so

$$\vec{F}_1^R = \frac{4a}{3}\ \vartheta_0^4 \vec{u}_0 - \frac{1}{3\sigma_a(\vartheta_0)}\ \nabla_x(a\vartheta_0^4). \tag{4.12}$$

Finally

$$\mathbb{P}_0^R = \int_0^\infty \int_{\mathcal{S}^2} \vec{\omega}\otimes\vec{\omega} B(\nu,\vartheta_0)\ d\vec{\omega}\ d\nu = \frac{a}{3}\vartheta_0^4\ \mathbb{I},$$

where $\mathbb{I}$ is the unit tensor. The limit momentum equation is then

$$\partial_t(\varrho_0\vec{u}_0) + \mathrm{div}_x(\varrho_0\vec{u}_0\otimes\vec{u}_0) + \nabla_x \mathbf{p}(\varrho_0,\vartheta_0) = \mathrm{div}_x \mathbb{S}(\varrho_0,\vartheta_0),$$

where $\mathbf{p}(\varrho_0,\vartheta_0) = p(\varrho_0,\vartheta_0) + \frac{a}{3}\vartheta_0^4$ and the limit energy equation is

$$\partial_t\left(\varrho \mathbf{e}(\varrho_0,\vartheta_0)\right) + \mathrm{div}_x\left(\varrho_0 e(\varrho_0,\vartheta_0)\vec{u}_0\right) + \mathrm{div}_x\left(\mathbf{k}(\varrho_0,\vartheta_0)\nabla_x\vartheta_0\right) = \mathbb{S}(\varrho_0,\vartheta_0) : \nabla_x\vec{u}_0 - p(\varrho_0,\vartheta_0)\mathrm{div}_x\vec{u}_0,$$

where $\mathbf{e}(\varrho_0,\vartheta_0) = e(\varrho_0,\vartheta_0) + \frac{a\vartheta_0^4}{\varrho_0}$, and $\mathbf{k}(\vartheta_0) = \kappa(\vartheta_0) + \frac{4a}{3\sigma_a}\ \vartheta_0^3$.

Hence omitting the 0 index, we finally obtain the decoupled limit system in $(0,T)\times\Omega$

$$\partial_t\varrho + \mathrm{div}_x(\varrho\vec{u}) = 0, \tag{4.13}$$

$$\partial_t(\varrho\vec{u}) + \mathrm{div}_x(\varrho\vec{u}\otimes\vec{u}) + \nabla_x\mathbf{p} = \mathrm{div}_x\mathbb{S}, \tag{4.14}$$

$$\partial_t\left(\frac{1}{2}\ \varrho|\vec{u}_0|^2 + \varrho\mathbf{e}\right) + \mathrm{div}_x\left(\left(\frac{1}{2}\ \varrho|\vec{u}|^2 + \varrho\mathbf{e} + \mathbf{p}\right)\vec{u} + \vec{\mathbf{q}} - \mathbb{S}\vec{u}\right) = 0, \tag{4.15}$$

$$\partial_t\left(\varrho\mathbf{s}\right) + \mathrm{div}_x\left(\varrho\mathbf{s}\vec{u}\right) + \mathrm{div}_x\left(\frac{\vec{\mathbf{q}}}{\vartheta}\right) = \frac{1}{\vartheta}\left(\mathbb{S}:\nabla_x\vec{u} - \frac{\vec{\mathbf{q}}\cdot\nabla_x\vartheta}{\vartheta}\right), \tag{4.16}$$

$$I = B(\nu,\vartheta), \tag{4.17}$$

where $\mathbf{p}(\varrho,\vartheta) = p(\varrho,\vartheta) + \frac{a}{3}\vartheta^4$, $\mathbf{e}(\varrho,\vartheta) = e(\varrho,\vartheta) + \frac{a}{\varrho}\vartheta^4$, $\mathbf{k}(\vartheta_0) = \kappa(\vartheta) + \frac{4a}{3\sigma_a}\vartheta^3$, $\vec{\mathbf{q}} = -\mathbf{k}(\vartheta)\nabla_x\vartheta$ and $\varrho\mathbf{s} = \varrho s + \frac{4}{3}a\vartheta^3$.

We also get boundary conditions

$$\vec{u}|_{\partial\Omega} = 0,\ \nabla\vartheta\cdot\vec{n}|_{\partial\Omega} = 0, \tag{4.18}$$

and initial conditions

$$(\varrho(x,t),\ \vec{u}(x,t),\ \vartheta(x,t))|_{t=0} = \left(\varrho^0(x),\ \vec{u}^0(x),\ \vartheta^0(x)\right), \tag{4.19}$$

for any $x \in \Omega$ with the following compatibility conditions

$$\vec{u}^0(x)|_{\partial\Omega} = 0,\ \nabla\vartheta^0 \cdot \vec{n}|_{\partial\Omega} = 0. \tag{4.20}$$

As expected, this system corresponds to a viscous compressible heat-conductive fluid at local thermodynamical equilibrium with radiation, equilibrium being achieved between matter and radiation with radiative intensity $I = B(\nu,\vartheta)$, corresponding to the black-body radiation at temperature ϑ with radiative energy $E_R(\vartheta) = a\vartheta^4$.

From the classical results of Matsumura and Nishida [23] (see also Jiang [17]) let us quote an existence result for this system. Let $(\overline{\varrho},\overline{\vartheta})$ be two given constants with $\overline{\varrho} > 0$ and $\overline{\vartheta} > 0$. We note

$$e_0 := \|\varrho_0 - \overline{\varrho}\|_{L^\infty(\Omega)} + \|\vec{u}_0\|_{H^1(\Omega)} + \|\vartheta_0 - \overline{\vartheta}\|_{H^1(\Omega)} + \|\mathbb{T}_0\|_{L^2(\Omega)} + \|\mathbb{V}_0\|_{L^4(\Omega)}, \tag{4.21}$$

where $\mathbb{V}_0$ is the initial vorticity (recall that $\mathbb{V}_{ij} = \partial_j u_i - \partial_i u_j$), and

$$E_0 := e_0 + \|\nabla_x\varrho_0\|_{L^2(\Omega)} + \|\nabla_x\vartheta_0\|_{L^q(\Omega)} + \|\nabla_x\mathbb{T}_0\|_{L^2(\Omega)}, \tag{4.22}$$

for an arbitrary fixed q such that $3 < q < 6$. The following result holds

Theorem 4.1. *1. (Local solution) There exists a positive constant T_* such that $(\varrho,\vec{u},\vartheta,I)$ is the unique classical solution to the problem (4.13)-(4.15) with boundary conditions (4.18), initial conditions (4.19), and compatibility conditions (4.20) in $(0,T)\times\Omega$ for any $T < T_*$ such that*

$$(\varrho,\vec{u},\vartheta) \in C([0,T],H^3(\Omega)),$$

$$\partial_t\varrho \in C([0,T],H^2(\Omega)),\quad \partial_t\vec{u},\partial_t\vartheta \in C([0,T],H^1(\Omega)),$$

$$\partial_t\varrho,\partial_t\vec{u},\partial_t\vartheta \in L^2([0,T],H^2(\Omega)).$$

2. (Global solution) Let $(\varrho_0 - \overline{\varrho},\vec{u}_0,\vartheta_0 - \overline{\vartheta}) \in (H^3(\Omega))^5$ and $\inf\vartheta_0 > 0$.

There exists positive constants $\eta \le 1$ and $\Gamma > 0$ depending on the data such that if $E_0 \le \Gamma\eta$, $(\varrho,\vec{u},\vartheta,I)$ is the unique classical solution to the problem (4.13)-(4.15) with boundary conditions (4.18),initial conditions (4.19) and compatibility conditions (4.20) in $(0,T)\times\Omega$ for any $T > 0$, such that

$$(\varrho - \overline{\varrho},\vec{u},\vartheta - \overline{\vartheta}) \in C([0,T],H^3(\Omega)),$$

$$\sup_{t\ge0}\|\varrho - \overline{\varrho}\|_{L^\infty(\Omega)} \le \overline{\varrho}/2,\qquad \inf_{x\in\overline{\Omega},t\ge0}\overline{\vartheta} > 0,$$

$$\partial_t\varrho \in C([0,T],H^2(\Omega)),\quad \partial_t\vec{u},\partial_t\vartheta \in C([0,T],H^1(\Omega)),$$

$$\partial_t\varrho,\partial_t\vec{u},\partial_t\vartheta \in L^2([0,T],H^2(\Omega)).$$

Moreover if $e_0 \le \eta$

$$\sup_{0\le t\le T}\|\varrho - \overline{\varrho}\|^2_{L^2(\Omega)} + \|\vec{u}\|^2_{L^2(\Omega)} + \|\vartheta - \overline{\vartheta}\|^2_{L^2(\Omega)} + \|\nabla_x\vartheta\|^2_{L^2(\Omega)} \le \Gamma e_0^2,$$

and

$$\sup_{0\le t\le T}\left(\|\varrho - \overline{\varrho}\|_{L^\infty(\Omega)} + \|\vartheta - \overline{\vartheta}\|_{L^\infty(\Omega)}\right) \le \Gamma e_0.$$

Remark 1. *As mentioned previously it is worth to note that when one considers the formal "nonconducting at rest" situation where $\kappa = 0$ and $\vec{u} = 0$ and in the no-scattering case ($\sigma_s \equiv 0$), one obtains from (4.1)- (4.4) a simplified system introduced by Bardos, Golse and Perthame [2] for which they proved global existence and diffusion limit (called "Rosseland approximation") under assumptions much more general than ours.*

4.2 The non-equilibrium diffusion regime

Expanding as above using (3.12) in (4.7) and evaluating the lowest orders terms we get

$$\frac{1}{4\pi}\int_{\mathcal{S}^2} I_0 \, \mathrm{d}\vec{\omega} = I_0, \tag{4.23}$$

$$\vec{\omega}\cdot\nabla_x I_0 = \sigma_s(\vartheta_0)\left(\frac{1}{4\pi}\int_{\mathcal{S}^2} I_1 \, \mathrm{d}\vec{\omega} - I_1\right), \tag{4.24}$$

and

$$\begin{aligned}\partial_t I_0 + \vec{\omega}\cdot\nabla_x I_1 &= \sigma_a(\vartheta_0)(B(\vartheta_0,\nu) - I_0)\\ +\sigma_s(\vartheta_0)\left(\frac{1}{4\pi}\int_{\mathcal{S}^2} I_2 \, \mathrm{d}\vec{\omega} - I_2\right) &+ \partial_\vartheta\sigma_s(\vartheta_0)\left(\frac{1}{4\pi}\int_{\mathcal{S}^2} I_1 \, \mathrm{d}\vec{\omega} - I_1\right)\vartheta_1.\end{aligned} \tag{4.25}$$

Plugging the first two relations into the last one, we find

$$\partial_t I_0 + \vec{\omega}\cdot\nabla_x \tilde{I}_1 - \vec{\omega}\otimes\vec{\omega}\ \mathrm{div}_x\left(\frac{1}{\sigma_s(\vartheta_0)}\nabla_x I_0\right)$$

$$= \sigma_a(\vartheta_0)(B(\vartheta_0,\nu) - I_0) + \sigma_s(\vartheta_0)\left(\frac{1}{4\pi}\int_{\mathcal{S}^2} I_2 \, \mathrm{d}\vec{\omega} - I_2\right) + \partial_\vartheta\sigma_s(\vartheta_0)\left(\frac{1}{4\pi}\int_{\mathcal{S}^2} I_1 \, \mathrm{d}\vec{\omega} - I_1\right)\vartheta_1.$$

Integrating in ν and $\vec{\omega}$ and using (4.23), (4.24) we get a diffusion equation for $N := \int_0^\infty I_0 \, d\nu$

$$\partial_t N - \frac{1}{3}\ \mathrm{div}_x\left(\frac{1}{\sigma_s(\vartheta_0)}\nabla_x N\right) = \sigma_a(\vartheta_0)\left(\frac{a}{3}\vartheta_0^4 - N\right), \tag{4.26}$$

and we get also

$$E_0^R = N, \quad \vec{F}_1^R = -\frac{1}{3\sigma_s}\ \nabla_x N, \quad \mathbb{P}_0^R = \frac{1}{3}\ N\ \mathbb{I}.$$

Keeping the zero order term in equations (4.2),(4.3)(4.4) we finally obtain a compressible Navier-Stokes-Fourier system with sources coupled to a diffusion equation for N. Omitting the 0 index, we get finally the system

$$\partial_t\varrho + \mathrm{div}_x(\varrho\vec{u}) = 0, \tag{4.27}$$

$$\partial_t(\varrho\vec{u}) + \mathrm{div}_x(\varrho\vec{u}\otimes\vec{u}) + \nabla_x\mathbf{p} = \mathrm{div}_x\mathbb{S}, \tag{4.28}$$

$$\partial_t\left(\frac{1}{2}\ \varrho|\vec{u}_0|^2 + \varrho\mathbf{e}\right) + \mathrm{div}_x\left(\left(\frac{1}{2}\ \varrho|\vec{u}|^2 + \varrho\mathbf{e} + \mathbf{p}\right)\vec{u} + \vec{\mathbf{q}} - \mathbb{S}\vec{u}\right) = 0, \tag{4.29}$$

$$\partial_t\left(\varrho s\right) + \mathrm{div}_x\left(\varrho s\vec{u}\right) + \mathrm{div}_x\left(\frac{\vec{q}}{\vartheta}\right) = \frac{1}{\vartheta}\left(\mathbb{S}:\nabla_x\vec{u} - \frac{\vec{q}\cdot\nabla_x\vartheta}{\vartheta}\right) + \frac{1}{3}\frac{\nabla_x N\cdot\vec{u}}{\vartheta} - \frac{\sigma_a(\vartheta)}{\vartheta}\left(a\vartheta^4 - N\right), \tag{4.30}$$

$$\partial_t N - \frac{1}{3}\ \mathrm{div}_x\left(\frac{1}{\sigma_s(\vartheta)}\nabla_x N\right) = \sigma_a(\vartheta)\left(a\vartheta^4 - N\right), \tag{4.31}$$

where $\mathbf{p} = p + \frac{1}{3}N$, $\mathbf{e} = e + \frac{N}{\varrho}$ and $\vec{\mathbf{q}} = \kappa\nabla_x\vartheta + \frac{1}{3\sigma_s}\nabla_x N$ with boundary conditions

$$\vec{u}|_{\partial\Omega} = 0,\ \nabla\vartheta\cdot\vec{n}|_{\partial\Omega} = 0,\ N|_{\partial\Omega} = 0, \tag{4.32}$$

initial conditions

$$(\varrho(x,t),\ \vec{u}(x,t),\ \vartheta(x,t), N(x,t))|_{t=0} = \left(\varrho^0(x),\ \vec{u}^0(x),\ \vartheta^0(x), N^0(x)\right), \tag{4.33}$$

for any $x\in\Omega$, with $N^0(x) = \int_0^\infty\int_{\mathcal{S}^2} I^0(x,\nu,\vec{\omega})\ d\vec{\omega}\ d\nu$ and the compatibility conditions

$$\vec{u}^2|_{\partial\Omega} = 0,\ \nabla\vartheta^0\cdot\vec{n}|_{\partial\Omega} = 0,\ N^0\big|_{\partial\Omega} = 0. \tag{4.34}$$

It will be useful as in [6] to define the non equilibrium temperature θ_r by

$$N = a\theta_r^4. \tag{4.35}$$

In analogy with previous works on asymptotic analysis of radiative transfer equation (see [2], [3]) we call (4.27)-(4.33) the Navier-Stokes-Rosseland system. As in the equilibrium case, we have a global existence result for solutions of this problem for small data.

