Continuum Mechanics, part 21

Kinematics_1

- Point in space, material point
- Displacement, velocity, acceleration
- Eulerian, Lagrangian coordinates
- Deformation gradient, Jacobian
- Material, spatial displacement gradient
- Green-Lagrange and Almansi strain tensors

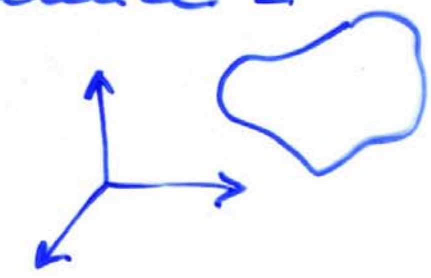
KINEMATICS

At first we deal with kinematics.

Kinematics is study of motion without regard to the forces which produce it.

A body which occupies a finite region of Euclidean space will be considered.

All motion will be 'measured' relative to the chosen fixed frame of reference – – a Cartesian coordinate system



cm0026b

Note

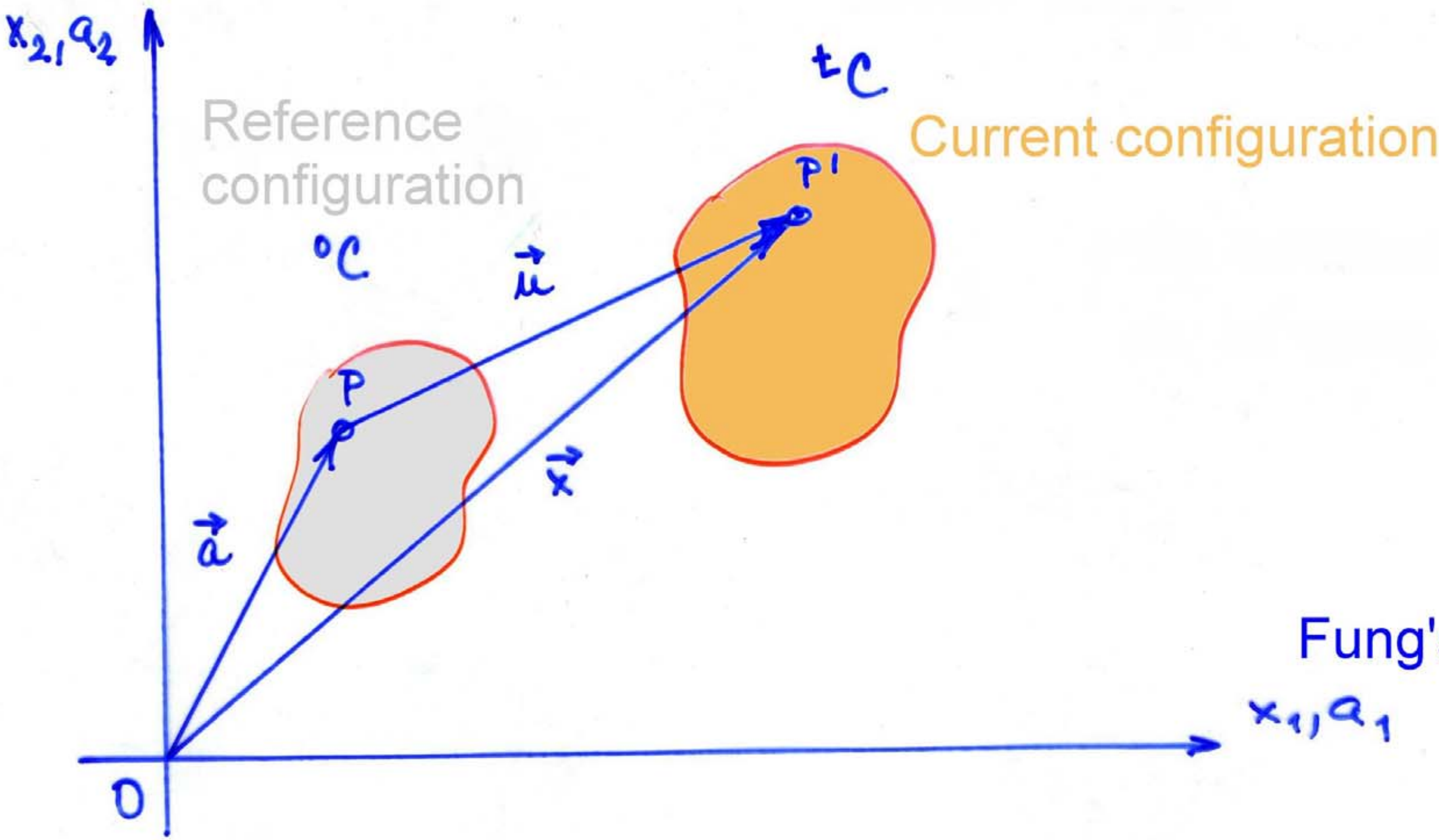
The term deformation refers to a change in the shape of the continuum between some initial (reference) configuration and a subsequent (current, sometimes final) deformed configuration.

