INSTITUTE of MATHEMATICS

ACADEMY of SCIENCES of the CZECH REPUBLIC

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Preprint No. 51-2014

PRAHA 2014

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Abstract. We consider the asymptotic regime for a barotropic model of a compressible fluid coupled to the radiation when the radiative intensity is driven to a diffusion limit and we study the convergence of the system toward the asymptotic limit.

Contents

1. Introduction

We consider a barotropic model in *radiation fluid dynamics* in the asymptotic non-equilibrium diffusion limit proposed by Buet and Desprès [**6**]. We suppose that the motion of the fluid is governed by the standard equations of classical fluid dynamics describing the evolution of the mass density $\varrho = \varrho(t,x)$ and the velocity field $\vec{u} = \vec{u}(t,x)$, considered as functions of the time $t > 0$ and the spatial (Eulerian) coordinate $x \in \Omega$, where $\Omega \subset \mathbb{R}^3$ is a bounded domain. The effect of radiation is incorporated in the system through the *radiative intensity* $I = I(t,x,\vec{\omega},\nu)$, depending, besides the variables t, x, on the direction vector $\vec{\omega} \in \mathcal{S}^2$, where $\mathcal{S}^2$ denotes the unit sphere in $\mathbb{R}^3$, and the frequency $\nu \geq 0$. The action of radiation is then expressed in terms of integral average with respect to the variables ω and ν of quantities depending on I. Finally the evolution of the compressible viscous heat

1991 *Mathematics Subject Classification.* Primary 76N10, ; Secondary 35Q30.

Key words and phrases. Radiation hydrodynamics, Navier-Stokes equations, weak solution, diffusion limit.

Šárka Nečasová acknowledges the support of the GAČR (Czech Science Foundation) project P201-13-00522S in the framework of RVO: 67985840.

conductive flow is coupled to radiation through *radiative transfer equation* [**7**] which reads

$$\frac{1}{c}\partial_t I + \vec{\omega}\cdot\nabla_x I = S, \tag{1.1}$$

where c is the speed of light. The radiative source $S := S_a + S_s$ is the sum of an emission-absorption term $S_{a,e} := \sigma_a\,(B(\nu,\varrho) - I)$ and a scattering contribution $S_s := \sigma_s\left(\tilde{I} - I\right)$ where $\tilde{I} := \frac{1}{4\pi}\int_{\mathcal{S}^2} I\ d\omega$. S takes the form

$$S = \sigma_a(B - I) + \sigma_s\left(\frac{1}{4\pi}\int_{S^2} I\ \mathrm{d}\vec{\omega} - I\right), \tag{1.2}$$

In what follows, we assume:

- **Isotropy:** The coefficients σ_a, σ_s are independent of $\vec{\omega}$.
- **Grey hypothesis:** The coefficients σ_a, σ_s are independent of ν.

The function $B = B(\nu,\varrho)$ measures the departure from equilibrium and is a barotropic equivalent of the Planck function. We also denote by b the frequency average of $B(\nu,\varrho)$ given by

$$b(\varrho) := \int_0^\infty B(\nu,\varrho)\ d\nu. \tag{1.3}$$

The time evolution of the density ϱ and the velocity $\vec{u}$ is governed by the standard *barotropic Navier-Stokes system*:

$$\partial_t\varrho + \mathrm{div}_x(\varrho\vec{u}) = 0, \tag{1.4}$$

$$\partial_t(\varrho\vec{u}) + \mathrm{div}_x(\varrho\vec{u}\otimes\vec{u}) + \nabla_x p(\varrho) = \mu\Delta_x\vec{u} + (\lambda+\mu)\nabla_x\mathrm{div}_x\vec{u} - \vec{S}_F, \tag{1.5}$$

where the (constant) viscosity coefficients satisfy

$$\mu > 0,\ \ \lambda + \frac{2}{3}\mu \geq 0, \tag{1.6}$$

and

$$\vec{S}_F = (\sigma_a + \sigma_s)\int_0^\infty\int_{S^2}\vec{\omega} I\ \mathrm{d}\vec{\omega}\ \mathrm{d}\nu. \tag{1.7}$$

The system of equations (1.1 - 1.7) is supplemented with the (dissipative) boundary conditions

$$\vec{u}|_{\partial\Omega} = 0, \tag{1.8}$$

$$I(t,x,\vec{\omega},\nu) = 0 \text{ for } (x,\vec{\omega}) \in \Gamma_- \equiv \left\{(x,\vec{\omega})\ \middle|\ (x,\vec{\omega}) \in \partial\Omega\times S^2,\ \vec{\omega}\cdot\vec{n} \leq 0\right\}, \tag{1.9}$$

where $\vec{n}$ denotes the outer normal vector to $\partial\Omega$, and initial conditions

$$(\varrho(x,t),\ \vec{u}(x,t), I(x,t,\vec{\omega},\nu))|_{t=0} = \left(\varrho^0(x),\ \vec{u}^0(x), I^0(x,\vec{\omega},\nu)\right), \tag{1.10}$$

for $x \in \Omega$, $\vec{\omega} \in \mathcal{S}^2$ and $\nu \in (0,\infty)$.

The coupled system (1.1 - 1.10) can be viewed as a simplified model in radiation hydrodynamics [**31**], [**27**]. More realistic systems (including an energy equation) appear in astrophysical applications [**27**] [**31**] and their asymptotic regimes have been proposed by Lowrie, Morel and Hittinger [**26**] and revisited recently by Buet and Després [**6**] (see also Dubroca and Feugeas [**17**], Lin [**28**] and Lin, Coulombel and Goudon [**29**] for related numerical issues). For the "complete system" including temperature, a global existence result has also recently been proved in [**10**] under some cut-off hypotheses on transport coefficients and also in the steady case see [**25**]. Let us mention for completeness that existence of local-in-time solutions in the inviscid case was obtained by Zhong and Jiang [**32**] and that a number of results in one-dimensional geometry are available (see [**2**], [**12**], [**13**], [**14**] and references therein). Let us finally mention that singular limits in low Mach number regime and diffusion regime for the full Navier- Stokes- Fourier system coupled with radiation were also investigated see [**15, 16**].

Our goal in this paper is to study the asymptotic behavior of solutions to the problem (1.1 - 1.10) under the scaling

$$c \approx \frac{1}{\varepsilon},\ \sigma_a \approx \varepsilon \sigma_a(\varrho),\ \sigma_s \approx \frac{1}{\varepsilon}\sigma_s(\varrho),$$

where $\varepsilon \to 0$ is a small positive parameter.

In fact this asymptotic regime corresponds to a (non-equilibrium) diffusion limit of the system in the sense of [**26**] or [**6**]. From a physical point of view, it is well known that, when the mean free-path of photons is small, the radiative transfer equation (1.1) is well approximated by a diffusion equation, which drastically simplifies numerical simulations used for example in inertial confinement fusion or astrophysical purposes (see [**1**] and [**5**] for more complete introductions). Such asymptotic regimes have yet been studied in [**3**] and [**4**] and our aim is to extend this perspective to the simplified coupled system (1.1)(1.4)(1.5).