Let $(\overline{\varrho}, 0, \overline{\vartheta}, \overline{N})$ be a given constant state with $\overline{\varrho} > 0$, $\overline{\vartheta} > 0$ and $\overline{N} = B(\overline{\vartheta})$. We note

$$e_0 := \|\varrho^0 - \overline{\varrho}\|_{L^\infty(\Omega)} + \|\vec{u}^0\|_{H^1(\Omega)} + \|\vartheta^0 - \overline{\vartheta}\|_{H^1(\Omega)} + \|N^0 - \overline{N}\|_{H^1(\Omega)} + \|\mathbb{T}^0\|_{L^2(\Omega)} + \|\mathbb{V}^0\|_{L^4(\Omega)}, \tag{4.36}$$

and

$$E_0 := e_0 + \|\nabla_x \varrho^0\|_{L^2(\Omega)} + \|\nabla_x \vartheta^0\|_{L^q(\Omega)} + \|\nabla_x \mathbb{T}^0\|_{L^2(\Omega)}, \tag{4.37}$$

for an arbitrary fixed q such that $3 < q < 6$. The following result holds

Theorem 4.2. *1. (Local solution) There exists a positive constant T_* such that $(\varrho, \vec{u}, \vartheta, N)$ is the unique classical solution to the problem (4.27)-(4.31) with boundary conditions (4.32), initial conditions (4.33) and the compatibility conditions (4.34) in $(0,T) \times \Omega$ for any $T < T_*$ such that*

$$(\varrho, \vec{u}, \vartheta, N) \in C([0,T], H^3(\Omega)),$$

$$\partial_t \varrho \in C([0,T], H^2(\Omega)), \quad \partial_t \vec{u}, \partial_t \vartheta, \partial_t N \in C([0,T], H^1(\Omega)),$$
$$\partial_t \varrho, \partial_t \vec{u}, \partial_t \vartheta, \partial_t N \in L^2([0,T], H^2(\Omega)).$$

2. (Global solution) Let $(\varrho^0 - \overline{\varrho}, \vec{u}^0, \vartheta^0 - \overline{\vartheta}, N^0 - \overline{N}) \in (H^3(\Omega))^6$ and $\inf \vartheta^0, \inf N^0 > 0$.

There exists positive constants $\eta \le 1$ and $\Gamma > 0$ depending on the data such that if $E_0 \le \Gamma\eta$, $(\varrho, \vec{u}, \vartheta, N)$ is the unique classical solution to the problem (4.27)-(4.31) with boundary conditions (4.32), initial conditions (4.33) and the compatibility conditions (4.34) in $(0,T) \times \Omega$ for any $T > 0$, such that

$$(\varrho - \overline{\varrho}, \vec{u}, \vartheta - \overline{\vartheta}, N - \overline{N}) \in C([0,T], H^3(\Omega)),$$

$$\sup_{t \ge 0} \|\varrho - \overline{\varrho}\|_{L^\infty(\Omega)} \le \overline{\varrho}/2, \qquad \inf_{x \in \overline{\Omega}, t \ge 0} \overline{\vartheta} > 0,$$

$$\partial_t \varrho \in C([0,T], H^2(\Omega)), \quad \partial_t \vec{u}, \partial_t \vartheta, \partial_t N \in C([0,T], H^1(\Omega)),$$
$$\partial_t \varrho, \partial_t \vec{u}, \partial_t \vartheta, \partial_t N \in L^2([0,T], H^2(\Omega)).$$

Moreover if $e_0 \le \eta$

$$\sup_{0 \le t \le T} \|\varrho - \overline{\varrho}\|^2_{L^2(\Omega)} + \|\vec{u}\|^2_{L^2(\Omega)} + \|\vartheta - \overline{\vartheta}\|^2_{L^2(\Omega)} + \|N - \overline{N}\|^2_{L^2(\Omega)} + \|\nabla_x \vartheta\|^2_{L^2(\Omega)} + \|\nabla_x N\|^2_{L^2(\Omega)} \le \Gamma e_0^2.$$

and

$$\sup_{0 \le t \le T} \left(\|\varrho - \overline{\varrho}\|_{L^\infty(\Omega)} + \|\vartheta - \overline{\vartheta}\|_{L^\infty(\Omega)} + \|N - \overline{N}\|_{L^\infty(\Omega)} \right) \le \Gamma e_0.$$

The proof of this result consists in a direct adaptation of the technique used in [11] and is omitted.

4.3 Relative entropy inequality

We can rephrase the existence result of Theorem 2.1 in the rescaled context as follows

Proposition 4.1. *Suppose that the conditions of Theorem 2.1 are satisfied. Then for any $\varepsilon > 0$ small enough there exists a weak solution $(\varrho_\varepsilon, \vec{u}_\varepsilon, \vartheta_\varepsilon, I_\varepsilon)$ to the radiative Navier-Stokes systems (1.1-1.4) for $(t, x, \vec{\omega}, \nu) \in [0,T]\times\Omega\times\mathcal{S}^2\times\mathbb{R}_+$, with boundary conditions (1.10 - 1.11) and initial conditions $(\varrho_{0,\varepsilon}, \vec{u}_{0,\varepsilon}, \vartheta_{0,\varepsilon}, I_{0,\varepsilon})$. More precisely we have*

$$\int_\Omega \varrho_\varepsilon(\tau,\cdot)\phi(\tau,\cdot)\ dx - \int_\Omega \varrho_{0,\varepsilon}\phi(0,\cdot)\ dx = \int_0^\tau \int_\Omega \varrho_\varepsilon\left(\partial_t\phi + \vec{u}_\varepsilon\cdot\nabla_x\phi\right)\ dx\ dt \tag{4.38}$$

for any $\phi \in C^1([0,T)\times\overline{\Omega})$, and any $\tau \in [0,T]$,

$$\int_\Omega \varrho_\varepsilon\vec{u}_\varepsilon(\tau,\cdot)\cdot\vec{\phi}(\tau,\cdot)\ dx - \int_\Omega \varrho_{0,\varepsilon}\vec{u}_{0,\varepsilon}\cdot\vec{\phi}(0,\cdot)\ dx$$

$$= \int_0^\tau \int_\Omega \left(\left(\varrho_\varepsilon\vec{u}_\varepsilon + \varepsilon\vec{F}^R_\varepsilon\right)\cdot\partial_t\vec{\phi} + \left(\varrho_\varepsilon\vec{u}_\varepsilon\otimes\vec{u}_\varepsilon + \mathbb{P}^R_\varepsilon\right):\nabla_x\vec{\phi} + p_\varepsilon\ \mathrm{div}_x\vec{\phi} - \mathbb{S}_\varepsilon:\nabla_x\vec{\phi}\right)\ dx\ dt = 0, \tag{4.39}$$

for any $\vec{\phi} \in C^1([0,T)\times\overline{\Omega};\mathbb{R}^3)$, and any $\tau \in [0,T]$, such that $\phi\cdot n|_{\partial\Omega} = 0$, with $p_\varepsilon = p(\varrho_\varepsilon, \vartheta_\varepsilon)$ and $\mathbb{S}_\varepsilon = \mathbb{S}(\vec{u}_\varepsilon, \vartheta_\varepsilon)$,

$$\int_0^\tau \int_\Omega \left(\frac{1}{2}\ \varrho_\varepsilon|\vec{u}_\varepsilon|^2 + \varrho_\varepsilon e_\varepsilon + \varepsilon E^R_\varepsilon\right)\ dx\ dt + \int_0^T \int_{\Gamma_+} \vec{\omega}\cdot\vec{n}_x I_\varepsilon(t,x,\vec{\omega},\nu)\ d\Gamma\ d\nu\ dt$$

$$= \int_\Omega \left(\frac{1}{2}\ \varrho_{0,\varepsilon}|\vec{u}_{0,\varepsilon}|^2 + \varrho_{0,\varepsilon}e_{0,\varepsilon} + \varepsilon E^R_{0,\varepsilon}\right)\ dx =: \mathcal{E}_0, \tag{4.40}$$

for a.a. $t \in [0,T]$ with $\Gamma_+ = \{(x,\vec{\omega}) \in \partial\Omega\times\mathcal{S}^2\ :\ \vec{\omega}\cdot\vec{n}_x \geq 0\}$ and with $e_\varepsilon = e(\varrho_\varepsilon, \vartheta_\varepsilon)$ and $E^R_\varepsilon(t,x) = \int_0^\infty\int_{\mathcal{S}^2} I_\varepsilon(t,x,\vec{\omega},\nu)\ d\vec{\omega}\ d\nu$

$$\int_0^\tau \int_\Omega \left(\left(\varrho_\varepsilon s_\varepsilon + \varepsilon s^R_\varepsilon\right)\partial_t\varphi + \left(\varrho_\varepsilon s_\varepsilon\vec{u}_\varepsilon + \vec{q}^R_\varepsilon\right)\cdot\nabla_x\varphi\right)\ dx\ dt + \int_0^\tau \int_\Omega \frac{\vec{q}_\varepsilon}{\vartheta_\varepsilon}\cdot\nabla_x\varphi\ dx\ dt + \left\langle\varsigma^m_\varepsilon + \varsigma^R_\varepsilon;\phi\right\rangle_{[\mathcal{M};C]([0,T\times\overline{\Omega})}$$

$$\leq -\int_\Omega \left(\varrho s_{0,\varepsilon} + \varepsilon s^R_{0,\varepsilon}\right)\varphi(0,\cdot)\ dx + \int_\Omega \left(\varrho s_\varepsilon + \varepsilon s^R_\varepsilon\right))(\tau,\cdot)\varphi(\tau,\cdot)\ dx, \tag{4.41}$$

where

$$\varsigma^m_\varepsilon = \frac{1}{\vartheta_\varepsilon}\left(\mathbb{S}_\varepsilon:\nabla_x\vec{u}_\varepsilon - \frac{\vec{q}_\varepsilon\cdot\nabla_x\vartheta_\varepsilon}{\vartheta_\varepsilon}\right), \tag{4.42}$$

and

$$\varsigma^R_\varepsilon \geq \int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu}\left[\log\frac{n(I_\varepsilon)}{n(I_\varepsilon)+1} - \log\frac{n(B_\varepsilon)}{n(B_\varepsilon)+1}\right]\sigma_{a_\varepsilon}^{(j)}(B_\varepsilon - I_\varepsilon)\ d\vec{\omega}d\nu$$

$$+ \int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu}\left[\log\frac{n(I_\varepsilon)}{n(I_\varepsilon)+1} - \log\frac{n(\tilde{I}_\varepsilon)}{n(\tilde{I}_\varepsilon)+1}\right]\sigma_{s_\varepsilon}^{(j)}(\tilde{I}_\varepsilon - I_\varepsilon)\ d\vec{\omega}d\nu, \tag{4.43}$$

for $\varphi \in C^\infty_c([0,T)\times\overline{\Omega})$ and any $\tau \in [0,T]$, with $\varsigma^m_\varepsilon \in \mathcal{M}^+([0,T)\times\overline{\Omega})$ and $\varsigma^R_\varepsilon \in \mathcal{M}^+([0,T)\times\overline{\Omega})$, where $\mathcal{M}(X)$ is the set of signed Borel measures on X and $\mathcal{M}^+(X)$ is the cone of non-negative elements of $\mathcal{M}(X)$.

We consider two possible values for the transport coefficients in the two cases $j = 1$ (equilibrium case) or $j = 2$ (non-equilibrium case)

$$\sigma_{a_\varepsilon}^{(j)} = \left|\begin{array}{l} \frac{1}{\varepsilon}\ \sigma_a(\vartheta_\varepsilon)\quad \text{if } j = 1, \\ \varepsilon\sigma_a(\vartheta_\varepsilon)\quad \text{if } j = 2, \end{array}\right. \tag{4.44}$$

and

$$\sigma_{s_\varepsilon}^{(j)} = \left|\begin{array}{l} \varepsilon\sigma_s(\vartheta_\varepsilon)\quad \text{if } j = 1, \\ \frac{1}{\varepsilon}\ \sigma_s(\vartheta_\varepsilon)\quad \text{if } j = 2. \end{array}\right. \tag{4.45}$$

Denoting $B_\varepsilon = B(\nu, \vec{\omega}, \vec{u}_\varepsilon, \vartheta_\varepsilon)$, $\vec{q}_\varepsilon = \kappa(\vartheta_\varepsilon)\nabla_x\vartheta_\varepsilon$, $s_\varepsilon = s(\varrho_\varepsilon, \ \vartheta_\varepsilon)$, $s^R_\varepsilon = s^R(I_\varepsilon)$, $\vec{q}^R_\varepsilon = \vec{q}^R(I_\varepsilon)$, *and* $\tilde{I}_\varepsilon := \frac{1}{4\pi}\int_{\mathcal{S}^2} I_\varepsilon(t, x, \nu, \vec{\omega})\,\mathrm{d}\vec{\omega}$, *we have finally*

$$\int_0^\tau \int_\Omega \int_0^\infty \int_{\mathcal{S}^2} (\varepsilon\partial_t\psi + \vec{\omega}\cdot\nabla_x\psi)\, I_\varepsilon\ d\vec{\omega}\ d\nu\ dx\ dt$$

$$+\int_0^\tau \int_\Omega \int_0^\infty \int_{\mathcal{S}^2} \left[\sigma_{a_\varepsilon}^{(j)}\,(B_\varepsilon - I_\varepsilon) + \sigma_{s_\varepsilon}^{(j)}\left(\tilde{I}_\varepsilon - I_\varepsilon\right)\right]\psi\ d\vec{\omega}\ d\nu\ dx\ dt,$$

$$= \int_\Omega \int_0^\infty \int_{\mathcal{S}^2} \varepsilon I_{0,\varepsilon}\psi(0, x, \vec{\omega}, \nu)\ d\vec{\omega}\ d\nu\ dx - \int_\Omega \int_0^\infty \int_{\mathcal{S}^2} \varepsilon I_\varepsilon\psi(\tau, x, \vec{\omega}, \nu)\ d\vec{\omega}\ d\nu\ dx$$

$$+\int_0^\tau \int_{\Gamma_+} \int_0^\infty \vec{\omega}\cdot\vec{n}_x I_\varepsilon\psi\ d\Gamma\ d\nu\ dt, \tag{4.46}$$

for any $\psi \in C^1([0,T) \times \overline{\Omega} \times \mathcal{S}^2 \times \mathbb{R}_+)$, *any* $\tau \in [0, T]$, *for* $j = 1, 2$.

Just mention that any weak solution $(\varrho_\varepsilon, \vec{u}_\varepsilon, \vartheta_\varepsilon, I_\varepsilon)$ enjoys all of the regularity and integrability properties given in Theorem 2.1.

Following now the lines of [15] we introduce a relative entropy inequality satisfied by any weak solution $(\varrho, \vec{u}, \vartheta, I)$ of the radiative Navier-Stokes system (see also [27] in a more general context). Let us consider a set $\{r, \Theta, \vec{U}\}$ of arbitrary smooth functions such that r and Θ are bounded below away from zero and $\vec{U}\Big|_{\partial\Omega} = 0$. We call *ballistic free energy* the thermodynamical potential given by $H_\Theta(\varrho, \vartheta) = \varrho e(\varrho, \vartheta) - \Theta\varrho s(\varrho, \vartheta)$, and *radiative ballistic free energy* the potential $H^R_\Theta(I) = E^R(I) - \Theta s^R(I)$. The *relative entropy* is then defined by

$$\mathcal{E}(\varrho, \vartheta | r, \Theta) := H_\Theta(\varrho, \vartheta) - \partial_\rho H_\Theta(r, \Theta)(\varrho - r) - H_\Theta(r, \Theta).$$

One observes that, after thermodynamical stability, $\rho \to H_\Theta(\rho, \Theta)$ is strictly convex and $\theta \to H_\Theta(\rho, \theta)$ attains its global minimum at $\theta = \Theta$.

