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On the existence of global strong solutions to the equations modeling a motion of a rigid body around a viscous fluid

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Abstract: *The paper deals with the global existence of strong solution to the equations modeling a motion of a rigid body fluid around viscous fluid. Moreover, the estimates of second gradients of velocity and pressure are given.*

Keywords: *Navier-Stokes equations, moving body in fluid motion, global strong solutions, regularity*

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1 Introduction. Equations and Notations

Let $\mathscr{B}$ denote a bounded C^2 domain, representing a rigid body in a fluid motion. While the fluid is governed by the usual Navier-Stokes equations, the evolution of body is determinate by external forces acting on the surface of the body. By changing of coordinates with respect to the movement of the body we are led to a coupled system in $\mathscr{D} := \mathbb{R}^3 \setminus \overline{\mathscr{B}}$, where $\mathscr{D}$ appears to be an exterior domain in $\mathbb{R}^3$ with C^2 boundary $\Sigma = \partial\mathscr{D} = \partial\mathscr{B}$.

Given $T > 0$, by Q_T we denote the cylinder $\mathscr{D} \times (0, T)$. The velocity of the body at a specific point of the body at time $t \in [0, T]$ will be denoted by $\boldsymbol{\xi}(t)$, while the angular velocity of the body at time $t \in [0, T]$, will be denoted by $\boldsymbol{\omega}(t)$. The velocity of the fluid $\boldsymbol{u} = (u^1, u^2, u^3)$ and the pressure p will be governed by the following equations

$$(1.1) \qquad \begin{cases} \operatorname{div} \boldsymbol{u} = 0, \\ \partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = (\boldsymbol{U} \cdot \nabla)\boldsymbol{u} - \boldsymbol{\omega} \times \boldsymbol{u} + \nu \Delta \boldsymbol{u} - \nabla p \quad \text{in} \quad Q_T \end{cases}$$

fulfilling the initial and boundary conditions

$$(1.2) \qquad \boldsymbol{u} = \boldsymbol{U} \quad \text{on} \quad \Sigma \times (0, T), \quad \boldsymbol{u} = 0 \quad \text{as} \quad |x| \to +\infty,$$

$$(1.3) \qquad \boldsymbol{u} = \boldsymbol{u}_0 \quad \text{on} \quad \mathscr{D} \times \{0\}.$$

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Here,

$$\boldsymbol{U} = \boldsymbol{\xi} + \boldsymbol{\omega} \times y \quad \text{in} \quad (0,T).$$

The equations of the rigid body are given by the momentum equations for $\boldsymbol{\xi}$ and $\boldsymbol{\omega}$,

$$\begin{cases} m\dot{\boldsymbol{\xi}} + m\boldsymbol{\omega} \times \boldsymbol{\xi} = -\boldsymbol{\Phi}_{\rm tra}(\boldsymbol{u},p) \quad \text{in} \quad (0,T), \\ \boldsymbol{J}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \boldsymbol{J}\boldsymbol{\omega} = -\boldsymbol{\Phi}_{\rm rot}(\boldsymbol{u},p) \quad \text{in} \quad (0,T), \end{cases} \tag{1.4}$$

where

$$\boldsymbol{\Phi}_{\rm tra}(\boldsymbol{u},p) := \int_{\Sigma} \mathbb{T}(\boldsymbol{u},p) \cdot \boldsymbol{n} dS,$$

$$\boldsymbol{\Phi}_{\rm rot}(\boldsymbol{u},p) := \int_{\Sigma} (y \times \mathbb{T}(\boldsymbol{u},p)) \cdot \boldsymbol{n} dS.$$

Here $\boldsymbol{n}$ stands for the inward unit normal on $\partial\mathscr{B}$ [1)] while $\mathbb{T} = (T_{ij})$ stands for the full stress, i. e.

$$\mathbb{T} = 2\nu \boldsymbol{D}(\boldsymbol{u}) - \boldsymbol{I}p, \quad \text{where} \quad \boldsymbol{D}(\boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^t)$$

($\nu = \text{const} > 0$ denotes the viscosity of the fluid).

The system (1.4) will be completed by the following initial condition

$$\boldsymbol{\xi}(0) = \boldsymbol{\xi}_0, \qquad \boldsymbol{\omega}(0) = \boldsymbol{\omega}_0. \tag{1.5}$$

Remark 1.1. 1. Since $\mathscr{B}$ is bounded there exists $R_0 > 0$, such that $\mathscr{B} \subset B_{R_0/2}(0)$. Define

$$\mathscr{D}_0 := \mathscr{D} \cap B_{2R_0}(0).$$

Let $(\boldsymbol{v}, \pi) \in \boldsymbol{C}^2(\overline{\mathscr{D}}_0) \times C^1(\overline{\mathscr{D}}_0)$. By a direct application of Gaußian theorem one calculates

$$\boldsymbol{\Phi}_{\rm tra}(\boldsymbol{v},\pi) := \int_{\Sigma} \mathbb{T}(\boldsymbol{v},\pi) \cdot \boldsymbol{n} dS = \int_{\mathscr{D}_0} \operatorname{div}(\phi \mathbb{T}(\boldsymbol{v},\pi)) dy,$$

$$\boldsymbol{\Phi}_{\rm rot}(\boldsymbol{v},\pi) := \int_{\Sigma} (y \times \mathbb{T}(\boldsymbol{v},\pi)) \cdot \boldsymbol{n} dS = \int_{\mathscr{D}_0} \operatorname{div}(\phi (y \times \mathbb{T}(\boldsymbol{v},\pi))) dy$$

for all $\phi \in C_0^\infty(B_{2R_0})$ with $\phi = 1$ on Σ. By a standard density argument the above identity yields that both $\boldsymbol{\Phi}_{\rm tra}$ and $\boldsymbol{\Phi}_{\rm rot}$ are continuous linear functional on

[1)] Note, $\boldsymbol{n}$ outward unit normal on $\partial\mathscr{D}$ since $\mathscr{D} = \mathbb{R}^3 \setminus \mathscr{B}$.

$\boldsymbol{W}^{2,1}(\mathscr{D}_0) \times W^{1,1}(\mathscr{D}_0)$. Moreover, we can estimate $\boldsymbol{\Phi}_{\rm tra}(\boldsymbol{v}, \pi)$ ($\boldsymbol{\Phi}_{\rm rot}(\boldsymbol{v}, \pi)$ respectively) by the multiplicative inequality stated in Lemma 3.1 (see Section 3 below).

2. Given $\boldsymbol{U} = \boldsymbol{\xi} + \boldsymbol{\omega} \times y$ such that $\boldsymbol{\xi}, \boldsymbol{\omega} \in \boldsymbol{W}^{1,1}(0,T)$ by an elementary calculus we see that

$$\int_{\Sigma} \boldsymbol{U} \cdot \mathbb{T}(\boldsymbol{v}, \pi) \cdot \boldsymbol{n} dS = \boldsymbol{\xi} \cdot \boldsymbol{\Phi}_{\rm tra}(\boldsymbol{v}, \pi) + \boldsymbol{\omega} \cdot \boldsymbol{\Phi}_{\rm rot}(\boldsymbol{v}, \pi), \tag{1.6}$$

$$\int_{\Sigma} \dot{\boldsymbol{U}} \cdot \mathbb{T}(\boldsymbol{v}, \pi) \cdot \boldsymbol{n} dS = \dot{\boldsymbol{\xi}} \cdot \boldsymbol{\Phi}_{\rm tra}(\boldsymbol{v}, \pi) + \dot{\boldsymbol{\omega}} \cdot \boldsymbol{\Phi}_{\rm rot}(\boldsymbol{v}, \pi) \tag{1.7}$$

holds for all $(\boldsymbol{v} \times \pi) \in \boldsymbol{W}^{2,1}(\mathscr{D}_0) \times W^{1,1}(\mathscr{D}_0)$ a.e. in $(0,T)$.

Remark 1.2. The system (1.1)-(1.5) is obtained from the original system (e.g. see [9]) by transforming to the coordinate system with respect to the moving body $\mathscr{B}$. For more details see [9].

Remark 1.3. Given $(\boldsymbol{u}, p) \in L^1(0,T; \boldsymbol{W}^{2,1}(\mathscr{D}_0)) \times L^1(0,T; W^{1,1}(\mathscr{D}_0))$ and $\boldsymbol{\xi}, \boldsymbol{\omega} \in \boldsymbol{W}^{1,1}(0,T)$ such that (1.4) is fulfilled. Then inserting $(\boldsymbol{u}(t), p(t))$ into (1.6) and (1.7) respectively we see that

$$\int_{\Sigma} \boldsymbol{U} \cdot \mathbb{T}(\boldsymbol{u}, p) \cdot \boldsymbol{n} dS = -m \dot{\boldsymbol{\xi}} \cdot \boldsymbol{\xi} - \boldsymbol{J} \dot{\boldsymbol{\omega}} \cdot \boldsymbol{\omega} \tag{1.8}$$

a.e. in $(0,T)$ and

$$\int_{\Sigma} \dot{\boldsymbol{U}} \cdot \mathbb{T}(\boldsymbol{u}, p) \cdot \boldsymbol{n} dS = -m |\dot{\boldsymbol{\xi}}|^2 - |\boldsymbol{R} \dot{\boldsymbol{\omega}}|^2 - m \dot{\boldsymbol{\xi}} \cdot (\boldsymbol{\omega} \times \boldsymbol{\xi}) - \dot{\boldsymbol{\omega}} \cdot \boldsymbol{\omega} \times \boldsymbol{J} \boldsymbol{\omega} \tag{1.9}$$

a.e. in $(0,T)$ respectively, where $\boldsymbol{R}$ denotes the square root of $\boldsymbol{J}$ [2).]

The existence of a global weak solution of the Leray-Hopf type has been proved by Borchers see [1] (see also [21]).
In [13] Hishida obtained the local existence and uniqueness of a strong solution locally in time if the initial velocity belongs to $H^{1/2}$. This regularity assumption concides with that in the famous paper of Fujita, Kato [6]. An essential part for the proof is the deduction of a certain smoothing property together with estimates near $t = 0$, although the semigroup generated by the operator $\mathcal{L}u = -\mathbf{P}[\Delta u + (\omega \times x) \cdot \nabla u - \omega \times u]$ [3)], is not an analytic one in the L^2 space.

The generalization of Hishida's results in L^p spaces has been worked out by Hieber, Heck and Geissert in [11], where the authors proved the existence of a unique local mild solution to the Navier-Stokes problem. The existence of global strong solutions in the L^2 framework under a smallness assumption of the data has been studied by Galdi and Silvestre [7, 8] and by Takahashi and Tucsnak [23] for a rigid body being a disk

[2)] Recalling that $\boldsymbol{J}$ is positive and self adjoined, there exists a positive matrix $\boldsymbol{R}$ such that $\boldsymbol{R}^2 = \boldsymbol{J}$.
[3)] Here $\mathbf{P}$ denotes the projection associated with the Helmholtz decomposition.

in the two-dimensional situation. Local in time existence and uniqueness of the strong solution has been proved by Cumsille and Tucsnak [3]. The global time existence and uniqueness was investigated in work of Cumsille and Takahashi [4]. However in three dimensional case the uniqueness is valid only under a smallness assumption of data. L^p-L^q estimates for Stokes-system in the rotating framework can be found in [14].

Alternatively, the problem has been studied in [3, 24, 23, 5] by using the local transformation introduced by Inue and Wakimoto in [15], and in domains depending on time in [16, 18, 19, 20].

The paper is organized as follows. In Section 2 we introduce the notion of a weak solution and a strong solution to system (1.1)–(1.5) belonging to appropriate Sobolev spaces and Lebesgue spaces. Based on these definitions we state our main result on the existence of a global strong solution (cf. Theorem 2.2 and Theorem 2.3). In Section 3 we list few preliminary Lemmas being used in the sequel of the paper. In Section 4 appealing to [17] we provide a global strong solution to the approximate system based on the L^2 theory. Then Section 5 deals with a-priori estimates of the solution to the approximate system by using the $L^p - L^q$ theory of the Stokes equations in an exterior domain (cf. [12]) combined with the linear theory of the equation of a moving body studied in [17]. Furthermore, using the pressure estimate from [25] we get the key estimate (5.25) which finally leads to the crucial estimate (5.33). The proof of the existence of strong solutions will be completed in Section 6 by carrying out the passage to the limit in the approximate system using the a-priori estimate (5.33). Finally, Section 7 is devoted to the proof of the second main result concerning removing the weight in estimate of the second gradient together with the higher regularity of the pressure.

2 Notion of a weak solution

For the statement of our main result it requires to introduce both the notion of a weak and a strong solution to the system under consideration. Here and subsequently, by $W^{k,s}(\Omega), W_0^{k,s}(\Omega)$, etc. $(k \in \mathbb{N}, 0 \le s \le +\infty)$ we denote the usual Sobolev spaces. In case if Ω is unbounded, for instance an exterior domain, then by $\widehat{W}_0^{1,s}(\Omega)$ we denote the corresponding homogeneous Sobolev space in general to be defined by taking the closure of $C_0^\infty(\Omega)$ under the norm

$$\|\boldsymbol{u}\|_{\widehat{W}_0^{1,s}} = \left(\int_\Omega |\nabla u|^s dx \right)^{1/s}.$$

In what follows spaces of vector-valued functions as well as vector-functions will be designed by bold letters, i.e. we write $\boldsymbol{L}^s(\Omega), \boldsymbol{W}_0^{1,s}(\Omega)$ etc. instead of $[L^s(\Omega)]^n$, $[W^{1,s}(\Omega)]^n$ etc.
Given any Banach space X with norm $\|\cdot\|_X$ by $L^s(a,b;X)$ we denote the space of

Bochner measurable functions $f:(a,b)\to X$, such that

$$\begin{cases} \|f\|^s_{L^s(a,b;X)} := \int\limits_a^b \|f(t)\|^s_X d < +\infty & \text{if} \quad 1\le s<+\infty, \\ \|f\|_{L^\infty(a,b;X)} := \operatorname{ess\,sup}_{t\in(a,b)} \|f(t)\|_X < +\infty & \text{if} \quad s=+\infty. \end{cases}$$

Let $\tau:\Omega\to\mathbb{R}$ be a measurable positive function. We define the Lebesgue space $L^q_\tau(\Omega) = \left\{ f\in L^1_{loc}(\overline{\Omega}) \ s.t.\ \int_\Omega |f|^q\tau\,dx < +\infty \right\}$, with the norm $\|f\|_{L^q_\tau} := \left(\int_\Omega |f|^q \tau\,dx\right)^{1/q}$.