2. Hypothesis and Mathematical Preliminaries

Hypotheses imposed on constitutive relations and transport coefficients are motivated by the existence theory for the compressible Navier-Stokes system developped in [**18**] and reasonable physical assumptions [**31**]. We suppose that the pressure satisfies the following assumptions

- p is a C^1 function on $[0,\infty)$ such that $p(0) = 0$,
- $p \in C[0,\infty) \cap C^2(0,\infty)$,
- $p'(\rho) > 0$ for all $\rho > 0$, such that

$$\frac{p'(\rho)}{\rho^{\gamma-1}} = p_\infty > 0,\ \gamma > \frac{3}{2}, \tag{2.1}$$

Let us mention that such a behavior includes the case of monoatomic gases $\gamma = 5/3$ but one can check that all of our results also hold for more general fluids, in particular for non-monotone equations of state met in nuclear physics [**8**] and considered in [**9**] and [**23**].

We also assume the following bounds for radiative quantities

$$0 \le \sigma_s(\varrho),\ \sigma_a(\varrho) \le c_1, \tag{2.2}$$

$$\sigma_a(\varrho)B^m(\nu,\varrho) \le h(\nu),\ \ h \in L^1(0,\infty) \ \ \text{for } m = 1,2, \tag{2.3}$$

for any $\varrho \ge 0$. Note that relations (2.2 - 2.3) represent "cut-off" hypotheses at large density.

The equation of continuity (1.4) is replaced by the integral identity

$$\int_\Omega \varrho(\tau,\cdot)\psi(\tau,\cdot)\ dx - \int_\Omega \varrho_0\psi(0,\cdot)\ dx = \int_0^\tau \int_\Omega \varrho\partial_t\psi + \varrho\vec{u}\cdot\nabla_x\psi\ dx\ dt \tag{2.4}$$

satisfied for any $\psi \in C^1([0,T]\times\overline{\Omega})$ and any $\tau \in [0,T]$, and for $\varrho(0,\cdot) = \varrho_0$. It is customary to replace the equation of continuity (1.4) by its (weak) *renormalized* version represented by a family of integral identities

(2.5)

$$\int_0^T \int_\Omega \Big(\Big(\varrho + \beta(\varrho)\Big)\partial_t\psi + \Big(\varrho + \beta(\varrho)\Big)\vec{u}\cdot\nabla_x\psi + \Big(\beta(\varrho) - \beta'(\varrho)\varrho\Big)\mathrm{div}_x\vec{u}\psi\Big)\ \mathrm{d}x\ \mathrm{d}t$$

$$= -\int_\Omega \Big(\varrho_0 + \beta(\varrho_0)\Big)\psi(0,\cdot)\ \mathrm{d}x$$

satisfied for any $\psi \in C_c^\infty([0,\infty)\times\overline{\Omega})$, and any $\beta \in C^\infty[0,\infty)$, $\beta' \in C_c^\infty[0,\infty)$. Note that (2.5) implicitly includes satisfaction of the initial condition

$$\varrho(0,\cdot) = \varrho_0.$$

The momentum equation (1.5) is replaced by

$$\int_\Omega \varrho\vec{u}(\tau,\cdot)\phi(\tau,\cdot)\ dx - \int_\Omega \varrho_0\vec{u}_0\phi(0,\cdot)\ dx$$

$$= \int_0^\tau \int_\Omega \varrho\vec{u}\cdot\partial_t\phi + \varrho\vec{u}\otimes\vec{u} : \nabla_x\phi + p\,\mathrm{div}_x\phi - \mathbb{S} : \nabla_x\phi - \vec{S}_F\cdot\phi\ dx\ dt, \tag{2.6}$$

for any $\phi \in C^1([0,T]\times\overline{\Omega};\mathbb{R}^3)$ with $\phi|_{\partial\Omega} = 0$, any $\tau \in [0,T]$.

Definition: *We say that* $(\varrho,\vec{u},I)$ *is a weak solution of problem* (1.1) − (1.10) *on* $(0,T)$ *if the density* ϱ *is a non negative measurable function and if*

$$\rho \in C_{\text{weak}}(0,T;L^\gamma(\Omega)), \tag{2.7}$$

$$\vec{u} \in L^2(0,T;W^{1,2}(\Omega)), \tag{2.8}$$

$$\varrho\vec{u} \in C_{\text{weak}}(0,T;L^{\frac{2\gamma}{\gamma+1}}(\Omega;\mathbb{R}^3)), \tag{2.9}$$

$$I \in L^\infty((0,T)\times\Omega\times\mathcal{S}^2\times(0,\infty)), \tag{2.10}$$

$$I \in L^\infty(0,T;L^1(\Omega\times\mathcal{S}^2\times(0,\infty)) \tag{2.11}$$

and if $(\varrho,\vec{u},I)$ *satisfy the integral identities (2.5),(2.6) together with the transport equation (1.1).*

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class $C^{2,\nu}$, $\nu > 0$. Assume that the pressure p, the transport coefficients σ_a, σ_s and the equilibrium function B comply with (2.1 - 2.3).*

Let $(\varrho, , \vec{u}, I)$ be a weak solution to radiative Navier-Stokes system (1.1)-(1.10) for $(t,x) \in [0,T] \times \Omega$, and $(\vec{\omega}, \nu) \in \mathcal{S}^2 \times R_+$.

Then problem (1.1)-(1.10) has a weak solution $(\varrho, \vec{u}, I)$ such that the density ϱ is a non negative measurable function,

$$\rho \in C_{\text{weak}}(0,T; L^\gamma(\Omega)), \tag{2.12}$$

$$\vec{u} \in L^2(0,T; W^{1,2}(\Omega)), \tag{2.13}$$

$$\varrho\vec{u} \in C_{\text{weak}}(0,T; L^{\frac{2\gamma}{\gamma+1}}(\Omega; \mathbb{R}^3)), \tag{2.14}$$

$$I \in L^\infty((0,T) \times \Omega \times \mathcal{S}^2 \times (0,\infty)), \tag{2.15}$$

$$I \in L^\infty(0,T; L^1(\Omega \times \mathcal{S}^2 \times (0,\infty)), \tag{2.16}$$

possesses a finite energy weak solution $(\varrho, \vec{u}, I)$ for $(t,x) \in [0,T] \times \Omega$, and $(\vec{\omega}, \nu) \in \mathcal{S}^2 \times \mathbb{R}_+$ and satisfying the integral identities (2.4-2.6) together with the transport equation (1.1).

Proof. See the proof in the Appendix. □

3. Formal scaling analysis

In order to identify the appropriate limit regime we perform a general scaling, denoting by L_{ref}, T_{ref}, U_{ref}, ρ_{ref}, p_{ref}, the reference hydrodynamical quantities (length, time, velocity, density, pressure) and by I_{ref}, ν_{ref}, $\sigma_{a,ref}$, $\sigma_{s,ref}$, B_{ref}, the reference radiative quantities (radiative intensity, frequency, absorption and scattering coefficients and equilibrium function). We denote by $Sr := \frac{L_{ref}}{T_{ref}U_{ref}}$, $Ma = \frac{U_{ref}}{\sqrt{\rho_{ref}p_{ref}}}$, $Re = \frac{U_{ref}\rho_{ref}L_{ref}}{\mu_{ref}}$, the Strouhal, Mach, Reynolds (dimensionless) numbers corresponding to hydrodynamics, and by $\mathcal{C} = \frac{c}{U_{ref}}$, $\mathcal{L} = L_{ref}\sigma_{a,ref}$, $\mathcal{L}_s = \frac{\sigma_{s,ref}}{\sigma_{a,ref}}$, $\mathcal{P} = \frac{L_{ref}\nu_{ref}S_{ref}}{c\ \rho_{ref}U_{ref}^2}$, various dimensionless numbers corresponding to radiation.