Testing equation (4.38) with $\phi = \frac{1}{2}\ |\vec{U}|^2$, we get

$$\int_\Omega \frac{1}{2}\ \varrho_\varepsilon |\vec{U}|^2(\tau, \cdot)\ dx - \int_\Omega \frac{1}{2}\ \varrho_{0,\varepsilon}|\vec{U}(0, \cdot)|^2\ dx = \int_0^\tau \int_\Omega \varrho_\varepsilon \left(\vec{U}\cdot\partial_t\vec{U} + \vec{u}_\varepsilon\cdot\nabla_x\vec{U}\cdot\vec{U}\right)\ dx\ dt. \tag{4.47}$$

Testing now equation (4.39) with $\phi = \vec{U}$, we get

$$\int_\Omega \varrho_\varepsilon\vec{u}_\varepsilon(\tau, \cdot)\cdot\vec{U}(\tau, \cdot)\ dx - \int_\Omega \varrho_{0,\varepsilon}\vec{u}_{0,\varepsilon}\cdot\vec{U}(0, \cdot)\ dx.$$

$$= \int_0^\tau \int_\Omega \left(\left(\varrho_\varepsilon\vec{u}_\varepsilon + \varepsilon\vec{F}^R_\varepsilon\right)\cdot\partial_t\vec{U} + (\varrho_\varepsilon\vec{u}_\varepsilon\otimes\vec{u}_\varepsilon + \mathbb{P}_\varepsilon) : \nabla_x\vec{U} + p_\varepsilon\ \mathrm{div}_x\vec{U} - \mathbb{S}_\varepsilon : \nabla_x\vec{U}\right)\ dx\ dt. \tag{4.48}$$

Combining (4.47), (4.48) and (4.40), we get

$$\int_0^\tau \int_\Omega \left(\frac{1}{2}\ \varrho_\varepsilon|\vec{u}_\varepsilon - \vec{U}|^2 + \varrho_\varepsilon e_\varepsilon + \varepsilon E^R_\varepsilon\right)(\tau, \cdot)\ dx\ dt + \int_0^T \int_{\Gamma_+} \vec{\omega}\cdot\vec{n}_x I_\varepsilon(t, x, \vec{\omega}, \nu)\ d\Gamma\ d\nu\ dt$$

$$= \int_\Omega \left(\frac{1}{2}\ \varrho_{0,\varepsilon}|\vec{u}_{0,\varepsilon} - \vec{U}(0, \cdot)|^2 + \varrho_{0,\varepsilon}e_{0,\varepsilon} + \varepsilon E^R_{0,\varepsilon}\right)\ dx$$

$$+\int_0^\tau \int_\Omega \left(\left(\varrho_\varepsilon\partial_t\vec{U} + \varrho_\varepsilon\vec{u}_\varepsilon\cdot\nabla_x\vec{U}\right)\cdot\left(\vec{U} - \vec{u}_\varepsilon\right) - p_\varepsilon\ \mathrm{div}_x\vec{U} + \mathbb{S}_\varepsilon : \nabla_x\vec{U} - \varepsilon\vec{F}^R_\varepsilon\cdot\partial_t\vec{U} - \mathbb{P}_\varepsilon : \nabla_x\vec{U}\right)\ dx\ dt. \tag{4.49}$$

Testing now equation (4.41) with $\varphi = \Theta$, we get

$$\int_\Omega \left(\varrho_{0,\varepsilon} s_{0,\varepsilon} + \varepsilon s^R_{0,\varepsilon}\right) \Theta(0,\cdot)\ dx - \int_\Omega \left(\varrho_\varepsilon s_\varepsilon + \varepsilon s^R_\varepsilon\right) \Theta(\tau,\cdot)\ dx + \int_0^\tau \int_\Omega \frac{\Theta}{\vartheta_\varepsilon} \left(\mathbb{S}_\varepsilon : \nabla_x \vec{u}_\varepsilon - \frac{\vec{q}_\varepsilon \cdot \nabla_x \vartheta_\varepsilon}{\vartheta_\varepsilon}\right) \mathrm{d}x\ dt$$
$$+ \int_0^\tau \int_\Omega \Theta \left\{ \int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu} \left[\log \frac{n(I_\varepsilon)}{n(I_\varepsilon)+1} - \log \frac{n(B_\varepsilon)}{n(B_\varepsilon)+1}\right] \sigma_{a_\varepsilon}^{(j)} (B_\varepsilon - I_\varepsilon)\ d\vec{\omega}\ d\nu \right.$$
$$\left. + \int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu} \left[\log \frac{n(I_\varepsilon)}{n(I_\varepsilon)+1} - \log \frac{n(\tilde{I}_\varepsilon)}{n(\tilde{I}_\varepsilon)+1}\right] \sigma_{s_\varepsilon}^{(j)} (\tilde{I}_\varepsilon - I_\varepsilon)\ d\vec{\omega}\ d\nu \right\}\ dx\ dt$$
$$\leq -\int_0^\tau \int_\Omega \left(\left(\varrho_\varepsilon s_\varepsilon + \varepsilon s^R_\varepsilon\right) \partial_t \Theta + \left(\varrho_\varepsilon s_\varepsilon \vec{u}_\varepsilon + \vec{q}^R_\varepsilon\right) \cdot \nabla_x \Theta\right)\ dx\ dt - \int_0^\tau \int_\Omega \frac{\vec{q}_\varepsilon}{\vartheta_\varepsilon} \cdot \nabla_x \Theta\ dx\ dt. \tag{4.50}$$

From (4.49) and (4.50) we get

$$\int_\Omega \left(\frac{1}{2}\ \varrho_\varepsilon |\vec{u}_\varepsilon - \vec{U}|^2 + \varrho_\varepsilon e_\varepsilon + \varepsilon E^R_\varepsilon - (\varrho_\varepsilon s_\varepsilon + \varepsilon s^R_\varepsilon)\Theta\right)(\tau,\cdot)\ dx + \int_0^T \int_{\Gamma_+} \vec{\omega} \cdot \vec{n}_x I_\varepsilon(t,x,\vec{\omega},\nu)\ d\Gamma\ d\nu\ dt$$
$$+ \int_0^\tau \int_\Omega \frac{\Theta}{\vartheta_\varepsilon} \left(\mathbb{S}_\varepsilon : \nabla_x \vec{u}_\varepsilon - \frac{\vec{q}_\varepsilon \cdot \nabla_x \vartheta_\varepsilon}{\vartheta_\varepsilon}\right) \mathrm{d}x\ dt$$
$$+ \int_0^\tau \int_\Omega \Theta \left\{ \int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu} \left[\log \frac{n(I_\varepsilon)}{n(I_\varepsilon)+1} - \log \frac{n(B_\varepsilon)}{n(B_\varepsilon)+1}\right] \sigma_{a_\varepsilon}^{(j)} (B_\varepsilon - I_\varepsilon)\ d\vec{\omega}\ d\nu \right.$$
$$\left. + \int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu} \left[\log \frac{n(I_\varepsilon)}{n(I_\varepsilon)+1} - \log \frac{n(\tilde{I}_\varepsilon)}{n(\tilde{I}_\varepsilon)+1}\right] \sigma_{s_\varepsilon}^{(j)} (\tilde{I}_\varepsilon - I_\varepsilon)\ d\vec{\omega}\ d\nu \right\}\ dx\ dt$$
$$\leq \int_\Omega \left(\frac{1}{2}\ \varrho_{0,\varepsilon} |\vec{u}_{0,\varepsilon} - \vec{U}(0,\cdot)|^2 + \varrho_{0,\varepsilon} e_{0,\varepsilon} + \varepsilon E^R_{0,\varepsilon} - (\varrho_0 s_{0,\varepsilon} + \varepsilon s^R_{0,\varepsilon})\Theta(0,\cdot)\right)\ dx$$
$$+ \int_0^\tau \int_\Omega \left(\left(\varrho_\varepsilon \partial_t \vec{U} + \varrho_\varepsilon \vec{u}_\varepsilon \cdot \nabla_x \vec{U}\right) \cdot \left(\vec{U} - \vec{u}_\varepsilon\right) - p_\varepsilon\ \mathrm{div}_x \vec{U} + \mathbb{S}_\varepsilon : \nabla_x \vec{U} - \varepsilon \vec{F}^R_\varepsilon \cdot \partial_t \vec{U} - \mathbb{P}_\varepsilon : \nabla_x \vec{U}\right)\ dx\ dt$$
$$- \int_0^\tau \int_\Omega \left(\left(\varrho_\varepsilon s_\varepsilon + \varepsilon s^R_\varepsilon\right) \partial_t \Theta + \left(\varrho_\varepsilon s_\varepsilon \vec{u}_\varepsilon + \vec{q}^R_\varepsilon\right) \cdot \nabla_x \Theta\right)\ dx\ dt - \int_0^\tau \int_\Omega \frac{\vec{q}_\varepsilon}{\vartheta_\varepsilon} \cdot \nabla_x \Theta\ dx\ dt. \tag{4.51}$$

Testing equation (4.38) with $\phi = \partial_\rho H_\Theta(r,\Theta)$, we get

$$\int_\Omega \varrho_\varepsilon \partial_\rho H_\Theta(r,\Theta)(\tau,\cdot)\ dx - \int_\Omega \varrho_{0,\varepsilon} \partial_\rho H_\Theta\left(r(0,\cdot),\Theta(0,\cdot)\right)\ dx$$
$$= \int_0^\tau \int_\Omega \left(\varrho_\varepsilon \partial_t \left(\partial_\rho H_\Theta(r,\Theta)\right) + \varrho_\varepsilon \vec{u}_\varepsilon \cdot \nabla_x \left(\partial_\rho H_\Theta(r,\Theta)\right)\right)\ dx\ dt. \tag{4.52}$$

From (4.51) and (4.52) we get

$$\int_\Omega \left(\frac{1}{2}\ \varrho_\varepsilon |\vec{u}_\varepsilon - \vec{U}|^2 + H_\Theta(\varrho_\varepsilon, \vartheta_\varepsilon) - H_\Theta(r,\Theta) - \partial_\rho H_\Theta(r,\Theta)(\varrho_\varepsilon - r) + \varepsilon H^R_\Theta(I_\varepsilon)\right)(\tau,\cdot)\ dx$$
$$+ \int_0^T \int_{\Gamma_+} \vec{\omega} \cdot \vec{n}_x I_\varepsilon(t,x,\vec{\omega},\nu)\ d\Gamma\ d\nu\ dt + \int_0^\tau \int_\Omega \frac{\Theta}{\vartheta_\varepsilon} \left(\mathbb{S}_\varepsilon : \nabla_x \vec{u}_\varepsilon - \frac{\vec{q}_\varepsilon \cdot \nabla_x \vartheta_\varepsilon}{\vartheta_\varepsilon}\right) \mathrm{d}x\ dt$$
$$+ \int_0^\tau \int_\Omega \Theta \left\{ \int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu} \left[\log \frac{n(I_\varepsilon)}{n(I_\varepsilon)+1} - \log \frac{n(B_\varepsilon)}{n(B_\varepsilon)+1}\right] \sigma_{a_\varepsilon}^{(j)} (B_\varepsilon - I_\varepsilon)\ d\vec{\omega}\ d\nu \right.$$
$$\left. + \int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu} \left[\log \frac{n(I_\varepsilon)}{n(I_\varepsilon)+1} - \log \frac{n(\tilde{I}_\varepsilon)}{n(\tilde{I}_\varepsilon)+1}\right] \sigma_{s_\varepsilon}^{(j)} (\tilde{I}_\varepsilon - I_\varepsilon)\ d\vec{\omega}\ d\nu \right\}\ dx\ dt$$

$$\le \int_\Omega \frac{1}{2}\ \varrho_{0,\varepsilon}|\vec{u}_{0,\varepsilon} - \vec{U}(0,\cdot)|^2\ dx$$

$$+ \int_\Omega \left(H_{\Theta(0,\cdot)}(\varrho_{0,\varepsilon},\vartheta_{0,\varepsilon}) - H_{\Theta(0,\cdot)}(r(0,\cdot),\Theta(0,\cdot)) - \partial_{\varrho_{0,\varepsilon}} H_{\Theta(0,\cdot)}(r(0,\cdot),\Theta(0,\cdot))(\varrho_{0,\varepsilon} - r(0,\cdot)) \right.$$

$$\left. + \varepsilon H^R_{\Theta(0,\cdot)}(I_{0,\varepsilon})\right)\ dx$$

$$+ \int_0^\tau \int_\Omega \left(\left(\varrho_\varepsilon \partial_t \vec{U} + \varrho_\varepsilon \vec{u}_\varepsilon \cdot \nabla_x \vec{U}\right)\cdot\left(\vec{U} - \vec{u}_\varepsilon\right) - p_\varepsilon\ \mathrm{div}_x \vec{U} + \mathbb{S}_\varepsilon : \nabla_x \vec{U} - \varepsilon \vec{F}^R_\varepsilon \cdot \partial_t \vec{U} - \mathbb{P}^R_\varepsilon : \nabla_x \vec{U}\right)\ dx\ dt$$

$$- \int_0^\tau \int_\Omega \left(\varrho_\varepsilon s_\varepsilon \partial_t \Theta + \varrho_\varepsilon s_\varepsilon \vec{u}_\varepsilon \cdot \nabla_x \Theta + \frac{\vec{q}_\varepsilon}{\vartheta_\varepsilon}\cdot \nabla_x \Theta\right)\ dx\ dt$$

$$- \int_0^\tau \int_\Omega \left(\varepsilon s^R_\varepsilon \partial_t \Theta + \vec{q}^R_\varepsilon \cdot \nabla_x \Theta\right)\ dx\ dt - \int_0^\tau \int_\Omega \left(\varrho_\varepsilon \partial_t \Big(\partial_{\varrho_\varepsilon} H_\Theta(r,\Theta)\Big) + \varrho_\varepsilon \vec{u}_\varepsilon \cdot \nabla_x \Big(\partial_\varrho H_\Theta(r,\Theta)\Big)\right)\ dx\ dt$$

$$+ \int_0^\tau \int_\Omega \partial_t \Big(r \partial_\varrho H_\Theta(r,\Theta) - H_\Theta(r,\Theta)\Big)\ dx\ dt. \tag{4.53}$$

Observing finally that for $D = \partial_t$ or $D = \nabla_x$ one has

$$D\partial_\varrho H_\Theta(r,\Theta) = -s(r,\Theta) D\Theta - r\partial_\varrho s(r,\Theta) D\Theta + \partial^2_{\varrho,\varrho} H_\Theta(r,\Theta) D\varrho + \partial^2_{\varrho,\vartheta} H_\Theta(r,\Theta) D\vartheta,$$

and using the thermodynamical relations $\partial^2_{\varrho,\varrho} H_\Theta(r,\Theta) = \frac{1}{r}\ \partial_\varrho p(r,\Theta)$, $r\partial_\varrho s(r,\Theta) = -\frac{1}{r}\ \partial_\vartheta p(r,\Theta)$, and $\partial^2_{\varrho,\vartheta} H_\Theta(r,\Theta) = \partial_\varrho \Big(\varrho(\vartheta - \Theta)\partial_\vartheta s\Big)(r,\Theta) = (\vartheta - \Theta)\partial_\vartheta \Big(\varrho \partial_\vartheta s(\varrho,\vartheta)\Big)(r,\Theta) = 0$, equation (4.53) rewrites after some algebraic rearrangements (see [15] for details)

$$\int_\Omega \left(\frac{1}{2}\ \varrho_\varepsilon |\vec{u}_\varepsilon - \vec{U}|^2 + \mathcal{E}\left(\varrho_\varepsilon,\vartheta_\varepsilon | r,\Theta\right) + \varepsilon H^R(I_\varepsilon)\right)(\tau,\cdot)\ dx + \int_0^\tau \int_{\Gamma_+} \vec{\omega}\cdot \vec{n}_x I_\varepsilon(t,x,\vec{\omega},\nu)\ d\Gamma\ d\nu\ dt$$

$$+ \int_0^\tau \int_\Omega \frac{\Theta}{\vartheta_\varepsilon}\left(\mathbb{S}_\varepsilon : \nabla_x \vec{u}_\varepsilon - \frac{\vec{q}_\varepsilon \cdot \nabla_x \vartheta_\varepsilon}{\vartheta_\varepsilon}\right)\ dx\ dt$$

$$+ \int_0^\tau \int_\Omega \int_0^\infty \int_{\mathcal{S}^2} \frac{\Theta}{\nu}\left[\log \frac{n(I_\varepsilon)}{n(I_\varepsilon)+1} - \log \frac{n(B_\varepsilon)}{n(B_\varepsilon)+1}\right] \sigma_{a_\varepsilon}{}^{(j)} (B_\varepsilon - I_\varepsilon)\ d\vec{\omega}\ d\nu\ dx\ dt$$

