# Pitfalls of FE Computing

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# Governing equations

- Cauchy equations of motion

$$\frac{\partial\, {}^t_t\sigma_{ji}}{\partial\, {}^t x_j} + {}^t f_i = {}^t\rho\, {}^t\ddot{x}_i$$

- Kinematic relations (strain – displacement relations)

$$\varepsilon_{ij}^{\text{engineering}} = \tfrac{1}{2}\left(\frac{\partial\, {}^t u_i}{\partial\, {}^0 x_j} + \frac{\partial\, {}^t u_j}{\partial\, {}^0 x_i}\right) \quad \varepsilon_{ij}^{\text{Green\_Lagrange}} = \tfrac{1}{2}\left(\frac{\partial\, {}^t u_i}{\partial\, {}^0 x_j} + \frac{\partial\, {}^t u_j}{\partial\, {}^0 x_i} + \frac{\partial\, {}^t u_k}{\partial\, {}^0 x_i}\frac{\partial\, {}^t u_k}{\partial\, {}^0 x_j}\right)$$

- Constitutive relations

$$\sigma_{ij}^{\text{engineering}} = C_{ijkl}\,\varepsilon_{kl}^{\text{engineering}} \qquad \dot{S}_{ij} = D_{ijkl}\;\dot{\varepsilon}_{kl}^{\text{Green\_Lagrange}}$$

**Methods of solution**

- Linearization – small rotations, small strains, linear constitutive relations
- Discretization
  - Finite difference method
  - Transfer matrix method
  - Matrix method
    - Displacement formulation
    - Force formulation
  - Finite element method
    - Displacement formulation
    - Force formulation
    - Hybrid finite element method
    - Mixed finite element method
  - Boundary element method

**Development of finite element method formulations and technology**

- Method of weighted residuals
- Galerkin – weighted f. = basis f.
- Ritz

# Numerical methods in FEA

- Equilibrium problems

$$\mathbf{K}(\mathbf{q})\,\mathbf{q} = \mathbf{Q}$$

solution of algebraic equations

- Steady-state vibration problems

$$\left(\mathbf{K} - \Omega^2\mathbf{M}\right)\overline{\mathbf{q}} = \mathbf{0}$$

generalized eigenvalue problem

- Propagation problems

$$\mathbf{M}\ddot{\mathbf{q}} = \mathbf{F}^{\mathrm{int}} - \mathbf{F}^{\mathrm{ext}}, \quad \mathbf{F}^{\mathrm{int}} = \int_V \mathbf{B}^{\mathrm{T}}\boldsymbol{\sigma}\,\mathrm{d}V$$

step by step integration in time

In linear cases we have

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{P}(t)$$

# Now a few examples

from

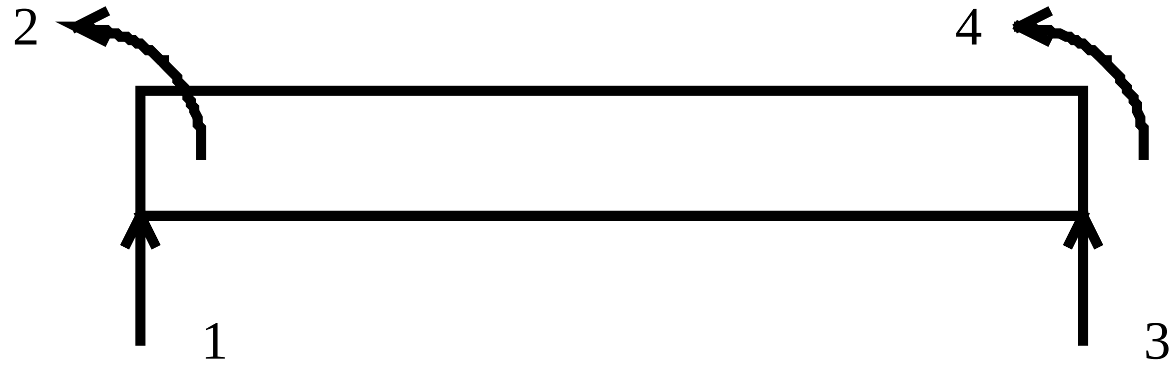
- Statics
- Steady-state vibration
- Transient dynamics

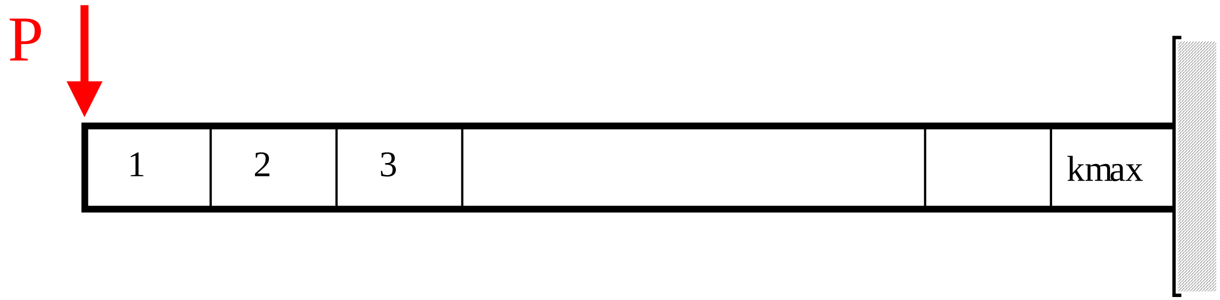
using

- Rod elements
- Beam elements
- Bilinear (L) and biquadratic (Q) plane elements

**Static loading of a cantilever beam by a vertical force acting at the free end.**

Beam 4-dof elements (Euler-Bernoulli) are used.






$$\mathbf{k} = \frac{2EI}{l^3}\begin{bmatrix} 6 & 3l & -6 & 3l \\ & 2l^2 & -3l & l^2 \\ & & 6 & -3l \\ \text{sym.} & & & 2l^2 \end{bmatrix}$$

$$\mathbf{Kq} = \mathbf{P}$$

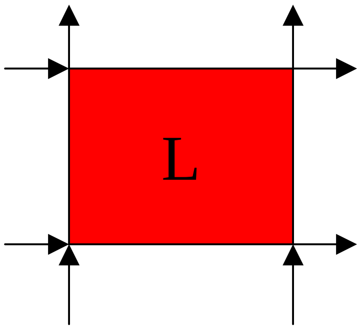
'Exact' formula for thin beams

$$v_{\text{tip}} = \frac{PL^3}{3EI}$$

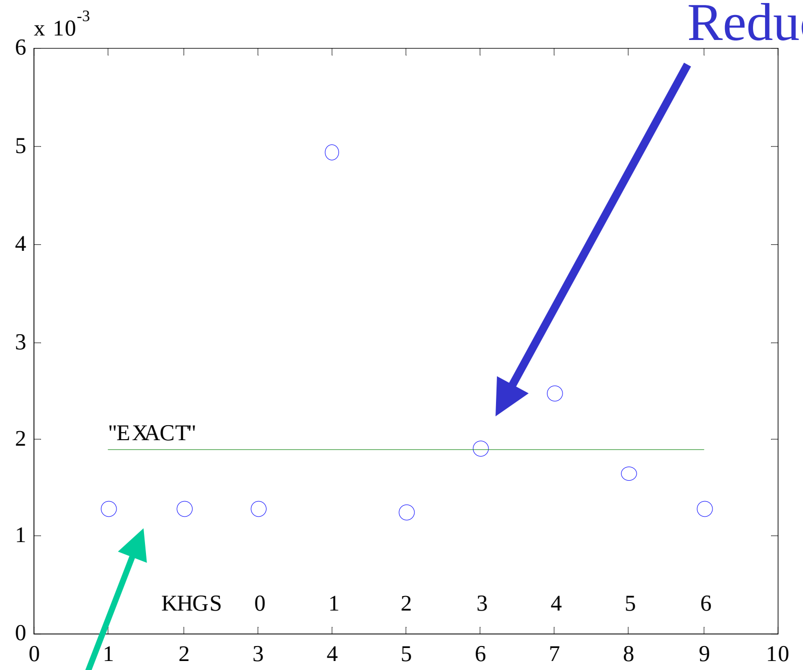
1 1.904761904761905e-003 1.904761904761904e-003

10 1.90476190476043e-003 1.904761904761904e-003

A thin cantilever beam,
vertical point load at the free end,
four-node plane stress elements
FE technology




Default in ANSYS
Reduced integration in Marc




Data obtained with 10 elements

Exact integration

$$\mathbf{k} = \int_V \mathbf{B}^{\mathrm{T}} \mathbf{E} \mathbf{B} \, \mathrm{d}V$$

Exact integration does not yield ‘exact’ results

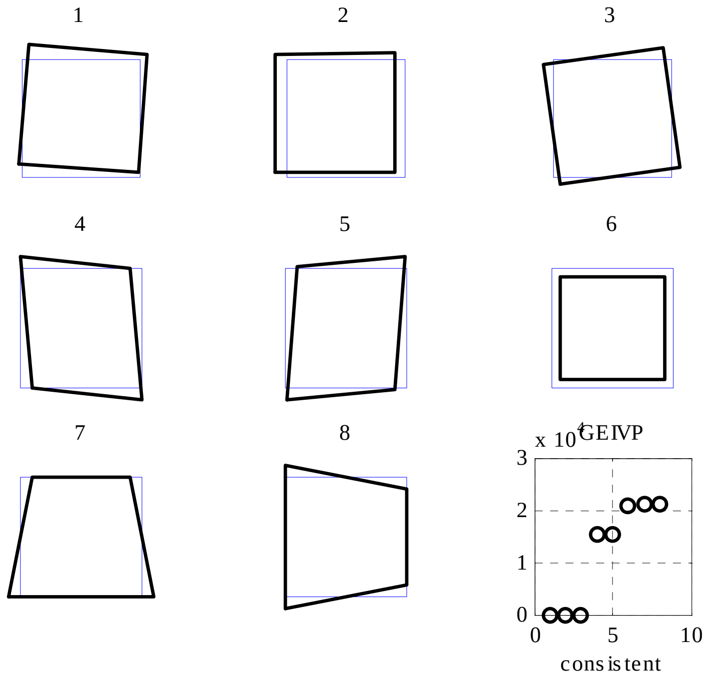
full integration 1 … anal, 2 … Gauss q.,
different types of underintegration