Using these scalings, using carets to symbolize renormalized variables and choosing $B_{ref} = I_{ref}$ we get $S = \frac{I_{ref}}{L_{ref}}\ \hat{S}$, where

$$\hat{S} = \mathcal{L}\hat{\sigma}_a\left(\mathcal{B}(\hat{\nu}, \hat{\varrho}) - \hat{I}\right) + \mathcal{L}\mathcal{L}_s\hat{\sigma}_s\left(\frac{1}{4\pi}\int_{\mathcal{S}^2} \hat{I}(\cdot, \vec{\omega})\ \mathrm{d}\vec{\omega} - \hat{I}\right).$$

Omitting the carets in the following, we get first the scaled equation for I, in the region $(0,T) \times \Omega \times (0,\infty) \times \mathcal{S}^2$

$$\frac{Sr}{\mathcal{C}}\ \partial_t I + \vec{\omega} \cdot \nabla_x I = S = \mathcal{L}\sigma_a\left(B - I\right) + \mathcal{L}\mathcal{L}_s\sigma_s\left(\frac{1}{4\pi}\int_{\mathcal{S}^2} I\ \mathrm{d}\vec{\omega} - I\right). \tag{3.1}$$

We also denote by $E_R = \int_0^\infty \int_{\mathcal{S}^2} I\ \mathrm{d}\vec{\omega}\ \mathrm{d}\nu$ the renormalized energy and $\vec{S}_F = \int_0^\infty \int_{\mathcal{S}^2} \vec{\omega} S\ \mathrm{d}\vec{\omega}\ \mathrm{d}\nu$.

The continuity equation is now

$$Sr\ \partial_t \varrho + \mathrm{div}_x(\varrho \vec{u}) = 0, \tag{3.2}$$

and the momentum equation

$$Sr\ \partial_t(\varrho \vec{u}) + \mathrm{div}_x(\varrho \vec{u} \otimes \vec{u}) + \frac{1}{Ma^2}\ \nabla_x p(\varrho) - \frac{1}{Re}\left(\mu \Delta \vec{u} + (\lambda+\mu)\nabla_x \mathrm{div}_x \vec{u}\right) = -\mathcal{P}\vec{S}_F. \tag{3.3}$$

Supposing that a moderate amount of radiation is present ($\mathcal{P} = O(1)$) in our strongly under-relativistic flow ($\mathcal{C} = O(\varepsilon^{-1})$), where ε is a small positive number, we obtain the " non-equilibrium diffusion regime" defined by

$$Ma = Sr = Pe = Re = 1,\ \mathcal{P} = 1,\ \mathcal{C} = \varepsilon^{-1},\ \mathcal{L} = \varepsilon \text{ and } \mathcal{L}_s = \varepsilon^{-2}.$$

The new system reads finally

$$\varepsilon\ \partial_t I + \vec{\omega} \cdot \nabla_x I = \varepsilon \sigma_a (B - I) + \frac{1}{\varepsilon}\sigma_s \left(\frac{1}{4\pi}\int_{\mathcal{S}^2} I\ \mathrm{d}\vec{\omega} - I\right), \tag{3.4}$$

$$\partial_t \varrho + \mathrm{div}_x(\varrho \vec{u}) = 0, \tag{3.5}$$

$$\partial_t(\varrho \vec{u}) + \mathrm{div}_x(\varrho \vec{u} \otimes \vec{u}) + \nabla_x p(\varrho)$$
$$= \mu \Delta \vec{u} + (\lambda+\mu)\nabla_x \mathrm{div}_x \vec{u} + \left(\varepsilon \sigma_a + \frac{1}{\varepsilon}\sigma_s\right)\int_0^\infty \int_{\mathcal{S}^2} \vec{\omega} I\ d\vec{\omega}\ d\nu. \tag{3.6}$$

3.1. Formal computation of the diffusion regime. In order to compute the limit system, we consider the formal expansions

$$\begin{cases} I = I_0 + \varepsilon I_1 + \varepsilon^2 I_2 + O(\varepsilon^3), \\ \varrho = \rho_0 + \varepsilon \varrho_1 + \varepsilon^2 \varrho_2 + O(\varepsilon^3), \\ \vec{u} = \vec{u}_0 + \varepsilon \vec{u}_1 + \varepsilon^2 \vec{u}_2 + O(\varepsilon^3). \end{cases} \tag{3.7}$$

Plugging (3.7) in (3.4) and evaluating the lowest orders terms we get

$$\frac{1}{4\pi}\int_{\mathcal{S}^2} I_0\ \mathrm{d}\vec{\omega} = I_0, \tag{3.8}$$

$$\vec{\omega} \cdot \nabla_x I_0 = \sigma_s(\varrho_0, \nu)\left(\frac{1}{4\pi}\int_{\mathcal{S}^2} I_1\ \mathrm{d}\vec{\omega} - I_1\right), \tag{3.9}$$

and

$$\partial_t I_0 + \vec{\omega} \cdot \nabla_x I_1 = \sigma_a(\varrho_0)(B(\varrho_0, \nu) - I_0) + \sigma_s(\varrho_0)\left(\frac{1}{4\pi}\int_{\mathcal{S}^2} I_2\ \mathrm{d}\vec{\omega} - I_2\right)$$
$$+\partial_\varrho \sigma_s(\varrho_0)\left(\frac{1}{4\pi}\int_{\mathcal{S}^2} I_1\ \mathrm{d}\vec{\omega} - I_1\right)\varrho_1. \tag{3.10}$$

Integrating on $\mathcal{S}^2$ and plugging the first two relations into the last one, we find

$$\partial_t I_0 + \vec{\omega} \cdot \nabla_x \tilde{I}_1 - \vec{\omega} \otimes \vec{\omega}\ \mathrm{div}_x\left(\frac{1}{\sigma_s(\varrho_0)}\nabla_x I_0\right)$$
$$= \sigma_a(\varrho_0)(B(\varrho_0, \nu) - I_0) + \sigma_s(\varrho_0)\left(\frac{1}{4\pi}\int_{\mathcal{S}^2} I_2\ \mathrm{d}\vec{\omega} - I_2\right)$$
$$+\partial_\varrho \sigma_s(\varrho_0, \nu)\left(\frac{1}{4\pi}\int_{\mathcal{S}^2} I_1\ \mathrm{d}\vec{\omega} - I_1\right)\varrho_1.$$

Integrating in ν and using (3.8)(3.9), we get a diffusion equation for $N := \int_0^\infty I_0 \, d\nu$

$$\partial_t N - \frac{1}{3} \, \mathrm{div}_x \left(\frac{1}{\sigma_s(\varrho_0)} \nabla_x N \right) = \sigma_a(\varrho_0)(b(\varrho_0) - N), \tag{3.11}$$

where $b(\varrho_0) := \int_0^\infty B(\varrho_0, \nu) \, d\nu$.

We finally obtain a compressible Navier-Stokes type system for ϱ and $\vec{u}$ coupled to a diffusion equation for N.