Next, by $\boldsymbol{C}^\infty_{0,\sigma}(\Omega)$ we denote the space of all smooth solenoidal fields having compact support in Ω. Furthermore, by $\mathcal{C}(\Omega)$ we denote the space of all smooth vector fields $\boldsymbol{\varphi}\in\boldsymbol{C}^\infty_{0,\sigma}(\Omega)$, such that

$$\boldsymbol{\varphi} = \boldsymbol{\Phi}_1 + \boldsymbol{\Phi}_2\times x \quad \text{in a neighbourhood of } \partial\Omega$$

for some constants vectors $\boldsymbol{\Phi}_1,\boldsymbol{\Phi}_2$ in $\mathbb{R}^3$. Then $\mathcal{V}(\Omega)$ will be defined as closure of $\mathcal{C}(\Omega)$ with respect to the norm in $\boldsymbol{W}^{1,2}(\Omega)$. Clearly, by means of Sobolev's embedding theorem we have $\mathcal{V}(\Omega)\hookrightarrow \boldsymbol{L}^6(\Omega)$.

From now on let $\mathscr{D}$ denote an exterior domain. Without loss of generality throughout we assume that $\nu=1$.

Definition 2.1. 1. Let $\boldsymbol{u}_0\in\mathcal{V}(\mathscr{D})$ with $\boldsymbol{u}_0 = \boldsymbol{\xi}_0 + \boldsymbol{\omega}_0\times x$ on $\partial\mathscr{D}$, where $\boldsymbol{\xi}_0,\boldsymbol{\omega}_0\in\mathbb{R}^3$ are given. Then $(\boldsymbol{u},\boldsymbol{\xi},\boldsymbol{\omega})$ is called a *weak solution* to (1.1)–(1.5) if

(i) $\boldsymbol{u}\in L^2(0,T;\mathcal{V}(\mathscr{D}))\cap C_w([0,T);\boldsymbol{L}^2(\mathscr{D}))$,

(ii) $\boldsymbol{\xi},\boldsymbol{\omega}\in\boldsymbol{C}([0,T])$,

(iii) for every $\boldsymbol{\varphi}\in C^\infty([0,T];\mathcal{C}(\mathscr{D}))$ there holds the identity

$$\begin{aligned} &\int\limits_0^t\int\limits_{\mathscr{D}} (-\boldsymbol{u}\cdot\partial_t\boldsymbol{\varphi} + \boldsymbol{D}(\boldsymbol{u}):\boldsymbol{D}(\boldsymbol{\varphi}) + ((\boldsymbol{u}-\boldsymbol{U})\cdot\nabla\boldsymbol{u})\cdot\boldsymbol{\varphi} + (\boldsymbol{\omega}\times\boldsymbol{u})\cdot\boldsymbol{\varphi})dxds \\ &\qquad + \int\limits_{\mathscr{D}} \boldsymbol{u}(t)\cdot\boldsymbol{\varphi}(t)dx + m\boldsymbol{\xi}(t)\cdot\boldsymbol{\Phi}_1(t) + (\boldsymbol{J}\boldsymbol{\omega}(t))\cdot\boldsymbol{\Phi}_2(t)) \\ &= \int\limits_0^t m\boldsymbol{\xi}(s)\cdot\dot{\boldsymbol{\Phi}}_1(s) + (\boldsymbol{J}\boldsymbol{\omega}(s))\cdot\dot{\boldsymbol{\Phi}}_2(s)ds \\ &\qquad + \int\limits_{\mathscr{D}} \boldsymbol{u}_0\cdot\boldsymbol{\varphi}(0)dx + m\boldsymbol{\xi}_0\cdot\boldsymbol{\Phi}_1(0) + (\boldsymbol{J}\boldsymbol{\omega}_0)\cdot\boldsymbol{\Phi}_2(0) \end{aligned} \tag{2.1}$$

for all $0<t<T$.

(iv) For all $0<t\le T$ there holds the energy inequality

$$\frac{1}{2}\|\boldsymbol{u}(t)\|^2_{\boldsymbol{L}^2} + \int_0^t \int_{\mathscr{D}} |\boldsymbol{D}(\boldsymbol{u})|^2 dxds + \frac{m}{2}|\boldsymbol{\xi}(t)|^2 + \frac{1}{2}|\boldsymbol{R}\boldsymbol{\omega}(t)|^2$$

$$\le \frac{1}{2}\|\boldsymbol{u}_0\|^2_{\boldsymbol{L}^2} + \frac{m}{2}|\boldsymbol{\xi}_0|^2 + \frac{1}{2}|\boldsymbol{R}\boldsymbol{\omega}_0|^2. \tag{2.2}$$

2. A weak solution $(\boldsymbol{u}, \boldsymbol{\xi}, p, \boldsymbol{\omega})$ is called a *strong solution* if

$$\partial_i\partial_j \boldsymbol{u}, \partial_t \boldsymbol{u} \in L^s(0,T; \boldsymbol{L}^q_\tau(\mathscr{D})) \quad (i,j = 1,2,3), \tag{2.3}$$

$$\nabla p \in L^s(0,T; \boldsymbol{L}^q(\mathscr{D}) + \boldsymbol{L}^2(\mathscr{D})) \tag{2.4}$$

for all $s, q \in (1,2)$ with $\frac{2}{s} + \frac{3}{q} = 4$, where $\tau = (1+|x|)^{-1}$ and there holds

$$\dot{\boldsymbol{\xi}}, \dot{\boldsymbol{\omega}} \in \boldsymbol{L}^\alpha(0,T) \quad \forall\, \alpha \in [1,2). \tag{2.5}$$

Our main result is the following

Theorem 2.2. *Let $\boldsymbol{u}_0 \in \mathcal{V}(\mathscr{D})$ and $\boldsymbol{\xi}_0, \boldsymbol{\omega}_0 \in \mathbb{R}^3$. Then there exists a strong solution $(\boldsymbol{u}, p, \boldsymbol{\xi}, \boldsymbol{\omega})$ to the system* (1.1)–(1.5)*. In addition, there holds*

$$\|\nabla^2 \boldsymbol{u}\|_{L^s(0,T\boldsymbol{L}^q_\tau)} + \|\partial_t \boldsymbol{u}\|_{L^s(0,T\boldsymbol{L}^q_\tau)} + \|\nabla p\|_{L^s(0,T\boldsymbol{L}^q + \boldsymbol{L}^2)} \le c_1(1 + K_0^{16}), \tag{2.6}$$

$$\|\dot{\boldsymbol{\xi}}\|_{\boldsymbol{L}^\alpha(0,T)} + \|\dot{\boldsymbol{\omega}}\|_{\boldsymbol{L}^\alpha(0,T)} \le c_2(1 + K_0^{16}), \tag{2.7}$$

where $c_1 = \text{const} > 0$ depending on $s, q, \mathscr{D}$, while $c_2 = \text{const} > 0$ depending on α and $\mathscr{D}$. Here $K_0 = \|\boldsymbol{u}_0\|_{\boldsymbol{L}^2} + |\boldsymbol{\xi}_0| + |\boldsymbol{\omega}_0|$.

Our second main result will give additional estimates on the second gradients of both the velocity and the pressure

Theorem 2.3. *Under the assumption of Theorem 2.2 we have*

$$\frac{\nabla^2 \boldsymbol{u}}{(1+|\nabla \boldsymbol{u}|)^{(1+\delta)/2}} \in L^2(0,T; \boldsymbol{L}^2(\mathscr{D}')) \quad \forall\, 0 < \delta < 1 \tag{2.8}$$

and

$$\nabla^2 p \in L^1(0,T; \boldsymbol{L}^1(\mathscr{D}') + \boldsymbol{L}^\alpha(\mathscr{D}')) \quad \forall\, \alpha \in (1,2) \tag{2.9}$$

for every open set $\mathscr{D}' \subset \mathscr{D}$ with $\operatorname{dist}(\mathscr{D}', \partial\mathscr{D}) > 0$.

Remark 2.4. The global existence is shown on time interval $t \in (0,T)$, $T < \infty$. The global existence of strong solution with respect to our Definition 2.1 is also valid in the case where $T = \infty$.

3 Auxiliary lemmas

Based on trace theorem and complex interpolation we have the following multiplicative inequality

Lemma 3.1. *Let* $1 < q < 3$. *For every* $\frac{1}{q} < \theta \le 1$ *there exists a constant* $c = c(\theta, \Sigma) > 0$, *such that*

$$\|f\|_{L^{2q/(3-\theta q)}(\Sigma)} \le c\|f\|^{1-\theta}_{L^q(\mathscr{D}_0)}\|f\|^{\theta}_{W^{1,q}(\mathscr{D}_0)} \quad \forall\, f \in W^{1,q}(\mathscr{D}_0). \tag{3.1}$$

Proof First, assume that $f \in C^1(\overline{\mathscr{D}}_0)$. By using Sobolev's embedding theorem and the well-known trace theorem we see that

$$\|f\|_{L^{2q/(3-\theta q)}(\Sigma)} \le c\|f\|_{W^{\theta-1/q,\,q}(\Sigma)} \le c\|f\|_{W^{\theta,\,q}(\mathscr{D}_0)}.$$

Since $W^{\theta,q}(\mathscr{D}_0)$ is continuously embedded into the interpolation space $[L^q(\mathscr{D}_0), W^{1,q}(\mathscr{D}_0)]_\theta$ we get

$$\|f\|_{W^{\theta,q}(\mathscr{D}_0)} \le c\|f\|^{1-\theta}_{L^q(\mathscr{D}_0)}\|f\|^{\theta}_{W^{1,q}(\mathscr{D}_0)}.$$

Inserting the latter estimate into the former we obtain (3.1).

The general case follows by applying a standard density argument. ■

Remark 3.2. In Lemma 3.1 letting $\int\limits_{\mathscr{D}_0} f dx = 0$ by Poincaré's inequality we obtain

$$\|f\|_{L^{2q/(3-\theta q)}(\Sigma)} \le c\|f\|^{1-\theta}_{L^q(\mathscr{D}_0)}\|\nabla f\|^{\theta}_{L^q(\mathscr{D}_0)} \;\; \forall\, f \in W^{1,q}(\mathscr{D}_0) \text{ with } \int\limits_{\mathscr{D}_0} f dx = 0. \tag{3.2}$$

Applying Lemma 3.1 to the functional $\boldsymbol{\Phi}_{\mathrm{tra}}$ and $\boldsymbol{\Phi}_{\mathrm{rot}}$ we get

Lemma 3.3. *Let* $1 < q < 3$. *For every* $\frac{1}{q} < \theta < 1$ *there exists a constant* $c > 0$, *such that*

$$\begin{aligned} &|\boldsymbol{\Phi}_{\mathrm{tra}}(\boldsymbol{u}, p)| + |\boldsymbol{\Phi}_{\mathrm{rot}}(\boldsymbol{u}, p)| \\ &\quad \le c\Big(\|\boldsymbol{D}(\boldsymbol{u})\|_{\boldsymbol{L}^q(\mathscr{D}_0)} + \|p - p_{\mathscr{D}_0}\|_{L^q(\mathscr{D}_0)}\Big)^{1-\theta}\Big(\|\nabla^2\boldsymbol{u}\|_{\boldsymbol{L}^q(\mathscr{D}_0)} + \|\nabla p\|_{L^q(\mathscr{D}_0)}\Big)^{\theta} \end{aligned} \tag{3.3}$$

for all $(\boldsymbol{u}, p) \in \boldsymbol{W}^{2,q}(\mathscr{D}_0) \times W^{1,q}(\mathscr{D}_0)$.

Proof Fix $\frac{1}{q} < \theta < 1$. Applying Jensen's inequality we get

$$\begin{aligned} &|\boldsymbol{\Phi}_{\mathrm{tra}}(\boldsymbol{u}, p)| + |\boldsymbol{\Phi}_{\mathrm{rot}}(\boldsymbol{u}, p)| \\ &\quad \le c\|\boldsymbol{D}(\boldsymbol{u}) - (\boldsymbol{D}(\boldsymbol{u}))_{\mathscr{D}_0}\|_{L^{4/(3-2\theta)}(\Sigma)} + c\|p - p_{\mathscr{D}_0}\|_{L^{4/(3-2\theta)}(\Sigma)}\;{}^{4)}\,. \end{aligned} \tag{3.4}$$

Thus, the assertion immediately follows from (3.2) (cf. Remark 3.2) ■

For the evolutionary case we get the following multiplicative inequality

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4) Notice, $\int\limits_{\Sigma} \boldsymbol{A} \cdot n dS = 0$ for any constant matrix.

Lemma 3.4. *There exists a constant $c > 0$, depending only on $\mathscr{D}$, such that*

$$\int_0^T \int_\Sigma fg dS dt \le c\|f\|_{L^{40/21}(0,T;W^{1,2}(\mathscr{D}_0))} \times$$

$$\times \|g\|^{3/20}_{L^2(0,T;L^{4/3}(\mathscr{D}_0))}\|g\|^{1/20}_{L^\infty(0,T;L^{4/3}(\mathscr{D}_0))}\|g\|^{4/5}_{L^2(0,T;W^{1,4/3}(\mathscr{D}_0))} \tag{3.5}$$

for all $f \in L^{40/21}(0,T;W^{1,2}(\mathscr{D}_0))$ and $g \in L^\infty(0,T;L^{4/3}(\mathscr{D}_0)) \cap L^2(0,T;W^{1,4/3}(\mathscr{D}_0))$.