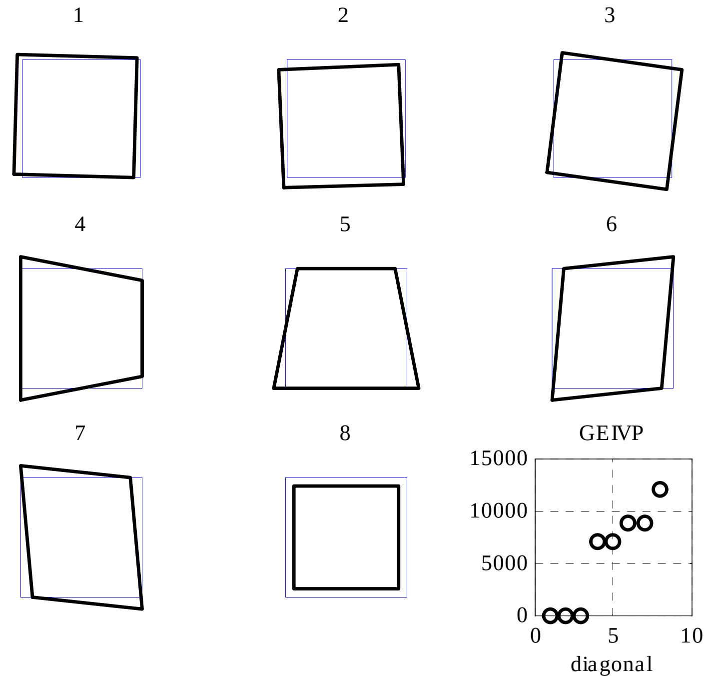
## Comparison of beam and L elements used for modelling of a static loading of a cantilever beam

- Beam … even one element gives a negligible error
- L … too stiff in bending, tricks have to be employed to get correct results

# A single four-node plane stress element
# Generalized eigenvalueproblem
# Full integration

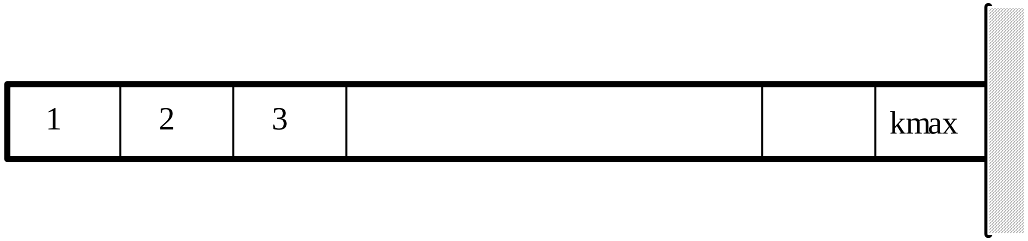
$$(\mathbf{K} - \lambda \mathbf{M})\mathbf{q} = \mathbf{0}$$



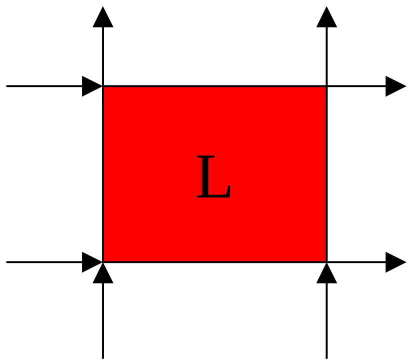
# Free transverse vibration of a thin elastic cantilever beam
# FE vs. continuum approach (Bernoulli_Euler theory)
# Generalized eigenvalue problem

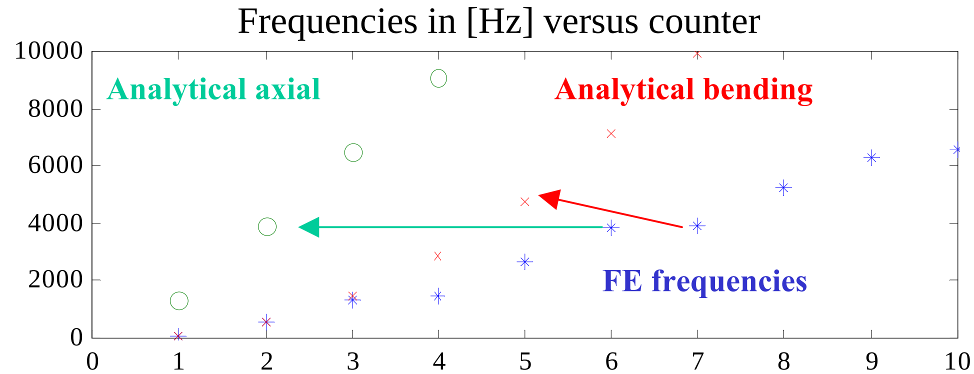
$$(\mathbf{K} - \lambda \mathbf{M})\mathbf{q} = \mathbf{0}$$