Omitting the 0 index, we get finally the system

$$\partial_t \varrho + \mathrm{div}_x(\varrho \vec{u}) = 0, \tag{3.12}$$

$$\partial_t(\varrho \vec{u}) + \mathrm{div}_x(\varrho \vec{u} \otimes \vec{u}) + \nabla_x \left[p(\varrho) + \frac{1}{3} \, N \right] = \mu \Delta \vec{u} + (\lambda + \mu) \nabla_x \mathrm{div}_x \vec{u}, \tag{3.13}$$

$$\partial_t N - \frac{1}{3} \, \mathrm{div}_x \left(\frac{1}{\sigma_s(\varrho)} \nabla_x N \right) = \sigma_a(\varrho) \left(b(\varrho) - N \right), \tag{3.14}$$

with the boundary conditions

$$\vec{u}|_{\partial\Omega} = 0, \tag{3.15}$$

the extra boundary condition on N

$$N|_{\partial\Omega} = 0. \tag{3.16}$$

and initial conditions

$$(\varrho(x,t), \ \vec{u}(x,t), N(x,t))|_{t=0} = \left(\varrho^0(x), \ \vec{u}^0(x), N^0(x) \right), \tag{3.17}$$

for any $x \in \Omega$, with $N^0(x) = \int_0^\infty \int_{\mathcal{S}^2} I^0(x, \nu, \vec{\omega}) \, d\vec{\omega} \, d\nu$.

One observes that in the limit regime, hydrodynamics is coupled to radiation through the effective pressure $\pi := p + \frac{1}{3} \, N$.

The main theorem reads

Theorem 3.1. *(Main Theorem) Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class $C^{2,\nu}$. Let p is a C^1 function on $[0, \infty)$ such that $p(0) = 0$, $p'(\rho) > 0$ for all $\rho > 0$ and (2.2-2.3) are satisfied. Let $(\varrho_\varepsilon, \vec{u}_\varepsilon, I_\varepsilon)$ be a weak solution of rescaled system of equations (1.1-1.10) with*

$$\varrho_{0,\varepsilon} \to \varrho_0 \ \text{in} \ L^\gamma(\Omega), \tag{3.18}$$

$$\int_\Omega \frac{|(\varrho \vec{u})_{0,\varepsilon}|^2}{\varrho_{0,\varepsilon}} \, \mathrm{d}x \le c, \tag{3.19}$$

$$|I_{0,\varepsilon}(\cdot, \nu)| \le h(\nu), \ \ h \in L^1 \cap L^\infty(0, \infty). \tag{3.20}$$

Then up to subsequences

$$\varrho_\varepsilon \to \varrho \ \text{in} \ C([0,T]; L^1(\Omega)) \ \text{and in} \ C_{\mathrm{weak}}([0,T]; L^\gamma(\Omega)), \tag{3.21}$$

$$\vec{u}_\varepsilon \to \vec{u} \ \text{weakly in} \ L^2(0,T; W^{1,2}(\Omega; \mathbb{R}^3)), \tag{3.22}$$

$$I_\varepsilon \to I \ \text{weakly} \ * \ \text{in} \ L^\infty(0,T; \Omega \times \mathcal{S}^2 \times (0, \infty)) \tag{3.23}$$

where $\varrho, \vec{u}, I$ is a weak solution satisfying

$$\partial_t \varrho + \mathrm{div}_x(\varrho \vec{u}) = 0 \tag{3.24}$$

$$\partial_t(\varrho\vec{u}) + \mathrm{div}_x(\varrho\vec{u}\otimes\vec{u}) + \nabla_x\left(p(\varrho) + \frac{1}{3}N\right) = \mu\Delta\vec{u} + (\lambda+\mu)\nabla_x\mathrm{div}_x\vec{u} \tag{3.25}$$

$$\partial_t N - \frac{1}{3}\mathrm{div}_x\left(\frac{1}{\sigma_s(\varrho)}\nabla_x N\right) = \sigma_a(\varrho)(b(\varrho) - N),\ b(\varrho) = \int_0^\infty B(\varrho,\nu)\ \mathrm{d}\nu. \tag{3.26}$$

4. Uniform estimates

Multiplying (1.1) on I we get

$$\frac{\varepsilon}{2}\partial_t I^2 + \frac{1}{2}\vec{\omega}\cdot\nabla_x I^2 = \varepsilon\sigma_a(b_\varepsilon - I)I + \frac{\sigma_s}{\varepsilon}\left(\frac{1}{4\pi}\int_{S^2} I\ \mathrm{d}\vec{\omega} - I\right)I.$$

Consequently, denoting

$$\tilde{I}(t,x,\nu) = \frac{1}{4\pi}\int_{S^2} I(t,x,\vec{\omega},\nu)\ \mathrm{d}\vec{\omega},$$

we deduce, integrating the above expression, that

$$\frac{1}{2}\int_\Omega \varepsilon\int_{S^2} I^2(\tau,\cdot)\ \mathrm{d}\vec{\omega}\ \mathrm{d}x + \frac{1}{2}\int_0^\tau\int_\Omega\sigma_a\int_{S^2}(b_\varepsilon - I)^2\ \mathrm{d}\vec{\omega}\ \mathrm{d}x\ \mathrm{d}t + \frac{1}{\varepsilon^2}\int_0^\tau\int_\Omega\sigma_s\int_{S^2}\left(I-\tilde{I}\right)^2\ \mathrm{d}\vec{\omega}\ \mathrm{d}x\ \mathrm{d}t$$
$$\leq \frac{1}{2}\int_\Omega\varepsilon\int_{S^2} I_{0,\varepsilon}^2\ \mathrm{d}\vec{\omega}\ \mathrm{d}x + 4\pi\varepsilon\int_0^\tau\int_\Omega\sigma_a b_\varepsilon^2\ \mathrm{d}x\ \mathrm{d}t. \tag{4.1}$$

$$\|\sigma_{a,\varepsilon}^{1/2}\,(b_\varepsilon - I_\varepsilon)\,\|_{L^2(\Omega\times\mathcal{S}^2\times(0,\infty))} \leq C, \tag{4.2}$$

$$\|\sigma_{s,\varepsilon}^{1/2}\left(\tilde{I}_\varepsilon - I_\varepsilon\right)\|_{L^2(\Omega\times\mathcal{S}^2\times(0,\infty))} \leq C\varepsilon, \tag{4.3}$$

and

$$\|\varepsilon\ \partial_t I_\varepsilon +\ \vec{\omega}\cdot\nabla_x I_\varepsilon\|_{L^2(\Omega\times\mathcal{S}^2\times(0,\infty))} \leq C. \tag{4.4}$$

Using the Fourier argument of [**4**] (see Lemma 3 in [**4**]) we also get that for any $T > 0$ the quantity $\left(\tilde{I}_\varepsilon^\alpha\right)^{1/\alpha}$ is bounded in $L^q(0,T;W^{\beta,q}(\Omega))$ where $q = \frac{2p}{p+1}$, $\alpha = 1 + \frac{1}{2p}$ and for any $\beta < \frac{p-1}{2p+1}$.