$$+ \int_0^\tau \int_\Omega \int_0^\infty \int_{\mathcal{S}^2} \frac{\Theta}{\nu}\left[\log \frac{n(I_\varepsilon)}{n(I_\varepsilon)+1} - \log \frac{n(\tilde{I}_\varepsilon)}{n(\tilde{I}_\varepsilon)+1}\right] \sigma_{s_\varepsilon}{}^{(j)} (\tilde{I}_\varepsilon - I_\varepsilon)\ d\vec{\omega}\ d\nu\ dx\ dt,$$

$$\le \int_\Omega \frac{1}{2}\left(\varrho_{0,\varepsilon}|\vec{u}_{0,\varepsilon} - \vec{U}(0,\cdot)|^2 + \mathcal{E}\left(\varrho_{0,\varepsilon},\vartheta_{0,\varepsilon} | r(0,\cdot),\Theta(0,\cdot)\right) + \varepsilon H^R(I_{0,\varepsilon})\right)\ dx$$

$$+ \int_0^\tau \int_\Omega \varrho_\varepsilon (\vec{u}_\varepsilon - \vec{U})\cdot \nabla_x \vec{U} \cdot \left(\vec{U} - \vec{u}_\varepsilon\right)\ dx\ dt + \int_0^\tau \int_\Omega \varrho_\varepsilon\left(s_\varepsilon - s(r,\Theta)\right)\left(\vec{U} - \vec{u}_\varepsilon\right)\cdot \nabla_x \Theta\ dx\ dt$$

$$+ \int_0^\tau \int_\Omega \left(\varrho_\varepsilon \left(\partial_t \vec{U} + \vec{U}\cdot \nabla_x \vec{U}\right)\cdot \left(\vec{U} - \vec{u}_\varepsilon\right)\right)\ dx\ dt$$

$$- \int_0^\tau \int_\Omega \left(p_\varepsilon\ \mathrm{div}_x \vec{U} - \mathbb{S}_\varepsilon : \nabla_x \vec{U}\right)\ dx\ dt - \int_0^\tau \int_\Omega \left(\varepsilon s^R_\varepsilon \partial_t \Theta + \vec{q}^R_\varepsilon \cdot \nabla_x \Theta\right)\ dx\ dt$$

$$- \int_0^\tau \int_\Omega \left(\varrho_\varepsilon\left(s_\varepsilon - s(r,\Theta)\right)\partial_t \Theta\right)\ dx\ dt - \int_0^\tau \int_\Omega \varrho_\varepsilon\left(s_\varepsilon - s(r,\Theta)\right)\vec{U}\cdot \nabla_x \Theta\ dx\ dt$$

$$- \int_0^\tau \int_\Omega \frac{\vec{q}_\varepsilon}{\vartheta_\varepsilon}\cdot \nabla_x \Theta\ dx\ dt + \int_0^\tau \int_\Omega \left(\left(1 - \frac{\varrho_\varepsilon}{r}\right)\partial_t p(r,\Theta) - \frac{\varrho_\varepsilon}{r}\ \vec{u}_\varepsilon \nabla_x p(r,\Theta)\right)\ dx\ dt$$

$$- \int_0^\tau \int_\Omega \left(\varepsilon \vec{F}^R_\varepsilon \cdot \partial_t \vec{U} + \mathbb{P}^R_\varepsilon : \nabla_x \vec{U}\right)\ dx\ dt =: K_0 + \sum_{j=1}^{10} \int_0^\tau K_j(t)\ dt. \tag{4.54}$$

It will be the goal of our next Section 4.4 to provide a bound for the right-hand side of (4.54).

4.4 Uniform estimates

Our intention is to apply the previous relative entropy inequality (4.54) with $(\varrho, \vec{u}, \vartheta)$ is the classical solution of the target system (in the equilibrium case or in the non equilibrium case), in order to bound the quantities $\varrho_\varepsilon - r, \vec{u}_\varepsilon - \vec{U}, \vartheta_\varepsilon - \Theta, E^R - N$. Note that in the equilibrium case: $N = \int_0^\infty B(\nu, \vartheta)\, d\nu = a\Theta^4$ while accordingly in the non-equilibrium case, N is the solution of the diffusion equation (4.31).

Just mention that the existence of classical solutions of the previous target systems is either local (for $T < T_*$ small enough) or corresponds to a small departure from an equilibrium state. This last possibility corresponding to the kind of regime we are interested in (diffusion limits), we suppose in the following that the data of the problem satisfy the smallness requirements of Theorems 4.1 and 4.2.

Following the definitions of Section 2, we choose positive numbers $(\underline{\varrho}, \overline{\varrho}, \underline{\vartheta}, \overline{\vartheta}, \underline{E}, \overline{E})$ in the equilibrium case and $(\underline{\varrho}, \overline{\varrho}, \underline{\vartheta}, \overline{\vartheta}, \underline{N}, \overline{N})$ in the non-equilibrium case, such that

$$0 < \underline{\varrho} \leq \frac{1}{2} \min_{(t,x)\in[0,T]\times\overline{\Omega}} r(t,x) \leq 2 \max_{(t,x)\in[0,T]\times\overline{\Omega}} r(t,x) \leq \overline{\varrho},$$

$$0 < \underline{\vartheta} \leq \frac{1}{2} \min_{(t,x)\in[0,T]\times\overline{\Omega}} \Theta(t,x) \leq 2 \max_{(t,x)\in[0,T]\times\overline{\Omega}} \Theta(t,x) \leq \overline{\vartheta},$$

$$0 < \underline{E} \leq \frac{1}{2} \min_{(t,x)\in[0,T]\times\overline{\Omega}} E^R(t,x) \leq 2 \max_{(t,x)\in[0,T]\times\overline{\Omega}} E^R(t,x) \leq \overline{E},$$

$$0 < \underline{N} \leq \frac{1}{2} \min_{(t,x)\in[0,T]\times\overline{\Omega}} N(t,x) \leq 2 \max_{(t,x)\in[0,T]\times\overline{\Omega}} N(t,x) \leq \overline{N},$$

and we split any measurable function h as $h = h_{ess} + h_{res}$, where $h_{ess}(t,x) = h(t,x)$ if $(\varrho, \vartheta, E^R) \in [\underline{\varrho}, \overline{\varrho}] \times [\underline{\vartheta}, \overline{\vartheta}] \times [\underline{\vartheta}, \overline{\vartheta}]$ (equilibrium) (or if $(\varrho, \vartheta, E^R) \in [\underline{\varrho}, \overline{\varrho}] \times [\underline{\vartheta}, \overline{\vartheta}] \times [\underline{N}, \overline{E}]$ (non-equilibrium), and $h_{ess}(t,x) = 0$ otherwise.

Then applying Lemma 3.1, we see that there exist positive constants C_j for $j = 1, \cdots, 6$ such that

$$C_1 \left(|\varrho_\varepsilon - r|^2 + |\vartheta_\varepsilon - \Theta|^2\right) \leq H_\Theta(\varrho_\varepsilon, \vartheta_\varepsilon) - (\varrho_\varepsilon - r)\partial_\Theta H_\Theta(r, \Theta) - H_\Theta(r, \Theta)$$
$$\leq C_2 \left(|\varrho_\varepsilon - r|^2 + |\vartheta_\varepsilon - \Theta|^2\right), \tag{4.55}$$

for all $(\varrho_\varepsilon, \vartheta_\varepsilon) \in [\underline{\varrho}, \overline{\varrho}] \times [\underline{\vartheta}, \overline{\vartheta}]$,

$$H_\Theta(\varrho_\varepsilon, \vartheta_\varepsilon) - (\varrho_\varepsilon - r)\partial_\Theta H_\Theta(r, \Theta) - H_\Theta(r, \Theta) \geq C_3 \left(1 + \varrho e(\varrho_\varepsilon, \vartheta_\varepsilon) + \varrho |s(\varrho_\varepsilon, \vartheta_\varepsilon)|\right), \tag{4.56}$$

otherwise. In the same stroke we have

$$C_4 \int_0^\infty \int_{\mathcal{S}^2} |I_\varepsilon - B(\nu, \vec{\omega}, \vec{U}, \Theta)|^2 d\vec{\omega}\, d\nu \leq H^R(I_\varepsilon) \leq C_5 \int_0^\infty \int_{\mathcal{S}^2} |I_\varepsilon - B(\nu, \vec{\omega}, \vec{U}, \Theta)|^2 d\vec{\omega}\, d\nu, \tag{4.57}$$

for all $E^R_\varepsilon \in [\underline{E}, \overline{E}]$,

$$H^R(I_\varepsilon) \geq C_6(1 + E^R_\varepsilon + |s^R_\varepsilon|), \tag{4.58}$$

otherwise, with Θ replaced by Θ_r in the non equilibrium case. We have now the crucial inequality

Lemma 4.1. *Let $(\varrho^{(e)}, \vec{u}^{(e)}, \theta^{(e)})$ be the solution of problem (4.13-4.19) satisfying the conditions of Theorem 4.1 (equilibrium case) and let $(\varrho^{(ne)}, \vec{u}^{(ne)}, \theta^{(ne)}, \theta_r^{(ne)})$ be the solution of problem (4.13-4.19) satisfying the conditions of Theorem 4.2 (non equilibrium case) and choose $(r, \vec{U}, \Theta) = (\varrho^{(e)}, \vec{u}^{(e)}, \theta^{(e)})$ in the equilibrium case or $(r, \vec{U}, \Theta, \Theta_r) = (\varrho^{(ne)}, \vec{u}^{(ne)}, \theta^{(ne)}, \theta_r^{(ne)})$ in the non equilibrium case. One has the following relative entropy inequality*

$$\int_\Omega \left(\frac{1}{2}\, \varrho_\varepsilon |\vec{u}_\varepsilon - \vec{U}|^2 + \mathcal{E}\left(\varrho_\varepsilon, \vartheta_\varepsilon | r, \Theta\right) + H^R(I_\varepsilon)\right)(t, \cdot)\, dx \tag{4.59}$$

$$\leq \frac{1}{\varepsilon}\left[\mathcal{C}e_0 + \int_\Omega \Big(\frac{1}{2}\varrho_{0,\varepsilon}|\vec{u}_{0,\varepsilon} - \vec{U}(0,\cdot)|^2 + \mathcal{E}\left(\varrho_{0,\varepsilon}, \vartheta_{0,\varepsilon}|r(0,\cdot), \Theta(0,\cdot)\right) + H^R(I_{0,\varepsilon})\Big)dx\right] e^{\frac{\mathcal{C}'}{\varepsilon}t},$$

where $\mathcal{C}$ and $\mathcal{C}'$ are positive constant depending on $(r, \vec{U}, \Theta, \Theta_r)$ and e_0 is the same as in Theorems 4.1 and 4.2.

The lengthy proof of this result is given in the Appendix.

Remark 2. *The price to pay in order to get rid of any ε in the left-hand side of the inequality (4.59) (compare with (4.54)) is the large factor $\frac{\mathcal{C}'}{\varepsilon}$ which makes the temporal range of validity of (4.59) very small.*

Using this inequality the following estimates hold

Lemma 4.2. *Suppose that $e_0 = O(\varepsilon^2)$ (initial data of the target systems are "well pepared") and suppose also that initial data of the primitive system and any of the target systems are close in the following sense*

$$\|\varrho_{0,\varepsilon} - \varrho_0\|_{L^2(\Omega)} \leq C\varepsilon, \|\vartheta_{0,\varepsilon} - \vartheta_0\|_{L^2(\Omega)} \leq C\varepsilon, \|\sqrt{\varrho_{0,\varepsilon}}\ (\vec{u}_{0,\varepsilon} - \vec{u})\|_{L^2(\Omega;\mathbb{R}^3)} \leq C\varepsilon.$$

Then the following estimates hold

$$\Big(\varsigma^m_\varepsilon + \varsigma^R_\varepsilon\Big)\Big[[0,T] \times \overline{\Omega}\Big] \leq C\varepsilon, \tag{4.60}$$

$$\text{ess} \sup_{t\in(0,T)} |\mathcal{M}^\varepsilon_{res}(t)| \leq C\varepsilon, \tag{4.61}$$

$$\text{ess} \sup_{t\in(0,T)} \|\varrho_\varepsilon - \varrho]_{ess}(t)\|_{L^2(\Omega)} \leq C\sqrt{\varepsilon}, \tag{4.62}$$

$$\text{ess} \sup_{t\in(0,T)} \|[\vartheta_\varepsilon - \vartheta]_{ess}(t)\|_{L^2(\Omega)} \leq C\sqrt{\varepsilon}, \tag{4.63}$$

$$\text{ess} \sup_{t\in(0,T)} \|\sqrt{\varrho_\varepsilon}\left(\vec{u}_\varepsilon(t) - \vec{u}(t)\right)\|_{L^2(\Omega;\mathbb{R}^3)} \leq C\sqrt{\varepsilon}, \tag{4.64}$$

$$\text{ess} \sup_{t\in(0,T)} \|[E^R_\varepsilon - E^R(I)]_{ess}(t)\|_{L^2(\Omega)} \leq C\sqrt{\varepsilon}, \tag{4.65}$$

$$\text{ess} \sup_{t\in(0,T)} \|[\varrho_\varepsilon e(\varrho_\varepsilon, \vartheta_\varepsilon)]_{res}(t)\|_{L^1(\Omega)} \leq C\sqrt{\varepsilon}, \tag{4.66}$$

$$\text{ess} \sup_{t\in(0,T)} \|[\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon)]_{res}(t)\|_{L^1(\Omega)} \leq C\sqrt{\varepsilon}, \tag{4.67}$$

$$\text{ess} \sup_{t\in(0,T)} \|[E^R(I_\varepsilon)]_{res}(t)\|_{L^1(\Omega)} \leq C\sqrt{\varepsilon}, \tag{4.68}$$

$$\text{ess} \sup_{t\in(0,T)} \|[s^R(I_\varepsilon)]_{res}(t)\|_{L^1(\Omega)} \leq C\sqrt{\varepsilon}. \tag{4.69}$$

Proof: Bound (4.60) follows after the proof of (4.59) and implies (4.61). Bounds (4.62),(4.63),(4.64) and (4.68) follow after (3.39), (3.42) and (4.59). Bounds (4.66) and (4.67) follow after (3.41) and finally (4.68) and (4.69) follow after (3.44) □

Let us finally quote the following result which is a straightforward application of Proposition 5.2 of [14] (the proof is omitted)

Proposition 4.2. *Let $\{\varrho_\varepsilon\}_{\varepsilon>0}, \{\vartheta_\varepsilon\}_{\varepsilon>0}\{I_\varepsilon\}_{\varepsilon>0}$ three sequences of non-negative measurable functions such that*

$$\Big[\varrho^{(1)}_\varepsilon\Big]_{ess} \to \varrho^{(1)}\ weakly-(*)\ in\ L^\infty(0,T;L^2(\Omega)),$$

$$\Big[\vartheta^{(1)}_\varepsilon\Big]_{ess} \to \vartheta^{(1)}\ weakly-(*)\ in\ L^\infty(0,T;L^2(\Omega)),$$

$$\left[I_\varepsilon^{(1)}\right]_{ess} \to I^{(1)} \ weakly-(*) \ in \ L^\infty(0,T;L^2(\Omega)), \ a.e. \ in \ \mathcal{S}^2\times\mathbb{R}_+,$$

where

$$\varrho_\varepsilon^{(1)} = \frac{\varrho_\varepsilon - \varrho}{\varepsilon}, \ \vartheta_\varepsilon^{(1)} = \frac{\vartheta_\varepsilon - \vartheta}{\varepsilon}, \ I_\varepsilon^{(1)} = \frac{I_\varepsilon - I}{\varepsilon}.$$

Suppose that

$$\operatorname{ess}\sup_{t\in(0,T)} |\mathcal{M}^\varepsilon_{res}(t)| \le C\varepsilon^2. \tag{4.70}$$

Let $G, G^R \in C^1(\overline{\mathcal{O}_{ess}})$ be given functions. Then

$$\frac{[G(\varrho_\varepsilon,\vartheta_\varepsilon)]_{ess} - G(\varrho,\vartheta)}{\varepsilon} \to \frac{\partial G(\varrho,\vartheta)}{\partial\varrho}\ \varrho^{(1)} + \frac{\partial G(\varrho,\vartheta)}{\partial\vartheta}\ \vartheta^{(1)},$$

weakly $-(*)$ *in* $L^\infty(0,T;L^2(\Omega))$, *and if we note*

$$\left[G^R(I_\varepsilon)\right]_{ess} := \left[G^R(I_\varepsilon(\cdot,\cdot,\vec\omega,\nu))\right]_{ess} = G^R(I_\varepsilon)\cdot\mathbb{I}_{\mathcal{M}^\varepsilon_{ess}}, \ for\ a.a.\ (\vec\omega,\nu)\in\mathcal{S}^2\times\mathbb{R}_+,$$

we have

$$\frac{\left[G^R(I_\varepsilon)\right]_{ess} - G^R(I)}{\varepsilon} \to \frac{\partial G(I)}{\partial I}\ I^{(1)},$$

weakly $-(*)$ *in* $L^\infty(0,T;L^2(\Omega))$, *a.e. in* $\mathcal{S}^2\times\mathbb{R}_+$.