Proof An iterative application of Hölder's inequality implies

$$\int_0^T \int_\Sigma fg dS dt \le \|f\|_{L^{40/21}(0,T;L^{40/11}(\Sigma))}\|g\|_{L^{40/19}(0,T;L^{40/29}(\Sigma))}.$$

In view of Lemma 3.1 with $q = 2$ and $\theta = \frac{19}{20}$ we immediately get

$$\|f\|_{L^{40/21}(0,T;L^{40/11}(\Sigma))} \le c\|f\|_{L^{40/21}(0,T;W^{1,2}(\mathscr{D}_0))}. \tag{3.6}$$

Thus, it only remains to estimate the norm involving g. Applying Lemma 3.1 with $q = \frac{4}{3}$ and $\theta = \frac{4}{5}$ we get

$$\|g(t)\|_{L^{40/29}(\Sigma)} \le c\|g(t)\|^{1/5}_{L^{4/3}(\mathscr{D}_0)}\|g(t)\|^{4/5}_{W^{1,4/3}(\mathscr{D}_0)} \tag{3.7}$$

for a. e. $t \in (0,T)$. Taking both sides of (3.7) to the $\frac{40}{19}$-th power integrating the obtained inequality over $(0,T)$ and applying Hölder's inequality we infer

$$\begin{aligned} \|g\|_{L^{40/19}(0,T;L^{40/29}(\Sigma))} &\le c\|g\|^{1/5}_{L^{8/3}(0,T;L^{4/3}(\mathscr{D}_0))}\|g\|^{4/5}_{L^2(0,T;W^{1,4/3}(\mathscr{D}_0))} \\ &\le \|g\|^{3/20}_{L^2(0,T;L^{4/3}(\mathscr{D}_0))}\|g\|^{1/20}_{L^\infty(0,T;L^{4/3}(\mathscr{D}_0))}\|g\|^{4/5}_{L^2(0,T;W^{1,4/3}(\mathscr{D}_0))}. \end{aligned} \tag{3.8}$$

Whence, the assertion of the lemma follows by means of (3.6) and (3.8). ■

4 The approximate system

In order to approximate the system under consideration we will truncate both the convective term $(\boldsymbol{u} \cdot \nabla)\boldsymbol{u}$ and the unbounded term $\boldsymbol{U}$. This requires to introduce an appropriate cut-off function ζ_ρ $(R_0 < \rho < +\infty)$ such that $\zeta_\rho \to 1$ as $\rho \to +\infty$. To begin with, let $\eta \in C^\infty(\mathbb{R})$, such that $0 \le \eta \le 1$ in $\mathbb{R}$, $\eta \equiv 1$ in $(-\infty, 1]$ and $\eta = 0$ in $[2, +\infty)$. We set

$$\zeta_\rho(x) = \eta\Big(\frac{|x|}{\rho}\Big), \quad x \in \mathbb{R}^3 \quad (0 < \rho < +\infty).$$

Then, we define the truncation

$$[\boldsymbol{a}]_\rho := \boldsymbol{a}\zeta_\rho(\boldsymbol{a}), \quad \boldsymbol{a} \in \mathbb{R}^3.$$

From this definition it is immediately clear that

$$(4.1)\qquad [\boldsymbol{a}]_\rho = \boldsymbol{a} \quad \text{if} \quad |\boldsymbol{a}| \le \rho; \qquad [\boldsymbol{a}]_\rho = 0 \quad \text{if} \quad |\boldsymbol{a}| \ge 2\rho.$$

In particular, in view of $\partial_i[\boldsymbol{v}]_\rho = (\partial_i \boldsymbol{v})\eta\Big(\frac{|\boldsymbol{v}|}{\rho}\Big) + \frac{\boldsymbol{v}}{\rho}\frac{\partial_i \boldsymbol{v}\cdot\boldsymbol{v}}{|\boldsymbol{v}|}\eta'\Big(\frac{|\boldsymbol{v}|}{\rho}\Big)\chi_{\{\rho<|\boldsymbol{v}|<2\rho\}}$ it follows

$$(4.2)\qquad |\nabla[\boldsymbol{v}]_\rho| \le (1 + 2\max|\eta'|)|\nabla \boldsymbol{v}| \quad \forall\, \boldsymbol{v} \in \boldsymbol{C}^1(\mathbb{R}^3).$$

This shows that $[\,\cdot\,]_\rho : \boldsymbol{v} \mapsto [\boldsymbol{v}]_\rho$ is bounded linear mapping from $\boldsymbol{W}^{1,q}(\mathscr{D})$ into itself uniformly in $\rho > 0$.

Let $R_0 < \rho < +\infty$ be fixed. We consider the following approximate system.

$$(1.1)_\rho \qquad \begin{cases} \operatorname{div} \boldsymbol{u}_\rho = 0 \quad \text{in} \quad Q_T, \\ \partial_t \boldsymbol{u}_\rho + (\boldsymbol{u}_\rho \cdot \nabla)[\boldsymbol{u}_\rho]_\rho = (\boldsymbol{U}_\rho \cdot \nabla)[\boldsymbol{u}_\rho]_\rho - \boldsymbol{\omega} \times \boldsymbol{u}_\rho + \Delta \boldsymbol{u}_\rho - \nabla p_\rho \quad \text{in} \quad Q_T, \end{cases}$$

where

$$\boldsymbol{U}_\rho := \frac{1}{2}\mathbf{rot}\,(\zeta_\rho \boldsymbol{\xi}_\rho \times y - \zeta_\rho \boldsymbol{\omega}_\rho |y|^2).$$

The system $(1.1)_\rho$ will be completed by the boundary condition (1.2) and the initial condition (1.3), i. e.

$$(1.2)_\rho \qquad \boldsymbol{u}_\rho = \boldsymbol{U}_\rho \quad \text{on} \quad \Sigma \times (0,T), \quad \boldsymbol{u}_\rho = 0 \quad \text{as} \quad |x| \to +\infty,$$

$$(1.3)_\rho \qquad \boldsymbol{u}_\rho = \boldsymbol{u}_0 \quad \text{on} \quad \mathscr{D} \times \{0\}.$$

The equation of translation $\boldsymbol{\xi}_\rho$ and $\boldsymbol{\omega}_\rho$ are the following

$$(1.4)_\rho \qquad \begin{cases} m\dot{\boldsymbol{\xi}}_\rho + m\boldsymbol{\omega}_\rho \times \boldsymbol{\xi}_\rho = -\boldsymbol{\Phi}_{\rm tra}(\boldsymbol{u}_\rho, p_\rho), \\ \boldsymbol{J}\dot{\boldsymbol{\omega}}_\rho + \boldsymbol{\omega}_\rho \times \boldsymbol{J}\boldsymbol{\omega}_\rho = -\boldsymbol{\Phi}_{\rm rot}(\boldsymbol{u}_\rho, p_\rho) \quad \text{in} \quad (0,T), \end{cases}$$

together with the initial condition

$$(1.5)_\rho \qquad \boldsymbol{\xi}_\rho(0) = \boldsymbol{\xi}_0, \quad \boldsymbol{\omega}_\rho(0) = \boldsymbol{\omega}_0.$$

Remark 4.1. In what follows for notational simplicity we write $\boldsymbol{\xi}$ and $\boldsymbol{\omega}$ instead of $\boldsymbol{\xi}_\rho$ and $\boldsymbol{\omega}_\rho$ respectively. Set $\boldsymbol{U} = \boldsymbol{\xi} + \boldsymbol{\omega} \times y$. Calculating

$$\frac{1}{2}\mathbf{rot}\,(\boldsymbol{\xi} \times y - \boldsymbol{\omega}|y|^2) = \boldsymbol{\xi} + \boldsymbol{\omega} \times y = \boldsymbol{U}$$

by using the product rule we obtain

$$\begin{aligned} \boldsymbol{U}_\rho &= \zeta_\rho \boldsymbol{U} + \frac{1}{2}\nabla\zeta_\rho \times (\boldsymbol{\xi} \times y - \boldsymbol{\omega}|y|^2) \\ (4.3)\qquad &= \zeta_\rho \boldsymbol{U} + \eta'\Big(\frac{|y|}{\rho}\Big)\frac{|y|}{2\rho}\boldsymbol{U} - \eta'\Big(\frac{|y|}{\rho}\Big)\frac{y}{2\rho}\Big(\boldsymbol{\xi} \cdot \frac{y}{|y|}\Big). \end{aligned}$$

In particular, we have

$$\boldsymbol{U}_\rho = \boldsymbol{U} \quad \text{on} \quad B_\rho, \qquad |\boldsymbol{U}_\rho| \le |\boldsymbol{U}| + |\boldsymbol{\xi}| \quad \text{in} \quad \mathbb{R}^3, \tag{4.4}$$

and thus, $\boldsymbol{U}_\rho \to \boldsymbol{U}$ uniformly as $\rho \to +\infty$ on each compact subset of $\mathbb{R}^3$.

Secondly, applying the chain rule and the product rule using identity (4.3) for $\boldsymbol{U}_\rho$ we find

$$|\nabla \boldsymbol{U}_\rho| \le c\Big(\frac{|\boldsymbol{U}|}{(1+|y|)} + |\nabla \boldsymbol{U}| + |\boldsymbol{\xi}|\Big) \le c(|\boldsymbol{\xi}| + |\boldsymbol{\omega}|), \tag{4.5}$$

where $c > 0$ denotes an absolute constant independent on ρ.

Similar as for the system (1.1)–(1.5) we introduce the notion of a weak solution to the approximate system

Definition 4.2. Let $\boldsymbol{u}_0 \in \mathcal{V}(\mathscr{D})$ with $\boldsymbol{u}_0 = \boldsymbol{\xi}_0 + \boldsymbol{\omega}_0 \times x$ on $\partial\mathscr{D}$, where $\boldsymbol{\xi}_0, \boldsymbol{\omega}_0 \in \mathbb{R}^3$ are given. Then $(\boldsymbol{u}_\rho, \boldsymbol{\xi}_\rho, \boldsymbol{\omega}_\rho)$ is called a *weak solution* to $(1.1)_\rho$–$(1.5)_\rho$ if

(i) $\boldsymbol{u}_\rho \in L^2(0,T;\mathcal{V}(\mathscr{D})) \cap C([0,T);\boldsymbol{L}^2(\mathscr{D}))$,

(ii) $\boldsymbol{\xi}_\rho, \boldsymbol{\omega}_\rho \in \boldsymbol{C}([0,T])$,

(iii) for every $\boldsymbol{\varphi} \in C^\infty([0,T];\mathcal{C}(\mathscr{D}))$ there holds the identity

$$\begin{aligned}
&\int_0^t \int_{\mathscr{D}} -\boldsymbol{u}_\rho \cdot \partial_t \boldsymbol{\varphi} + \boldsymbol{D}(\boldsymbol{u}_\rho) : \boldsymbol{D}(\boldsymbol{\varphi}) + ((\boldsymbol{u}_\rho - \boldsymbol{U}_\rho) \cdot \nabla [\boldsymbol{u}_\rho]_\rho) \cdot \boldsymbol{\varphi} + (\boldsymbol{\omega}_\rho \times \boldsymbol{u}_\rho) \cdot \boldsymbol{\varphi} dxds \\
&\qquad + \int_{\mathscr{D}} \boldsymbol{u}_\rho(t) \cdot \boldsymbol{\varphi}(t) dx + m\boldsymbol{\xi}_\rho(t) \cdot \boldsymbol{\Phi}_1(t) + (\boldsymbol{J}\boldsymbol{\omega}_\rho(t)) \cdot \boldsymbol{\Phi}_2(t) \\
&= \int_0^t m\boldsymbol{\xi}_\rho(s) \cdot \dot{\boldsymbol{\Phi}}_1(s) + (\boldsymbol{J}\boldsymbol{\omega}_\rho(s)) \cdot \dot{\boldsymbol{\Phi}}_2(s) ds \\
&\qquad + \int_{\mathscr{D}} \boldsymbol{u}_0 \cdot \boldsymbol{\varphi}(0) dx + m\boldsymbol{\xi}_0 \cdot \boldsymbol{\Phi}_1(0) + (\boldsymbol{J}\boldsymbol{\omega}_0) \cdot \boldsymbol{\Phi}_2(0)
\end{aligned} \tag{4.6}$$

for all $0 < t < T$.

Remark 4.3. From the integral identity in (4.6) by well-know arguments from the theory of parabolic systems we obtain the following energy equality

$$\begin{aligned}
&\frac{1}{2}\|\boldsymbol{u}_\rho(t)\|_{\boldsymbol{L}^2}^2 + \frac{m}{2}|\boldsymbol{\xi}_\rho(t)|^2 + \frac{1}{2}|\boldsymbol{R}\boldsymbol{\omega}_\rho(t)|^2 + \int_t^0 \int_{\mathscr{D}} |\boldsymbol{D}(\boldsymbol{u}_\rho)|^2 dxds \\
&\qquad = \frac{1}{2}\|\boldsymbol{u}_0\|_{\boldsymbol{L}^2}^2 + \frac{m}{2}|\boldsymbol{\xi}_0|^2 + \frac{1}{2}|\boldsymbol{R}\boldsymbol{\omega}_0|^2
\end{aligned} \tag{4.7}$$

for all $t \in [0,T]$. From (4.7) we infer the following a-priori estimate

$$\|\boldsymbol{u}_\rho\|_{L^\infty(0,T;\boldsymbol{L}^2)} + \|\boldsymbol{\xi}_\rho\|_{\boldsymbol{L}^\infty(0,T)} + \|\boldsymbol{\omega}_\rho\|_{\boldsymbol{L}^\infty(0,T)} + \|\boldsymbol{D}(\boldsymbol{u}_\rho)\|_{\boldsymbol{L}^2} \le cK_0, \tag{4.8}$$

where

$$K_0 = \|\boldsymbol{u}_0\|_{\boldsymbol{L}^2} + |\boldsymbol{\xi}_0| + |\boldsymbol{\omega}_0|.$$

Applying the standard monotone operator theory we get the following existence result

Lemma 4.4. (**Existence of a weak solution to the approximate system**) *Let $\boldsymbol{u}_0 \in \mathcal{V}(\mathscr{D})$ with $\boldsymbol{u}_0 = \boldsymbol{\xi}_0 + \boldsymbol{\omega}_0 \times x$ on $\partial\mathscr{D}$, where $\boldsymbol{\xi}_0, \boldsymbol{\omega}_0 \in \mathbb{R}^3$ are given. Then there exists a unique weak solution $(\boldsymbol{u}_\rho, \boldsymbol{\xi}_\rho, \boldsymbol{\omega}_\rho)$ to* $(1.1)_\rho$–$(1.5)_\rho$*, satisfying the identity* (4.7) *and the a-priori estimate* (4.8).