Integrating (3.4) over $\vec{\omega}$, we get first

$$\partial_t\ \widetilde{I_\varepsilon} + \frac{1}{\varepsilon}\ \mathrm{div}_x\ \widetilde{\vec{\omega}I_\varepsilon} = \sigma_{a,\varepsilon}\left(b_\varepsilon - \tilde{I}_\varepsilon\right), \tag{4.5}$$

and multiplying (3.4) by $\vec{\omega}$ and integrating over $\vec{\omega}$, we also have

$$\partial_t\ \widetilde{\vec{\omega}I_\varepsilon} + \frac{1}{\varepsilon}\ \mathrm{div}_x\ (\widetilde{\vec{\omega}\otimes\vec{\omega}I_\varepsilon}) = -\left(\frac{1}{\varepsilon^2}\ \sigma_{s,\varepsilon} + \sigma_{a,\varepsilon}\right)\widetilde{\vec{\omega}I_\varepsilon}. \tag{4.6}$$

Then we get the equation

$$\partial_t\ \widetilde{I_\varepsilon} - \mathrm{div}_x\left(\frac{1}{\sigma_{s,\varepsilon} + \varepsilon^2\sigma_{a,\varepsilon}}\left[\varepsilon\partial_t\ \widetilde{\vec{\omega}I_\varepsilon} +\ \mathrm{div}_x\ (\widetilde{\vec{\omega}\otimes\vec{\omega}I_\varepsilon})\right]\right)$$
$$= \sigma_{a,\varepsilon}\left(b_\varepsilon - \tilde{I}_\varepsilon\right)\quad \text{in } \mathcal{D}'((0,T)\times\Omega\times\mathcal{S}^2)\times(0,\infty). \tag{4.7}$$

Using (4.4) and (2.3), we conclude that the sequence $\{\partial_t\widetilde{I_\varepsilon}\}_\varepsilon$ is bounded in $L^q(0,T;W^{-1,q}(\Omega))$.

Setting $J_\varepsilon := \left(\tilde{I}_\varepsilon^\alpha\right)^{1/\alpha}$, we deduce that

$$J_\varepsilon \in L^q([0,T]; W^{\beta,q}(\Omega)),$$

$$\|\tilde{I}_\varepsilon - J_\varepsilon\|_{L^q((0,T)\times\Omega)} \to 0 \ \text{ for } \varepsilon \to 0,$$

and

$$\partial_t \tilde{I}_\varepsilon \in L^q([0,T]; W^{-1,q}(\Omega)).$$

Applying a variant of the Aubin-Lions Lemma (see Lemma in [**4**]), we deduce from these last estimates that there exists a subsequence $\widetilde{I_\varepsilon}$ converging in $L^q((0,T)\times\Omega)$.

4.1. Boundedness of the forcing term in the momentum equation. We show that the forcing terms $\vec{S}_F$ in the momentum equation is bounded in $L^2((0,T)\times\Omega;\mathbb{R}^3)$ uniformly for $\varepsilon \to 0$. Indeed we have

$$\int_0^T \int_\Omega \vec{S}_F \cdot \vec{u} \ \mathrm{d}x \ \mathrm{d}t = \int_0^\infty \int_0^T \int_\Omega \left(\varepsilon\sigma_a + \frac{1}{\varepsilon}\sigma_s\right) \vec{u} \cdot \int_{S^2} \vec{\omega} I \ \mathrm{d}\vec{\omega} \ \mathrm{d}x \ \mathrm{d}t \ \mathrm{d}\nu$$

$$= \int_0^\infty \int_0^T \int_\Omega \varepsilon\sigma_a \vec{u} \cdot \int_{S^2} \vec{\omega} I \ \mathrm{d}\vec{\omega} \ \mathrm{d}x \ \mathrm{d}t \ \mathrm{d}\nu + \int_0^\infty \int_0^T \int_\Omega \frac{1}{\varepsilon}\sigma_s \vec{u} \cdot \int_{S^2} \vec{\omega}(I - \tilde{I}) \ \mathrm{d}\vec{\omega} \ \mathrm{d}x \ \mathrm{d}t \ \mathrm{d}\nu,$$

where

$$\left| \int_0^\infty \int_0^T \int_\Omega \varepsilon\sigma_a \vec{u} \cdot \int_{S^2} \vec{\omega} I \ \mathrm{d}\vec{\omega} \ \mathrm{d}x \ \mathrm{d}t \ \mathrm{d}\nu \right| \le \varepsilon \, \|\sqrt{\sigma_a}\vec{u}\|_{L^2((0,T)\times\Omega)} \int_0^\infty \left\| \sqrt{\sigma_a} \int_{S^2} \vec{\omega} I \ \mathrm{d}\omega \right\|_{L^2((0,T)\times\Omega;\mathbb{R}^3)} \mathrm{d}\nu,$$

while

$$\left| \int_0^\infty \int_0^T \int_\Omega \frac{1}{\varepsilon}\sigma_s \vec{u} \cdot \int_{S^2} \vec{\omega}(I - \tilde{I}) \ \mathrm{d}\vec{\omega} \ \mathrm{d}x \ \mathrm{d}t \ \mathrm{d}\nu \right|$$

$$\le \|\sqrt{\sigma_s}\vec{u}\|_{L^2((0,T)\times\Omega;\mathbb{R}^3)} \int_0^\infty \left\| \sqrt{\sigma_s} \int_{S^2} \vec{\omega} \frac{I - \tilde{I}}{\varepsilon} \ \mathrm{d}\vec{\omega} \right\|_{L^2((0,T)\times\Omega;\mathbb{R}^3)} \mathrm{d}\nu.$$

As a consequence of (4.1), we have

$$\int_0^\infty \left\| \sqrt{\sigma_a} \int_{S^2} \vec{\omega} I \ \mathrm{d}\omega \right\|_{L^2((0,T)\times\Omega;\mathbb{R}^3)} \mathrm{d}\nu, \ \int_0^\infty \left\| \sqrt{\sigma_s} \int_{S^2} \vec{\omega} \frac{I - \tilde{I}}{\varepsilon} \ \mathrm{d}\vec{\omega} \right\|_{L^2((0,T)\times\Omega;\mathbb{R}^3)} \mathrm{d}\nu \le c$$

uniformly for $\varepsilon \to 0$ as soon as

$$0 \le \sigma_a(\varrho), \sigma_s(\varrho) \le \overline{\sigma}, \ |B(\varrho,\nu)|, |I_0(\cdot,\nu)| \le h(\nu), \ h \in L^1 \cap L^\infty(0,\infty). \tag{4.8}$$

Thus we conclude that

$$\vec{S}_F \to \vec{g} \text{ weakly in } L^2((0,T)\times\Omega;\mathbb{R}^3), \tag{4.9}$$

where

$$\vec{g} = \text{weak} \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\sigma_s \int_0^\infty \int_{S^2} \vec{\omega} I \ \mathrm{d}\vec{\omega} \ \mathrm{d}\nu \ \text{in } L^2((0,T)\times\Omega;\mathbb{R}^3). \tag{4.10}$$

5. Compactness for the Navier-Stokes system

It follows from the abstract compactness results on the solution set of the compressible Navier-Stokes system, see e.g. [**18**, Chapter 6], that

$$\varrho_\varepsilon \to \varrho \text{ in } C([0,T];L^1(\Omega)) \text{ and in } C_{\rm weak}([0,T];L^\gamma(\Omega)), \tag{5.1}$$

$$\vec{u}_\varepsilon \to \vec{u} \text{ weakly in } L^2(0,T;W^{1,2}(\Omega;R^3)) \tag{5.2}$$

as soon as we assume that

$$\varrho_{0,\varepsilon} \to \varrho_0 \text{ in } L^\gamma(\Omega),\ \frac{|(\varrho\vec{u})_{0,\varepsilon}|^2}{\varrho_{0,\varepsilon}} \text{ bounded in } L^1(\Omega), \tag{5.3}$$

where the limit is a weak solution of the Navier-Stokes system

$$\partial_t\varrho + {\rm div}_x(\varrho\vec{u}) = 0, \tag{5.4}$$

$$\partial_t(\varrho\vec{u}) + {\rm div}_x(\varrho\vec{u}\otimes\vec{u}) + \nabla_x p(\varrho) = \mu\Delta\vec{u} + (\lambda+\mu)\nabla_x{\rm div}_x\vec{u} + \vec{g}. \tag{5.5}$$

Thus it remains to identify the function $\vec{g}$ determined through (4.10).