Moreover if $G, G^R \in C^2(\overline{\mathcal{O}_{ess}})$ then

$$\left\| \frac{[G(\varrho_\varepsilon,\vartheta_\varepsilon)]_{ess} - G(\varrho,\vartheta)}{\varepsilon} - \frac{\partial G(\varrho,\vartheta)}{\partial\varrho}\ \left[\varrho^{(1)}\right]_{ess} - \frac{\partial G(\varrho,\vartheta)}{\partial\vartheta}\ \left[\vartheta^{(1)}\right]_{ess} \right\|_{L^\infty(0,T;L^1(\Omega))} \le C\varepsilon,$$

and

$$\left\| \frac{\left[G^R(I_\varepsilon)\right]_{ess} - G^R(I)}{\varepsilon} - \frac{\partial G(I)}{\partial I}\ \left[I^{(1)}\right]_{ess} \right\|_{L^\infty(0,T;L^1(\Omega))} \le C\varepsilon,$$

for a.a. $(\vec\omega,\nu)\in\mathcal{S}^2\times\mathbb{R}_+$.

4.5 Convergence toward the target systems

We are now in position to prove that the *equilibrium diffusion target system* (4.13)-(4.15) and the *non-equilibrium diffusion target system* (4.27)-(4.31)) are the limit in a suitable sense, of the primitive system (3.30)-(3.34) when $\varepsilon\to 0$.

• The convergence result in the equilibrium case goes as follows

Theorem 4.1. *Let $\Omega\subset\mathbb{R}^3$ be a bounded domain of class $C^{2,\nu}$. Assume that the thermodynamic functions p, e, s satisfy hypotheses (2.1 - 2.6) with $P\in C^1[0,\infty)\cap C^2(0,\infty)$, and that the transport coefficients μ, η, κ, σ_a, σ_s and the equilibrium function B comply with (2.7) - (2.10).*

Let $(\varrho_\varepsilon,\vec u_\varepsilon,\vartheta_\varepsilon,I_\varepsilon)$ be a weak solution to the scaled radiative Navier-Stokes system (3.5 - 3.8) for $(t,x,\vec\omega,\nu)\in[0,T]\times\Omega\times\mathcal{S}^2\times\mathbb{R}_+$, supplemented with boundary conditions (1.10 - 1.11) and initial conditions $(\varrho_{0,\varepsilon},\vec u_{0,\varepsilon},\vartheta_{0,\varepsilon},I_{0,\varepsilon})$ such that

$$\varrho_\varepsilon(0,\cdot) = \varrho_0 + \sqrt{\varepsilon}\varrho^{(1)}_{0,\varepsilon}, \quad \vec u_\varepsilon(0,\cdot) = \vec u_{0,\varepsilon}, \quad \vartheta_\varepsilon(0,\cdot) = \vartheta_0 + \sqrt{\varepsilon}\vartheta^{(1)}_{0,\varepsilon},$$

where $(\varrho_0,\vec u,\vartheta_0)\in H^3(\Omega)$ are smooth functions such that (ϱ_0,ϑ_0) belong to the set $\mathcal{O}^H_{ess}$ where $\overline{\varrho}>0$, $\overline{\vartheta}>0$, are two constants and $\int_\Omega \varrho^{(1)}_{0,\varepsilon}\ dx = 0$, $\int_\Omega \vartheta^{(1)}_{0,\varepsilon}\ dx = 0$.

Suppose also that

$$\vec u_{0,\varepsilon} \to \vec u_0 \ \ strongly\ in\ L^\infty(\Omega;\mathbb{R}^3),$$

$$\varrho_{0,\varepsilon}^{(1)} \to \varrho_0^{(1)} \text{ strongly in } L^2(\Omega),$$

$$\vartheta_{0,\varepsilon}^{(1)} \to \vartheta_0^{(1)} \text{ strongly in } L^2(\Omega).$$

Then up to subsequences

$$\varrho_\varepsilon \to \varrho \ \text{ strongly in } L^\infty(0,T;L^{\frac{5}{3}}(\Omega)),$$

$$\vec{u}_\varepsilon \to \vec{u} \text{ strongly in } L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3)),$$

$$\vartheta_\varepsilon \to \vartheta \ \text{ strongly in } L^\infty(0,T;L^4(\Omega)),$$

$$I_\varepsilon \to B(\nu,\theta) \ \text{ strongly in } L^\infty((0,T)\times\Omega\times\mathcal{S}^2)\times(0,\infty)),$$

where $(\varrho,\vec{u},\vartheta)$ is the smooth solution of the equilibrium decoupled system (4.13)-(4.15) on $[0,T]\times\Omega$, with initial data $(\varrho_0,\vec{u}_0,\vartheta_0)$.

Proof: Let us observe that after Theorem 2.1 bounds (2.9), (2.10) and relative entropy inequality (4.59), the temperature ϑ_ε is bounded in $L^2(0,T;W^{1,2}(\Omega))$ then after extraction of a subsequence

$$\vartheta_\varepsilon \to \vartheta \ \text{ in } L^2([0,T]\times\Omega). \tag{4.71}$$

1. For the continuity equation, one observes after Lemma 4.2 that

$$\int_0^T \left\| \nabla_x \vec{u}_\varepsilon + \nabla_x^T \vec{u}_\varepsilon - \frac{2}{3}\,\mathrm{div}_x \vec{u}_\varepsilon \mathbb{I} \right\|_{L^2(\Omega;\mathbb{R}^3)} dt \le C.$$

Using this fact together with bounds in Lemma 4.2, we see that

$$\int_0^T \|\vec{u}_\varepsilon(t) - \vec{u}(t)\|^2_{W^{1,2}(\Omega;\mathbb{R}^3)} \, dt \le C,$$

so, passing to the limit after possible extraction of a subsequence, we have $\vec{u}_\varepsilon \to \vec{u}$ weakly in $L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3))$. In the same stroke $\varrho_\varepsilon \to \varrho$, weakly in $L^\infty(0,T;L^{5/3}(\Omega;\mathbb{R}^3))$. So we can pass to the limit in the weak continuity equation (4.38) which rewrites as (4.13), together with the boundary condition $\vec{u}\cdot n_x|_{\partial\Omega} = 0$, provided $\partial\Omega$ is regular.

2. For the radiative transfer equation we have shown in the previous sections, using the result of Bardos, Golse, Perthame and Sentis [3] that $I_\varepsilon \to I$, weakly in $L^\infty((0,T)\times\Omega\times\mathcal{S}^2\times\mathbb{R}_+)$, and that $\vartheta_\varepsilon \to \vartheta$, weakly in $L^2(0,T;W^{1,2}(\Omega))$. As the radiative transfer equation (3.34) is linear in I, we can pass to the limit in the weak formulation of radiative transfer equation which gives

$$\int_\Omega \int_0^\infty \int_{\mathcal{S}^2} \sigma_a(\vartheta)\,(B(\nu,\vartheta) - I)\,\psi \, d\vec{\omega}\, d\nu\, dx = 0,$$

for any test function $\psi \in C_c^\infty((0,T)\times\overline{\Omega}\times\mathcal{S}^2\times\mathbb{R}_+)$ which is the weak formulation of the equation $I(t,x,\nu,\vec{\omega}) = B(\nu,\vartheta(t,x))$.

3. For the momentum equation, one knows that due to possible strong time oscillations of the gradient component of velocity, one has only $\varrho_\varepsilon\vec{u}_\varepsilon\otimes\vec{u}_\varepsilon \to \overline{\varrho\vec{u}\otimes\vec{u}}$, weakly in $L^2(0,T;L^{\frac{30}{29}}(\Omega;\mathbb{R}^3))$, however one can show after the analysis of [14] (see [10]) that one can pass to the limit in the convective term and obtain

$$\int_0^T \int_\Omega \overline{\varrho_\epsilon\, \vec{u}_\epsilon \otimes \vec{u}_\epsilon} : \nabla_x \phi \, dx\, dt \to \int_0^T \int_\Omega \varrho\, \vec{u}\otimes\vec{u} : \nabla_x\phi\, dx\, dt.$$

Moreover after the hypotheses on pressure, ϑ_ε is bounded in $L^\infty((0,T);L^4(\Omega)) \cap L^2(0,T;L^6(\Omega))$, which implies that $\mathbb{S}_\varepsilon \to \mu(\vartheta)(\nabla_x\vec{u} + \nabla_x^t\vec{u})$, weakly in $L^q(0,T;L^q(\Omega;\mathbb{R}^3))$ for a $q > 1$.

Using (4.62) and (4.63)

$$\operatorname*{ess\,sup}_{t\in(0,T)} \|p_\varepsilon - p]_{ess}(t)\|_{L^2(\Omega)} \le C\varepsilon,$$

then $\nabla_x p_\varepsilon \to \nabla_x p$ in $\mathcal{D}'$.

As we know that $E^R_\varepsilon \to a\vartheta^4$, weakly in $L^\infty((0,T)\times\Omega\times\mathcal{S}^2\times\mathbb{R}_+)$ then $\mathbb{P}^R_\varepsilon \to a\vartheta^4\,\mathbb{I}$ and $\varepsilon\vec{F}^R_\varepsilon \to 0$, so we can pass to the limit in all the terms of the momentum equation (4.39) and obtain (4.14).

4. For the entropy balance we rewrite equation (3.33) in the form

$$\int_0^\tau\int_\Omega\Big((\varrho_\varepsilon s_\varepsilon + s^R_\varepsilon)\partial_t\varphi + \varrho_\varepsilon s_\varepsilon\vec{u}_\varepsilon\cdot\nabla_x\varphi + \frac{\vec{q}_\varepsilon}{\vartheta_\varepsilon}\cdot\nabla_x\varphi\Big)\,dx\,dt + \frac{1}{\varepsilon}\int_0^\tau\int_\Omega\frac{\vec{q}^R_\varepsilon}{\vartheta_\varepsilon}\cdot\nabla_x\varphi\,dx\,dt$$

$$+\langle\varsigma^m;\phi\rangle_{[\mathcal{M};C]([0,T\times\overline{\Omega})} + \big\langle\varsigma^R_\varepsilon;\phi\big\rangle_{[\mathcal{M};C]([0,T\times\overline{\Omega})} - \int_\Omega\big(\varrho s + s^R\big))(0,\cdot)\varphi(0,\cdot)\,dx$$

$$\le \int_\Omega\big[\big(\varrho_{0,\varepsilon}s_{0,\varepsilon} + s^R_{0,\varepsilon}\big) - \big(\varrho_0 s_0 + s^R_0\big)\big]\,\varphi(0,\cdot)\,dx - \int_0^\tau\int_\Omega\big\{\varrho_\varepsilon(s_\varepsilon - s) + (\varrho_\varepsilon - \varrho)s + s^R_\varepsilon - s^R\big)\big\}\,\partial_t\varphi\,dx\,dt$$

$$-\int_0^\tau\int_\Omega\{\varrho_\varepsilon(s_\varepsilon - s)\vec{u}_\varepsilon + (\varrho_\varepsilon\vec{u}_\varepsilon - \varrho\vec{u})s\}\cdot\nabla_x\varphi\,dx\,dt - \int_0^\tau\int_\Omega\left[\frac{\vec{q}_\varepsilon}{\vartheta_\varepsilon} - \frac{\vec{q}}{\vartheta}\right]\cdot\nabla_x\varphi\,dx\,dt + \langle\varsigma^m_\varepsilon - \varsigma^m;\phi\rangle_{[\mathcal{M};C]([0,T\times\overline{\Omega})}.$$

for any $\varphi\in C^\infty_c([0,T]\times\overline{\Omega})$.

Using Proposition 4.2, one computes first

$$\frac{1}{\varepsilon}\int_0^\tau\int_\Omega\frac{\vec{q}^R_\varepsilon}{\vartheta_\varepsilon}\cdot\nabla_x\varphi\,dx\,dt \to \int_0^\tau\int_\Omega\frac{\vec{f}_1}{\vartheta}\cdot\nabla_x\varphi\,dx\,dt,$$

as $\varepsilon\to 0$, where $\vec{f}_1$ is given by formula (4.12).

In the same stroke, we find

$$\big\langle\varsigma^R_\varepsilon;\phi\big\rangle_{[\mathcal{M};C]([0,T\times\overline{\Omega})} \to \int_0^\tau\int_\Omega\frac{\vec{F}^R_1\cdot\nabla_x\vartheta}{\vartheta^2}\,\varphi\,dx\,dt.$$

as $\varepsilon\to 0$, by using once more Proposition 4.2.

After the conditions on the data and the estimates in Lemma 3.2 and using verbatim the techniques of [14](Chap. 5) one concludes that all of the integrals in the right hand side converge to zero as $\varepsilon\to 0$, which proves that the limit entropy inequality (4.16) is obtained $\square$

- The convergence result in the non-equilibrium case goes as follows

Theorem 4.2. *Let $\Omega\subset\mathbb{R}^3$ be a bounded domain of class $C^{2,\nu}$. Assume that the thermodynamic functions p, e, s satisfy hypotheses (2.1 - 2.6) with $P\in C^1[0,\infty)\cap C^2(0,\infty)$, and that the transport coefficients μ, λ, κ, σ_a, σ_s and the equilibrium function B comply with (2.7) - (2.10).*

Let $(\varrho_\varepsilon,\vec{u}_\varepsilon,\vartheta_\varepsilon,I_\varepsilon)$ be a weak solution to the system (3.5 - 3.8) for $(t,x,\vec{\omega},\nu)\in[0,T]\times\Omega\times\mathcal{S}^2\times\mathbb{R}_+$, supplemented with the boundary conditions (1.10 - 1.11) and the initial conditions $(\varrho_{0,\varepsilon},\vec{u}_{0,\varepsilon},\vartheta_{0,\varepsilon},I_{0,\varepsilon})$ such that

$$\varrho_\varepsilon(0,\cdot) = \varrho_0 + \sqrt{\varepsilon}\varrho^{(1)}_{0,\varepsilon},\quad \vec{u}_\varepsilon(0,\cdot) = \vec{u}_{0,\varepsilon},\quad \vartheta_\varepsilon(0,\cdot) = \vartheta_0 + \sqrt{\varepsilon}\vartheta^{(1)}_{0,\varepsilon},\quad I_\varepsilon(0,\cdot) = I_0 + \sqrt{\varepsilon}I^{(1)}_{0,\varepsilon},$$

where the functions $(\varrho_0,\vec{u},\vartheta_0)$ and $x\to I_0(x,\vec{\omega},\nu)$ belong to $H^3(\Omega)$ and are such that $(\varrho_0,\vartheta_0,E_R(I_0))$ belong to the set $\mathcal{O}_{ess}$. Suppose also that

$$\vec{u}_{0,\varepsilon}\to\vec{u}_0\ \ \textit{strongly in}\ L^\infty(\Omega;\mathbb{R}^3),$$

$$\varrho^{(1)}_{0,\varepsilon}\to\varrho^{(1)}_0\ \textit{strongly in}\ L^2(\Omega),$$

$$\vartheta^{(1)}_{0,\varepsilon}\to\vartheta^{(1)}_0\ \textit{strongly in}\ L^2(\Omega),$$

$$I^{(1)}_{0,\varepsilon}\to I^{(1)}_0\ \textit{strongly in}\ L^\infty((0,T)\times\Omega\times(0,\infty)).$$

Then up to subsequences

$$\varrho_\varepsilon \to \varrho \ \ \textit{strongly in } L^\infty(0,T;L^{\frac{5}{3}}(\Omega)),$$

$$\vec{u}_\varepsilon \to \vec{u} \textit{ strongly in } L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3)),$$

$$\vartheta_\varepsilon \to \vartheta \ \ \textit{strongly in } L^\infty(0,T;L^4(\Omega)),$$

and

$$N_\varepsilon \to N \ \ \textit{strongly in } L^\infty((0,T)\times\Omega),$$

where $N_\varepsilon = \int_0^\infty \int_{\mathcal{S}^2} I_\varepsilon \, d\vec{\omega} \ \ d\nu$ *and* $(\varrho, \vec{u}, \vartheta, N)$ *is the smooth solution of the Navier-Stokes-Rosseland system (4.27)-(4.31) on* $[0,T]\times\Omega$ *with initial data* $(\varrho_0, \vec{u}_0, \vartheta_0, N_0)$.