Two cases will be studied

- L (bilinear, 4-node, plane stress elements)
- Beam elements



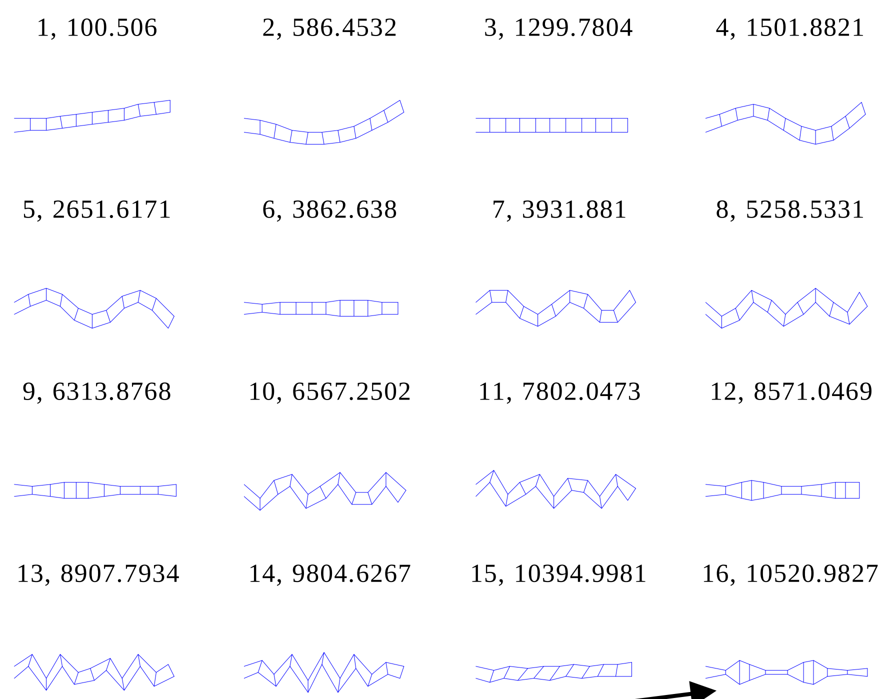

# Natural frequencies and modes of a cantilever beam … diagonal mass m.

Four-node plane stress elements, full integration

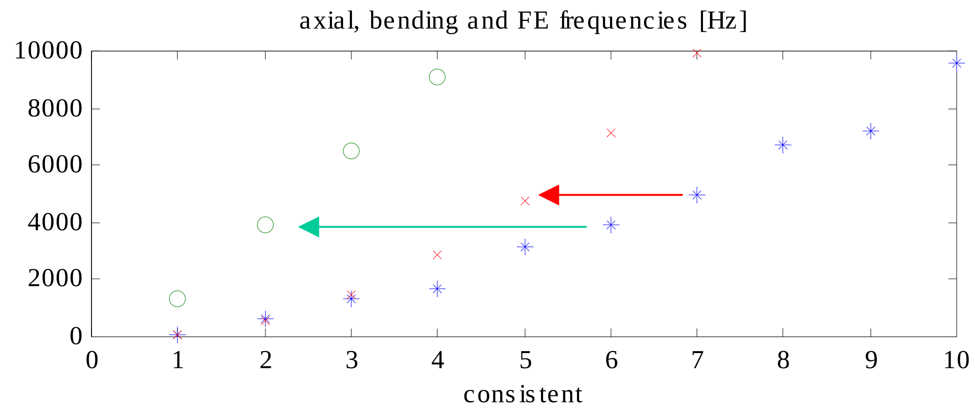
Sixth FE frequency is the second axial

Seventh FE frequency is the fifth bending

**This we cannot say without looking at eigenmodes**

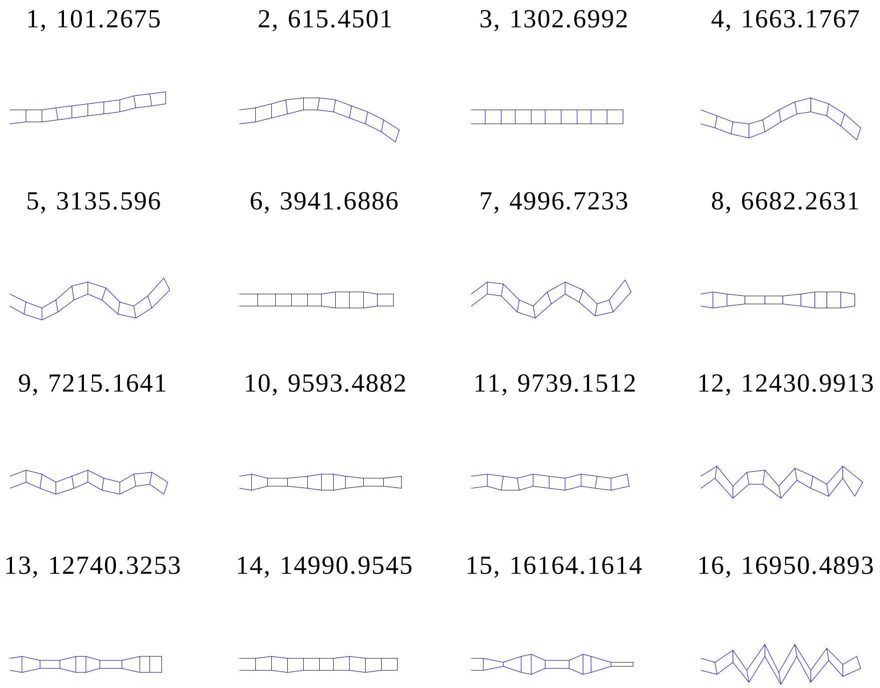



Higher frequencies are useless due to discretization errors



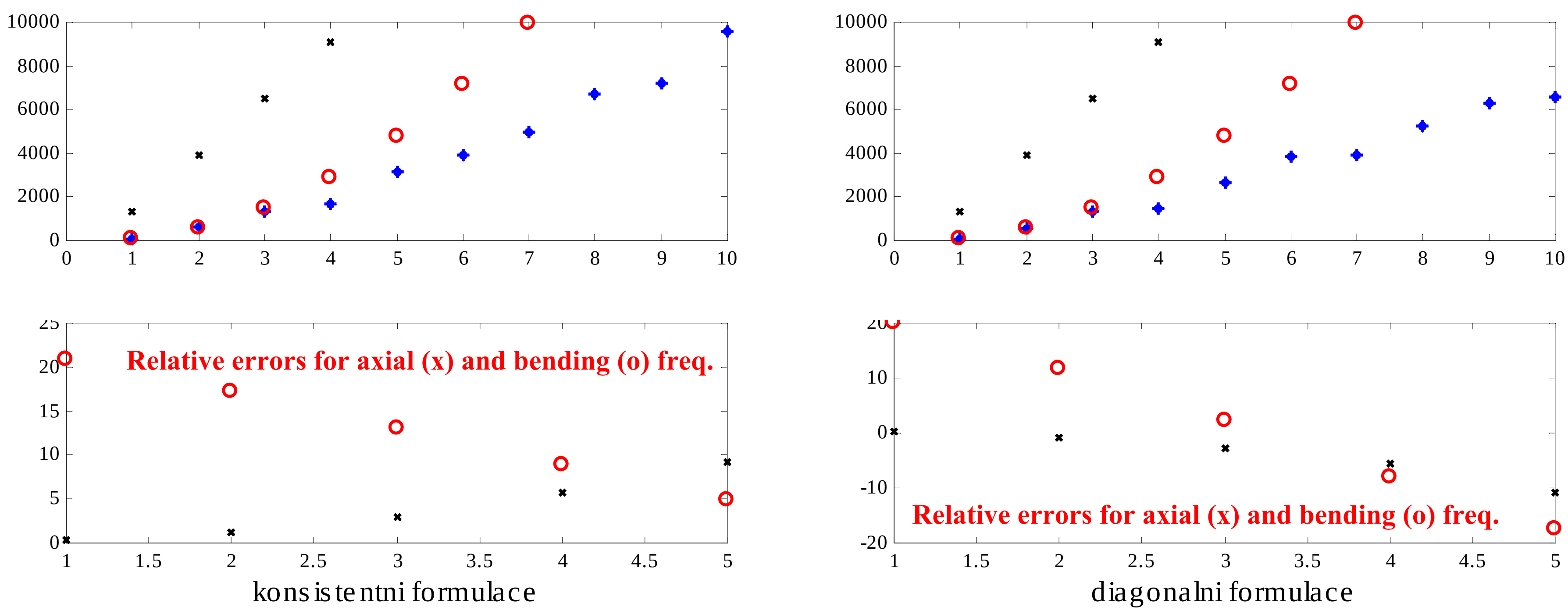

# Natural frequencies and modes of a cantilever beam … consistent mass m.

Four-node plane stress elements, full integration



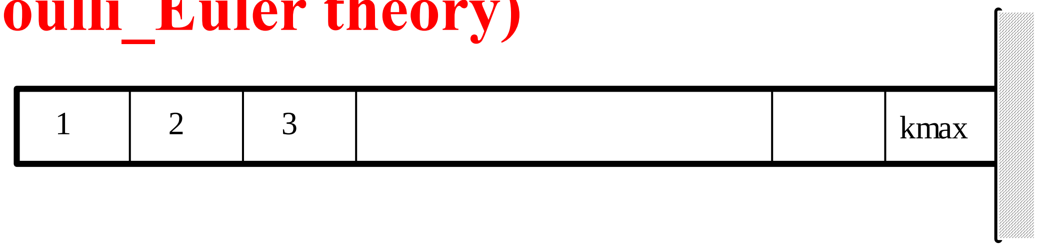
# Eigenfrequencies of a cantilever beam
# Four-node bilinear element, plane strain
# Relative errors [%] of FE frequencies

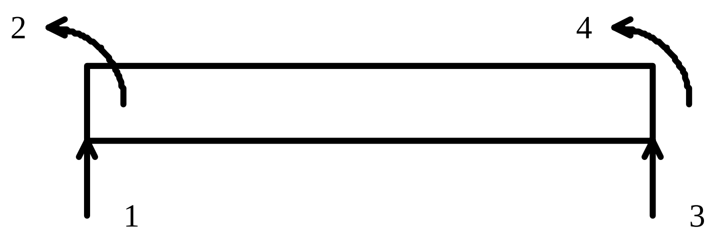
**x** … axial – continuum, **o** … bending – continuum, * … FE frequencies



# Free transverse vibration of a thin elastic cantilever beam

# FE (beam element) vs. continuum approach (Bernoulli_Euler theory)




$$\mathbf{m} = \frac{\rho l A}{420}\begin{bmatrix} 156 & 22l & 54 & -13l \\ & 4l^2 & 13l & -3l^2 \\ & & 156 & -22l^2 \\ \text{sym.} & & & 4l^2 \end{bmatrix}$$

$$\mathbf{k} = \frac{2EI}{l^3}\begin{bmatrix} 6 & 3l & -6 & 3l \\ & 2l^2 & -3l & l^2 \\ & & 6 & -3l \\ \text{sym.} & & & 2l^2 \end{bmatrix}$$

$$(\mathbf{K} - \lambda \mathbf{M})\mathbf{q} = \mathbf{0}$$

# Free transverse vibration of a thin elastic beam

## ANALYTICAL APPROACH

The equation of motion of a long thin beam considered as *continuum* undergoing transverse vibration is derived under Bernoulli-Euler assumptions, namely

- there is an axis, say x, of the beam that undergoes no extension,
- the x-axis is located along the neutral axis of the beam,
- cross sections perpendicular to the neutral axis remain planar during the deformation – transverse shear deformation is neglected,
- material is linearly elastic and homogeneous,
- the y-axis, perpendicular to the x-axis, together with x-axis form a principal plane of the beam.

These assumptions are acceptable for thin beams – the model ignores shear deformations of a beam element and rotary inertia forces.

For more details see

Craig, R.R.: Structural Dynamics. John Wiley, New York, 1981 or

Clough, R.W. and Penzien, J.: Dynamics of Structures, McGraw-Hill, New York, 1993.

The equation is usually presented in the form

$$\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 v}{\partial x^2}\right)+\rho A\frac{\partial^2 v}{\partial t^2}=p(x,t)$$

where $x$ is a longitudinal coordinate, $v$ is a transversal displacement of the beam in $y$ direction, which is perpendicular to $x$, $t$ is time, $E$ is the Young's modulus, $I$ is the planar moment of inertia of the cross section, $A$ is the cross sectional area and $\rho$ is the density. On the right hand side of the equation there is the loading $p(x,t)$ - generally a function of space and time - acting in the $xy$ plane.For *free* transverse *vibrations* we have zero on the right-hand side of Eq. (1). If the bending stiffness $EI$ is independent of time and space coordinates we can write

$$\frac{\partial^4 v}{\partial x^4} + \frac{\rho A}{EI}\frac{\partial^2 v}{\partial t^2} = 0 \quad .$$

(4a)

Assuming the *steady state vibration* in a *harmonic* form

$$v(x,t) = V(x)\cos(\omega t - \varphi)$$

we get

$$\frac{\mathrm{d}^4 V(x)}{\mathrm{d}x^4} - \lambda^4 V(x) = 0$$

(4b)

where we have introduced an auxiliary variable by

$$\lambda^4 = \rho A \omega^2 / (EI) \qquad (5)$$

The general solution of Eq. (4) can be assumed (see Kreysig, E.: Advanced Engineering Mathematics, John Wiley & Sons, New York, 1993) in the form

$$V(x) = C_1 \sinh \lambda x + C_2 \cosh \lambda x + C_3 \sin \lambda x + C_4 \cos \lambda x \tag{6}$$

where constants $C_1$ to $C_4$ depend on boundary conditions.