The meaning of the word point must be clearly understood, since it may be used to refer either to a point in space or to a point of a body. To avoid misunderstanding the term

point will be used to designate a location in space

particle will denote a small volumetric element of a body or material point

cm0027a



Fung's notation

Particles of the body are identified by a generic point P, whose position vector relative to the reference frame at time t=0 is $\vec{a}$

cm0027b

Let's assume that body undergoes the motion so that at a subsequent time t it occupies a new configuration ${}^{t}C$.

We make an assumption that we can identify individual particles of the body throughout its motion so the particle P moves to a new position P' and can be specified by the position vector $\vec{x}$ with components x_i.

Alternative notation

Eulerian coordinates	x_i	$\cdots$	x_i	${}^t x_i$	at	${}^t C$
Lagrangian coordinates	a_i	$\cdots$	X_i	${}^0 x_i$	at	${}^0 C$

the motion of the body can be described by specifying the dependence of the positions $\vec{x}$ of the particles of the body at time t on their positions $\vec{a}$ at time $t=0$.
This can be expressed in indicial, symbolic and matrix notation respectively

$$x_i = x_i(a_j, t) \qquad \vec{x} = \vec{x}(\vec{a}, t) \qquad \{x\} = \{x(\{a\}, t)\} \quad (1)$$

<u>Note</u>

x_i ... Eulerian, space coordinates

a_i ... Lagrangian, material coord., labels

A particular particle retains the same value of a_i throughout its motion.
They have the meaning of coordinates at time $t=0$ only

cm0029b

For physically realizable motion it is possible to solve (1) for a_i in terms of x_j which gives

$$a_i = a_i(x_j, t) \qquad (2)$$

<u>Note:</u>

a) $x_i = x_i(a_j, t)$ ... Lagrangian approach, material description

b) $a_i = a_i(x_j, t)$ ... Eulerian approach, spatial description

Attention is focussed on what is happening

a) at a particular material particle

b) at a particular point in space

the equations (1) and (2) are single-valued continuous functions if the Jacobian of the transformation (1), i.e. the determinant

$$J = \left| \frac{\partial x_i}{\partial a_j} \right| > 0 .$$

the quantity

$$F_{ij} = \frac{\partial x_i}{\partial a_j} \qquad [F] = \left[\frac{\partial x_i}{\partial a_j} \right] \qquad (3)$$

is called deformation gradient or material d.g.
As already mentioned

$$J = \det [F] = \left| \frac{\partial x_i}{\partial a_j} \right| \qquad (3a)$$

In classical mechanics mass is assumed to be conserved. It is further assumed that the mass is continuous function of volume.