Proof: Exactly as in the equilibrium limit, the temperature ϑ_ε is bounded in $L^2(0,T;W^{1,2}(\Omega))$ then (4.71) holds.

1. As in the equilibrium limit, we can pass to the limit in the weak continuity equation (4.38) which gives (4.27) and we can also pass to the limit in momentum equation (4.39) and obtain (4.28).

1. For the radiative transfer equation, one can adapt the result of Bardos, Golse, Perthame and Sentis [3].

As we consider the "grey hypothesis", we use the average notation I_ε instead of $N_\varepsilon := \int_0^\infty I_\varepsilon \, d\nu$ in all this subsection. We start with

$$\partial_t I_\varepsilon + \frac{1}{\varepsilon}\ \vec{\omega}\cdot\nabla_x I_\varepsilon = \sigma_{a,\varepsilon}\left(B_\varepsilon - I_\varepsilon\right) + \frac{1}{\varepsilon^2}\ \sigma_{s,\varepsilon}\left(\tilde{I}_\varepsilon - I_\varepsilon\right), \tag{4.72}$$

with $B_\varepsilon := B(\nu, \vec{\omega}, \vec{u}_\varepsilon, \vartheta_\varepsilon)$, and

$$I_\varepsilon|_{t=0} = I_0, \tag{4.73}$$

where $\tilde{I}_\varepsilon = \frac{1}{4\pi}\int_{\mathcal{S}^2} I_\varepsilon \, \mathrm{d}\vec{\omega}$, $\sigma_{a,\varepsilon} = \sigma_a(\vartheta_\varepsilon, \vec{u}_\varepsilon)$ and $\sigma_{s,\varepsilon} = \sigma_s(\vartheta_\varepsilon)$. After [10] we see that

$$\|I_\varepsilon\|_{L^\infty(\Omega\times\mathcal{S}^2)} \le C(T)\left(1 + \|I_0\|_{L^\infty(\Omega\times\mathcal{S}^2)}\right).$$

Multiplying (4.72) by I_ε, integrating over the whole phase space and using (2.10), we get

$$\|\sigma_{a,\varepsilon}^{1/2}\left(B_\varepsilon - I_\varepsilon\right)\|_{L^2(\Omega\times\mathcal{S}^2)} \le C\varepsilon, \tag{4.74}$$

$$\|\sigma_{s,\varepsilon}^{1/2}\left(\tilde{I}_\varepsilon - I_\varepsilon\right)\|_{L^2(\Omega\times\mathcal{S}^2)} \le C\varepsilon, \tag{4.75}$$

and

$$\left\|\varepsilon\ \partial_t I_\varepsilon + \frac{1}{\varepsilon}\ \vec{\omega}\cdot\nabla_x I_\varepsilon\right\|_{L^2(\Omega\times\mathcal{S}^2)} \le C. \tag{4.76}$$

Using the Fourier argument of [3] (see Lemma 3 in [3]) we also get that for any $T>0$ $\left(\tilde{I}_\varepsilon^r\right)^{1/r}$ is bounded in $L^q(0,T;W^{s,q}(\Omega))$ where $q = \frac{2p}{p+1}$, $r = 1 + \frac{1}{2p}$ and for any $s < \frac{p-1}{2p+1}$.

Integrating (4.72) over $\vec{\omega}$, we get first

$$\partial_t\ \widetilde{I_\varepsilon} + \frac{1}{\varepsilon}\ \mathrm{div}_x\ \widetilde{\vec{\omega} I_\varepsilon} = \sigma_{a,\varepsilon}\left(B_\varepsilon - \tilde{I}_\varepsilon\right), \tag{4.77}$$

and multiplying (4.72) by ω and integrating over $\vec{\omega}$, we also have

$$\partial_t\ \widetilde{\vec{\omega} I_\varepsilon} + \frac{1}{\varepsilon}\ \mathrm{div}_x\ (\widetilde{\vec{\omega}\otimes\vec{\omega} I_\varepsilon}) = -\left(\frac{1}{\varepsilon^2}\ \sigma_{s,\varepsilon} + \varepsilon\sigma_{a,\varepsilon}\right)\widetilde{\vec{\omega} I_\varepsilon}. \tag{4.78}$$

Then we get the equation

$$\partial_t\ \widetilde{I_\varepsilon} - \mathrm{div}_x\left(\frac{1}{\sigma_{s,\varepsilon} + \varepsilon\sigma_{a,\varepsilon}}\left[\varepsilon\partial_t\ \widetilde{\vec{\omega} I_\varepsilon} +\ \mathrm{div}_x\ (\widetilde{\vec{\omega}\otimes\vec{\omega} I_\varepsilon})\right]\right) = \sigma_{a,\varepsilon}\left(B_\varepsilon - \tilde{I}_\varepsilon\right), \tag{4.79}$$

in $\mathcal{D}'((0,T)\times\Omega\times\mathcal{S}^2)$. Using (4.76) and (2.10), we conclude that the sequence $\{\partial_t\ \widetilde{I_\varepsilon}\}_\varepsilon$ is bounded in $L^q(0,T;W^{-1,q}(\Omega))$.

Setting $J_\varepsilon := \left(\tilde{I}^r_\varepsilon\right)^{1/r}$, we deduce that

$$J_\varepsilon \in L^q([0,T];W^{s,q}(\Omega)),$$

$$\|\tilde{I}_\varepsilon - J_\varepsilon\|_{L^q((0,T)\times\Omega)} \to 0 \ \text{ for } \varepsilon \to 0,$$

and

$$\partial_t \tilde{I}_\varepsilon \in L^q([0,T];W^{-1,q}(\Omega)).$$

Applying a variant of the Aubin-Lions Lemma given in [3], there exists a subsequence $\widetilde{I_\varepsilon}$ converging in $L^q((0,T)\times\Omega)$.

Now we can pass to the limit in (4.72). In fact from (4.74) and (4.76) we see that there exists a $g \in L^2((0,T)\times\Omega\times\mathcal{S}^2)$ such that

$$(\sigma_{s,\varepsilon}+\varepsilon\sigma_{a,\varepsilon})^{-1/2}\,\mathrm{div}_x\,(\vec{\omega}\otimes\vec{\omega}\ I_\varepsilon) \to g \ \text{ weakly in } L^2((0,T)\times\Omega\times\mathcal{S}^2).$$

Multiplying by $(\sigma_{s,\varepsilon}+\varepsilon\sigma_{a,\varepsilon})^{1/2}\,I_\varepsilon$ and using (2.9)-(2.10) and (4.71) we obtain

$$I_\varepsilon\mathrm{div}_x\,(\vec{\omega}\otimes\vec{\omega}\ I_\varepsilon) \to g\sigma_s^{1/2}I \ \text{ weakly in } L^1((0,T)\times\Omega\times\mathcal{S}^2),$$

with $\sigma_s = \sigma_s(\vartheta)$.

Now we see from above that

$$(\sigma_{s,\varepsilon}+\varepsilon\sigma_{a,\varepsilon})^{1/2}\,I_\varepsilon \to \sigma_s^{1/2}I \ \text{ weakly in } L^2((0,T)\times\Omega\times\mathcal{S}^2),$$

so

$$\frac{1}{2}\ \mathrm{div}_x\,(\vec{\omega}\otimes\vec{\omega}\ I^2_\varepsilon) \to g\sigma_s^{1/2}I \ \text{ weakly in } L^1((0,T)\times\Omega\times\mathcal{S}^2),$$

and that

$$\frac{1}{2}\ \mathrm{div}_x\,(\vec{\omega}\otimes\vec{\omega}I^2_\varepsilon) \to \frac{1}{2}\ \mathrm{div}_x\,(\vec{\omega}\otimes\vec{\omega}I^2) \ \text{ weakly in } \mathcal{D}'((0,T)\times\Omega\times\mathcal{S}^2).$$

Therefore $g\sigma_s^{1/2}I = \frac{1}{2}\ \mathrm{div}_x\,(\vec{\omega}\otimes\vec{\omega}I^2)$.

Exactly as in [3], one can now check that $\sigma_s^{-1/2}\tilde{g} = \frac{1}{3}\frac{1}{\sigma_s}\ \nabla_x I$, and therefore one can pass to the limit in the second term in the left hand side of (4.79)

$$\frac{1}{\sigma_{s,\varepsilon}+\varepsilon\sigma_{a,\varepsilon}}\ \nabla_x(\widetilde{\vec{\omega}\otimes\vec{\omega}I_\varepsilon}) = \frac{1}{(\sigma_{s,\varepsilon}+\varepsilon\sigma_{a,\varepsilon})^{1/2}}\frac{1}{(\sigma_{s,\varepsilon}+\varepsilon\sigma_{a,\varepsilon})^{1/2}}\ \ \nabla_x(\widetilde{\vec{\omega}\otimes\vec{\omega}I_\varepsilon})$$
$$\to \sigma_s^{-1/2}\tilde{g} = \frac{1}{3}\frac{1}{\sigma_s}\ \nabla_x I. \tag{4.80}$$

As the term in the right hand side of (4.79) clearly converges to $\sigma_a(\vartheta)\,[B(\vartheta)-I]$, this finally proves that N satisfies the limit equation (4.31).

The argument of [3] shows finally that I satisfies the Dirichlet boundary condition $N|_{\partial\Omega} = 0$. In fact from the fact that $\vec{\omega}\cdot\nabla_x I^2_\varepsilon$ is bounded in $L^2((0,T)\times\Omega\times\mathbb{R}_+)$ we deduce that N_ε has a trace which holds at the limit.

2. For the entropy balance we rewrite equation (4.41) in the form

$$\int_0^\tau\int_\Omega\left(\varrho s_\varepsilon\partial_t\varphi+\varrho_\varepsilon s_\varepsilon\vec{u}_\varepsilon\cdot\nabla_x\varphi+\frac{\vec{q}_\varepsilon}{\vartheta_\varepsilon}\cdot\nabla_x\varphi\right)dx\,dt+\langle\varsigma^m;\phi\rangle_{[\mathcal{M}_\varepsilon;C]([0,T\times\overline{\Omega})}+\int_0^\tau\int_\Omega\frac{S_{E\varepsilon}}{\vartheta_\varepsilon}\,\varphi\,dx\,dt-\int_\Omega\varrho_{0,\varepsilon}s_{0,\varepsilon}\varphi(0,\cdot)\,dx$$
$$\le\int_0^\tau\int_\Omega\Big(\frac{S_E(I_\varepsilon)}{\vartheta_\varepsilon}-\frac{S_E(I)}{\vartheta}\Big)\varphi\ dx\ dt.+\int_\Omega\left[\varrho_{0,\varepsilon}s_{0,\varepsilon}-\varrho_0 s_0\right]\varphi(0,\cdot)\ dx$$

$$-\int_0^\tau\int_\Omega \{\varrho_\varepsilon(s_\varepsilon - s)+(\varrho_\varepsilon-\varrho)s\}\,\partial_t\varphi\ dx\ dt - \int_0^\tau\int_\Omega\{\varrho_\varepsilon(s_\varepsilon-s)\vec{u}_\varepsilon + (\varrho_\varepsilon\vec{u}_\varepsilon - \varrho\vec{u})s\}\cdot\nabla_x\varphi\ dx\ dt$$

$$-\int_0^\tau\int_\Omega\left[\frac{\vec{q}_\varepsilon}{\vartheta_\varepsilon}-\frac{\vec{q}}{\vartheta}\right]\cdot\nabla_x\varphi\ dx\ dt + \langle\varsigma_\varepsilon^m - \varsigma^m;\phi\rangle_{[\mathcal{M};C]([0,T\times\overline{\Omega})}$$

for any $\varphi\in C_c^\infty([0,T]\times\overline{\Omega})$.

We first observe that the first term in the right-hand side converge to zero, by applying the same argument as [10](see Proposition 4.1) based on the average Lemma of Bournaveas and Perthame [5].

Finally, after the hypotheses made on the data and the estimates in Lemma 3.2 and using once more verbatim the techniques of [14](Chap. 5) one concludes that all of the remaining integrals in the right hand side converge to zero as $\varepsilon\to 0$, which proves that the limit entropy inequality (4.30) is obtained □

Appendix: Proof of Lemma 4.1

Let us denote $\overline{\varrho}\in\mathbb{R}_+$, $\overline{\vartheta}\in\mathbb{R}_+$ and $\overline{E}\in\mathbb{R}_+$, the numbers appearing in Theorems 4.1 and 4.2 and use the previous definitions (see Section 3.2) of essential and residual quantities.

After Lemma 3.1, all of the terms in the left-hand side of (4.54) are positive. Then we have to estimate the contributions in the right-hand-side.

$$|K_1|\le\int_\Omega\varrho_\varepsilon\left|\vec{u}_\varepsilon-\vec{U}\right|^2|\nabla_x\vec{U}|\ dx\le 2\|\nabla_x\vec{U}\|_{L^\infty(\Omega;\mathbf{R}^9)}\int_\Omega\frac{1}{2}\ \varrho_\varepsilon\left|\vec{u}_\varepsilon-\vec{U}\right|^2\ dx.$$

$$|K_2|\le\left|\int_\Omega\varrho_\varepsilon\left(s_\varepsilon - s(r,\Theta)\right)\left(\vec{U}-\vec{u}_\varepsilon\right)\cdot\nabla_x\Theta\ dx\right|$$

$$\le\|\nabla_x\Theta\|_{L^\infty(\Omega;\mathbf{R}^3)}\left[2\overline{\rho}\int_\Omega|[s(\varrho_\varepsilon,\vartheta_\varepsilon)-s(r,\Theta)]_{ess}|\left|\vec{U}-\vec{u}_\varepsilon\right|\ dx+\int_\Omega|[\varrho_\varepsilon\left(s(\varrho_\varepsilon,\vartheta_\varepsilon)-s(r,\Theta)\right)]_{res}|\left|\vec{U}-\vec{u}_\varepsilon\right|\ dx\right].$$

From Lemma 3.1 we have

$$\int_\Omega|[s(\varrho_\varepsilon,\vartheta_\varepsilon)-s(r,\Theta)]_{ess}|\left|\vec{U}-\vec{u}_\varepsilon\right|\ dx\le\delta\left\|\vec{U}-\vec{u}_\varepsilon\right\|^2_{L^2(\Omega;\mathbf{R}^3)}+C(\delta)\int_\Omega\mathcal{E}\left(\varrho_\varepsilon,\vartheta_\varepsilon|r,\Theta\right)\ dx,$$

for $\delta>0$ and using interpolation we get

$$\int_\Omega|[\varrho_\varepsilon\left(s(\varrho_\varepsilon,\vartheta_\varepsilon)-s(r,\Theta)\right)]_{res}|\left|\vec{U}-\vec{u}_\varepsilon\right|\ dx\le\delta\left\|\vec{U}-\vec{u}_\varepsilon\right\|^2_{L^6(\Omega;\mathbf{R}^3)}+C(\delta)\left\|[\varrho_\varepsilon\left(s(\varrho_\varepsilon,\vartheta_\varepsilon)-s(r,\Theta)\right)]_{res}\right\|^2_{L^{6/5}(\Omega)}.$$