Applying boundary conditions for a thin cantilever beam (clamped – free) we get a

# frequency determinant

```
[0,                   1,                   0,                   1                 ]
[lam,                 0,                   lam,                 0                 ]
[sinh(lam*L)*lam^2, cosh(lam*L)*lam^2, -sin(lam*L)*lam^2, -cos(lam*L)*lam^2]
[cosh(lam*L)*lam^3, sinh(lam*L)*lam^3, -cos(lam*L)*lam^3,  sin(lam*L)*lam^3]
```

From the condition that the frequency determinant is equal to zero we get the frequency equation in he form

$$\cosh \lambda L \cos \lambda L + 1 = 0$$

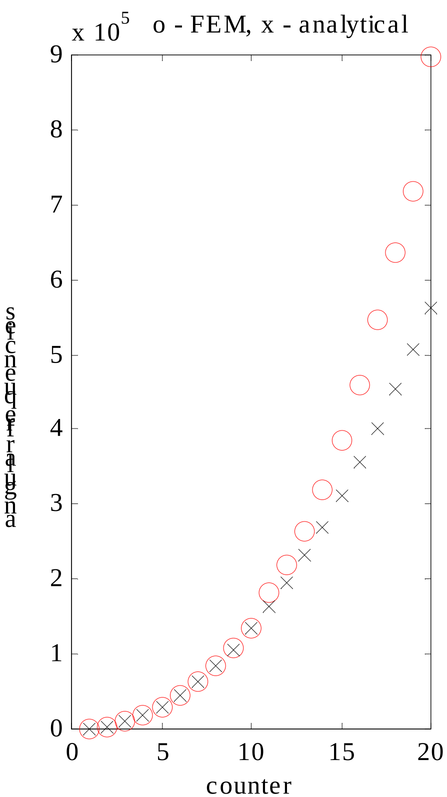
Roots of this equation can only be found numerically, Denoting $\bar{x}_i = \lambda_i L$ we get the natural frequencies in the form

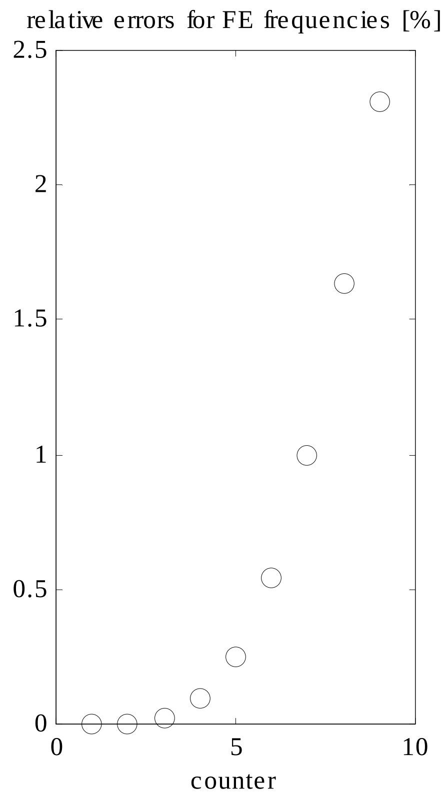
$$\omega_i = \frac{\bar{x}_i}{L^2} \sqrt{\frac{I}{A}} \sqrt{\frac{E}{\rho}} \qquad i = 1,2,3, \ldots$$

# Comparison of analytical and FE results

| counter | continuum frequencies | FE frequencies |
|---|---|---|
| 1 | **5.26650** 4690912090e+002 | **5.26650** 9194371887e+002 |
| 2 | **3.300** 46215172696 5e+003 | **3.300** 571391657554e+003 |
| 3 | **9.24** 1389593048039e+003 | **9.24** 3742518773286e+003 |
| 4 | **1.81** 0943523875022e+004 | **1.81** 2669270993247e+004 |
| 5 | **2.99** 3619402962561e+004 | **3.00** 1165614576545e+004 |
| 6 | **4.4** 7194902323343 9e+004 | **4.4** 9608739337132 7e+004 |
| 7 | **6.** 245945376065551e+004 | **6.** 308228786109306e+004 |
| 8 | **8.** 315607746908118e+004 | **8.** 451287572802173e+004 |
| 9 | **1.0** 68093617279631e+005 | **1.0** 92740977881639e+005 |

FE computation with
10 beam elements
Consistent mass matrix
Full integration




Are analytically computed frequencies exact to be used as an etalon for error analysis?

To answer this you have to recall the assumptions used for the thin beam theory

# Comparison of Beam and Bilinear Elements Used for Cantilever Beam Vibration

- 10 beam elements … the ninth bending frequency with 2.5% error
- 10 beam elements … this element does not yield axial frequencies
- 10 bilinear elements … the first bending frequency with 20% error, the errors goes down with increasing frequency counter
- 10 bilinear elements … the errors of axial frequencies are positive for consistent mass matrix, negative for diagonal mass matrix

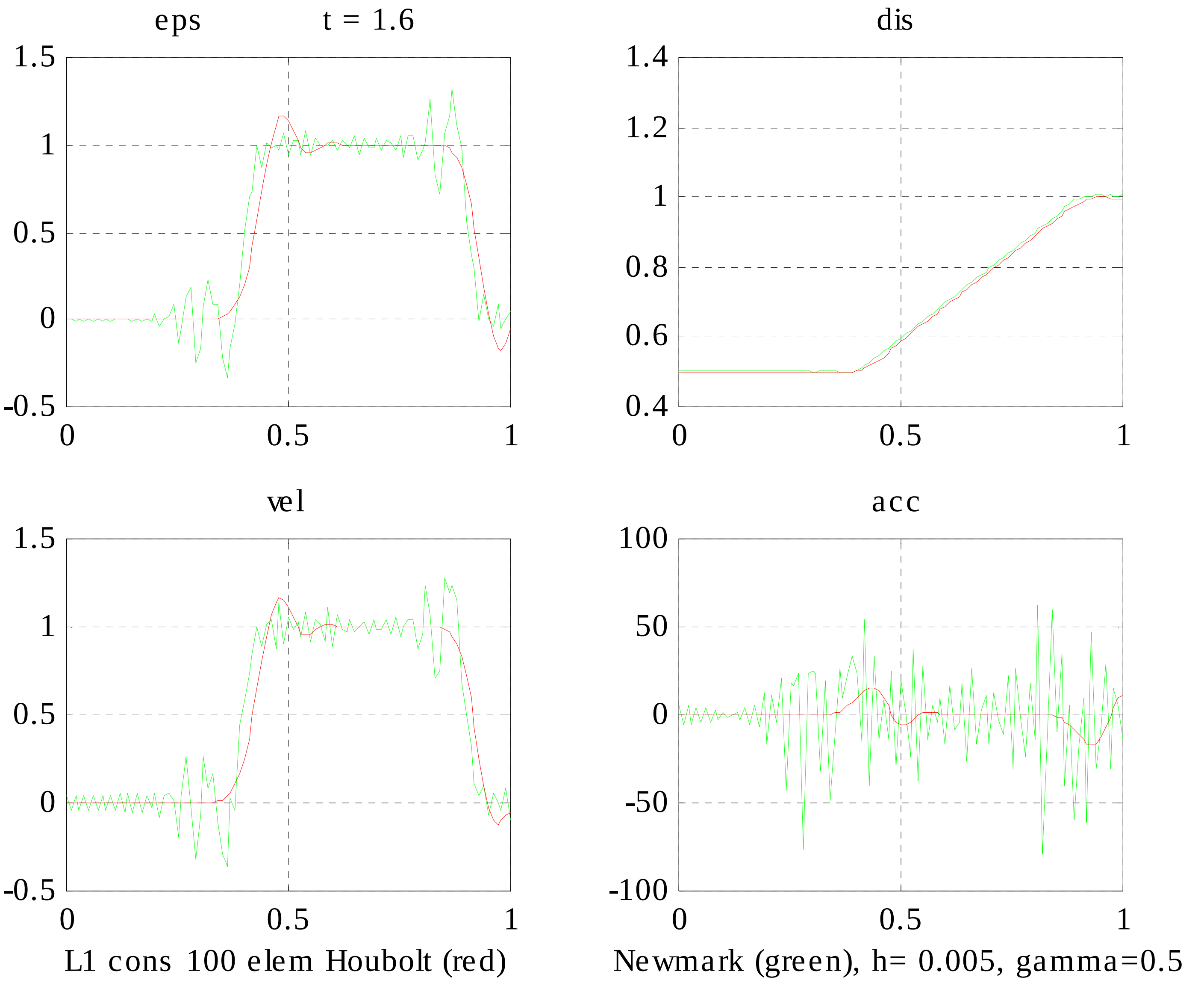
- Where is the truth?