Remark 4.5. It is readily seen that the weak solution $(\boldsymbol{u}_\rho, \boldsymbol{\xi}_\rho, \boldsymbol{\omega}_\rho)$ to $(1.1)_\rho$–$(1.5)_\rho$ solves the linear problem discussed in [17] with right hand side

$$\boldsymbol{f}_\rho = ((\boldsymbol{U}_\rho - \boldsymbol{u}_\rho) \cdot \nabla)[\boldsymbol{u}_\rho]_\rho - \boldsymbol{\omega}_\rho \times \boldsymbol{u}_\rho$$

which clearly belongs to $\boldsymbol{L}^2(Q_T)$. Thus, from [17, Lemma 3.2] we get the existence of a global pressure p_ρ such that $(\boldsymbol{u}_\rho, p_\rho, \boldsymbol{\xi}_\rho, \boldsymbol{\omega}_\rho)$ is a strong solution to $(1.1)_\rho$–$(1.5)_\rho$, i. e.

$$\partial_i\partial_j \boldsymbol{u}_\rho, \partial_t \boldsymbol{u}_\rho \in \boldsymbol{L}^2(Q_T) \quad (i,j = 1,2,3), \quad \dot{\boldsymbol{\xi}}_\rho, \dot{\boldsymbol{\omega}}_\rho \in \boldsymbol{L}^2(0,T),$$

while $p_\rho \in L^2(0,T; L^2_{\mathrm{loc}}(\overline{\mathscr{D}}))$ satisfying

$$\nabla p_\rho \in \boldsymbol{L}^2(Q_T).$$

5 A-priori estimates for $(\boldsymbol{u}_\rho, p_\rho, \boldsymbol{\xi}_\rho, \boldsymbol{\omega}_\rho)$

Fix $\rho > R_0$. Let $(\boldsymbol{u}_\rho, p_\rho, \boldsymbol{\xi}_\rho, \boldsymbol{\omega}_\rho)$ denote the strong solution to the approximate system $(1.1)_\rho$–$(1.5)_\rho$.

Throughout this section, for notational simplicity we write $\boldsymbol{u}, p, \boldsymbol{\xi}, \boldsymbol{\omega}$ instead of $\boldsymbol{u}_\rho, p_\rho, \boldsymbol{\xi}_\rho, \boldsymbol{\omega}_\rho$ respectively.

We write $\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{w}$ and $p = \pi + P$, where $(\boldsymbol{v}, \pi)$ is a strong solution to the following Stokes problem

$$\begin{cases} \operatorname{div} \boldsymbol{v} = 0 \quad \text{in} \quad Q_T, \\ \partial_t \boldsymbol{v} - \Delta \boldsymbol{v} = -\nabla\pi + \boldsymbol{f} \quad \text{in} \quad Q_T, \end{cases} \tag{5.1}$$

with

$$\boldsymbol{f} = -(\boldsymbol{u} \cdot \nabla)[\boldsymbol{u}]_\rho + (\boldsymbol{U}_\rho \cdot \nabla)[\boldsymbol{u}]_\rho - \boldsymbol{\omega} \times \boldsymbol{u} \qquad \text{a. e. in } Q_T$$

satisfying the boundary and initial conditions

$$\boldsymbol{v} = 0 \quad \text{on} \quad \Sigma \times (0,T), \qquad \boldsymbol{v} \to 0 \quad \text{as} \quad |x| \to +\infty, \tag{5.2}$$

$$\boldsymbol{v} = 0 \quad \text{on} \quad \mathscr{D} \times \{0\}. \tag{5.3}$$

5.1 Estimation of $(\boldsymbol{v}, \pi)$

We start our discussion by decomposing $\boldsymbol{f}$ into the sum $\boldsymbol{f}_1 + \boldsymbol{f}_2 + \boldsymbol{f}_3$, where

$$\begin{aligned}
\boldsymbol{f}_1 &= -(\boldsymbol{u} \cdot \nabla)[\boldsymbol{u}]_\rho + (\operatorname{div}[\boldsymbol{u}]_\rho)\boldsymbol{U}_\rho,\\
\boldsymbol{f}_2 &= ([\boldsymbol{u}]_\rho \cdot \nabla)\boldsymbol{U}_\rho - \boldsymbol{\omega} \times \boldsymbol{u},\\
\boldsymbol{f}_3 &= (\boldsymbol{U}_\rho \cdot \nabla)[\boldsymbol{u}]_\rho - ([\boldsymbol{u}]_\rho \cdot \nabla)\boldsymbol{U}_\rho - (\operatorname{div}[\boldsymbol{u}]_\rho)\boldsymbol{U}_\rho.
\end{aligned}$$

Clearly, we can divide $\boldsymbol{v}$ and π into the sum $\boldsymbol{v}_1 + \boldsymbol{v}_2 + \boldsymbol{v}_3$ and $\pi_1 + \pi_2 + \pi_3$ respectively, where $(\boldsymbol{v}_i, \pi_i)$ denotes the strong solution to the Stokes system with the right hand side $\boldsymbol{f}_i$ $(i = 1, 2, 3)$ satisfying the boundary and initial conditions (5.2), (5.3).

1° *Estimation of* $(\boldsymbol{v}_1, \pi_1)$. First recalling a-priori estimate (4.8) by the aid of Sobolev's embedding theorem, we deduce

$$\|\boldsymbol{u}\|_{L^\infty(0,T;\boldsymbol{L}^2)} + \|\boldsymbol{u}\|_{L^2(0,T;\boldsymbol{L}^6)} \le cK_0, \tag{5.4}$$

where $c = \text{const} > 0$ depending only on the geometric property of $\mathscr{D}$. From (5.4) and using the Hölder's inequality we find

$$\|\boldsymbol{u}\|_{L^\alpha(0,T;\boldsymbol{L}^\beta)} \le cK_0 \quad \forall\, \alpha, \beta \in [2, +\infty] : \quad \frac{2}{\alpha} + \frac{3}{\beta} = \frac{3}{2}. \tag{5.5}$$

By the definition of the truncation $[\,\cdot\,]_\rho$ together with $\operatorname{div} \boldsymbol{u} = 0$ we have

$$\operatorname{div}[\boldsymbol{u}]_\rho = \frac{1}{\rho} \frac{\nabla \boldsymbol{u} : \boldsymbol{u} \otimes \boldsymbol{u}}{|\boldsymbol{u}|} \eta'\Big(\frac{|\boldsymbol{u}|}{\rho}\Big).$$

Observing $\frac{\boldsymbol{U}_\rho}{\rho} \le cK_0$ we see that

$$|\operatorname{div}[\boldsymbol{u}]_\rho \boldsymbol{U}_\rho| \le c|\boldsymbol{u}|\, |\nabla \boldsymbol{u}| K_0.$$

Thus,

$$|\boldsymbol{f}_1| \le c(1 + K_0)|\boldsymbol{u}|\, |\nabla \boldsymbol{u}| \quad \text{a.e. in } Q_T.$$

Applying the Hölder's inequality, taking into account (4.8) and (5.5) we estimate

$$\begin{cases} \|\boldsymbol{f}_1\|_{L^\alpha(0,T;\boldsymbol{L}^\beta)} \le c(1 + K_0)\||\boldsymbol{u}|\, |\nabla \boldsymbol{u}|\|_{L^\alpha(0,T;\boldsymbol{L}^\beta)} \le c(1 + K_0^3) \\ \forall\, \alpha, \beta \in [1, 2] : \quad \frac{2}{\alpha} + \frac{3}{\beta} = 4. \end{cases} \tag{5.6}$$

Applying [12, Theorem 2.8] and by the aid of (5.6) we obtain the estimate

$$\|\partial_t \boldsymbol{v}_1\|_{L^\alpha(0,T;\boldsymbol{L}^\beta)} + \|\nabla^2 \boldsymbol{v}_1\|_{L^\alpha(0,T;\boldsymbol{L}^\beta)} + \|\nabla \pi_1\|_{L^\alpha(0,T;\boldsymbol{L}^\beta)} \le c(1 + K_0^3) \tag{5.7}$$

for all $\alpha, \beta \in (1, 2)$ with $\frac{2}{\alpha} + \frac{3}{\beta} = 4$. [5)]

[5)] Clearly, by the choice of α, β we have $\beta < \frac{3}{2}$.

2° *Estimation of* $(\boldsymbol{v}_2, \pi_2)$. By means of (4.3) we get

$$|\boldsymbol{f}_2| \le c(|\boldsymbol{\xi}| + |\boldsymbol{\omega}|)|\boldsymbol{u}| \quad \text{a. e. in } Q_T.$$

Thus, observing (4.8) using Hölder's inequality we get

$$\|\boldsymbol{f}_2\|_{L^2(0,T;\boldsymbol{L}^2)} \le cT^{1/2}K_0^2. \tag{5.8}$$

Applying standard L^2-theory of the Stokes equation thanks (5.8) we infer

$$\|\partial_t \boldsymbol{v}_2\|_{L^2(0,T;\boldsymbol{L}^2)} + \|\nabla^2 \boldsymbol{v}_2\|_{L^2(0,T;\boldsymbol{L}^2)} + \|\nabla \pi_2\|_{L^2(0,T;\boldsymbol{L}^2)} \le cT^{1/2}K_0^2, \tag{5.9}$$

where $c = \text{const}$ depends only on the geometric property of $\mathscr{D}$.

3° *Estimation of* $(\boldsymbol{v}_3, \pi_3)$. One easily calculates,

$$\boldsymbol{f}_3 = -\mathbf{rot}\,(\boldsymbol{U}_\rho \times [\boldsymbol{u}]_\rho).$$

Estimating

$$(1+|y|)^{-1}\boldsymbol{U}_\rho \times [\boldsymbol{u}]_\rho \le c(|\boldsymbol{\xi}| + |\boldsymbol{\omega}|)|\boldsymbol{u}| \quad \text{a. e. in } Q_T,$$

we see that $\boldsymbol{U}_\rho(t) \times [\boldsymbol{u}(t)]_\rho$ belongs to the weighted Sobolev space $\boldsymbol{W}^{1,2}_\tau(\mathscr{D})$ with $\tau = (1+|y|)^{-1}$ for a. e. $t \in (0,T)$. Arguing as in 2° appealing to (4.8) we see

$$\|\boldsymbol{U}_\rho \times [\boldsymbol{u}]_\rho\|_{L^2(0,T;\boldsymbol{W}^{1,2}_\tau)} \le c(\|\boldsymbol{\xi}\|_{\boldsymbol{L}^\infty(0,T)} + \|\boldsymbol{\omega}\|_{\boldsymbol{L}^\infty(0,T)})\|\boldsymbol{u}\|_{L^2(0,T;\boldsymbol{L}^2)} \le cT^{1/2}K_0^2,$$

where $c = \text{const} > 0$ is independent on ρ. Applying the result [17, Theorem 2.1] with $\boldsymbol{\xi}_0 = 0$, $\boldsymbol{\omega}_0 = 0$ and right hand side $\boldsymbol{f} = \mathbf{rot}\,(\boldsymbol{g})$ with $\boldsymbol{g} = \boldsymbol{U}_\rho \times [\boldsymbol{u}]_\rho$ we deduce

$$\|\partial_t \boldsymbol{v}_3\|_{L^2(0,T;\boldsymbol{L}^2_\tau)} + \|\nabla^2 \boldsymbol{v}_3\|_{L^2(0,T;\boldsymbol{L}^2_\tau)} + \|\nabla \pi_3\|_{\boldsymbol{L}^2} \le cK_0^2. \tag{5.10}$$

From (5.7), (5.9) and (5.10) we infer

$$\|\partial_t \boldsymbol{v}\|_{L^\alpha(0,T;\boldsymbol{L}^\beta_\tau)} + \|\nabla^2 \boldsymbol{v}\|_{L^\alpha(0,T;\boldsymbol{L}^\beta_\tau)} + \|\nabla \pi\|_{L^\alpha(0,T;\boldsymbol{L}^\beta + \boldsymbol{L}^2)} \le c(1+K_0^3) \tag{5.11}$$

for all $\alpha, \beta \in (1,2)$ satisfying $\frac{2}{\alpha} + \frac{3}{\beta} = 4$.

In addition, in view of (5.11) by Sobolev's embedding theorem and multiplicative inequalities we get

$$\|\boldsymbol{v}\|_{L^r(0,T;\boldsymbol{L}^q(\mathscr{D}_0))} + \|\nabla \boldsymbol{v}\|_{L^{\tilde{r}}(0,T;\boldsymbol{L}^{\tilde{q}}(\mathscr{D}_0))} \le c(1+K_0^3) \tag{5.12}$$

for all $r, q \in \left[\frac{3}{2}, +\infty\right]$ with $\frac{2}{r} + \frac{3}{q} = 2$ and $\widetilde{r}, \widetilde{q} \in \left(\frac{6}{5}, 4\right)$ $(\widetilde{q} \ne +\infty)$ with $\frac{2}{\widetilde{r}} + \frac{3}{\widetilde{q}} = 3$.