Using hypotheses (2.1)-(2.4) together with the property $t\to\int_\Omega\mathcal{E}\left(\varrho_\varepsilon,\vartheta_\varepsilon|r,\Theta\right)\ dx\in L^\infty(0,T)$, we conclude that

$$\|[\varrho_\varepsilon\left(s(\varrho_\varepsilon,\vartheta_\varepsilon)-s(r,\Theta)\right)]_{res}\|^2_{L^{6/5}(\Omega)}\le C\left(\int_\Omega\mathcal{E}\left(\varrho_\varepsilon,\vartheta_\varepsilon|r,\Theta\right)\ dx\right)^{5/3}.$$

So finally we end up with

$$|K_2|\le\delta\left\|\vec{U}-\vec{u}_\varepsilon\right\|^2_{W_0^{1,2}(\Omega;\mathbf{R}^3)}+\mathcal{C}(\delta;r,\vec{U},\Theta)\int_\Omega\mathcal{E}\left(\varrho_\varepsilon,\vartheta_\varepsilon|r,\Theta\right)\ dx.$$

Using (4.15) we get

$$K_3=\int_\Omega\left(\varrho_\varepsilon\left(\partial_t\vec{U}+\vec{U}\cdot\nabla_x\vec{U}\right)\cdot\left(\vec{U}-\vec{u}_\varepsilon\right)\right)\ dx=\int_\Omega\frac{\varrho_\varepsilon}{r}\ \left(\vec{U}-\vec{u}_\varepsilon\right)\left(\mathrm{div}_x\mathbb{S}(\Theta,\nabla_x\vec{U})-\nabla_x p(r,\Theta)\right)dx$$

$$=\int_\Omega\frac{\varrho_\varepsilon-r}{r}\ \left(\vec{U}-\vec{u}_\varepsilon\right)\left(\mathrm{div}_x\mathbb{S}(\Theta,\nabla_x\vec{U})-\nabla_x p(r,\Theta)\right)dx+\int_\Omega\left(\vec{U}-\vec{u}_\varepsilon\right)\left(\mathrm{div}_x\mathbb{S}(\Theta,\nabla_x\vec{U})-\nabla_x p(r,\Theta)\right)dx.$$

Estimating the first integral as for K_2, we have

$$\left| \int_\Omega \left[\frac{\varrho_\varepsilon - r}{r} \ \left(\vec{U} - \vec{u}_\varepsilon\right) \left({\rm div}_x \mathbb{S}(t, \nabla_x \vec{U}) - \nabla_x p(r,\Theta)\right)\right]_{ess} \ dx \right| \leq \mathcal{C}(\delta; r, \vec{U}, \Theta) \| \left[\varrho_\varepsilon - r\right]_{ess} \|^2_{L^2(\Omega)} + \delta \left\| \vec{U} - \vec{u}_\varepsilon \right\|^2_{L^2(\Omega;\mathbf{R}^3)}$$

$$\leq \mathcal{C}(\delta; r, \vec{U}, \Theta) \left(\| \left[\varrho_\varepsilon\right]_{ess} \|^2_{L^{6/5}(\Omega)} + \| \left[1\right]_{ess} \|^2_{L^{6/5}(\Omega)} \right) + \delta \left\| \vec{U} - \vec{u}_\varepsilon \right\|^2_{L^6(\Omega;\mathbf{R}^3)}.$$

Integrating by parts in the second integral, we have also

$$\int_\Omega \left(\vec{U} - \vec{u}_\varepsilon\right) \left({\rm div}_x \mathbb{S}(\Theta, \nabla_x \vec{U}) - \nabla_x p(r,\Theta)\right) dx = \int_\Omega \left(\mathbb{S}(\Theta, \nabla_x \vec{U}) : \nabla_x \left(\vec{U} - \vec{u}_\varepsilon\right) - p(r,\Theta) \, {\rm div}_x \left(\vec{U} - \vec{u}_\varepsilon\right) \right) dx.$$

So, using the embedding $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$, we end as above with

$$K_3 \leq \int_\Omega \left(\mathbb{S}(\Theta, \nabla_x \vec{U}) : \nabla_x \left(\vec{U} - \vec{u}_\varepsilon\right) - p(\varrho,\vartheta) \, {\rm div}_x \left(\vec{U} - \vec{u}_\varepsilon\right) \right) \ dx + \delta \left\| \vec{U} - \vec{u}_\varepsilon \right\|^2_{W^{1,2}(\Omega;\mathbf{R}^3)}$$

$$+ \mathcal{C}'(\delta; r, \vec{U}, \Theta) \int_\Omega \mathcal{E}\left(\varrho_\varepsilon, \vartheta_\varepsilon | r, \Theta\right) \ dx,$$

for any $\delta > 0$.

Now we have

$$K_6 = - \int_\Omega \varrho_\varepsilon \Big(s(\varrho_\varepsilon, \vartheta_\varepsilon) - s(r,\Theta) \partial_t \Theta \ dx = - \int_\Omega \varrho_\varepsilon \Big[s(\varrho_\varepsilon, \vartheta_\varepsilon) - s(r,\Theta) \Big]_{ess} \partial_t \Theta \ dx - \int_\Omega \varrho_\varepsilon \Big[s(\varrho_\varepsilon, \vartheta_\varepsilon) - s(r,\Theta) \Big]_{res} \partial_t \Theta \ dx,$$

where the second term is bounded as follows

$$\left| \int_\Omega \varrho_\varepsilon \Big[s(\varrho_\varepsilon, \vartheta_\varepsilon) - s(r,\Theta) \Big]_{res} \partial_t \Theta \ dx \right| \leq \| \partial_t \Theta \|_{L^\infty(\Omega)} \left(\int_\Omega \Big[\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon) \Big]_{res} \ dx + \| s(r,\Theta) \|_{L^\infty(\Omega)} \int_\Omega \Big[\varrho_\varepsilon \Big]_{res} \ dx \right)$$

$$\leq \mathcal{C}'(\delta; r, \vec{U}, \Theta) \int_\Omega \mathcal{E}\left(\varrho_\varepsilon, \vartheta_\varepsilon | r, \Theta\right) \ dx.$$

The remaining integral is bounded as follows

$$\int_\Omega \varrho_\varepsilon \Big[s(\varrho_\varepsilon, \vartheta_\varepsilon) - s(r,\Theta) \Big]_{ess} \partial_t \Theta \ dx = \int_\Omega \Big(\varrho_\varepsilon - r \Big) \Big[s(\varrho_\varepsilon, \vartheta_\varepsilon) - s(r,\Theta) \Big]_{ess} \partial_t \Theta \ dx + \int_\Omega r \Big[s(\varrho_\varepsilon, \vartheta_\varepsilon) - s(r,\Theta) \Big]_{ess} \partial_t \Theta \ dx.$$

Using Taylor formula

$$\left| \int_\Omega \Big(\varrho_\varepsilon - r \Big) \Big[s(\varrho_\varepsilon, \vartheta_\varepsilon) - s(r,\Theta) \Big]_{ess} \partial_t \Theta \ dx \right| \leq \mathcal{C}(\delta; r, \vec{U}, \Theta) \int_\Omega \mathcal{E}\left(\varrho_\varepsilon, \vartheta_\varepsilon | r, \Theta\right) \ dx.$$

Finally

$$\int_\Omega r \Big[s(\varrho_\varepsilon, \vartheta_\varepsilon) - s(r,\Theta) \Big]_{ess} \partial_t \Theta \ dx = \int_\Omega r \Big[s(\varrho_\varepsilon, \vartheta_\varepsilon) - \partial_\varrho s(r,\Theta)(\varrho_\varepsilon - r) - \partial_\Theta s(r,\Theta)(\vartheta_\varepsilon - \Theta) - s(r,\Theta) \Big]_{ess} \partial_t \Theta \ dx$$

$$- \int_\Omega \varrho \Big[\partial_\varrho s(r,\Theta)(\varrho_\varepsilon - r) - \partial_\Theta s(r,\Theta)(\vartheta_\varepsilon - \Theta) \Big]_{ess} \partial_t \Theta \ dx + \int_\Omega r \Big[\partial_r s(r,\Theta)(\varrho_\varepsilon - r) - \partial_\Theta s(r,\Theta)(\vartheta_\varepsilon - \Theta) \Big] \partial_t \Theta \ dx.$$

The first two integrals in the right-hand side can be estimated in the same way as before and we end with

$$K_6 \leq \mathcal{C}(\delta; r, \vec{U}, \Theta) \int_\Omega \mathcal{E}\left(\varrho_\varepsilon, \vartheta_\varepsilon | \varrho, \vartheta\right) \ dx - \int_\Omega \varrho \Big[\partial_r s(r,\Theta)(\varrho_\varepsilon - r) + \partial_\Theta s(r,\Theta)(\vartheta_\varepsilon - \Theta) \Big] \partial_t \Theta \ dx.$$

Accordingly, we have also

$$K_7 = -\int_\Omega \varrho_\varepsilon\Big(s(\varrho_\varepsilon,\vartheta_\varepsilon) - s(r,\Theta)\Big)\vec{u}\cdot\nabla_x\Theta\ dx \le \mathcal{C}(\delta; r, \vec{U}, \Theta)\int_\Omega \mathcal{E}\left(\varrho_\varepsilon,\vartheta_\varepsilon|r,\Theta\right)\ dx$$

$$-\int_\Omega \varrho\Big[\partial_r s(r,\Theta)(\varrho_\varepsilon - r) + \partial_\Theta s(r,\Theta)(\vartheta_\varepsilon - \Theta)\Big]\vec{U}\cdot\nabla_x\Theta\ dx.$$

Finally we find

$$K_9 = \int_\Omega \Big(\Big(1-\frac{\varrho_\varepsilon}{r}\Big)\partial_t p(r,\Theta) - \frac{\varrho_\varepsilon}{r}\ \vec{u}_\varepsilon\nabla_x p(r,\Theta)\Big)\ dx$$

$$\le \int_\Omega \Big(\Big(1-\frac{\varrho_\varepsilon}{r}\Big)\Big(\partial_t p(r,\Theta) + \vec{U}\nabla_x p(r,\Theta)\Big)\ dx + \int_\Omega p(r,\Theta)\mathrm{div}_x\vec{u}_\varepsilon\ dx + \int_\Omega \Big(\Big(1-\frac{\varrho_\varepsilon}{r}\Big)\nabla_x p(r,\Theta)\Big(\vec{u}_\varepsilon - \vec{U}\Big)\ dx,$$

and using the same argument used for K_2, we get

$$\left|\int_\Omega \Big(\Big(1-\frac{\varrho_\varepsilon}{r}\Big)\nabla_x p(r,\Theta)\Big(\vec{u}_\varepsilon - \vec{U}\Big)\ dx\right| \le \mathcal{C}'(\delta; r, \vec{U}, \Theta)\left[\delta\left\|\vec{U} - \vec{u}_\varepsilon\right\|^2_{W^{1,2}(\Omega;\mathbf{R}^3)} + \int_\Omega \mathcal{E}\left(\varrho_\varepsilon,\vartheta_\varepsilon|r,\Theta\right)\ dx\right],$$

for any $\delta > 0$, so we end with

$$K_9 \le \int_\Omega \Big(\Big(1-\frac{\varrho_\varepsilon}{r}\Big)\Big(\partial_t p(r,\Theta) + \vec{u}\nabla_x p(r,\Theta)\Big)\ dx + \int_\Omega p(r,\Theta)\mathrm{div}_x\vec{u}_\varepsilon\ dx$$

$$+\delta\left\|\vec{U} - \vec{u}_\varepsilon\right\|^2_{W^{1,2}(\Omega;\mathbf{R}^3)} + \mathcal{C}(\delta; r, \vec{U}, \Theta)\int_\Omega \mathcal{E}\left(\varrho_\varepsilon,\vartheta_\varepsilon|r,\Theta\right)\ dx.$$

Plugging all of the previous estimates into (4.54) we get

$$\int_\Omega \Big(\frac{1}{2}\ \varrho_\varepsilon|\vec{u}_\varepsilon - \vec{U}|^2 + \mathcal{E}\left(\varrho_\varepsilon,\vartheta_\varepsilon|r,\Theta\right) + \varepsilon H^R(I_\varepsilon)\Big)(\tau,\cdot)\ dx + \int_0^\tau\int_{\Gamma_+}\vec{\omega}\cdot\vec{n}_x I_\varepsilon(t,x,\vec{\omega},\nu)\ d\Gamma\ d\nu\ dt$$

$$+\int_0^\tau\int_\Omega\Big(\frac{\vartheta}{\vartheta_\varepsilon}\mathbb{S}(\varrho_\varepsilon,\nabla_x\vec{u}_\varepsilon):\nabla_x\vec{u}_\varepsilon - \mathbb{S}(r,\vec{U}):(\nabla_x\vec{u}_\varepsilon - \nabla_x\vec{U}) - \mathbb{S}(\varrho_\varepsilon,\nabla_x\vec{u}_\varepsilon):\nabla_x\vec{U}\Big)\ dx\ dt$$

$$+\int_0^\tau\int_\Omega\Big(\frac{\vec{q}(\varrho_\varepsilon,\vartheta_\varepsilon)\cdot\nabla_x\Theta}{\vartheta_\varepsilon} - \frac{\Theta}{\vartheta_\varepsilon}\ \frac{\vec{q}(\varrho_\varepsilon,\vartheta_\varepsilon)\cdot\nabla_x\vartheta_\varepsilon}{\vartheta_\varepsilon}\Big)\ dx\ dt$$

$$+\int_0^\tau\int_\Omega\int_0^\infty\int_{\mathcal{S}^2}\frac{\Theta}{\nu}\left[\log\frac{n(I_\varepsilon)}{n(I_\varepsilon)+1} - \log\frac{n(B_\varepsilon)}{n(B_\varepsilon)+1}\right]\sigma_{a_\varepsilon}^{(j)}(B_\varepsilon - I_\varepsilon)\ d\vec{\omega}\ d\nu\ dx\ dt$$

$$+\int_0^\tau\int_\Omega\int_0^\infty\int_{\mathcal{S}^2}\frac{\Theta}{\nu}\left[\log\frac{n(I_\varepsilon)}{n(I_\varepsilon)+1} - \log\frac{n(\tilde{I}_\varepsilon)}{n(\tilde{I}_\varepsilon)+1}\right]\sigma_{s_\varepsilon}^{(j)}(\tilde{I}_\varepsilon - I_\varepsilon)\ d\vec{\omega}\ d\nu\ dx\ dt$$

$$\le \int_\Omega\frac{1}{2}\Big(\varrho_{0,\varepsilon}|\vec{u}_{0,\varepsilon} - \vec{U}(0,\cdot)|^2 + \mathcal{E}\left(\varrho_{0,\varepsilon},\vartheta_{0,\varepsilon}|r(0,\cdot),\Theta(0,\cdot)\right) + H^R(I_{0,\varepsilon})\Big)\ dx$$

$$+\int_0^\tau\left[\delta\left\|\vec{U} - \vec{u}_\varepsilon\right\|^2_{W^{1,2}(\Omega;\mathbf{R}^3)} + \mathcal{C}(\delta; r, \vec{U}, \Theta)\int_\Omega\Big(\frac{1}{2}\ \varrho_\varepsilon|\vec{u}_\varepsilon - \vec{U}|^2 + \mathcal{E}\left(\varrho_\varepsilon,\vartheta_\varepsilon|r,\Theta\right)\Big)\ dx\right]\ dt$$

$$+\int_\Omega\Big(p(r,\Theta) - p(\varrho_\varepsilon,\vartheta_\varepsilon)\Big)\mathrm{div}_x\vec{U}\ dx + \int_\Omega\Big(\Big(1-\frac{\varrho_\varepsilon}{r}\Big)\Big(\partial_t p(r,\Theta) + \vec{u}\cdot\nabla_x p(r,\Theta)\Big)\ dx$$

$$-\int_\Omega\varrho\Big(\partial_r s(r,\Theta)(\varrho_\varepsilon - r) - \partial_\Theta s(r,\Theta)(\vartheta_\varepsilon - \Theta)\Big)\Big(\partial_t\Theta + \vec{U}\cdot\nabla_x\Theta\Big)\ dx$$

$$-\int_0^\tau\int_\Omega\Big(\varepsilon s_\varepsilon^R\partial_t\Theta + \vec{q}_\varepsilon^R\cdot\nabla_x\Theta\Big)\ dx\ dt - \int_0^\tau\int_\Omega(\varepsilon\vec{F}_\varepsilon^R\cdot\partial_t\vec{U} + \mathbb{P}_\varepsilon:\nabla_x\vec{U})\ dx\ dt. \tag{4.81}$$

We must now estimate the five last terms in the right-hand side.