Transient problems in linear dynamics, no damping

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{P}(t)$$

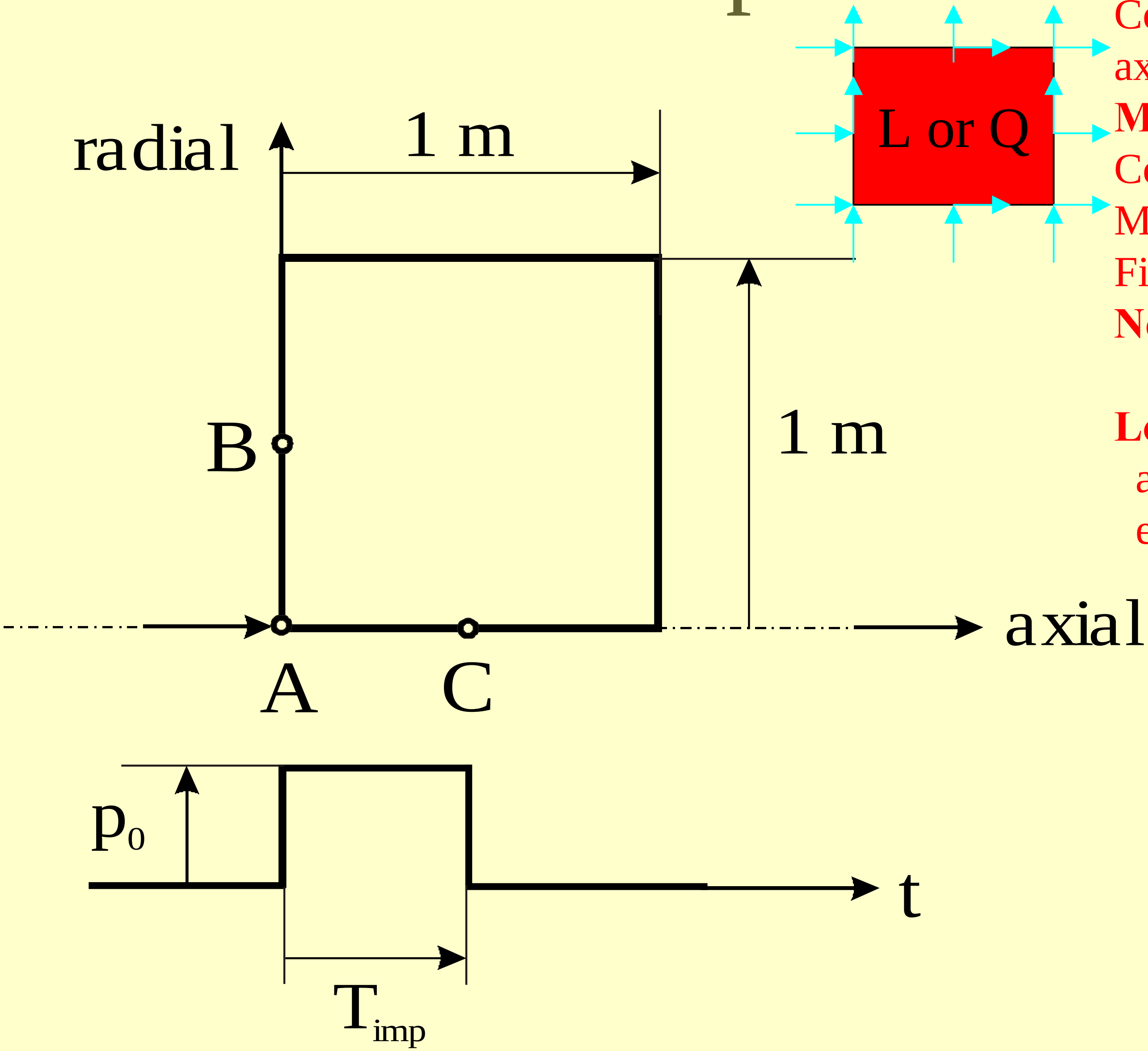
Modelling the 1D wave equation

$$\frac{\partial^2 u}{\partial t^2} = c_0^2 \frac{\partial^2 u}{\partial x^2}$$




Rod elements used here, the results depend on the method of integration

# Classical Lamb's problem




**Elements**
L, Q, full int.
Consistent mass
axisymmetric

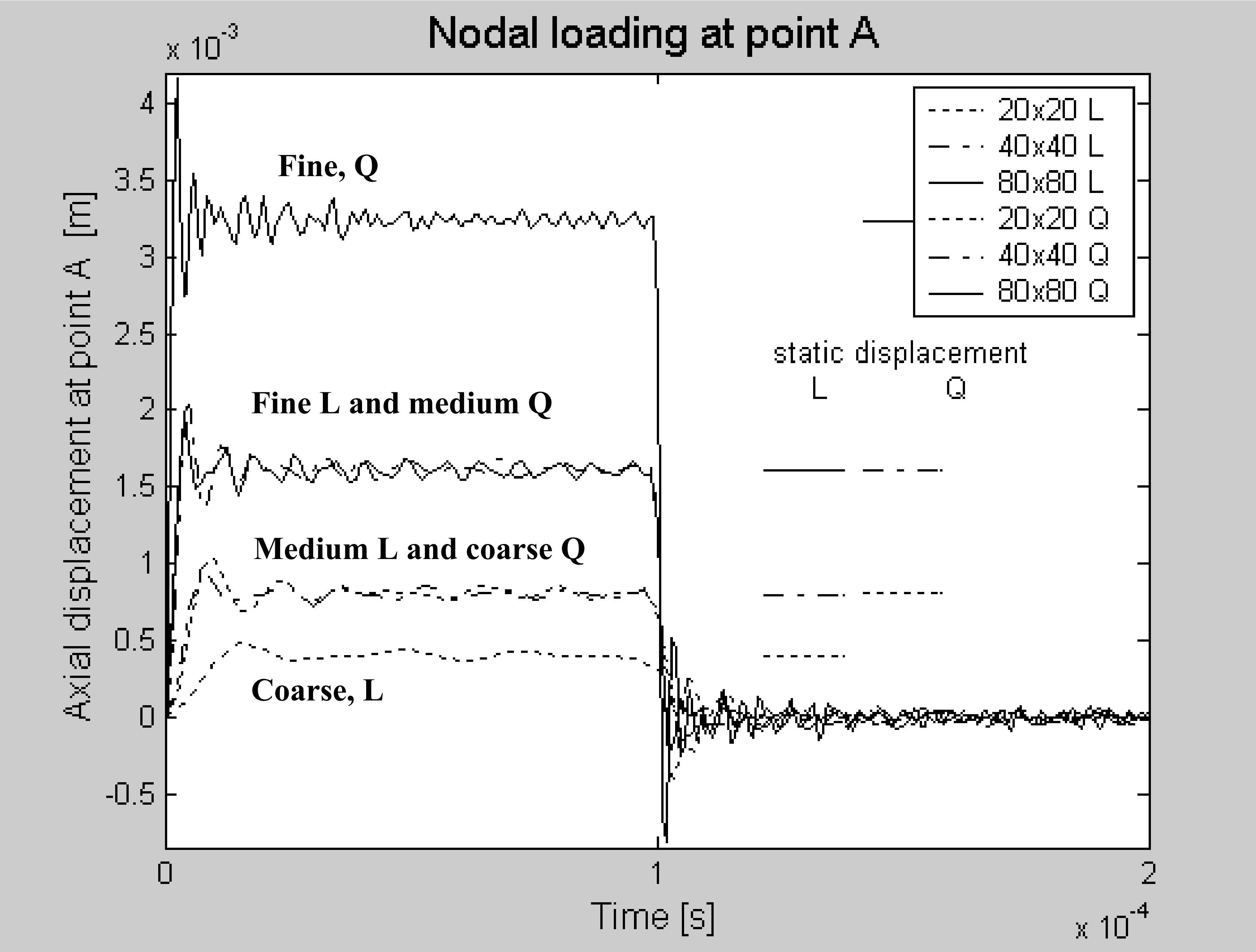
**Mesh**

| | |
|---|---|
| Coarse | 20x20 |
| Medium | 40x40 |
| Fine | 80x80 |

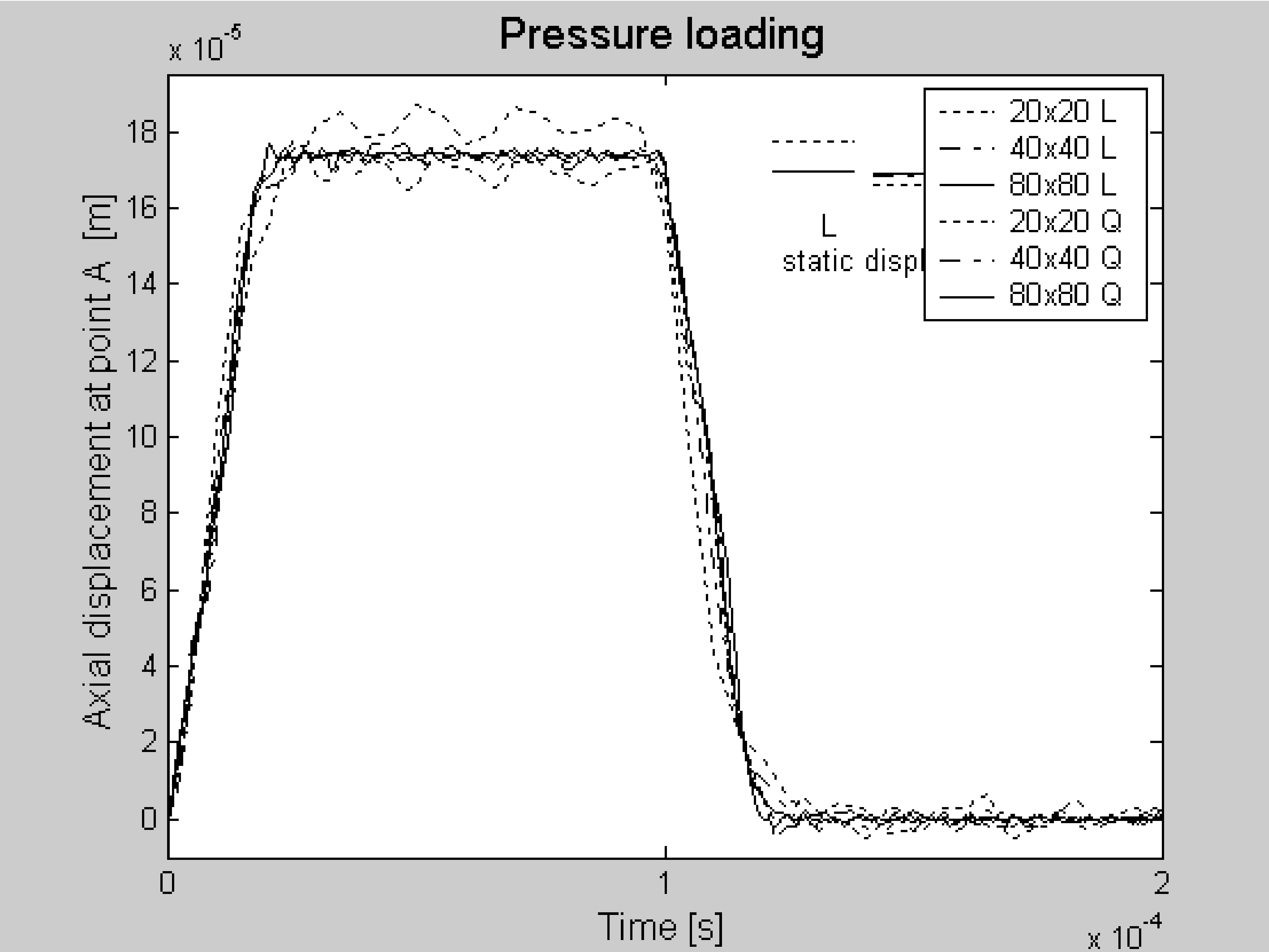
**Newmark**

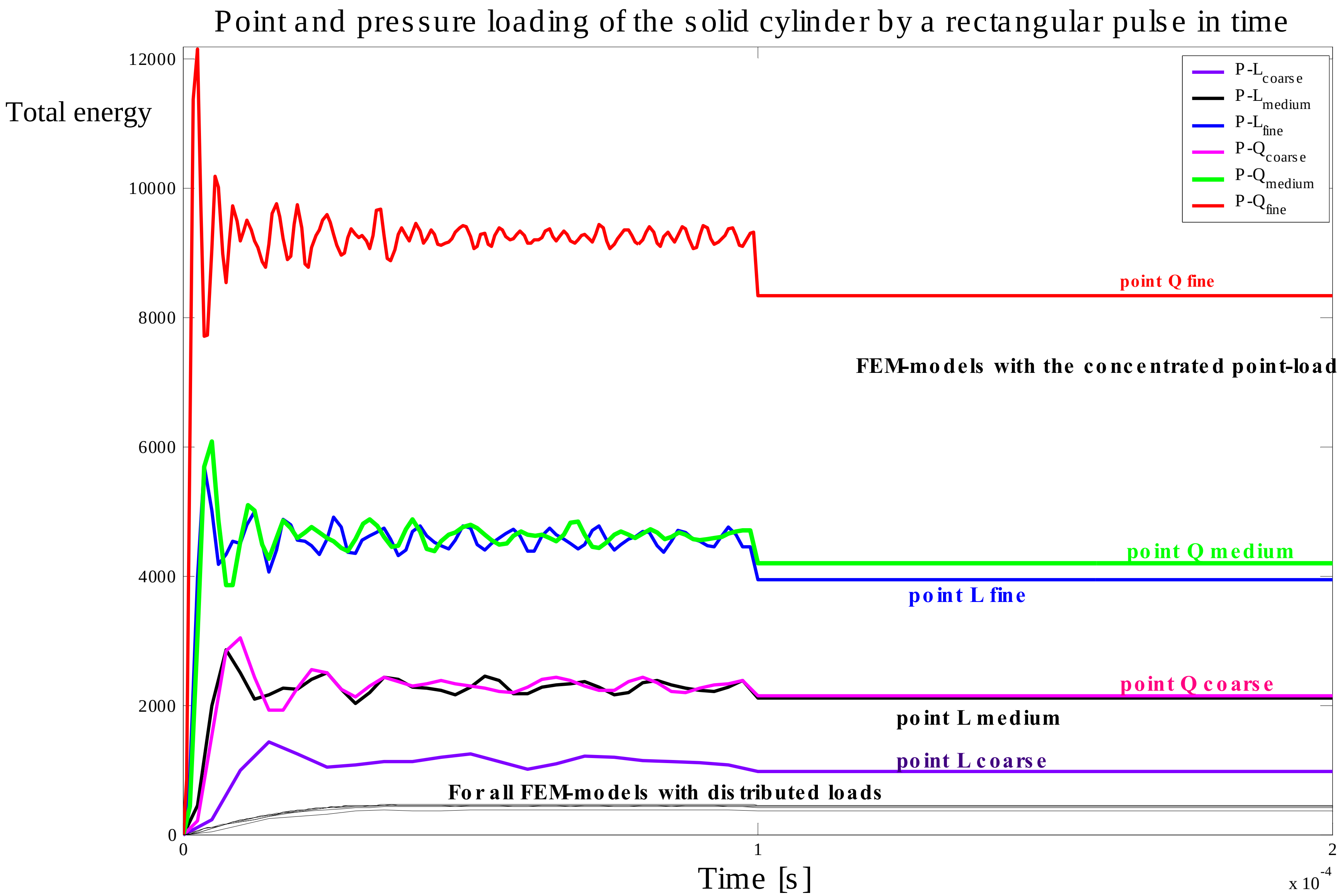
**Loading**
a point force
equiv. pressure

# Axial displacements for point force loading

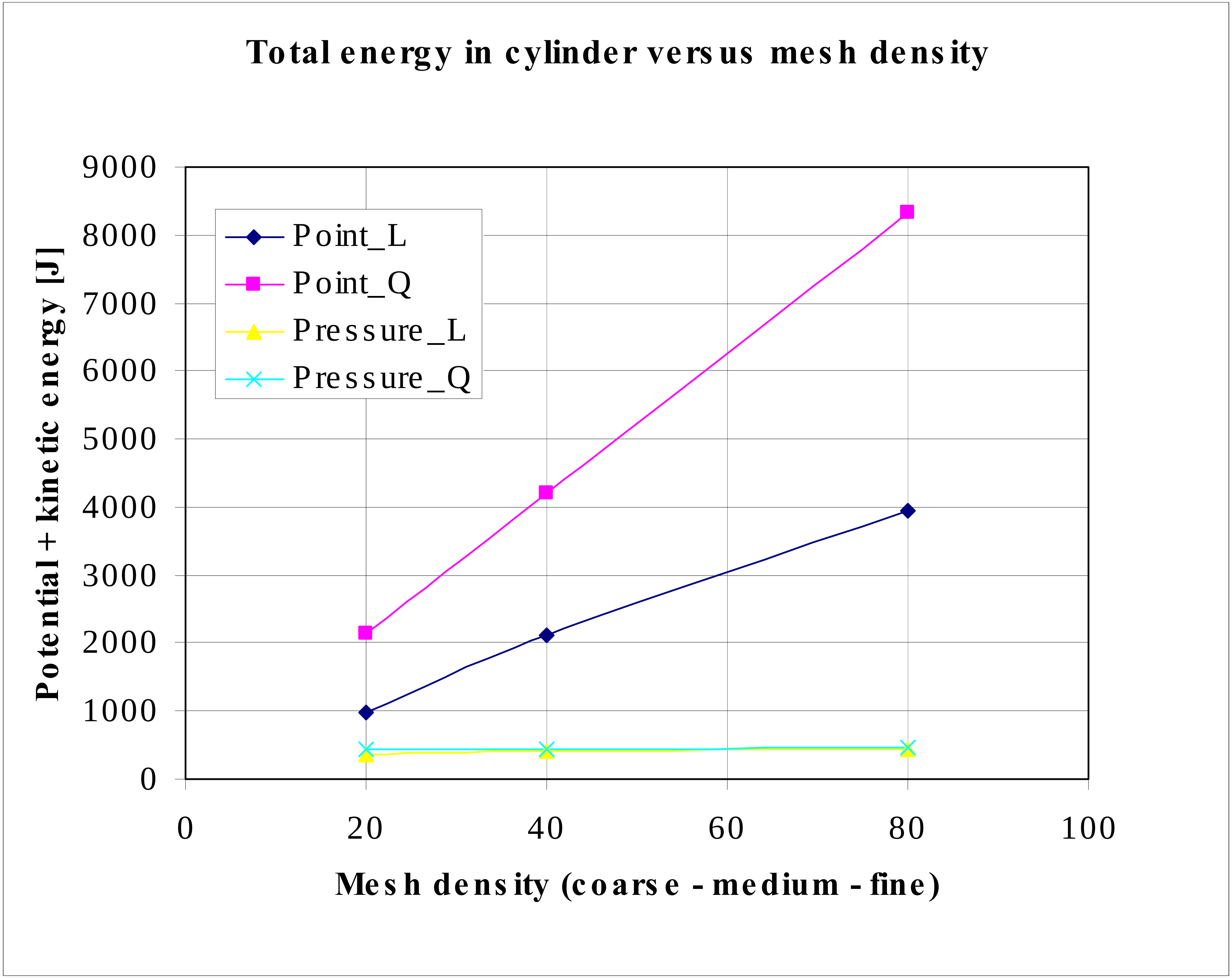


# Axial displacements for pressure loading






Pollution-free energy production by a proper misuse of FE analysis



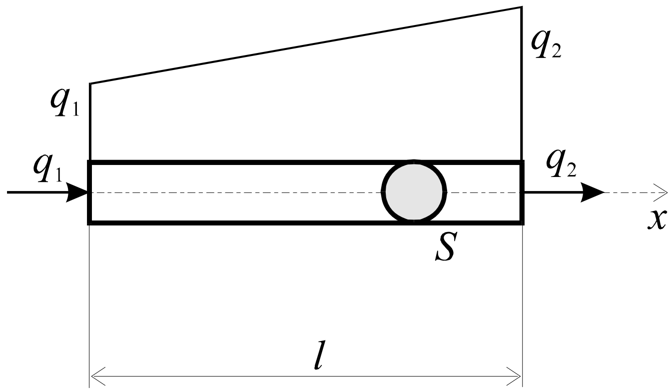
Total energy in cylinder versus mesh density
Potential + kinetic energy [J]
9000
8000
7000
6000
5000
4000
3000
2000
1000
0
Point_L
Point_Q
Pressure_L
Pressure_Q
0
20
40
60
80
100
Mesh density (coarse - medium - fine)

# 1D element for large strains and large deformations

Linear case

Non-linear case (material and geometry)

## Bar element, small strains, small displacements, linear material




Approximation of displacements $\{u\} = [A]\{q\}$ has the form

$$u_{\text{approx}} = u = c_1 + c_2 x = \begin{bmatrix} 1 & x \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} = [U]\{c\}$$

and must be valid at nodes as well $\quad u\big|_{x=0} = q_1$ a $u\big|_{x=l} = q_2$.

Substituting we get $\quad \{q\} = [S]\,\{c\},$

where

$$\{q\} = \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}, \quad [S] = \begin{bmatrix} 1 & 0 \\ 1 & l \end{bmatrix}, \quad \{c\} = \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix}.$$

If the length of element is greater than zero, then $\quad \{c\} = [S]^{-1}\,\{q\}$, kde $[S]^{-1} = \begin{bmatrix} 1 & 0 \\ -1/l & 1/l \end{bmatrix}$.