A positive quantity ρ called density can be defined at every point in the body.

at time $t = 0$, at the point a_i .... ${}^0\rho = {}^0\rho(a_i)$

t $\qquad$ x_i .... ${}^t\rho = {}^t\rho(x_i)$

the conservation of mass is expressed by

$$\int {}^t\rho \underbrace{dx_1\, dx_2\, dx_3}_{d\,{}^tV} = \int {}^0\rho \underbrace{da_1\, da_2\, da_3}_{d\,{}^0V}$$

cm0031a

The change of variable under left-hand side integral sign gives

$$\int \overset{t}{\rho}\, dx_1\, dx_2\, dx_3 = \int \overset{t}{\rho} \left| \frac{\partial x_i}{\partial a_j} \right| da_1\, da_2\, da_3$$

Comparing the last two equations and realizing that the relations should be valid for all volumes we get

cm0031b

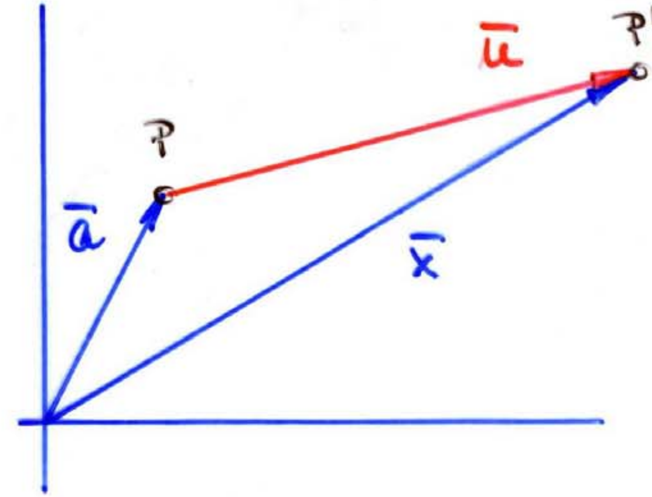
$$ {}^{0}\rho = {}^{t}\rho \left| \frac{\partial x_i}{\partial a_j} \right| = \rho^t \det [F] $$

$$ \boxed{\frac{{}^{0}\rho}{{}^{t}\rho} = \det [F]} = J $$

$$ x_i = x_i(a_j, t) $$

$$ \rho^t = {}^{t}\rho(x_i(a_i, t)) $$

Displacement



$$\bar{u} = \bar{x} - \bar{a}$$

In material description: $f(\bar{a},t)$

$$u_i = x_i(a_j,t) - a_i$$

In spatial description: $f(\bar{x},t)$

$$u_i = x_i - a_i(x_j,t)$$

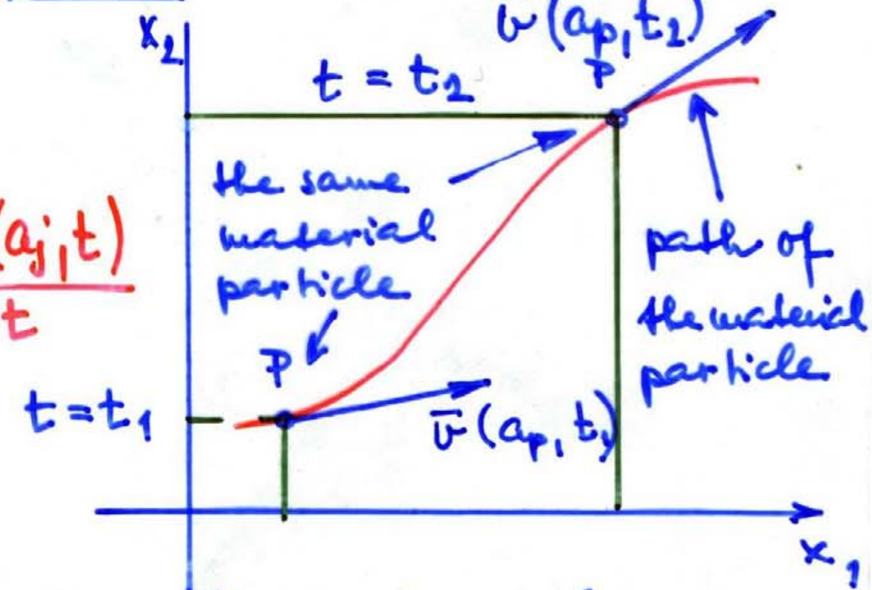
Displacement of a particle which presently – at time t – occupies position x_i