Finally, referring to Remark 1.1 from (5.11) we obtain

$$\|\boldsymbol{\Phi}_{\rm tra}(\boldsymbol{v}, \pi)\|_{\boldsymbol{L}^\alpha(0,T)} + \|\boldsymbol{\Phi}_{\rm rot}(\boldsymbol{v}, \pi)\|_{\boldsymbol{L}^\alpha(0,T)} \le c(1+K_0^3) \tag{5.13}$$

for all $1 \le \alpha < 2$.

5.2 Estimation of $(\boldsymbol{w}, P)$

For the sake of simplicity we assume $\boldsymbol{u}_0 = 0$. Clearly, recalling the definition of $(\boldsymbol{w}, P)$ this pair solves the equation

$$(5.14)\qquad \begin{cases} \operatorname{div} \boldsymbol{w} = 0 \quad \text{in} \quad Q_T, \\ \partial_t \boldsymbol{w} - \Delta \boldsymbol{w} = -\nabla P \quad \text{in} \quad Q_T, \end{cases}$$

satisfying the following initial and boundary conditions

$$(5.15)\qquad \boldsymbol{w} = \boldsymbol{U}_\rho \quad \text{on} \quad \Sigma \times (0,T), \quad \boldsymbol{w} = 0 \quad \text{as} \quad |x| \to +\infty,$$
$$(5.16)\qquad \boldsymbol{w} = 0 \quad \text{on} \quad \mathscr{D} \times \{0\},$$

accompanied by the equations of the body

$$(5.17)\qquad \begin{cases} m\dot{\boldsymbol{\xi}} + m\boldsymbol{\omega} \times \boldsymbol{\xi} = -\boldsymbol{\Phi}_{\text{tra}}(\boldsymbol{w}, P) - \boldsymbol{\Phi}_{\text{tra}}(\boldsymbol{v}, \pi) \quad \text{in} \quad (0,T), \\ \boldsymbol{J}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \boldsymbol{J}\boldsymbol{\omega} = -\boldsymbol{\Phi}_{\text{rot}}(\boldsymbol{w}, P) - \boldsymbol{\Phi}_{\text{rot}}(\boldsymbol{v}, \pi) \quad \text{in} \quad (0,T), \end{cases}$$

together with the initial condition

$$(5.18)\qquad \boldsymbol{\xi}(0) = \boldsymbol{\xi}_0, \quad \boldsymbol{\omega}(0) = \boldsymbol{\omega}_0.$$

The first a-priori bound on $\boldsymbol{w}$ can be obtained immediately from (5.12) with $r = +\infty, q = \frac{3}{2}$ and $\widetilde{r} = \frac{4}{3}, \widetilde{q} = 2$, respectively, together with the a-priori bound on $\boldsymbol{u}$ (cf. (4.8), (5.5)). Thus,

$$(5.19)\qquad \|\boldsymbol{w}\|_{L^\infty(0,T;\boldsymbol{L}^{3/2}(\mathscr{D}_0))} + \|\nabla \boldsymbol{w}\|_{L^{4/3}(0,T;\boldsymbol{L}^2(\mathscr{D}_0))} \leq c(1 + K_0^3),$$

where c depends on $\mathscr{D}_0$ and T.

5.2.1 Homogenization

Set $\boldsymbol{W} := \boldsymbol{w} - \boldsymbol{U}_{R_0}$. Recalling the definition of $\boldsymbol{U}_\rho$ (cf. $(1.1)_\rho$ in Section 4) since $\rho > R_0$ we have $\boldsymbol{U}_{R_0} = \boldsymbol{U}_\rho$ in a neighbourhood of $\mathscr{B}$. Since $\operatorname{div} \boldsymbol{U}_{R_0} = 0$ and $\boldsymbol{U}_{R_0} = \boldsymbol{U} = \boldsymbol{\xi} + \boldsymbol{\omega} \times y$ on $\Sigma \times (0,T)$ the pair $(\boldsymbol{W}, P)$ solves the Stokes like system

$$(5.20)\qquad \begin{cases} \operatorname{div} \boldsymbol{W} = 0 \quad \text{in} \quad Q_T, \\ \partial_t \boldsymbol{W} - \Delta \boldsymbol{W} = -\nabla P - \partial_t \boldsymbol{U}_{R_0} + \Delta \boldsymbol{U}_{R_0} \quad \text{in} \quad Q_T \end{cases}$$

satisfying the homogeneous Dirichlet boundary condition with zero initial data.

Since ζ_{R_0} is smooth, from the definition of $\boldsymbol{U}_{R_0}$ it follows that

$$(5.21)\qquad |D^\gamma \boldsymbol{U}_{R_0}| \leq c(|\boldsymbol{\xi}| + |\boldsymbol{\omega}|)\chi_{B_{2R_0}} \leq cK_0\chi_{B_{2R_0}} \quad \text{a.e. in } Q_T$$

for every multi-index γ, were $c = \text{const} > 0$ depending on $|\gamma|$ and R_0 only. Observing (5.19) with help of (5.21) we obtain

$$\|\boldsymbol{W}\|_{L^\infty(0,T;\boldsymbol{L}^{3/2}(\mathscr{D}_0))} + \|\nabla \boldsymbol{W}\|_{L^{4/3}(0,T;\boldsymbol{L}^2(\mathscr{D}_0))} \le c(1+K_0^3). \tag{5.22}$$

On the other hand, for every multi-index γ we get

$$|\partial_t D^\gamma \boldsymbol{U}_{R_0}| \le c(|\dot{\boldsymbol{\xi}}| + |\dot{\boldsymbol{\omega}}|)\chi_{B_{2R_0}} \quad \text{a. e. in } Q_T. \tag{5.23}$$

In particular, (5.23) implies

$$\|\partial_t \boldsymbol{U}_{R_0}\|_{L^{40/21}(0,T;\boldsymbol{W}^{2,2})} \le c\boldsymbol{\Lambda}, \tag{5.24}$$

where

$$\boldsymbol{\Lambda} = \|\dot{\boldsymbol{\xi}}\|_{\boldsymbol{L}^{40/21}(0,T)} + \|\dot{\boldsymbol{\omega}}\|_{\boldsymbol{L}^{40/21}(0,T)}.$$

Unfortunately, both $\dot{\boldsymbol{\xi}}$ and $\dot{\boldsymbol{\omega}}$ are not controlled yet, since it requires estimates on the second gradient of $\boldsymbol{w}$ and the pressure gradient ∇P. This will be achieved by using the L^p estimate of the higher order derivatives of the pressure P obtained in [25]. Accordingly, using (5.21) and (5.24) we find

$$\begin{aligned}\|\nabla P\|_{L^{40/21}(0,T;\boldsymbol{W}^{2,2})} &\le c\|\partial_t \boldsymbol{U}_{R_0} - \Delta \boldsymbol{U}_{R_0}\|_{L^{40/21}(0,T;\boldsymbol{W}^{2,2})}\\ &\le c(\|\dot{\boldsymbol{\xi}}\|_{\boldsymbol{L}^{40/21}(0,T)} + \|\dot{\boldsymbol{\omega}}\|_{\boldsymbol{L}^{40/21}(0,T)} + K_0)\\ &= c(\boldsymbol{\Lambda} + K_0).\end{aligned} \tag{5.25}$$

5.2.2 Estimates of the second gradient

Multiplying $(5.20)_2$ with $-2\Delta \boldsymbol{W}$ integrating the result over $\mathscr{D} \times (0,t)$ $(t \in (0,T))$ and applying integration by parts we end up with the following identity

$$\begin{aligned}&\int_{\mathscr{D}} |\nabla \boldsymbol{W}(t)|^2 dx + 2\int_0^t \int_{\mathscr{D}} |\Delta \boldsymbol{W}|^2 dx ds\\ &\quad\le 2\int_0^t \int_{\mathscr{D}} \partial_t \boldsymbol{U}_{R_0} \cdot \Delta \boldsymbol{W}\, dx ds - 2\int_0^t \int_{\mathscr{D}} \Delta \boldsymbol{U}_{R_0} \cdot \Delta \boldsymbol{W}\, dx ds.\\ &\qquad + 2\int_0^t \int_{\mathscr{D}} \nabla P \cdot \Delta \boldsymbol{W}\, dx ds\end{aligned}$$

for a. e. $t \in (0,T)$.

With help of Cauchy-Schwarz's inequality and Young's inequality making use of (5.21) we get

$$
\begin{aligned}
&\frac{1}{2}\int\limits_{\mathscr{D}} |\nabla \boldsymbol{W}(t)|^2 dx + \int\limits_0^t \int\limits_{\mathscr{D}} |\Delta \boldsymbol{W}|^2 dx ds \\
&\qquad \le cK_0^2 + 2\int\limits_0^t \int\limits_{\mathscr{D}} \partial_t \boldsymbol{U}_{R_0} \cdot \Delta \boldsymbol{W} dx ds + 2\int\limits_0^t \int\limits_{\mathscr{D}} \nabla P \cdot \Delta \boldsymbol{W} dx ds \\
&\qquad = cK_0^2 + I + II. \qquad (5.26)
\end{aligned}
$$

(i) To estimate I we apply Green's theorem, which yields

$$
\begin{aligned}
I &= 2\int\limits_0^t \int\limits_{\Sigma} \boldsymbol{n} \cdot (\nabla \boldsymbol{W} - (\nabla \boldsymbol{W})_{\mathscr{D}_0}) \cdot \dot{\boldsymbol{U}} dS ds - 2\int\limits_0^t \int\limits_{\Sigma} \boldsymbol{n} \cdot \nabla \dot{\boldsymbol{U}} \cdot (\boldsymbol{W} - \boldsymbol{W}_{\mathscr{D}_0}) dS ds \\
&\quad + 2\int\limits_0^t \int\limits_{\mathscr{D}} \partial_t \Delta \boldsymbol{U}_{R_0} \cdot (\boldsymbol{W} - \boldsymbol{W}_{\mathscr{D}_0}) dx ds \\
&= I_1 + I_2 + I_3.
\end{aligned}
$$

With help of Lemma 3.4 applying Poincaré's inequality and Young's inequality together with (5.22) we get

$$
\begin{aligned}
I_1 &\le c\|\dot{\boldsymbol{U}}\|_{L^{40/21}(0,T)} \|\nabla \boldsymbol{W}\|^{3/20}_{L^2(0,T;\boldsymbol{L}^{4/3}(\mathscr{D}_0))} \|\nabla \boldsymbol{W}\|^{1/20}_{L^\infty(0,T;\boldsymbol{L}^2(\mathscr{D}_0))} \|\nabla^2 \boldsymbol{W}\|^{4/5}_{L^2(0,T;\boldsymbol{L}^2(\mathscr{D}_0))} \\
&\le c(1+K_0^{9/20}) \|\dot{\boldsymbol{U}}\|_{L^{40/21}(0,T)} \Big\{ \|\nabla \boldsymbol{W}\|^2_{L^\infty(0,T;\boldsymbol{L}^2)} + \|\nabla^2 \boldsymbol{W}\|^2_{L^2(0,T;\boldsymbol{L}^2)} \Big\}^{17/40} \\
&\le c(1+K_0^{9/20}) \boldsymbol{\Lambda} \Big\{ \|\nabla \boldsymbol{W}\|^2_{L^\infty(0,T;\boldsymbol{L}^2)} + \|\Delta \boldsymbol{W}\|^2_{L^2(0,T;\boldsymbol{L}^2)} \Big\}^{17/40} {}^{6)} .
\end{aligned}
$$

For the estimation of I_2 we argue similar as above by using (5.22) and taking into account (5.24). This gives

$$
\begin{aligned}
I_2 &\le c\|\nabla \dot{\boldsymbol{U}}\|_{L^{40/21}(0,T)} \|\boldsymbol{W}\|^{1/5}_{L^\infty(0,T;\boldsymbol{L}^{4/3}(\mathscr{D}_0))} \|\nabla \boldsymbol{W}\|^{4/5}_{L^2(0,T;\boldsymbol{L}^{4/3}(\mathscr{D}_0))} \\
&\le c(1+K_0^3)\boldsymbol{\Lambda}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
I_3 &\le c(1+K_0^3) \|\partial_t \Delta \boldsymbol{U}_{R_0}\|_{L^{40/21}(0,T;\boldsymbol{L}^2(\mathscr{D}_0))} \\
&\le c(1+K_0^3)\boldsymbol{\Lambda}.
\end{aligned}
$$

Whence, inserting each estimates of I_1–I_3 into the right hand side of I we deduce

$$
I \le c(1+K_0^3)\boldsymbol{\Lambda} + c(1+K_0^{9/20})\boldsymbol{\Lambda} \Big\{ \|\nabla \boldsymbol{W}\|^2_{L^\infty(0,T;\boldsymbol{L}^2)} + \|\Delta \boldsymbol{W}\|^2_{L^2(0,T;\boldsymbol{L}^2)} \Big\}^{17/40}.
$$

---

[6] Notice, in view of (5.24) we have $\|\dot{\boldsymbol{U}}\|_{L^{40/21}(0,T)} \le c\boldsymbol{\Lambda}$.