6. The limit passage

We start by writing the rescaled equation (1.1):

$$\varepsilon\partial_t I_\varepsilon + \vec{\omega}\cdot\nabla_x I_\varepsilon = \varepsilon\sigma_a(B - I_\varepsilon) + \frac{1}{\varepsilon}\sigma_s(\tilde{I}_\varepsilon - I_\varepsilon). \tag{6.1}$$

In fact from (4.2) and (4.4) we see that there exists a $g \in L^2((0,T)\times\Omega\times\mathcal{S}^2)$ such that

$$\left(\sigma_{s,\varepsilon} + \varepsilon^2\sigma_{a,\varepsilon}\right)^{-1/2}{\rm div}_x\ (\vec{\omega}\otimes\vec{\omega}\ I_\varepsilon) \to g \ \text{ weakly in } L^2((0,T)\times\Omega\times\mathcal{S}^2\times(0,\infty)).$$

Multiplying by $\left(\sigma_{s,\varepsilon} + \varepsilon^2\sigma_{a,\varepsilon}\right)^{1/2} I_\varepsilon$ and using (2.2)-(2.3) we obtain

$$I_\varepsilon{\rm div}_x\ (\vec{\omega}\otimes\vec{\omega}\ I_\varepsilon) \to g\sigma_s^{1/2}I \ \text{ weakly in } L^1((0,T)\times\Omega\times\mathcal{S}^2\times(0,\infty)),$$

with $\sigma_s = \sigma_s(\varrho)$.

Now we see from above that

$$\left(\sigma_{s,\varepsilon} + \varepsilon^2\sigma_{a,\varepsilon}\right)^{1/2} I_\varepsilon \to \sigma_s^{1/2}I \ \text{ weakly in } L^2((0,T)\times\Omega\times\mathcal{S}^2\times(0,\infty)),$$

so

$$\frac{1}{2}\ {\rm div}_x\ (\vec{\omega}\otimes\vec{\omega}\ I_\varepsilon^2) \to g\sigma_s^{1/2}I \ \text{ weakly in } L^1((0,T)\times\Omega\times\mathcal{S}^2\times(0,\infty)),$$

and that

$$\frac{1}{2}\ {\rm div}_x\ (\vec{\omega}\otimes\vec{\omega}I_\varepsilon^2) \to \frac{1}{2}\ {\rm div}_x\ (\vec{\omega}\otimes\vec{\omega}I^2) \ \text{ weakly in } \mathcal{D}'((0,T)\times\Omega\times\mathcal{S}^2\times(0,\infty)).$$

Therefore

$$g\sigma_s^{1/2}I = \frac{1}{2}\ {\rm div}_x\ (\vec{\omega}\otimes\vec{\omega}I^2).$$

Exactly as in [**4**], one can now check that

$$\sigma_s^{-1/2}\tilde{g} = \frac{1}{3}\frac{1}{\sigma_s}\ \nabla_x I,$$

and therefore one can pass to the limit in the second term in the left hand side of (4.7)

$$\frac{1}{\sigma_{s,\varepsilon}+\varepsilon^2\sigma_{a,\varepsilon}}\ \nabla_x(\widetilde{\vec{\omega}\otimes\vec{\omega}I_\varepsilon}) = \frac{1}{(\sigma_{s,\varepsilon}+\varepsilon^2\sigma_{a,\varepsilon})^{1/2}}\frac{1}{(\sigma_{s,\varepsilon}+\varepsilon^2\sigma_{a,\varepsilon})^{1/2}}\ \nabla_x(\widetilde{\vec{\omega}\otimes\vec{\omega}I_\varepsilon})$$

$$\to \sigma_s^{-1/2}\tilde{g} = \frac{1}{3}\frac{1}{\sigma_s}\ \nabla_x I. \tag{6.2}$$

As the term in the right hand side of (4.7) clearly converges to $\sigma_a(\varrho)\left[b(\varrho)-\tilde{I}\right]$, this finally proves that $N := \int_{\mathcal{S}^2} I\ d\vec{\omega}$ satisfies the limit equation (3.14).

The same argument as in [**4**] shows finally that N satisfies the Dirichlet boundary condition $N|_{\partial\Omega} = 0$. In fact from the fact that $\vec{\omega}\cdot\nabla_x I_\varepsilon^2$ is bounded in $L^2((0,T)\times\Omega\times\mathbb{R}_+)$ we deduce that I_ε has a well-defined trace on $\partial\Omega$ which holds at the limit for I and then for N. Thus, introducing

$$N = \int_0^\infty I\ \mathrm{d}\nu,$$

we get the limit system in the form

$$\partial_t\varrho + \mathrm{div}_x(\varrho\vec{u}) = 0 \tag{6.3}$$

$$\partial_t(\varrho\vec{u}) + \mathrm{div}_x(\varrho\vec{u}\otimes\vec{u}) + \nabla_x\left(p(\varrho)+\frac{1}{3}N\right) = \mu\Delta\vec{u} + (\lambda+\mu)\nabla_x\mathrm{div}_x\vec{u} \tag{6.4}$$

$$\partial_t N - \frac{1}{3}\mathrm{div}_x\left(\frac{1}{\sigma_s(\varrho)}\nabla_x N\right) = \sigma_a(\varrho)(b(\varrho)-N),\ b(\varrho) = \int_0^\infty B(\varrho,\nu)\ \mathrm{d}\nu. \tag{6.5}$$

The convergence holds provided

$$\varrho_{0,\varepsilon} \to \varrho_0 \text{ in } L^\gamma(\Omega), \tag{6.6}$$

$$\int_\Omega \frac{|(\varrho\vec{u})_{0,\varepsilon}|^2}{\varrho_{0,\varepsilon}}\ \mathrm{d}x \le c, \tag{6.7}$$

$$|I_{0,\varepsilon}(\cdot,\nu)| \le h(\nu),\ h\in L^1\cap L^\infty(0,\infty). \tag{6.8}$$

Remark 6.1. The existence of a classical solution for the target system is an easy consequence of the existence of classical solution for the full compressible Navier- Stokes -Fourier system with diffusion see [**16**] and [**11**].