So the approximation of displacements is $\quad \{u\} = [U]\{c\} = [U][S]^{-1}\{q\} = [A]\{q\}$.

where

$$[A] = \begin{bmatrix} 1 & x \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1/l & 1/l \end{bmatrix} = \begin{bmatrix} 1 - x/l & x/l \end{bmatrix} = \begin{bmatrix} a_1(x) & a_2(x) \end{bmatrix}.$$

Approximation of strains $\{\varepsilon\} = [B]\{q\}$

$$\varepsilon = \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left([A]\{q\}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left[1 - x/l \quad x/l\right]\{q\} = \left[-1/l \quad 1/l\right]\{q\},$$

where $$[B] = \left[-1/l \quad 1/l\right].$$

The mass and stiffness matrices are

$$[m] = \rho \int_V [A]^{\mathrm{T}}[A]\,\mathrm{d}V = \rho S \int_0^l [A]^{\mathrm{T}}[A]\,\mathrm{d}l = \frac{\rho S l}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

$$[k] = \int_V [B]^{\mathrm{T}}[C][B]\,\mathrm{d}V = \int_0^l \begin{bmatrix} -1/l \\ 1/l \end{bmatrix} E \left[-1/l \quad 1/l\right] S\,\mathrm{d}x = \frac{ES}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix},$$

$[C] = E$ – the Young’s modulus.

# Summary for NL approach

$$\mathbf{K(q)\ \Delta q = P - F}$$

$$k^{\mathrm{L}} = \frac{{}^{0}A\,{}_{0}C}{{}^{0}l^{3}}\left({}^{0}l^{2} + 2q_{21}\,{}^{0}l + q_{21}^{2}\right)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} =$$

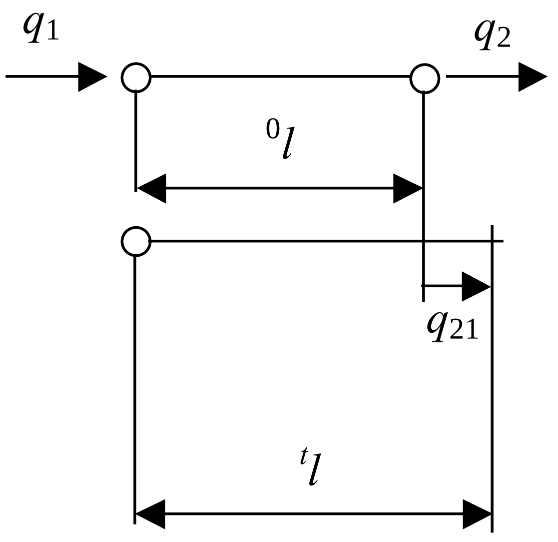
$$= \frac{{}^{0}A\,{}_{0}C}{{}^{0}l^{3}}\left({}^{0}l + q_{21}\right)^{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{{}^{0}A\,{}_{0}C}{{}^{0}l^{3}}\ {}^{t}l^{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{{}^{0}A\,{}_{0}C\xi^{2}}{{}^{0}l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

where

$$q_{21} = q_{2} - q_{1}$$

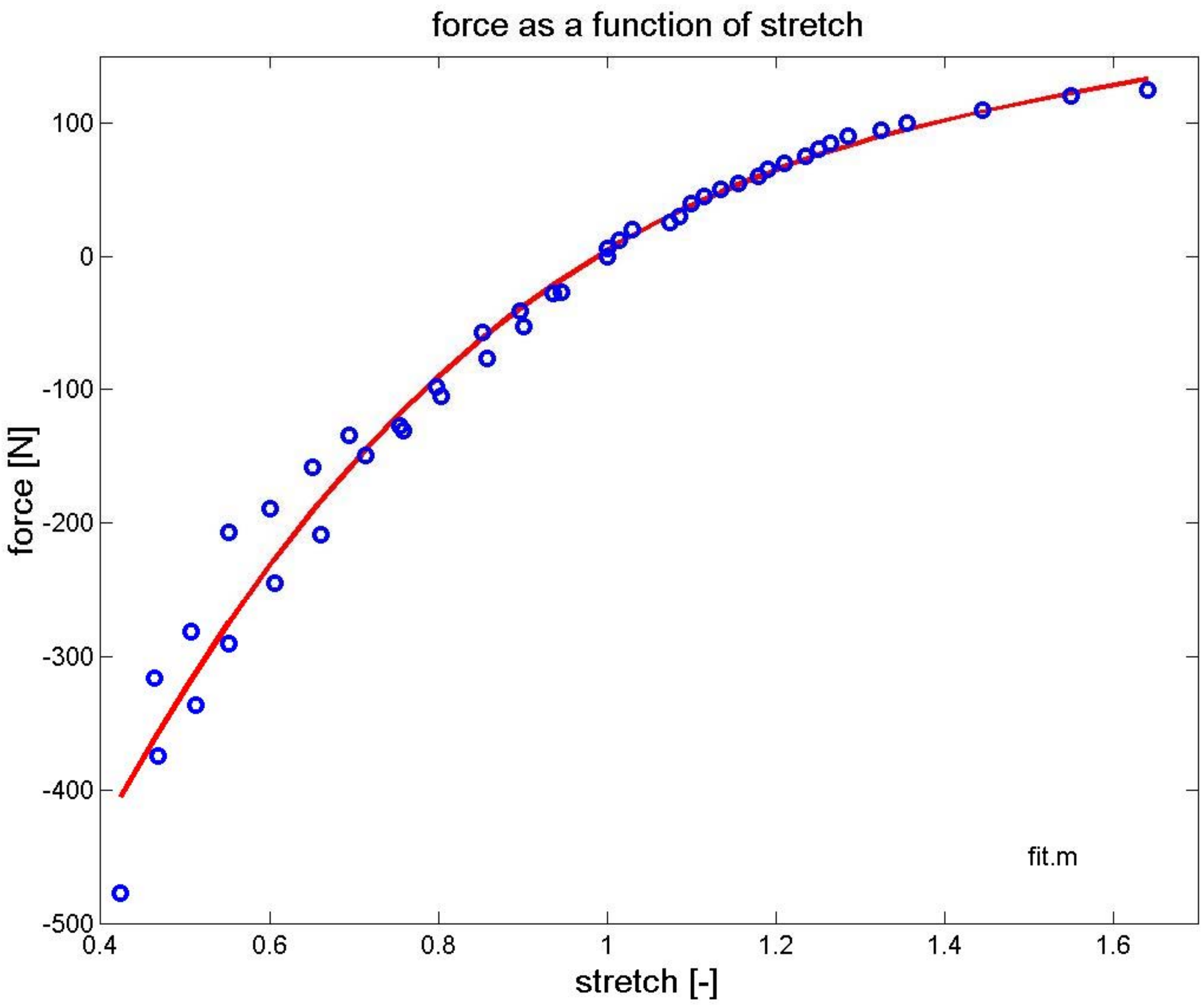
$${}^{0}l + q_{21} = {}^{t}l$$

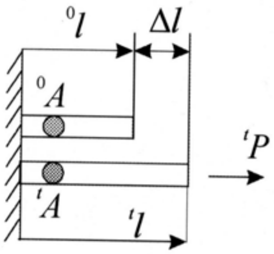
$$\xi = {}^{t}l\ /\ {}^{0}l$$



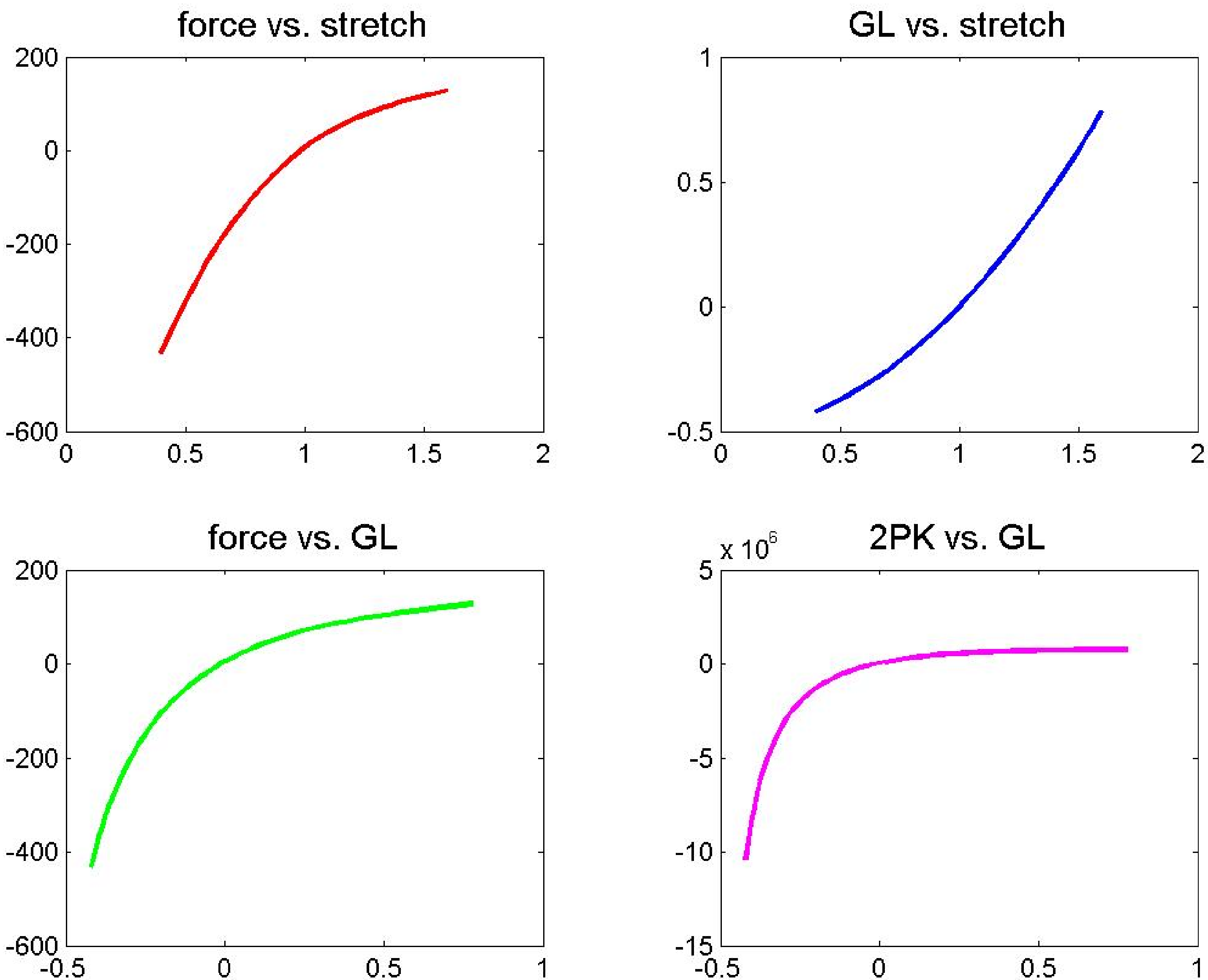

$$k^{\mathrm{N}} = \frac{{}^{0}A\,{}_{0}^{t}S}{{}^{0}l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