(ii) Secondly, by using Green's theorem observing $\Delta P = 0$, the integral II becomes

$$
\begin{aligned}
II = &\int_0^t \int_\Sigma \boldsymbol{n} \cdot (\nabla \boldsymbol{W} - (\nabla \boldsymbol{W})_{\mathscr{D}_0}) \cdot \nabla P dS dt \\
&\qquad - \int_0^t \int_\Sigma \boldsymbol{n} \cdot \nabla^2 P \cdot (\boldsymbol{W} - \boldsymbol{W}_{\mathscr{D}_0}) dS dt \\
= &\, II_1 + II_2.
\end{aligned}
$$

For the estimation of II_1 we proceed exactly as for the estimation of I_1 making use of Lemma 3.4. Thus,

$$
\begin{aligned}
II_1 \le &\, c(1 + K_0^{9/20}) \|\nabla P\|_{L^{40/21}(0,T;\boldsymbol{W}^{1,2})} \times \\
&\times \Big\{ \|\nabla \boldsymbol{W}\|^2_{L^\infty(0,T;\boldsymbol{L}^2)} + \|\Delta \boldsymbol{W}\|^2_{L^2(0,T;\boldsymbol{L}^2)} \Big\}^{17/40}.
\end{aligned}
$$

To estimate II_2 we argue as for I_2. Hence,

$$
\begin{aligned}
II_2 &\le c \|\nabla P\|_{L^{40/21}(0,T;\boldsymbol{W}^{2,2})} \|\boldsymbol{W}\|^{1/5}_{L^\infty(0,T;\boldsymbol{L}^{4/3}(\mathscr{D}_0))} \|\nabla \boldsymbol{W}\|^{4/5}_{L^2(0,T;\boldsymbol{L}^{4/3}(\mathscr{D}_0))} \\
&\le c(1 + K_0^3) \|\nabla P\|_{L^{40/21}(0,T;\boldsymbol{W}^{2,2})}.
\end{aligned}
$$

Inserting estimates of II_1 and II_2 into the right hand side of II making use of (5.25) we arrive at

$$
\begin{aligned}
II \le &\, c(1 + K_0^{9/20})(K_0 + \boldsymbol{\Lambda}) \Big\{ \|\nabla \boldsymbol{W}\|^2_{L^\infty(0,T;\boldsymbol{L}^2)} + \|\Delta \boldsymbol{W}\|^2_{L^2(0,T;\boldsymbol{L}^2)} \Big\}^{17/40} \\
&\qquad + c(1 + K_0^3)(K_0 + \boldsymbol{\Lambda}).
\end{aligned}
$$

Finally, inserting estimates for I and II into (5.26) applying Young's inequality we are led to

$$
\|\nabla \boldsymbol{W}\|^2_{L^\infty(0,T;\boldsymbol{L}^2)} + \|\Delta \boldsymbol{W}\|^2_{L^2(0,T;\boldsymbol{L}^2)} \le c(1 + K_0^4)(1 + \boldsymbol{\Lambda}^{40/23}). \tag{5.27}
$$

5.2.3 Estimation of $\boldsymbol{\Lambda}$

Fix $t \in (0, T)$ such that $\boldsymbol{w}(t), \boldsymbol{v}(t), \partial_t \boldsymbol{w}(t), \partial_t \boldsymbol{v}(t), P(t), \pi(t)$ are sufficiently regular. We multiply both sides of $(5.14)_2$ in t by $-\operatorname{div} \mathbb{T}(\boldsymbol{w}(t), P(t))$ and integrate the result over $\mathscr{D}$. Accordingly,

$$
\int_{\mathscr{D}} |\operatorname{div} \mathbb{T}(\boldsymbol{w}(t), P(t))|^2 dx = \int_{\mathscr{D}} \partial_t \boldsymbol{w}(t) \cdot \operatorname{div} \mathbb{T}(\boldsymbol{w}(t), P(t)) dx. \tag{5.28}
$$

We continue our discussion by evaluating the integral on the right. For, we use integration by parts and the fact that $\boldsymbol{w}(t) = \boldsymbol{U}(t)$ on Σ. Thus,

$$
\begin{aligned}
&\int_{\mathscr{D}} \partial_t \boldsymbol{w}(t) \cdot \operatorname{div} \mathbb{T}(\boldsymbol{w}(t), P(t)) dx \\
&\quad = \int_{\Sigma} \dot{\boldsymbol{U}}(t) \cdot \mathbb{T}(\boldsymbol{u}(t), \pi(t)) \cdot \boldsymbol{n} dS \\
&\qquad + \int_{\Sigma} \dot{\boldsymbol{U}}(t) \cdot \mathbb{T}(\boldsymbol{v}(t), \pi(t)) \cdot \boldsymbol{n} dS - \int_{\mathscr{D}} \partial_t \nabla \boldsymbol{w}(t) : \mathbb{T}(\boldsymbol{w}(t), P(t)) dx \\
&\quad = A + B + C.
\end{aligned}
\tag{5.29}
$$

(a) In view of (1.9) since $|\boldsymbol{J}\dot{\boldsymbol{\omega}}|^2 \geq \kappa |\dot{\boldsymbol{\omega}}|^2$ for a positive constant κ using Young's inequality together with (4.8) we see that

$$
\begin{aligned}
A \leq -\frac{m}{2}|\dot{\boldsymbol{\xi}}(t)|^2 - \frac{\kappa}{2}|\dot{\boldsymbol{\omega}}(t)|^2 + c(|\boldsymbol{\xi}(t)|^2 + |\boldsymbol{\omega}(t)|^2) \\
-\frac{m}{2}|\dot{\boldsymbol{\xi}}(t)|^2 - \frac{\kappa}{2}|\dot{\boldsymbol{\omega}}(t)|^2 + cK_0^2
\end{aligned}
$$

(b) Observing (1.7) we have

$$
B = \dot{\boldsymbol{\xi}}(t) \cdot \boldsymbol{\Phi}_{\text{tra}}(\boldsymbol{v}(t), \pi(t)) + \dot{\boldsymbol{\omega}}(t) \cdot \boldsymbol{\Phi}_{\text{rot}}(\boldsymbol{v}(t), \pi(t)).
$$

Thus, with help of Young's inequality we deduce

$$
B \leq \frac{m}{8}|\dot{\boldsymbol{\xi}}(t)|^2 + \frac{\kappa}{8}|\dot{\boldsymbol{\omega}}(t)|^2 + c(|\boldsymbol{\Phi}_{\text{tra}}(\boldsymbol{v}(t), \pi(t))|^2 + |\boldsymbol{\Phi}_{\text{tra}}(\boldsymbol{v}(t), \pi(t))|^2).
$$

(c) To estimate C we use integration by parts taking into account $\operatorname{div} \boldsymbol{w} = 0$. This shows that

$$
\begin{aligned}
C &= -2 \int_{\mathscr{D}} \partial_t \boldsymbol{D}(\boldsymbol{w})(t) : \boldsymbol{D}(\boldsymbol{w})(t) dx \\
&= \int_{\mathscr{D}} \partial_t \boldsymbol{w}(t) \cdot \Delta \boldsymbol{w}(t) dx - \int_{\mathscr{D}} \partial_t \boldsymbol{U}_{R_0}(t) \cdot \Delta \boldsymbol{w}(t) dx \\
&\qquad - 2 \int_{\mathscr{D}} \partial_t \nabla \boldsymbol{U}_{R_0}(t) : \boldsymbol{D}(\boldsymbol{w})(t) dx \\
&= C_1 + C_2 + C_3.
\end{aligned}
$$

Next, using Cauchy-Schwarz's inequality it follows

$$
C_1 \leq \|\partial_t \boldsymbol{w}(t)\|_{\boldsymbol{L}^2} \|\Delta \boldsymbol{w}(t)\|_{\boldsymbol{L}^2}.
$$

To estimate C_2 and C_3 make use of (5.23) along with Young's inequality. Consequently,

$$
C_2 + C_3 \leq \frac{m}{8}|\dot{\boldsymbol{\xi}}(t)|^2 + \frac{\kappa}{8}|\dot{\boldsymbol{\omega}}(t)|^2 + c\|\nabla \boldsymbol{w}(t)\|_{\boldsymbol{L}^2}^2 + c\|\Delta \boldsymbol{w}(t)\|_{\boldsymbol{L}^2}^2.
$$

Inserting the estimates of A, B and C into the right of (5.29), taking into account the a-priori bound for $\boldsymbol{\xi}, \boldsymbol{\omega}$ and moving the terms involving $\dot{\boldsymbol{\xi}}, \dot{\boldsymbol{\omega}}$ to the left, we are led to

$$\begin{aligned}&\frac{m}{4}|\dot{\boldsymbol{\xi}}(t)|^2 + \frac{\kappa}{4}|\dot{\boldsymbol{\omega}}(t)|^2 + \int_{\mathscr{D}} \partial_t \boldsymbol{w}(t) \cdot \operatorname{div} \mathbb{T}(\boldsymbol{w}(t), P(t)) dx \\ &\le cK_0^2 + c\Big\{|\boldsymbol{\Phi}_{\text{tra}}(\boldsymbol{v}(t), \pi(t))|^2 + |\boldsymbol{\Phi}_{\text{rot}}(\boldsymbol{v}(t), \pi(t))|^2 + \|\nabla \boldsymbol{w}(t)\|^2_{\boldsymbol{L}^2} + c\|\Delta \boldsymbol{w}(t)\|^2_{\boldsymbol{L}^2}\Big\} \\ &\qquad + c\|\partial_t \boldsymbol{w}(t)\|_{\boldsymbol{L}^2}\|\Delta \boldsymbol{w}(t)\|_{\boldsymbol{L}^2}. \end{aligned} \tag{5.30}$$

Taking both sides of the above inequality to the $\frac{20}{21}$-th power integrating the result over $(0, T)$ observing (5.13) and (5.28) making use of (5.27) we deduce

$$\begin{aligned}\boldsymbol{\Lambda}^2 &\le c\Big\{K_0^4 + \|\nabla \boldsymbol{w}\|^2_{\boldsymbol{L}^2} + \|\Delta \boldsymbol{w}\|^2_{\boldsymbol{L}^2}\Big\} + c\|\partial_t \boldsymbol{w}\|_{L^{40/21}(0,T;\boldsymbol{L}^2)}\|\Delta \boldsymbol{w}\|_{L^2(0,T;\boldsymbol{L}^2)} \\ &\le c(1 + K_0^4)(1 + \boldsymbol{\Lambda}^{40/23}) + c\|\partial_t \boldsymbol{w}\|_{L^{40/21}(0,T;\boldsymbol{L}^2)}(1 + K_0^2)(1 + \boldsymbol{\Lambda}^{20/23}).\end{aligned}$$

Then by the aid of Young's inequality the $\boldsymbol{\Lambda}$ terms on the right can be absorbed by the term on the left. Whence,

$$\boldsymbol{\Lambda} \le c(1 + K_0^{16}) + c(1 + K_0^{23/13})\|\partial_t \boldsymbol{w}\|^{23/26}_{L^{40/21}(0,T;\boldsymbol{L}^2)}. \tag{5.31}$$

5.2.4 A-priori bound of $\partial_t \boldsymbol{w}$ and $\boldsymbol{\Lambda}$

Let $t \in (0, T)$ be appropriately chosen as in the previous subsection. We are going to multiply both sides of $(5.14)_2$ in t with $\partial_t \boldsymbol{w}(t)$ and integrate the result over $\mathscr{D}$. This gives

$$\int_{\mathscr{D}} |\partial_t \boldsymbol{w}(t)|^2 dx = \int_{\mathscr{D}} \operatorname{div} \mathbb{T}(\boldsymbol{w}(t), P(t)) \cdot \partial_t \boldsymbol{w}(t) dx. \tag{5.32}$$

Using (5.30), arguing as in the proof of (5.31), we obtain

$$\|\partial_t \boldsymbol{w}\|_{L^{40/21}(0,T;\boldsymbol{L}^2)} \le c(1 + K_0^{16}) + c(1 + K_0^{23/13})\|\partial_t \boldsymbol{w}\|^{23/26}_{L^{40/21}(0,T;\boldsymbol{L}^2)}.$$

Whence, with help of Young's inequality together with (5.31) we deduce

$$\|\partial_t \boldsymbol{w}\|_{L^{40/21}(0,T;\boldsymbol{L}^2)} + \boldsymbol{\Lambda} \le c(1 + K_0^{16}). \tag{5.33}$$

5.2.5 A-priori bounds for $\nabla^2 \boldsymbol{w}$ and ∇P

Inserting a-priori estimate (5.33) for $\boldsymbol{\Lambda}$ into the right of (5.27), it follows that

$$\|\nabla \boldsymbol{w}\|_{L^\infty(0,T;\boldsymbol{L}^2)} + \|\Delta \boldsymbol{w}\|_{L^2(0,T;\boldsymbol{L}^2)} \le c(1 + K_0^{16}). \tag{5.34}$$

Next, by $(5.14)_2$,

$$\|\nabla P\|_{L^{40/21}(0,T;\boldsymbol{L}^2)} \le \|\partial_t \boldsymbol{w}\|_{L^{40/21}(0,T;\boldsymbol{L}^2)} + \|\Delta \boldsymbol{w}\|_{L^{40/21}(0,T;\boldsymbol{L}^2)}.$$

Hence, (5.27), (5.34) imply

$$\|\nabla P\|_{L^{40/21}(0,T;\boldsymbol{L}^2)} \le c(1+K_0^{16}). \tag{5.35}$$

5.2.6 Improvement of integrability in time

Let $1<\alpha<2$. Define,

$$\boldsymbol{\Lambda}_\alpha = \|\dot{\boldsymbol{\xi}}\|_{\boldsymbol{L}^\alpha(0,T)} + \|\dot{\boldsymbol{\omega}}\|_{\boldsymbol{L}^\alpha(0,T)}.$$

Taking both sides of (5.30) to the $\frac{\alpha}{2}$-th power and integrating the resultant inequality over $(0,T)$, taking into account (5.13) and (5.34) we are led to

$$\begin{aligned}\boldsymbol{\Lambda}_\alpha^2 &\le c\Big\{K_0 + \|\nabla \boldsymbol{w}\|^2_{\boldsymbol{L}^2} + \|\Delta \boldsymbol{w}\|^2_{\boldsymbol{L}^2}\Big\} + c\|\partial_t \boldsymbol{w}\|_{L^\alpha(0,T;\boldsymbol{L}^2)}\|\Delta \boldsymbol{w}\|_{\boldsymbol{L}^2}\\ &\le c(1+K_0^{32}) + c\|\partial_t \boldsymbol{w}\|_{L^\alpha(0,T;\boldsymbol{L}^2)}(1+K_0^{16}).\end{aligned}$$

Analogously, in view of (5.32) using (5.30) we get

$$\|\partial_t \boldsymbol{w}\|_{L^\alpha(0,T;\boldsymbol{L}^2)} \le c(1+K_0^{16}) + c\|\partial_t \boldsymbol{w}\|^{1/2}_{L^\alpha(0,T;\boldsymbol{L}^2)}(1+K_0^{8}).$$

Thus, inserting the latter into the former inequality using Young's inequality we arrive at

$$\|\partial_t \boldsymbol{w}\|_{L^\alpha(0,T;\boldsymbol{L}^2)} + \boldsymbol{\Lambda}_\alpha \le c(1+K_0^{16}). \tag{5.36}$$

Finally, by the equation $(5.14)_2$ using (5.34) and (5.36) we obtain

$$\|\nabla P\|_{L^{\alpha}(0,T;\boldsymbol{L}^2)} \le c(1+K_0^{16}). \tag{5.37}$$

6 Passage to the limit $\rho \to +\infty$

From (5.11), (5.34), (5.36) and (5.37) we get the a-priori estimate

$$\begin{aligned}&\|\partial_t \boldsymbol{u}_\rho\|_{L^s(0,T;\boldsymbol{L}^q_\tau)} + \|\nabla^2 \boldsymbol{u}_\rho\|_{L^s(0,T;\boldsymbol{L}^q_\tau)} + \|\nabla p_\rho\|_{L^s(0,T;\boldsymbol{L}^q+\boldsymbol{L}^2)}\\ &\quad + \|\dot{\boldsymbol{\xi}}_\rho\|_{\boldsymbol{L}^\alpha(0,T)} + \|\dot{\boldsymbol{\omega}}_\rho\|_{\boldsymbol{L}^\alpha(0,T)} \le c(1+K_0^{16})\end{aligned} \tag{6.1}$$

for all $s,q\in(1,2)$ with $\frac{2}{s}+\frac{3}{q}=4$ and $1\le\alpha<2$.