7. Appendix

Sketch of Proof of Theorem 2.1:

We will use three-level approximative system with parameters $n\to\infty$ (denoting the dimension of space of Galerkin approximations), $\eta\to 0$ (denoting the elliptic regularization of the continuity equation), $\delta\to 0$ (denoting the artificial pressure constant). We introduce *the approximative system* and give some remarks to the proof. We apply the approximation scheme introduced by Feireisl see [**20**] coupled together with the transport equation

$$\partial_t\varrho + \mathrm{div}_x(\varrho\vec{u}) = \eta\Delta\varrho, \tag{7.1}$$

$$\partial_t(\varrho u^i) + \mathrm{div}_x(\varrho u^i \vec{u}) + \partial_{x_i}\Big(p(\varrho) + \delta\varrho^\beta\Big) + \eta\nabla_x u^i.\nabla_x\varrho \tag{7.2}$$
$$= \mu\Delta u^i + (\lambda+\mu)(\mathrm{div}_x\ \vec{u})_{x_i} + S_F^i,\ i = 1,2,3$$

$$\varepsilon\ \partial_t I + \vec{\omega}\cdot\nabla_x I = S, \tag{7.3}$$

for $(t,x,\vec{\omega},\nu) \in (0,T)\times\Omega\times\mathcal{S}^2\times(0,\infty)$, complemented by the boundary conditions

$$\nabla_x\varrho.\vec{n}|_{\partial\Omega} = 0, \tag{7.4}$$

$$\vec{u}|_{\partial\Omega} = 0, \tag{7.5}$$

$$I|_{\Gamma_-} = 0, \tag{7.6}$$

and the initial conditions

$$\varrho(0) = \varrho_0 \in C^{2+\nu}(\overline{\Omega}\times S^2\times R^+),\ 0 < \underline{\varrho} \le \varrho_0(x) \le \overline{\varrho},\ \nabla_x\varrho_0.\vec{n}|_{\partial\Omega} = 0, \tag{7.7}$$

$$(\varrho\vec{u})(0) = \vec{q},\ \vec{q} = [q^1,q^2,q^3],\ q^i \in C^2(\overline{\Omega}),\ i = 1,2,3, \tag{7.8}$$

$$I(0) = I_0 \in C^{1+\nu}(\overline{\Omega}). \tag{7.9}$$

Here $S := \varepsilon\sigma_a\,(B-I) + \frac{1}{\varepsilon}\sigma_s\left(\frac{1}{4\pi}\int_{\mathcal{S}^2} I\ \mathrm{d}\vec{\omega} - I\right)$ and $\vec{S}_F = \int_0^\infty\int_{\mathcal{S}^2}\vec{\omega}S\ d\vec{\omega}d\nu$.

Let us fix $n \in N$, $\eta,\delta > 0$, ε and consider the orthogonal family of eigenfunctions ψ_n of the Dirichlet Laplacian on Ω given by

$$-\Delta\psi_n = \lambda_n\psi_n \text{ on } \Omega,\ \psi_n|_{\partial\Omega} = 0.$$

We consider a sequence of finite dimensional spaces

$$X_n = [\mathrm{span}\{\psi_j\}_{j=1}^n]^3,\ n = 1,2,...$$

The approximate solutions $\vec{u} \in C([0,T];X_n)$ we look for are required to satisfy the integral equation

$$\int_\Omega \varrho(t)\vec{u}(t).\psi\ dx - \int_\Omega \vec{q}.\psi\ dx = \tag{7.10}$$
$$\int_0^t\int_\Omega\Big[\mu\Delta\vec{u} - \mathrm{div}_x(\varrho\vec{u}\otimes\vec{u}) + \nabla\Big((\lambda+\mu)\mathrm{div}_x\ \vec{u} - p(\varrho) - \delta\varrho^\beta\Big)$$
-
$$\vec{S}_F + \eta\nabla_x\varrho.\nabla_x\vec{u}\Big].\psi\ dx\ \mathrm{d}s$$

for all $t \in [0,T]$ and any function $\psi \in X_n$.

$$\partial_t\varrho + \mathrm{div}_x(\varrho\vec{u}) = \eta\Delta\varrho, \tag{7.11}$$

and

$$\partial_t I + \vec{\omega}\cdot\nabla_x I = S, \tag{7.12}$$

with the initial and boundary conditions (7.4)-(7.9).

Then we consider the mapping

$$\mathcal{T}: X_n \to X_n,$$
$$\mathcal{T}(\vec{v}) \to \mathcal{T}(\vec{u})$$

defined in the following way: For a given v we firstly find ρ as a unique solution to the problem

$$\partial_t \varrho + \mathrm{div}_x(\varrho v) = \eta \Delta \varrho, \tag{7.13}$$

$$\nabla_x \varrho . \vec{n}|_{\partial\Omega} = 0. \tag{7.14}$$

$$\varrho(0) = \varrho_0 \in C^{2+\nu}(\overline{\Omega}),\ 0 < \underline{\varrho} \le \varrho_0(x) \le \overline{\varrho},\ \nabla_x \varrho_0 . \vec{n}|_{\partial\Omega} = 0, \tag{7.15}$$

Precisely, we will solve *the Neumann problem for the density*

The existence of a solution for the initial-boundary value problem (7.3), (7.4), (7.7), is standard and can be found in [**21**, Lemma 2.1, Lemma 2.2].

LEMMA 7.1. *Assume $\vec{v}$ is a given vector function belonging to the class*

$$\vec{v} \in C([0,T]; [C^2(\overline{\Omega})]^3),\ \vec{v}|_{\partial\Omega} = 0 \tag{7.16}$$

Then the initial-boundary value problem (7.13), (7.14) (7.15) possesses a unique classical solution $\varrho = \mathcal{S}(\vec{v})$ on the set $[0,T] \times \Omega$ such that $\varrho(t) \in C^{2+\nu}(\overline{\Omega})$ for any fixed $t \in [0,T]$. Moreover, assuming the initial datum ϱ_0 satisfies (7.15), the "solution" operator $\mathcal{S} : \vec{v} \mapsto \varrho$ enjoys the following properties:

(1) *$\varrho = \mathcal{S}(\vec{v})$ is the unique classical solution of (7.3), (7.4), (7.7) ;*
(2)

$$\underline{\varrho}\ \exp\Big(- \int_0^t \|\mathrm{div}_x\ \vec{v}(s)\|_{L^\infty(\Omega)}\ \mathrm{d}s \Big) \le \mathcal{S}(\vec{v})(t,x) \le \overline{\varrho}\ \exp\Big(\int_0^t \|\mathrm{div}_x\ \vec{v}(s)\|_{L^\infty(\Omega)}\ \mathrm{d}s \Big) \text{ for all } t \ge 0; \tag{7.17}$$

(3)

$$\|\mathcal{S}(\vec{v}^1) - \mathcal{S}(\vec{v}^2)\|_{C([0,T];W^{1,2}(\Omega))} \le Tc(\kappa, T)\|\vec{v}^1 - \vec{v}^2\|_{C([0,T];W_0^{1,2}(\Omega))}, \tag{7.18}$$

for any $\vec{v}^1$, $\vec{v}^2$ belonging to the set

$$M_\kappa = \{\vec{v} \in C([0,T]; W_0^{1,2}(\Omega)) \mid \|\vec{v}(t)\|_{L^\infty(\Omega)} + \|\nabla \vec{v}(t)\|_{L^\infty(\Omega)} \le \kappa \text{ for all } t\}.$$

Then we find I as a solution to the transport equations

$$\varepsilon\ \partial_t I + \vec{\omega} \cdot \nabla_x I = S, \tag{7.19}$$

for $(t, x, \vec{\omega}, \nu) \in (0,T) \times \Omega \times \mathcal{S}^2 \times (0,\infty)$

$$I|_{\Gamma_-} = 0, \tag{7.20}$$

$$I(0) = I_0 \in C^{1+\nu}(\overline{\Omega} \times S^2 \times R_+). \tag{7.21}$$

The compactness of the averages over sphere has to be used to get the existence of I. Precisely, we apply the following lemma