Displacement of a particle identified by material label a_i

cm0035a

Velocity in material description

$$v_i = v_i(a_j,t) = \frac{\partial u_i(a_j,t)}{\partial t} = \frac{\partial x_i(a_j,t)}{\partial t}$$



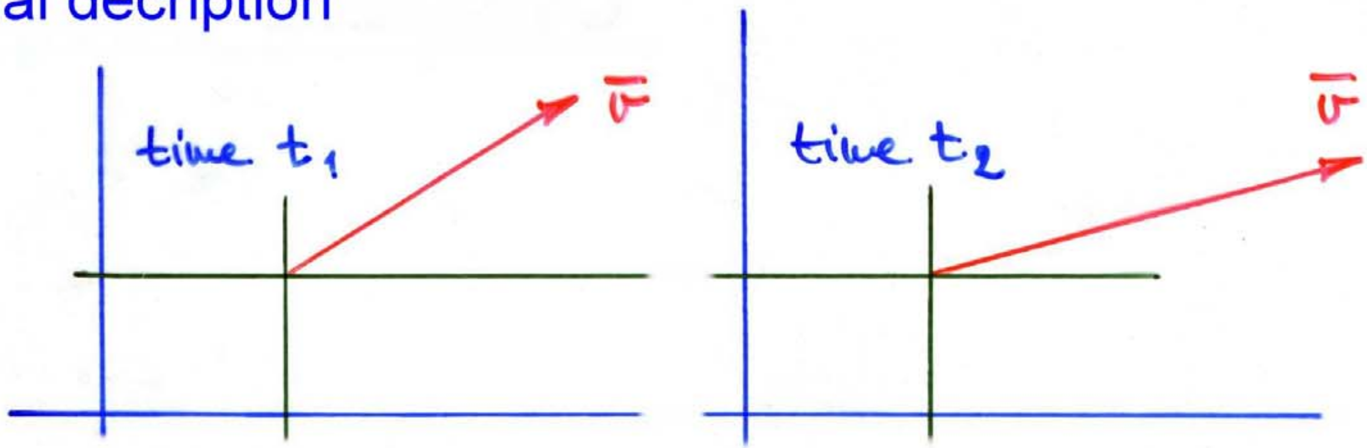
Velocity $\bar{v}$ of a particle is the rate of change of its displacement. Since a_i are constant at a fixed particle we can write ⇒. Differentiation was performed with a_j held constant.

This relation gives the velocity of the same particle as a function of time

cm0035b

Velocity in spatial decription

$v_i = v_i(x_j, t)$



- The same space point at different times
- There are different material points in it

The velocity vector field

cm0036a

Example Material Spatial

Let $x_i = x_i(a_j, t)$ is given by $\Rightarrow a_i = a_i(x_j, t)$

MUST BE DEFINED FOR EACH PARTICLE $\Rightarrow$ NEED FOR ANOTHER INDEX

Material	Spatial
$x_1 = a_1(1+c^2t^2)$	$a_1 = \dfrac{x_1}{1+c^2t^2}$
$x_2 = a_2$	$a_2 = x_2$
$x_3 = a_3$	$a_3 = x_3$

Displacement

$u_i = u_i(a_j, t)$ $\qquad$ $u_i = u_i(x_j, t)$

Material	Spatial
$u_1 = x_1 - a_1 = a_1(1+c^2t^2) - a_1 = a_1c^2t^2$	$u_1 = x_1 - a_1 = x_1\left(1 - \dfrac{1}{1+c^2t^2}\right) = \dfrac{x_1c^2t^2}{1+c^2t^2}$
$u_2 = 0$	$u_2 = 0$
$u_3 = 0$	$u_3 = 0$