We begin with the last two integrals. In the first one we observe that the first part is bounded as follows

$$\left|\int_0^\tau \int_\Omega \left(\varepsilon s_\varepsilon^R \partial_t \Theta\right) \ dx \ dt\right| \leq \int_0^\tau \int_\Omega \varepsilon H_\varepsilon^R |\partial_t \log \Theta| \ dx \ dt + \int_0^\tau \int_\Omega \varepsilon E_\varepsilon^R |\partial_t \log \Theta| \ dx \ dt$$

$$\leq \|\partial_t \log \Theta\|_{L^\infty(\Omega)} \left(\int_0^\tau \int_\Omega \varepsilon H_\varepsilon^R \ dx \ dt + \epsilon e_0\right),$$

and in the second part

$$\left|\int_0^\tau \int_\Omega \vec{q}_\varepsilon^R \cdot \nabla_x \Theta \ dx \ dt\right| \leq \left|\int_0^\tau \int_\Omega \Theta |s_\varepsilon^R \| \nabla_x \Theta\| \ dx \ dt\right| \leq C \|\nabla_x \Theta\|_{L^\infty(\Omega)} \int_0^\tau \int_\Omega H_\varepsilon^R \ dx \ dt,$$

then

$$\left|\int_0^\tau \int_\Omega \left(\varepsilon s_\varepsilon^R \partial_t \Theta + \vec{q}_\varepsilon^R \cdot \nabla_x \Theta\right) \ dx \ dt\right| \leq C \left(e_0 + \int_0^\tau \int_\Omega H_\varepsilon^R \ dx \ dt\right), \tag{4.82}$$

where we took into account (4.57) and (4.58).

In the second integral we check that the first part is bounded in the same way

$$\left|\int_0^\tau \int_\Omega \varepsilon \vec{F}_\varepsilon^R \partial_t \vec{U} \ dx \ dt\right| = \varepsilon \left|\int_0^\tau \int_\Omega \int_0^\infty \int_{\mathcal{S}^2} \vec{\omega} \left(I_\varepsilon - B(\nu, \Theta)\right) \cdot \partial_t \vec{U} \ d\vec{\omega} \ d\nu \ dx \ dt\right|$$

$$= \varepsilon \left|\int_0^\tau \int_\Omega \int_0^\infty \int_{\mathcal{S}^2} \vec{\omega} \left([I_\varepsilon - B(\nu, \Theta)]_{ess} + [I_\varepsilon - B(\nu, \Theta)]_{res}\right) \cdot \partial_t \vec{U} \ d\vec{\omega} \ d\nu \ dx \ dt\right|$$

$$\leq C\varepsilon \|\partial_t \vec{U}\|_{L^\infty} \left(\int_0^\tau \int_\Omega H_\varepsilon^R \ dx \ dt\right),$$

and in the second part

$$\left|\int_0^\tau \int_\Omega \mathbb{P}_\varepsilon^R \cdot \nabla_x \vec{U} \ dx \ dt\right| \leq \left|\int_0^\tau \int_\Omega \int_0^\infty \int_{\mathcal{S}^2} \vec{\omega} \otimes \vec{\omega} \left(I_\varepsilon - B(\nu, \Theta)\right) \cdot \nabla_x \vec{U} \ d\vec{\omega} \ d\nu \ dx \ dt\right|$$

$$+ \left|\int_0^\tau \int_\Omega \int_0^\infty \int_{\mathcal{S}^2} \vec{\omega} \otimes \vec{\omega} B(\nu, \Theta) \cdot \nabla_x \vec{U} \ d\vec{\omega} \ d\nu \ dx \ dt\right|$$

$$\leq \left|\int_0^\tau \int_\Omega \int_0^\infty \int_{\mathcal{S}^2} \vec{\omega} \otimes \vec{\omega} \left([I_\varepsilon - B(\nu, \Theta)]_{ess} + [I_\varepsilon - B(\nu, \Theta)]_{res}\right) \cdot \nabla_x \vec{U} \ d\vec{\omega} \ d\nu \ dx \ dt\right| + \frac{1}{3} \left|\int_0^\tau \int_\Omega \Theta^4 \mathrm{div}_x \vec{U} \ dx \ dt\right|$$

$$\leq C \|\nabla_x \vec{U}\|_{L^\infty} \left(\int_0^\tau \int_\Omega H_\varepsilon^R \ dx \ dt + e_0\right),$$

then finally

$$\left|\int_0^\tau \int_\Omega (\varepsilon \vec{F}_\varepsilon^R \cdot \partial_t \vec{U} + \mathbb{P}_\varepsilon : \nabla_x \vec{U}) \ dx \ dt\right| \leq C \left(e_0 + \int_0^\tau \int_\Omega H_\varepsilon^R \ dx \ dt\right). \tag{4.83}$$

Now using the previous thermodynamical identities for H_Θ and the continuity equation for the target system, we get rid of the remaining integrals in the right-hand side of (4.81) (see [15]) by observing that

$$\mathcal{A} := \int_\Omega \Big(p(r, \Theta) - p(\varrho_\varepsilon, \vartheta_\varepsilon)\Big) \mathrm{div}_x \vec{U} \ dx + \int_\Omega \left(\left(1 - \frac{\varrho_\varepsilon}{r}\right) \Big(\partial_t p(r, \Theta) + \vec{u} \cdot \nabla_x p(r, \Theta)\Big) \ dx\right.$$

$$- \int_\Omega \varrho \Big(\partial_r s(r, \Theta)(\varrho_\varepsilon - r) - \partial_\Theta s(r, \Theta)(\vartheta_\varepsilon - \Theta)\Big) \Big(\partial_t \Theta + \vec{U} \cdot \nabla_x \Theta\Big) \ dx$$

$$= \int_\Omega \Big(p(r, \Theta) - p(\varrho_\varepsilon, \vartheta_\varepsilon)\Big) \mathrm{div}_x \vec{u} \ dx + \int_\Omega \varrho \left(\Theta - \vartheta_\varepsilon\right) \partial_\Theta s(r, \Theta) \Big(\partial_t \Theta + \vec{U} \cdot \nabla_x \Theta\Big) \ dx$$

$$-\int_\Omega \Big(r-\varrho_\varepsilon\Big)\partial_r p(r,\Theta)\mathrm{div}_x\vec{U}\ dx.$$

Finally the second term in the right-hand side rewrites as follows

$$\int_\Omega \varrho\,(\Theta-\vartheta_\varepsilon)\,\partial_\Theta s(r,\Theta)\Big(\partial_t\Theta+\vec{U}\cdot\nabla_x\Theta\Big)\ dx$$

$$=\int_\Omega r\,(\Theta-\vartheta_\varepsilon)\,\Big[\partial_t s(r,\Theta)+\vec{U}\cdot\nabla_x s(r,\Theta)\Big]\ dx-\int_\Omega\Big(\Theta-\vartheta_\varepsilon\Big)\partial_\Theta p(r,\Theta)\mathrm{div}_x\vec{U}\ dx$$

$$=\int_\Omega (\vartheta-\vartheta_\varepsilon)\,\Big[\frac{1}{\Theta}\Big(\mathbb{S}(r,\vec{U}):\nabla_x\vec{U}-\frac{\vec{q}(\Theta,\nabla_x\Theta)\cdot\nabla_x\Theta}{\Theta}\Big)-\mathrm{div}_x\Big(\frac{\vec{q}(\Theta,\nabla_x\Theta)}{\Theta}\Big)\Big]\ dx$$

$$-\int_\Omega\Big(\Theta-\vartheta_\varepsilon\Big)\partial_\Theta p(r,\Theta)\mathrm{div}_x\vec{U}\ dx.$$

We deduce finally that

$$\mathcal{A}=\int_\Omega\Big(p(\varrho,\vartheta)-p(\varrho_\varepsilon,\vartheta_\varepsilon)-\partial_\varrho p(\varrho,\vartheta)(\varrho_\varepsilon-\varrho)-\partial_\vartheta p(\varrho,\vartheta)(\vartheta_\varepsilon-\vartheta)\Big)\mathrm{div}_x\vec{U}\ dx.$$

$$+\int_\Omega(\Theta-\vartheta_\varepsilon)\,\Big[\frac{1}{\Theta}\Big(\mathbb{S}(r,\vec{U}):\nabla_x\vec{U}-\frac{\vec{q}(\Theta,\nabla_x\Theta)\cdot\nabla_x\Theta}{\Theta}\Big)-\mathrm{div}_x\Big(\frac{\vec{q}(\Theta,\nabla_x\Theta)}{\Theta}\Big)\Big]\ dx.$$

Observing that

$$\Big|\int_\Omega\Big(p(r,\Theta)-p(\varrho_\varepsilon,\vartheta_\varepsilon)-\partial_r p(r,\Theta)(\varrho_\varepsilon-r)-\partial_\Theta p(r,\Theta)(\vartheta_\varepsilon-\Theta)\Big)\mathrm{div}_x\vec{U}\ dx\Big|$$

$$\leq C\left\|\mathrm{div}_x\vec{U}\right\|_{L^\infty(\Omega)}\int_\Omega\mathcal{E}\left(\varrho_\varepsilon,\vartheta_\varepsilon|r,\Theta\right)\ dx,$$

we see that (4.81) reduces finally to

$$\int_\Omega\Big(\frac{1}{2}\ \varrho_\varepsilon|\vec{u}_\varepsilon-\vec{U}|^2+\mathcal{E}\left(\varrho_\varepsilon,\vartheta_\varepsilon|r,\Theta\right)+\varepsilon H^R(I_\varepsilon)\Big)(\tau,\cdot)\ dx$$

$$+\int_0^\tau\int_\Omega\Big(\frac{\Theta}{\vartheta_\varepsilon}\mathbb{S}(\varrho_\varepsilon,\nabla_x\vec{u}_\varepsilon):\nabla_x\vec{u}_\varepsilon-\mathbb{S}(r,\vec{U}):(\nabla_x\vec{u}_\varepsilon-\nabla_x\vec{U})-\mathbb{S}(\varrho_\varepsilon,\nabla_x\vec{u}_\varepsilon):\nabla_x\vec{U}\Big)\ dx\ dt$$

$$+\int_0^\tau\int_\Omega\Big(\frac{\tilde{\vartheta}_\varepsilon-\Theta}{\tilde{\vartheta}_\varepsilon}\mathbb{S}(\varrho_\varepsilon,\nabla_x\vec{u}_\varepsilon):\nabla_x\vec{U}\Big)\ dx\ dt+$$

$$+\int_0^\tau\int_\Omega\Big(\frac{\vec{q}(\vartheta_\varepsilon,\nabla_x\vartheta_\varepsilon)\cdot\nabla_x\Theta}{\vartheta_\varepsilon}-\frac{\Theta}{\vartheta_\varepsilon}\ \frac{\vec{q}(\varrho_\varepsilon,\nabla_x\vartheta_\varepsilon)\cdot\nabla_x\vartheta_\varepsilon}{\vartheta_\varepsilon}\Big)\ dx\ dt$$

$$+\int_0^\tau\int_\Omega\Big((\Theta-\vartheta_\varepsilon)\frac{\vec{q}(\Theta,\nabla_x\Theta)\cdot\nabla_x\Theta}{\Theta^2}+\frac{\vec{q}(\Theta,\nabla_x\Theta)\cdot\nabla_x(\vartheta_\varepsilon-\Theta)}{\Theta}\Big)\ dx\ dt$$

$$\leq\int_\Omega\frac{1}{2}\Big(\varrho_{0,\varepsilon}|\vec{u}_{0,\varepsilon}-\vec{U}(0,\cdot)|^2+\mathcal{E}\left(\varrho_{0,\varepsilon},\vartheta_{0,\varepsilon}|r(0,\cdot),\Theta(0,\cdot)\right)+\varepsilon H^R(I_{0,\varepsilon})\Big)\ dx$$

$$+\int_0^\tau\Big[\delta\left\|\vec{U}-\vec{u}_\varepsilon\right\|^2_{W^{1,2}(\Omega;\mathbf{R}^3)}+\mathcal{C}'(\delta;r,\vec{U},\Theta)\int_\Omega\mathcal{E}\left(\varrho_\varepsilon,\vartheta_\varepsilon|r,\Theta\right)\ dx\Big]\ dt+\mathcal{C}''(\delta;r,\vec{U},\Theta)(e_0+\int_0^\tau\int_\Omega H^R_\varepsilon\ dx\ dt).\tag{4.84}$$

Finally we can control all of the dissipative terms (the three last integrals in the left-hand side), by using verbatim the computations in [15] which leads to the final inequality

$$\int_\Omega\Big(\frac{1}{2}\ \varrho_\varepsilon|\vec{u}_\varepsilon-\vec{U}|^2+\mathcal{E}\left(\varrho_\varepsilon,\vartheta_\varepsilon|r,\Theta\right)+\varepsilon H^R(I_\varepsilon)\Big)(\tau,\cdot)\ dx+k_1\int_0^\tau\int_\Omega\Big|\nabla_x\vec{u}_\varepsilon-\nabla_x\vec{U}\Big|^2\ dx\ dt$$

$$+k_2\int_0^\tau\int_\Omega\left|\nabla_x\vartheta_\varepsilon-\nabla_x\Theta\right|^2\ dx\ dt+k_3\int_0^\tau\int_\Omega\left|\nabla_x\log\vartheta_\varepsilon-\nabla_x\log\Theta\right|^2\ dx\ dt$$

$$\leq k_4e_0+\int_\Omega\Big(\frac{1}{2}\Big(\varrho_{0,\varepsilon}|\vec{u}_{0,\varepsilon}-\vec{U}(0,\cdot)|^2+\mathcal{E}\left(\varrho_{0,\varepsilon},\vartheta_{0,\varepsilon}|r(0,\cdot),\Theta(0,\cdot)\right)+\varepsilon H^R(I_{0,\varepsilon})\Big)\ dx$$

$$+k_5\int_0^\tau\int_\Omega\Big(\frac{1}{2}\ \varrho_\varepsilon\left|\vec{u}_\varepsilon-\vec{U}\right|^2+\mathcal{E}\left(\varrho_\varepsilon,\vartheta_\varepsilon|r,\Theta\right)+\varepsilon H^R(I_\varepsilon)\Big)\ dx\ dt, \tag{4.85}$$

where the positive constants k_j depend on $(r,\vec{U},\Theta,\Theta_r)$ through the norms involved in Theorems 4.1 and 4.2. Thus we end with

$$\varepsilon\int_\Omega\left(\frac{1}{2}\ \varrho_\varepsilon|\vec{u}_\varepsilon-\vec{U}|^2+\mathcal{E}\left(\varrho_\varepsilon,\vartheta_\varepsilon|r,\Theta\right)+H^R(I_\varepsilon)\right)(\tau,\cdot)\ dx$$

$$\leq k_4e_0+\int_\Omega\Big(\frac{1}{2}\Big(\varrho_{0,\varepsilon}|\vec{u}_{0,\varepsilon}-\vec{U}(0,\cdot)|^2+\mathcal{E}\left(\varrho_{0,\varepsilon},\vartheta_{0,\varepsilon}|r(0,\cdot),\Theta(0,\cdot)\right)+H^R(I_{0,\varepsilon})\Big)\ dx$$

$$+k_5\int_0^\tau\int_\Omega\Big(\frac{1}{2}\ \varrho_\varepsilon\left|\vec{u}_\varepsilon-\vec{U}\right|^2+\mathcal{E}\left(\varrho_\varepsilon,\vartheta_\varepsilon|r,\Theta\right)+H^R(I_\varepsilon)\Big)\ dx\ dt. \tag{4.86}$$

Using Gronwall's inequality we get finally the requested inequality (4.59).

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