LEMMA 7.2. *(C. Bardos, F. Golse, B. Perthame, R. Sentis) Let $I \in L^p(\Omega \times S \times (0,\infty))$ and $\partial_t I + \vec{\omega} \cdot \nabla_x I \in L^p(\Omega \times S \times (0,\infty))$ for some $1 < p < \infty$. Then*

$$\tilde{I} \equiv \frac{1}{4\pi} \int_S I(\cdot, \vec{\omega}, \cdot)\, d\vec{\omega} \tag{7.22}$$

belongs to the space $L^p((0,\infty); W^{s,p}(\Omega))$ for any $0 < s < \min\{\frac{1}{p}, 1 - \frac{1}{p}\}$, and

$$\|\tilde{I}(\cdot,\nu)\|_{W^{s,p}(\Omega)} \le C(\|I(\cdot,\cdot,\nu)\|_{L^p(\Omega\times S)} + \|\partial_t I + \vec{\omega} \cdot \nabla_x I(\cdot,\cdot,\nu)\|_{L^p(\Omega\times S)}). \tag{7.23}$$

PROOF. See [**24**, Theorem 4] □

Finally, we find $\vec{u}$ as a solution to

$$\int_\Omega \varrho(t)\vec{u}(t)\vec{\psi}\ dx - \int_\Omega \vec{q}\vec{\psi}\ dx \tag{7.24}$$

$$= \int_0^T \int_\Omega \Big\{ \mu\Delta\vec{v} - \mathrm{div}_x(\varrho\vec{v}\otimes\vec{v}) + \nabla_x\left((\lambda+\mu)\mathrm{div}_x\vec{v} - p(\varrho) - \delta\varrho^\beta\right) - \eta\nabla_x\varrho\nabla_x\vec{v} + \vec{S}_F \Big\}\vec{\psi} dx\, dt.$$

Applying the Schauder fixed point theorem and passing to the limit $n \to \infty$ we get

LEMMA 7.3. *Suppose $\beta > \max\{4,\gamma\}$. Assume the initial data ϱ_0, $\vec{q}$ satisfy (7.7), (7.8). Then there exists a weak solution ϱ, $\vec{u}$ of the problem (7.3) - (7.1) such that $\varrho \in L^{\beta+1}((0,T)\times\Omega)$ and the following estimates hold:*

$$\sup_{t\in[0,T]} \|\varrho(t)\|^\gamma_{L^\gamma(\Omega)} \le c(\varrho_0, \vec{q}, I_0, \delta), \tag{7.25}$$

$$\delta \sup_{t\in[0,T]} \|\varrho(t)\|^\beta_{L^\beta(\Omega)} \le c(\varrho_0, \vec{q}, I_0, \delta,), \tag{7.26}$$

$$\sup_{t\in[0,T]} \|\sqrt{\varrho}(t)\vec{u}(t)\|^2_{L^2(\Omega)} \le c(\varrho_0, \vec{q}, I_0, \delta), \tag{7.27}$$

$$\int_0^T \left(\|\vec{u}(t)\|^2_{L^2(\Omega)} + \|\nabla_x\vec{u}(t)\|^2_{L^2(\Omega)} \right)\ dt \le c(\varrho_0, \vec{q}, I_0, \delta), \tag{7.28}$$

and

$$\varepsilon \int_0^T \|\nabla_x\varrho(t)\|^2_{L^2(\Omega)}\ dt \le c(\varrho_0, \vec{q}, I_0, \delta). \tag{7.29}$$

Moreover, the modified energy inequality

$$\frac{d}{dt}\int_\Omega \left[\frac{1}{2}\varrho|\vec{u}|^2 + \Pi(\varrho) + \frac{\delta}{\beta-1}\varrho^\beta\right]\ dx + \int_\Omega \left[\mu|\nabla_x\vec{u}|^2 + (\lambda+\mu)|\mathrm{div}_x\ \vec{u}|^2\right]\ dx \tag{7.30}$$

$$\le \int_\Omega \left[p_s(\varrho)\ \mathrm{div}_x\ \vec{u} - \vec{S}_F\cdot\vec{u}\right]\ dx$$

holds in $\mathcal{D}'(0,T)$ along with its "integrated" version .

$$\int_\Omega \left[\frac{1}{2}\varrho|\vec{u}|^2(\tau) + \Pi(\varrho)(\tau) + \frac{\delta}{\beta-1}\varrho^\beta(\tau)\right]\ dx$$

$$+ \int_0^\tau \int_\Omega \left[\mu|\nabla_x\vec{u}|^2 + (\lambda+\mu)|\mathrm{div}_x\ \vec{u}|^2\right]\ dx\ \mathrm{dt} \tag{7.31}$$

$$\le \int_\Omega \left[\frac{1}{2}\frac{|\vec{q}|^2}{\varrho_0} + \Pi(\varrho_0) + \frac{\delta}{\beta-1}\varrho_0^\beta\right]\ dx + \int_0^\tau \int_\Omega \left[p_s(\varrho)\ \mathrm{div}_x\ \vec{u} - \vec{S}_F\cdot\vec{u}\right]\ dx\ \mathrm{dt}$$

for a.e. $\tau \in (0,T)$.

Finally, there exists $r > 1$ such that ϱ_t, $\Delta\varrho \in L^r((0,T)\times\Omega)$ and the equation (7.3) is satisfied a.a. on $(0,T)\times\Omega$.

The vanishing viscosity limit

Now we pass to the limit in (7.3), (7.2), (7.1) letting $\eta \to 0$. For more details see [**21**].

We get the following lemma

LEMMA 7.4. *Let β, $\delta > 0$, and $R > 0$ be given such that*

$$\beta > \max\{\gamma, 4\}.$$

Let the pressure p satisfy the constraints of hypothesis see Section 2. Then, given initial data ϱ_0, $\vec{q}$, I_0 as in (7.7), (7.8), there exists a finite energy weak solution ϱ, $\vec{u}$, I of the problem

$$\partial_t \varrho + \mathrm{div}_x(\varrho \vec{u}) = 0 \tag{7.32}$$

(7.33)
$$\partial_t(\varrho u^i) + \mathrm{div}_x(\varrho u^i\ \vec{u}) + \partial_{x_i}\Big(p(\varrho) + \delta\varrho^\beta\Big) = \mu\Delta u^i + (\lambda+\mu)(\mathrm{div}_x\ \vec{u})_{x_i} - \vec{S}_F,\ i = 1, 2, 3,$$

$$\varepsilon\ \partial_t I + \vec{\omega}\cdot\nabla_x I = \varepsilon\sigma_a\,(B - I) + \frac{1}{\varepsilon}\sigma_s\left(\frac{1}{4\pi}\int_{\mathcal{S}^2} I\ \mathrm{d}\vec{\omega} - I\right), \tag{7.34}$$

with boundary conditions

$$\vec{u}|_{\partial\Omega} = 0 \tag{7.35}$$

on $(0,T)\times\Omega$, and

$$I|_{\Gamma_-} = 0, \tag{7.36}$$

for $t \in (0,T)$, $(x,\vec{\omega}) \in \Gamma_- \equiv \Big\{(x,\vec{\omega})\ \Big|\ (x,\vec{\omega}) \in \partial\Omega\times S^2,\ \vec{\omega}\cdot\vec{n} \le 0\Big\}$ and for $\nu \in (0,\infty)$, and for initial conditions (1.5).

The artificial pressure limit

Finally, to conclude the proof of Theorem 2.1, one has to pass to the limit for $\delta \to 0$ to get rid of the artificial pressure term. For this step we refer to [**21**, Section 4] where this is done for the monotone pressure-density constitutive law and to [**22**, Theorem 1.1] where the necessary modifications how to accommodate the pressure satisfying the constraints of hypothesis see Section 2 can be found.

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