Velocity

$$v_i = v_i(a_j, t) = \left.\frac{\partial u_i}{\partial t}\right|_{a=\text{const}} = \left.\frac{\partial x_i}{\partial t}\right|_{a=\text{const}} \qquad v_i = v_i(x_j, t)$$

Material	Spatial
$v_1 = 2a_1c^2t$ $\longrightarrow$	$v_1 = \dfrac{2x_1c^2t}{1+c^2t^2}$
$v_2 = 0$	$v_2 = 0$
$v_3 = 0$	$v_3 = 0$

cm0036b

Deformation gradient in case is

$$[F] = \left[\frac{\partial x_i}{\partial a_j}\right] = \begin{bmatrix} \frac{\partial x_1}{\partial a_1} & \frac{\partial x_1}{\partial a_2} & \frac{\partial x_1}{\partial a_3} \\ \frac{\partial x_2}{\partial a_1} & \frac{\partial x_2}{\partial a_2} & \frac{\partial x_2}{\partial a_2} \\ \frac{\partial x_3}{\partial a_1} & \frac{\partial x_3}{\partial a_2} & \frac{\partial x_3}{\partial a_3} \end{bmatrix} = \begin{bmatrix} 1+c^2t^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The acceleration in material description can be expressed as

$$z_i = z_i(a_j,t) = \frac{dv_i}{dt} = \frac{d^2x_i}{dt^2} = \frac{d}{dt}v_i(a_j,t) = \frac{d^2}{dt^2}u_i(a_j,t)$$

since a_j is not time dependent

Expressing the acceleration in spatial description we must take into account the fact that variable x_j in $v_i(x_j,t)$ is a function of time.

Using the chain rule we get

$$z_i = z_i(x_j,t) = \frac{\partial v_i}{\partial x_k}\underbrace{\frac{dx_k}{dt}}_{v_k} + \frac{\partial v_i}{\partial t}\frac{dt}{dt}$$

Introducing a new notation D/Dt for so called material derivative we can rewrite the previous equation

$$z_i = z_i(x_j,t) = \frac{Dv_i}{Dt} = \underbrace{v_k(x_j,t)\,\frac{\partial v_i(x_j,t)}{\partial x_k}}_{\text{convective part}} + \underbrace{\frac{\partial v_i(x_j,t)}{\partial t}}_{\text{local part}}$$

contribution of the motion of a particle throughout the velocity field

time dependence of velocity field

relativní!

unášivé!

Example – cont.

Material

$$v_i = v_i(a_j, t) = 2a_1c^2t$$

$$z_i = z_i(a_j, t) = 2a_1c^2$$

Spatial

$$v_i = v_i(x_j, t) = \frac{2x_1c^2t}{1+c^2t^2}$$

$$z_i = z_i(x_j, t) = \frac{Dv_i}{Dt}$$

$$= v_k(x_j, t)\,\frac{\partial v_i(x_j, t)}{\partial x_k} + \frac{\partial v_i(x_j, t)}{\partial t}$$

$$= \begin{Bmatrix} \dfrac{2x_1c^2t}{1+c^2t^2} \cdot \dfrac{2c^2t}{1+c^2t^2} \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} \dfrac{2x_1c^2}{1+c^2t^2} \\ 0 \\ 0 \end{Bmatrix}$$