Observing (4.8) by a standard reflexivity argument, there exist a sequences $(\boldsymbol{u}_{\rho_j}, \boldsymbol{\xi}_{\rho_j}, \boldsymbol{\omega}_{\rho_j})$ with $\rho_j \to +\infty$ as $j\to+\infty$ and $\boldsymbol{u} \in L^\infty(0,T;\boldsymbol{L}^2(\mathscr{D})) \cap L^2(0,T;\mathcal{V}(\mathscr{D}))$ together with $\boldsymbol{\xi},\boldsymbol{\omega} \in C([0,T])$ such that

$$\boldsymbol{u}_{\rho_j} \rightharpoonup \boldsymbol{u} \quad \text{in} \quad L^2(0,T;\mathcal{V}(\mathscr{D})), \tag{6.2}$$

$$\boldsymbol{u}_{\rho_j} \overset{*}{\rightharpoonup} \boldsymbol{u} \quad \text{in} \quad L^\infty(0,T;\boldsymbol{L}^2(\mathscr{D})), \tag{6.3}$$

$$\boldsymbol{\xi}_{\rho_j} \to \boldsymbol{\xi}, \quad \boldsymbol{\omega}_\rho \to \boldsymbol{\omega} \quad \text{uniformly on } [0,T], \quad \text{as} \quad j\to+\infty. \tag{6.4}$$

In view of (6.1) applying a standard compactness argument we get

$$\boldsymbol{u}_{\rho_j} \to \boldsymbol{u} \quad \text{in} \quad \boldsymbol{L}^2(Q_T). \tag{6.5}$$

As it has been proved in case of the Navier-Stokes equation with help of (6.3) one verifies that $\boldsymbol{u}(\cdot,t)$ is weakly continuous with respect to the $\boldsymbol{L}^2$ norm. In addition there holds

$$\boldsymbol{u}(t) \to \boldsymbol{u}_0 \quad \text{in} \quad \boldsymbol{L}^2(\mathscr{D}) \quad \text{as} \quad t \to 0^+. \tag{6.6}$$

Now, in view of (6.2), (6.4) and (6.5) we are in a position to carry out the passage to the limit $\rho_j \to +\infty$ in the weak formulation of $(\boldsymbol{u}_\rho, \boldsymbol{\xi}_\rho, \boldsymbol{\omega}_\rho)$ to see that $(\boldsymbol{u}, \boldsymbol{\xi}, \boldsymbol{\omega})$ is a weak solution to (1.1)–(1.5).

Furthermore, by means of reflexivity from (6.1) we get $p \in L^1(0,T; L^1_{\text{loc}}(\overline{\mathscr{D}}))$ with $\nabla p \in L^s(0,T; L^q(\mathscr{D}))$ such that

$$\nabla p_{\rho_j} \to \nabla p \quad \text{in} \quad L^s(0,T; \boldsymbol{L}^q(\mathscr{D})) \quad \text{as} \quad j \to +\infty \tag{6.7}$$

for all $s, q \in [1,2)$ with $\frac{2}{s} + \frac{3}{q} = 4$. Thus, taking into account the lower semi continuity of the norm from (6.1), it follows

$$\begin{aligned} &\|\partial_t \boldsymbol{u}\|_{L^s(0,T;\boldsymbol{L}^r_\tau)} + \|\nabla^2 \boldsymbol{u}\|_{L^s(0,T;\boldsymbol{L}^q_\tau)} + \|\nabla p\|_{L^s(0,T;\boldsymbol{L}^q + \boldsymbol{L}^2)} \\ &\qquad + \|\dot{\boldsymbol{\xi}}\|_{\boldsymbol{L}^\alpha(0,T)} + \|\dot{\boldsymbol{\omega}}\|_{\boldsymbol{L}^\alpha(0,T)} \le c(1 + K_0^{16}) \end{aligned} \tag{6.8}$$

for all $s, q \in (1,2)$ with $\frac{2}{s} + \frac{3}{q} = 4$ and $1 \le \alpha < 2$. Accordingly, $\boldsymbol{u}$ is a strong solution to (1.1)–(1.5). This completes the proof of Theorem 2.2. ■

7 Proof of Theorem 2.3. Removing the weight

Let $(\boldsymbol{u}, p, \boldsymbol{\xi}, \boldsymbol{\omega})$ be a strong solution to (1.1)–(1.5), the existence which is guaranteed by Theorem 2.2. The aim of this section is to prove the statement of Theorem 2.3 removing the weight in estimating the norm of $\nabla^2 \boldsymbol{u}$.

We begin our discussion by a localization argument. Let $\mathscr{D}' \subset \mathscr{D}$ with $\operatorname{dist}(\mathscr{D}', \partial\mathscr{D}) > 0$ be arbitrarily chosen. Then, fix an open set $\mathscr{B}' \supset \overline{\mathscr{B}}$ such that $\overline{\mathscr{B}'} \subset \mathbb{R}^3 \setminus \mathscr{D}'$. In particular, there exists $0 < R_0 < +\infty$ such that $\mathbb{R}^3 \setminus \mathscr{D}' \subset B_{R_0}$. Now, let $R_0 < \rho < +\infty$. Take $\phi \in C_0^\infty(\mathscr{D})$ such that $0 \le \phi \le 1$ in $\mathbb{R}^3$, $\phi \equiv 1$ on $B_\rho \cap \mathscr{D}'$ and $\phi \equiv 0$ in $\mathbb{R}^3 \setminus B_{2\rho} \setminus \mathscr{B}'$. Furthermore we may assume that $|\nabla \phi| \le c$ and $|\nabla^2|\phi \le c$ in $\mathbb{R}^3$ with $c = \text{const}$ independent on ρ.

We multiply both sides of $(1.1)_2$ by ϕ taking into consideration the product rule we see that

$$\begin{aligned} &\partial_t(\phi \boldsymbol{u}) + ((\boldsymbol{u} - \boldsymbol{U}) \cdot \nabla)(\phi \boldsymbol{u}) - \Delta(\phi \boldsymbol{u}) \\ &\quad = (\boldsymbol{u} \cdot \nabla \phi)\boldsymbol{u} - 2\nabla\phi \cdot \nabla \boldsymbol{u} - \Delta(\phi \boldsymbol{u}) \\ &\qquad - \operatorname{div} \phi(\boldsymbol{U} \otimes \boldsymbol{u}) - \phi(\boldsymbol{\omega} \times \boldsymbol{u}) - \nabla(\phi p_\rho) + p_\rho \nabla \phi \end{aligned} \tag{7.1}$$

in $\mathbb{R}^3 \times (0,T)$, where

$$p_\rho = p - p_{B_{2\rho}\setminus B_\rho}.$$

Performing the divergence on both sides of the above equation we find

$$\begin{aligned}
-\Delta(\phi p_\rho) = (\nabla \boldsymbol{u})^t &: \nabla(\phi \boldsymbol{u}) + \boldsymbol{u}\cdot\nabla(\nabla\phi\cdot\boldsymbol{u}) + \nabla\phi\cdot\partial_t\boldsymbol{u} \\
&- \Delta(\nabla\phi\cdot\boldsymbol{u}) + 2\nabla^2\phi : \nabla\boldsymbol{u} + (\Delta\nabla\phi)\boldsymbol{u} \\
&\quad - \operatorname{div}(\boldsymbol{u}\cdot\nabla\phi\boldsymbol{U}) + \nabla^2\phi : (\boldsymbol{U}\otimes\boldsymbol{u}) \\
&\quad - \nabla\phi\cdot(\boldsymbol{\omega}\times\boldsymbol{u}) - \nabla p\cdot\nabla\phi - \Delta\phi p_\rho \\
&= (\nabla\boldsymbol{u})^t : \nabla(\phi\boldsymbol{u}) - \operatorname{div}\boldsymbol{g}_\rho + h_\rho,
\end{aligned}$$

where

$$\begin{aligned}
\boldsymbol{g}_\rho &= \nabla(\boldsymbol{u}\cdot\nabla\phi) + \boldsymbol{u}\cdot\nabla\phi\boldsymbol{U}, \\
h_\rho &= \boldsymbol{u}\cdot\nabla(\nabla\phi\cdot\boldsymbol{u}) + \nabla\phi\cdot\partial_t\boldsymbol{u} + 2\nabla^2\phi : \nabla\boldsymbol{u} + (\Delta\nabla\phi)\boldsymbol{u} \\
&\quad + \nabla^2\phi : (\boldsymbol{U}\otimes\boldsymbol{u}) - \nabla\phi\cdot(\boldsymbol{\omega}\times\boldsymbol{u}) - \nabla p\cdot\nabla\phi - \Delta\phi p_\rho.
\end{aligned}$$

In what follows by c we denote a positive constant which may change their numerical value line by line but does not depend on ρ. Clearly, in view of (2.2) we have $\boldsymbol{g}_\rho \in L^2(0,T;\boldsymbol{L}^2(\mathscr{D}))$ such that

$$\|\boldsymbol{g}_\rho\|_{\boldsymbol{L}^2} \le cK_0. \tag{7.2}$$

On the other hand, in view of $\nabla\phi\cdot\Delta\boldsymbol{u} \in L^{20/19}(0,T;L^{10/7}(\mathbb{R}^3))$ we infer $\operatorname{div}\boldsymbol{g}_\rho \in L^{20/19}(0,T;L^{10/7}(\mathbb{R}^3))$. By the aid of (2.6) and (2.2) we obtain

$$\|\operatorname{div}\boldsymbol{g}_\rho\|_{L^{20/19}(0,T;L^{10/7}(\mathbb{R}^3))} \le c(1+K_0^{16}). \tag{7.3}$$

Owing to $\nabla\phi\cdot\partial_t\boldsymbol{u} \in L^{20/19}(0,T;\boldsymbol{L}^{10/7}(\mathscr{D}))$ we have $h_\rho \in L^{20/19}(0,T;L^{10/7}(\mathbb{R}^3))$ and there holds

$$\|h_\rho\|_{L^{20/19}(0,T;L^{10/7}(\mathbb{R}^3))} \le c(1+K_0^{16}). \tag{7.4}$$

Introducing the Newton potential we may decompose ϕp_ρ into the sum $P_0+P_1+P_2$, where

$$\begin{aligned}
P_1(t) &:= -\frac{1}{4\pi}\int\limits_{\mathbb{R}^3} \boldsymbol{g}_\rho(y,t)\cdot\frac{x-y}{|x-y|^3}dx, \\
P_2(t) &:= \frac{1}{4\pi}\int\limits_{\mathbb{R}^3} \frac{h_\rho(y,t)}{|x-y|}dx
\end{aligned}$$

a.e. in $\mathbb{R}^3$ and for a.e. $t \in (0,T)$. Hence, in view of (7.2) one verifies $P_1 \in L^2(0,T;\widehat{W}^{1,2}(\mathbb{R}^3))$, $P_2 \in L^{20/19}(0,T;\widehat{W}^{2,10/7}(\mathbb{R}^3))$ together with the a-priori estimate

$$\|P_1\|_{L^2(0,T;\widehat{W}^{1,2}(\mathbb{R}^3))} + \|P_2\|_{L^{20/19}(0,T;\widehat{W}^{2,10/7}(\mathbb{R}^3))} \le c(1+K_0^{16}). \tag{7.5}$$

Furthermore, observing (7.3) it follows $\nabla^2 P_1 \in L^{20/19}(0,T; L^{10/7}(\mathbb{R}^3))$. Hence,

$$\|\nabla^2 P_1 + \nabla^2 P_2\|_{L^{20/19}(0,T;\boldsymbol{L}^{10/7}(\mathbb{R}^3))} \le c(1+K_0^{16}). \tag{7.6}$$

Clearly, since $P_1 + P_2$ solves the equation $-\Delta(P_1+P_2) = \operatorname{div} \boldsymbol{g}_\rho + h_\rho$ we deduce that

$$-\Delta P_0(t) = (\nabla \boldsymbol{u}(t))^t : \nabla(\phi \boldsymbol{u}(t)) \quad \text{in} \quad \mathbb{R}^3 \tag{7.7}$$

for a. e. $t \in (0,T)$.