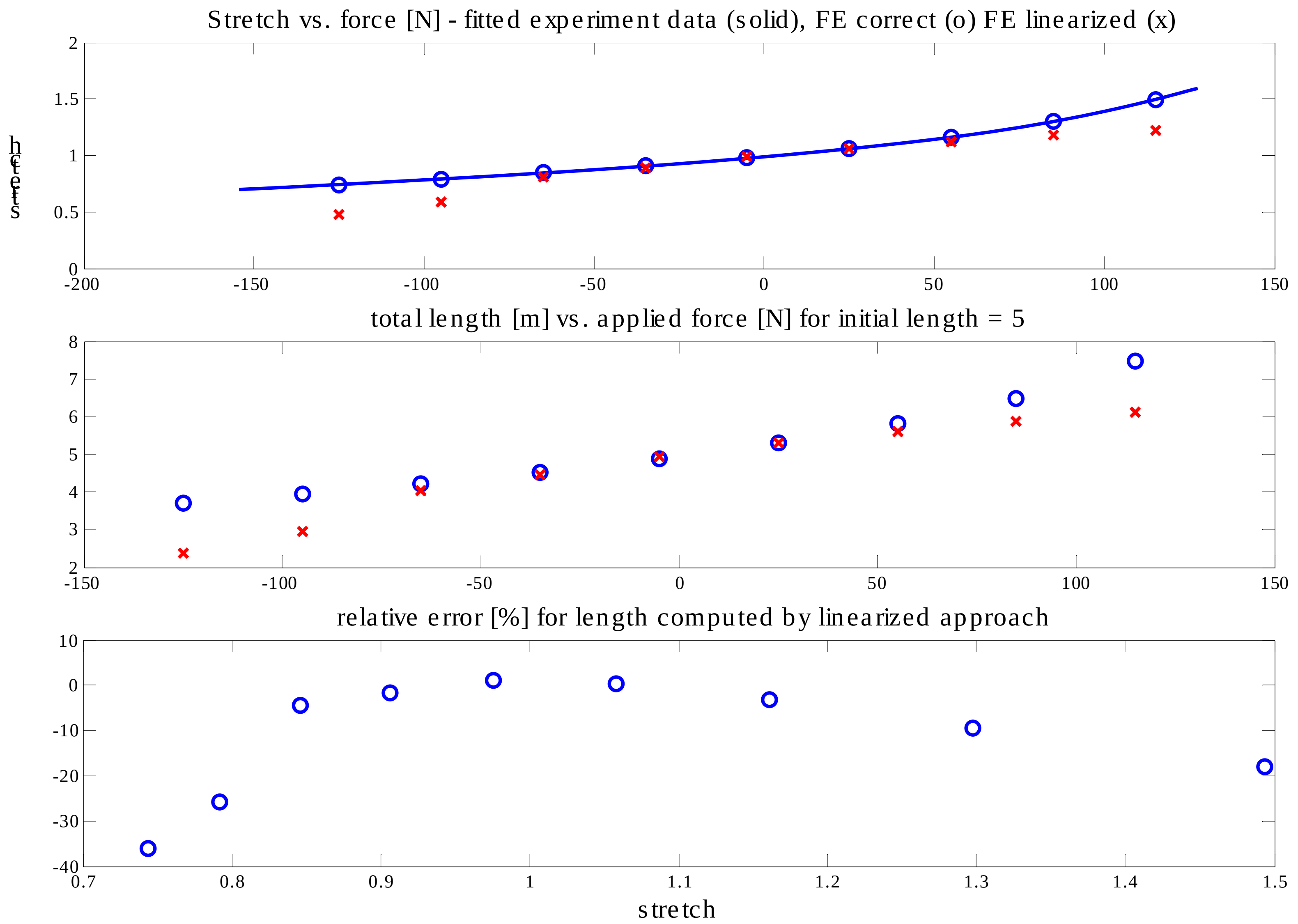
# 1D stretch experiment with rubber



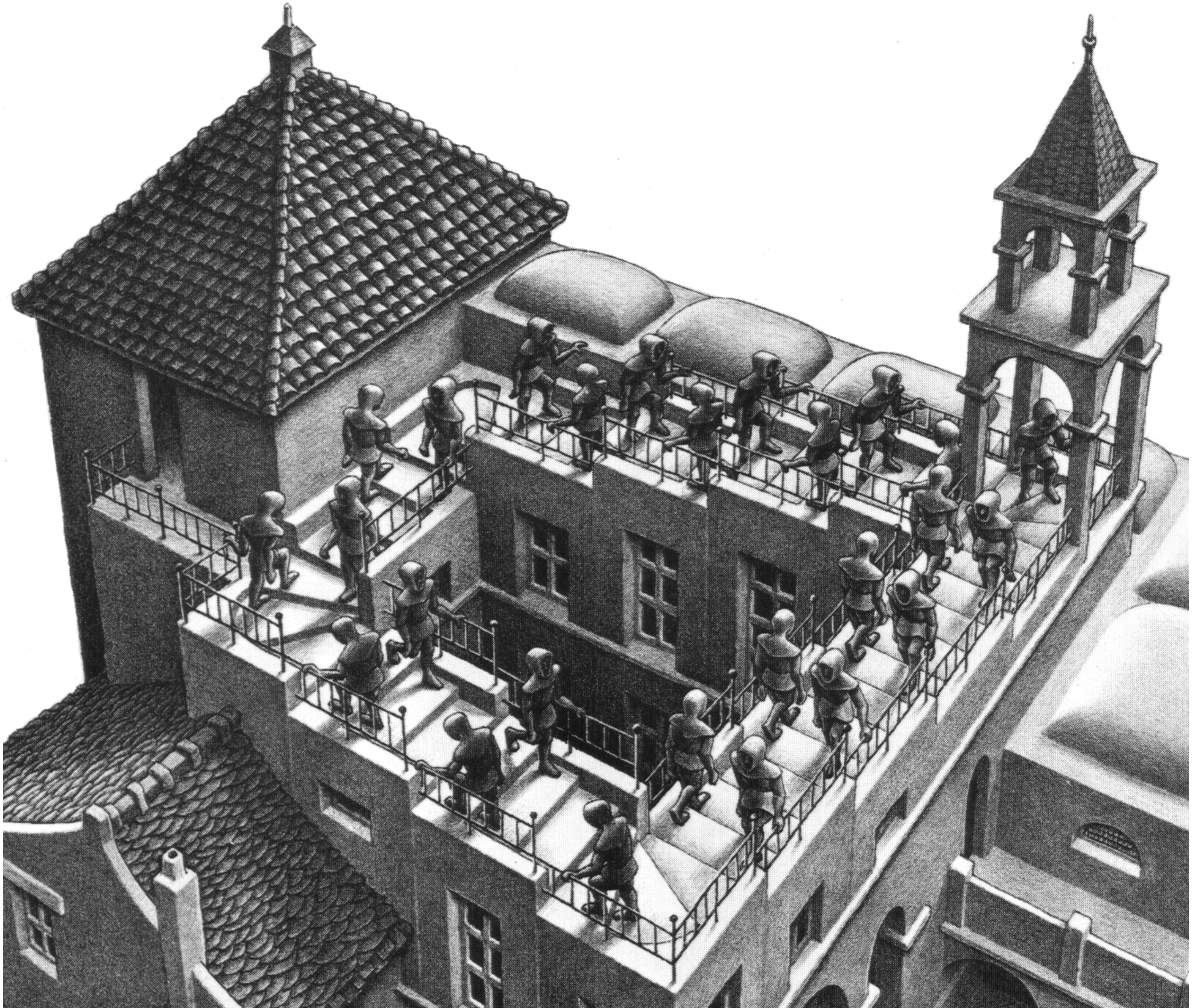



$$\xi = \frac{{}^{t}l}{{}^{0}l} \quad - \quad \text{stretch}$$


force vs. stretch
200
0
-200
-400
-600
0
0.5
1
1.5
2
GL vs. stretch
1
0.5
0
-0.5
0
0.5
1
1.5
2
force vs. GL
200
0
-200
-400
-600
-0.5
0
0.5
1
2PK vs. GL
x 10^6
5
0
-5
-10
-15
-0.5
0
0.5
1


Stretch vs. force [N] - fitted experiment data (solid), FE correct (o) FE linearized (x)
stretch
total length [m] vs. applied force [N] for initial length = 5
relative error [%] for length computed by linearized approach
stretch

# C.M. Esher: False perspectives



# How to avoid false perspectives?

- There are
  - Solid theoretical foundations
  - Efficient hardware
  - Quickly developing parallel and vector sw
- Goals
  - Ascertaining validity limits of models … material data are needed
  - Design of robust solvers running on parallel platforms
- Tools
  - Continuum mechanics theory
  - Engineering judgment
  - Common sense