FINITE DEFORMATION AND STRAIN TENSORS

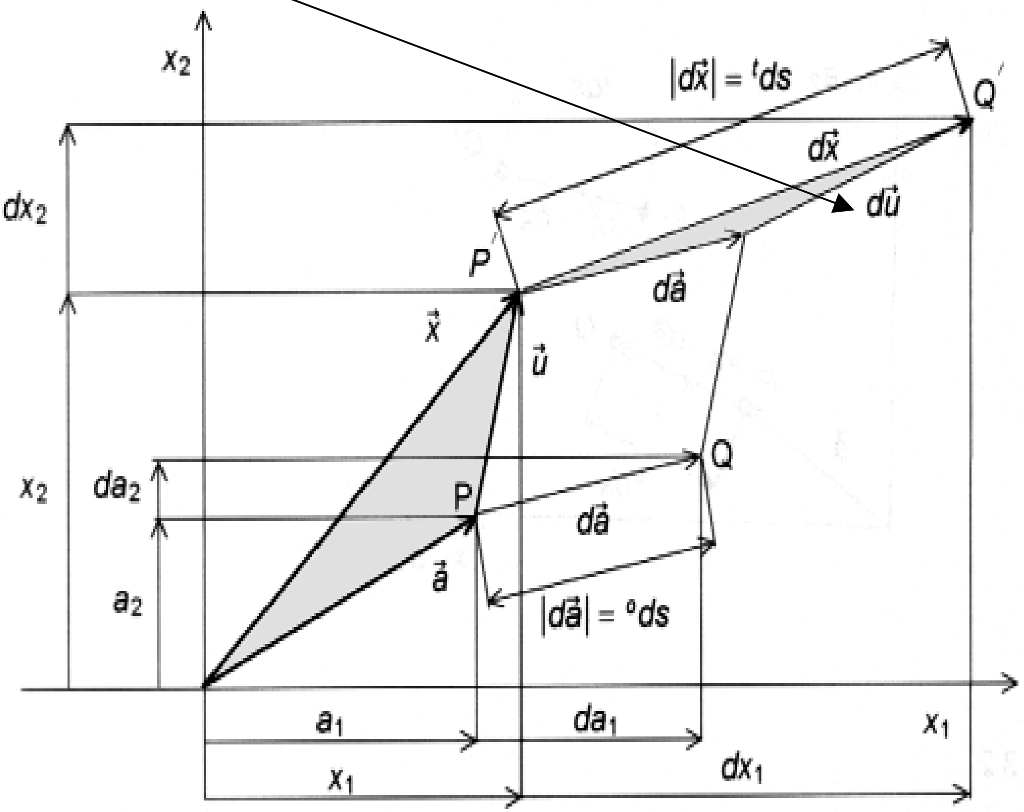
cm0101a

Describing the motion of continuum subjected to large deformations we are looking for certain invariant measures able to define deformation.

Such a measure could be based on the difference of square lengths of two infinitesimally close points P and Q at configurations ^{0}C and ^{t}C.

The measure we are looking for should be independent of rigid body rotation and the choice of coordinate system

Relative displacement vector



Configuration at time t = 0 ... ^{0}C:

P: a_i Q: $a_i + \mathrm{d}a_i$

Configuration at time t ... ^{t}C:

P': $x_i = a_i + u_i$ Q': $x_i + \mathrm{d}x_i = a_i + u_i + \mathrm{d}a_i + \mathrm{d}u_i$

total differential

Material displacement gradient [Z]

$$\mathrm{d}u_i = \frac{\partial u_i}{\partial a_j}\,\mathrm{d}a_j = Z_{ij}\,\mathrm{d}a_j$$

cm0102a

the coordinates of Q' can be expressed

$$Q': a_i + u_i + da_i + du_i = a_i + u_i + \underbrace{\delta_{ij} da_j}_{\text{substitution}} + \underbrace{Z_{ij}\, da_j}_{\text{Taylor}} =$$

$$= a_i + u_i + (\delta_{ij} + Z_{ij})\, da_j$$

We know that $\quad x_i = u_i + a_i$

$$\frac{\partial x_i}{\partial a_j} = \frac{\partial u_i}{\partial a_j} + \delta_{ij}$$

$$\boxed{F_{ij} = Z_{ij} + \delta_{ij}} \qquad [F] = [Z] + [I]$$

So the coordinates of Q' can be rewritten again

$$Q': a_i + u_i + F_{ij}\, da_j$$

And now the vectors corresponding to line segments $\overline{PQ}$ and $\overline{P'Q'}$ are