Fix $t \in (0,T)$ such that $\boldsymbol{u}(t) \in \boldsymbol{W}^{1,2}(\mathscr{D})$. Define,

$$\widetilde{\boldsymbol{u}}(t) = \begin{cases} \boldsymbol{u}(x,t) & \text{for } x \in \mathscr{D} \\ \boldsymbol{U}(t) & \text{for } x \in \overline{\mathscr{B}} \end{cases}$$

Recalling $(1.1)_1$ and (1.2) we see that $\widetilde{\boldsymbol{u}} \in \boldsymbol{W}^{1,2}(\mathbb{R}^3)$ satisfying $\operatorname{div} \widetilde{\boldsymbol{u}} = 0$. As it has been shown in [2] there holds

$$(\nabla \widetilde{\boldsymbol{u}}(t))^t : \nabla(\phi \boldsymbol{u}(t)) \in \mathscr{H}^1,$$

where $\mathscr{H}^1$ denotes the Hardy space (cf. [22]). Consulting [22, VII.3, Cor. 1] we obtain $P_0(t) \in \widehat{W}^{2,1}(\mathbb{R}^3))$ together with the estimate

$$\|P_0(t)\|_{\widehat{W}^{2,1}(\mathbb{R}^3)} \le c\|\nabla \boldsymbol{u}(t)\|^2_{\boldsymbol{W}^{1,2}}. \tag{7.8}$$

Integration of (7.8) over $(0,T)$ gives

$$\|P_0\|_{L^1(0,T;\widehat{W}^{2,1}(\mathbb{R}^3))} \le c(\|\nabla \boldsymbol{u}\|^2_{\boldsymbol{L}^2(0,T;\boldsymbol{L}^2)} + \|\boldsymbol{u}\|_{L^\infty(0,T;\boldsymbol{L}^2)}) \le cK_0^2. \tag{7.9}$$

Fix $j \in \{1,2,3\}$. Applying the ∂_j to both sides of (7.1) we see that $\boldsymbol{r} = \partial_j(\phi \boldsymbol{u})$ solves the transport equation

$$\partial_t \boldsymbol{r} + ((\boldsymbol{u} - \boldsymbol{U}_\rho)\cdot\nabla)\boldsymbol{r} - \Delta \boldsymbol{r} = \partial_j \boldsymbol{u} \cdot \nabla(\phi \boldsymbol{u}) + \partial_j \nabla P_0 + \boldsymbol{f}_1 + \boldsymbol{f}_2 \tag{7.10}$$

in $\mathbb{R}^3 \times (0,T)$, with $\boldsymbol{f}_1 \in L^2(0,T;\boldsymbol{L}^2(\mathbb{R}^3))$, $\boldsymbol{f}_2 \in L^{20/19}(0,T;\boldsymbol{L}^{10/7}(\mathbb{R}^3))$ such that

$$\|\boldsymbol{f}_1\|_{L^2(0,T;\boldsymbol{L}^2(\mathbb{R}^3))} + \|\boldsymbol{f}_2\|_{L^{20/19}(0,T;\boldsymbol{L}^{10/7}(\mathbb{R}^3))} \le c(1+K_0^{16}). \tag{7.11}$$

We divide $\boldsymbol{r}$ into the sum $\boldsymbol{s}+\boldsymbol{t}$, where $\boldsymbol{s}$ stands for the weak solution in $L^\infty(0,T;\boldsymbol{L}^2(\mathbb{R}^3)) \cap L^2(0,T;\boldsymbol{W}^{1,2}_0(\mathbb{R}^3))$ solving the transport problem

$$\partial_t \boldsymbol{s} + ((\boldsymbol{u} - \boldsymbol{U})\cdot\nabla)\boldsymbol{s} - \Delta \boldsymbol{s} = [\boldsymbol{f}_1 + \boldsymbol{f}_2]_1 \quad \text{in} \quad \mathbb{R}^3 \times (0,T), \tag{7.12}$$ [7)]

satisfying the following boundary and initial condition

$$\boldsymbol{s} = 0 \quad \text{on} \quad \Sigma \times (0,T), \quad \boldsymbol{s} = 0 \quad \text{as} \quad |x| \to +\infty, \tag{7.13}$$

$$\boldsymbol{s} = [\boldsymbol{r}(0)]_1 \quad \text{on} \quad \mathscr{D} \times \{0\}. \tag{7.14}$$

[7)] Recall that $[\boldsymbol{a}]_1 = \boldsymbol{a}\zeta_1(\boldsymbol{a})\ (\boldsymbol{a} \in \mathbb{R}^3)$.

As $[\boldsymbol{f}_1+\boldsymbol{f}_2]_1 \in L^2(0,T;\boldsymbol{L}^2(\mathbb{R}^3))$ by the standard energy method taking into account that $\|\boldsymbol{r}(0)\|_{\boldsymbol{L}^2} \le K_0$ along with (7.12) we get

$$\begin{aligned}
&\|\boldsymbol{s}\|^2_{L^\infty(0,T;\boldsymbol{L}^2(\mathbb{R}^3))} + 2\|\nabla \boldsymbol{s}\|^2_{L^2(0,T;\boldsymbol{L}^2(\mathbb{R}^3))} \\
&\qquad \le c\Big\{\|\boldsymbol{r}(0)\|^2_{\boldsymbol{L}^2(\mathbb{R}^3)} + \|[\boldsymbol{f}_1+\boldsymbol{f}_2]_1\|^2_{L^1(0,T;\boldsymbol{L}^2(\mathbb{R}^3))}\Big\} \\
&\qquad \le c(1+K_0^{16}) \qquad (7.15)
\end{aligned}$$

(7.16)

Furthermore, owing to $[\boldsymbol{r}(0)]_1 \le 2$ and $[\boldsymbol{f}_1+\boldsymbol{f}_2]_1 \le 2$ we claim that

$$\operatorname*{ess\,sup}_{\mathbb{R}^3\times(0,T)} |\boldsymbol{s}| \le c_*, \qquad (7.17)$$

where c_* denotes an absolute constant. Indeed, multiplying both sides of the i-th equation in (7.12) by $(s^i-h)^+$ $(h>2)$ we obtain

$$\operatorname*{ess\,sup}_{t\in(0,T)} \int_{\mathbb{R}^3} [(s^i-h)^+]^2 dx + 2\int_0^T\int_{\mathbb{R}^3} [\nabla(s^i-h)^+]^2 dxdt \le 4\int_{A_h} (s^i-h)^+ dxdt,$$

where

$$A_h = \{(x,t)\in\mathbb{R}^3\times(0,T)\,|\, s^i(x,t)>h\}.$$

By means of Sobolev's embedding theorem and Hölder's inequality we are led to

$$\int_{A_h} (s^i-h)^{10/3} dxdt \le c[\mathrm{mes}(A_h)]^{7/3}.$$

Now, let $2<h_1<h_2<+\infty$ be arbitrarily chosen. From the last inequality we deduce that

$$(h_2-h_1)\mathrm{mes}(A_{h_2}) \le \int_{A_{h_2}} (s^i-h_1)^{10/3} dxdt \le c[\mathrm{mes}(A_{h_1})]^{7/3}.$$

By a well-known algebraic argument due to Stampaccia we get $h_0=\mathrm{const}>2$, such that $\mathrm{mes}(A_{h_0})=0$, which implies $s^i\le h_0$. Analogously, one shows that s^i is bounded from below. Whence, (7.17).

Next, combining (7.10) and (7.12) we see that $\boldsymbol{t}$ is a weak solution to

$$\partial_t \boldsymbol{t} + ((\boldsymbol{u}-\boldsymbol{U})\cdot\nabla)\boldsymbol{t} - \Delta\boldsymbol{t} = \boldsymbol{f} \quad \text{in} \quad \mathbb{R}^3\times(0,T), \qquad (7.18)$$

where

$$\boldsymbol{f} = \partial_j\boldsymbol{u}\cdot\nabla(\phi\boldsymbol{u}) + \partial_j\nabla P_0 + \boldsymbol{f}_1+\boldsymbol{f}_2 - [\boldsymbol{f}_1+\boldsymbol{f}_2]_1,$$

fulfilling $\boldsymbol{t}(0) = \boldsymbol{r}(0) - [\boldsymbol{r}(0)]_1$. By the definition of the truncation $[\cdot]_1$ it follows that both $\boldsymbol{t}(0)$ and $\boldsymbol{f}$ are integrable, more precisely,

$$\int\limits_{\mathbb{R}^3} |\boldsymbol{t}(0)| dx + \int\limits_0^T \int\limits_{\mathbb{R}^3} |\boldsymbol{f}| dx dt \le c(1 + K_0^{16}). \tag{7.19}$$

Fix $i \in \{1, 2, 3\}$. Multiplying both sides of the i-th equation of (7.18) by sign $(t^i)\Big(1 - \frac{1}{(2c_* + |t^i|)^\delta}\Big)$ $(0 < \delta < 1)$ integrating the result over $\mathbb{R}^3 \times (0, t)$ $(t \in (0, T))$ using integration by parts and taking into account (7.19) we end up with

$$\begin{aligned} &\int\limits_0^t \frac{d}{dt} \int\limits_{\mathbb{R}^3} \varrho(t^i) dx ds + \delta \int\limits_0^t \int\limits_{\mathbb{R}^3} \frac{|\nabla t^i|^2}{(2c_* + |t^i|)^{2c_* + \delta}} dx ds \\ &\qquad \le - \int\limits_0^t \int\limits_{\mathbb{R}^3} (\boldsymbol{u} - \boldsymbol{U}) \cdot \nabla \varrho(t^i) dx ds + c(1 + K_0^{16}) \\ &\qquad = c(1 + K_0^{16}), \end{aligned} \tag{7.20}$$

where

$$\varrho(\tau) := \int\limits_0^{|\tau|} \Big(2c_* - \frac{1}{(1 + \xi)^\delta}\Big) d\xi, \quad \tau \in \mathbb{R}.$$

Again using integration by parts recalling $\|\boldsymbol{t}(0)\|_{\boldsymbol{L}^1} \le K_0$ we infer

$$\begin{aligned} \int\limits_0^t \frac{d}{dt} \int\limits_{\mathbb{R}^3} \varrho(t^i) dx ds &= \int\limits_{\mathbb{R}^3} \varrho(t^i)(x, t) dx - \int\limits_{\mathbb{R}^3} \varrho(t^i)(x, 0) dx \\ &\ge \int\limits_{\mathbb{R}^3} \varrho(t^i)(x, t) dx - cK_0. \end{aligned}$$

Thus,

$$\int\limits_0^T \int\limits_{\mathbb{R}^3} \frac{|\nabla \boldsymbol{t}|^2}{(2c_* + |\boldsymbol{t}|)^{1+\delta}} dx dt \le c\delta^{-1}(1 + K_0^{16}), \tag{7.21}$$

where c denotes a positive constant depending only on $\mathscr{D}$. Thanks to (7.17) having $|\boldsymbol{r}| \ge |\boldsymbol{t}| - c_*$ making use of (7.16) and (7.21) we obtain

$$\begin{aligned} \int\limits_0^T \int\limits_{\mathbb{R}^3} \frac{|\nabla \boldsymbol{r}|^2}{(2c_* + |\boldsymbol{r}|)^{1+\delta}} dx dt &\le \int\limits_0^T \int\limits_{\mathbb{R}^3} |\nabla \boldsymbol{s}|^2 dx dt + \int\limits_0^T \int\limits_{\mathbb{R}^3} \frac{|\nabla \boldsymbol{t}|^2}{(2c_* + |\boldsymbol{t}|)^{1+\delta}} dx dt \\ &\le c(1 + K_0^{16}). \end{aligned} \tag{7.22}$$

Recalling that $\boldsymbol{r} = \partial_j(\phi \boldsymbol{u})$ from (7.22) we deduce that

$$\int_0^T \int_{B_\rho \cap \mathscr{D}'} \frac{|\nabla^2 \boldsymbol{u}|^2}{(2c_* + |\nabla \boldsymbol{u}|)^{1+\delta}} dxdt \le c(1 + K_0^2).$$

Consequently, with help of Fatou's Lemma the passage to the limit $\rho \to +\infty$ leads to assertion (2.8).

It only remains to show (2.9). Let $\psi \in C^\infty(\mathbb{R}^3)$ such that $0 \le \psi \le 1$ in $\mathbb{R}^3$, $\psi \equiv 1$ on $\mathbb{R}^3 \setminus \mathscr{D}'$ and $\psi \equiv 0$ in $\mathbb{R}^3 \setminus \mathscr{B}'$. As above multiplying both sides of $(1.1)_2$ by ψ and applying the operator div we obtain the following Poisson equation

$$\begin{aligned} -\Delta(p\psi) &= (\nabla \widetilde{\boldsymbol{u}})^t : \nabla(\boldsymbol{u}\psi) \\ &\quad + \Big(\partial_t \boldsymbol{u} - \Delta \boldsymbol{u} + \boldsymbol{\omega} \times \boldsymbol{u} - \boldsymbol{U} \cdot \nabla \boldsymbol{u} - \nabla p\Big) \cdot \nabla \psi - p\Delta\psi \end{aligned}$$

in $\mathbb{R}^3 \times (0,T)$. Since the right hand side of the above equation belongs to $L^1(0,T;\mathscr{H}^1) + L^1(0,T;L^\alpha)$ for all $\alpha \in [1,2)$ by using [2] and Calderón-Zygmond theory (cf. [22]) we see that

$$\partial_i \partial_j p\psi \in L^1(0,T;L^1(\mathbb{R}^3)) + L^1(0,T;L^\alpha(\mathbb{R}^3)), \quad i,j = 1,2,3.$$

Hence, (2.9) holds true. This completes the proof of Theorem 2.3. ■

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