$\overline{PQ}$: $\{d\bar{a}\}$ with components da_i

$\overline{P'Q'}$: $\{dx\} = \{da\} + \{du\} = \{da\} + [Z]\{da\} =$

$$= ([I] + [Z])\{da\} = [F]\{da\}$$

with components $\boxed{dx_i = F_{ij}\, da_j}$

Remember

$$F_{ij} = \frac{\partial x_i}{\partial a_j}$$ deformation gradient

$$Z_{ij} = \frac{\partial u_i}{\partial a_j}$$ material displacement gradient

cm0103

the invariant we are looking for

$$(ds)^2 - (ds_0)^2 = dx_i\, dx_i - da_i\, da_i =$$

$$= \{da\}^T [F]^T [F] \{da\} - \{da\}^T [I] \{da\} =$$

$$= 2 \{da\}^T \underbrace{\tfrac{1}{2}\left([F]^T [F] - [I] \right)}_{[E]} \{da\}$$

Green-Lagrange strain tensor
sometimes Green

Quadratic form
scalar
invariant

A new quantity $[C] = [F]^T [F]$

right Cauchy-Green deformation tensor

The Green-Lagrange strain tensor is the measure of strain corresponding to change from configuration ${}^0C$ to ${}^tC$ with respect to initial reference configuration at ${}^0C$.

Green-Lagrange strain tensor can be expressed in various forms

$$[E] = \frac{1}{2}\left([C] - [I]\right) = \frac{1}{2}\left([F]^T[F] - [I]\right) = \qquad \text{using } [F] = [Z] + [I]$$

$$= \frac{1}{2}\left([Z] + [Z]^T + [Z]^T[Z]\right)$$

$$E_{ij} = \frac{1}{2}\left(Z_{ij} + Z_{ji} + Z_{ki} Z_{kj}\right) = \frac{1}{2}\left(\frac{\partial u_i}{\partial a_j} + \frac{\partial u_j}{\partial a_i} + \frac{\partial u_k}{\partial a_i}\frac{\partial u_k}{\partial a_j}\right) =$$

$$= \frac{1}{2}\left(u_{i,j} + u_{j,i} + u_{k,i} u_{k,j}\right) = \frac{1}{2}\left(F_{ki} F_{kj} - \delta_{ij}\right) =$$

$$= \frac{1}{2}\left(C_{ij} - \delta_{ij}\right) = \overset{\text{CAUCHY}}{\underset{\downarrow}{}}\!\!\varepsilon_{ij} + \eta_{ij}$$

LINEAR & NONLINEAR PARTS

cm0104b

$$[E] = \frac{1}{2}\left(\left(\{\nabla_0\}\{u\}^T\right)^T + \left(\{\nabla_0\}\{u\}^T\right) + \left(\{\nabla_0\}\{u\}^T\right)\left(\{\nabla_0\}\{u\}^T\right)^T\right)$$

where

$$\{\nabla_0\} = \begin{Bmatrix} \dfrac{\partial}{\partial a_1} \\ \dfrac{\partial}{\partial a_2} \\ \dfrac{\partial}{\partial a_3} \end{Bmatrix} \qquad \{u\} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

nabla operator

Remember $[Z] = \left(\{\nabla_0\}\{u\}^T\right)^T$

cm0105

green-Lagrange strain tensor – written in full

$$\varepsilon_{11} = \frac{\partial u_1}{\partial a_1} + \frac{1}{2}\left[\left(\frac{\partial u_1}{\partial a_1}\right)^2 + \left(\frac{\partial u_2}{\partial a_1}\right)^2 + \left(\frac{\partial u_3}{\partial a_1}\right)^2\right]$$

$$\varepsilon_{22} = \frac{\partial u_2}{\partial a_2} + \frac{1}{2}\left[\left(\frac{\partial u_1}{\partial a_2}\right)^2 + \left(\frac{\partial u_2}{\partial a_2}\right)^2 + \left(\frac{\partial u_3}{\partial a_2}\right)^2\right]$$

$$\varepsilon_{33} = \frac{\partial u_3}{\partial a_3} + \frac{1}{2}\left[\left(\frac{\partial u_1}{\partial a_3}\right)^2 + \left(\frac{\partial u_2}{\partial a_3}\right)^2 + \left(\frac{\partial u_3}{\partial a_3}\right)^2\right]$$

$$\varepsilon_{12} = \frac{1}{2}\left[\frac{\partial u_1}{\partial a_2} + \frac{\partial u_2}{\partial a_1} + \frac{\partial u_1}{\partial a_1}\frac{\partial u_1}{\partial a_2} + \frac{\partial u_2}{\partial a_1}\frac{\partial u_2}{\partial a_2} + \frac{\partial u_3}{\partial a_1}\frac{\partial u_3}{\partial a_2}\right]$$

$$\varepsilon_{23} = \frac{1}{2}\left[\frac{\partial u_2}{\partial a_3} + \frac{\partial u_3}{\partial a_2} + \frac{\partial u_1}{\partial a_3}\frac{\partial u_1}{\partial a_2} + \frac{\partial u_2}{\partial a_3}\frac{\partial u_2}{\partial a_2} + \frac{\partial u_3}{\partial a_3}\frac{\partial u_3}{\partial a_2}\right]$$

$$\varepsilon_{31} = \frac{1}{2}\left[\frac{\partial u_3}{\partial a_3} + \frac{\partial u_1}{\partial a_1} + \frac{\partial u_1}{\partial a_3}\frac{\partial u_1}{\partial a_1} + \frac{\partial u_2}{\partial a_3}\frac{\partial u_2}{\partial a_1} + \frac{\partial u_3}{\partial a_3}\frac{\partial u_3}{\partial a_1}\right]$$

quadratic term

linear part
known from linear elasticity

as CAUCHY INFINITESIMAL STRAIN TENSOR

Similarly we can derive another measure of strain, namely Almansi or Euler strain tensor which is related to the current configuration ${}^tC$.

$$a_i = x_i - u_i$$

Differentiating we get

$$da_i = \frac{\partial a_i}{\partial x_j} dx_j = dx_i - \frac{\partial u_i}{\partial x_j} dx_j =$$

F^{-1}_{ij}

$$= \left(\delta_{ij} - \frac{\partial u_i}{\partial x_j} \right) dx_j = \left(\delta_{ij} - \overline{Z}_{ij} \right) dx_j$$

where $\overline{Z}_{ij} = \dfrac{\partial u_i}{\partial x_j}$ is so called spatial displacement gradient

cm0106b

Comparing :

$$F_{ij}^{-1} = \delta_{ij} - \overline{Z}_{ij} \qquad [F]^{-1} = [I] - [\overline{Z}]$$

the quantity

$$H_{ij} = F_{ij}^{-1}$$

is called spatial deformation gradient.

So :

$$\{da\} = [F]^{-1}\{dx\}$$

cm0107a

the difference of squares of lengths can be expressed in the form

$$(ds)^2 - (ds_0)^2 = \{dx\}^T\{dx\} - \{da\}^T\{da\} =$$

$$= \{dx\}^T[I]\{dx\} - \{dx\}^T[F]^{-T}[F]^{-1}\{dx\} =$$

$$= 2\,\{dx\}^T \underbrace{\tfrac{1}{2}\left([I] - [F]^{-T}[F]^{-1}\right)}_{[A]\ \ldots\ \text{Almansi strain tensor}} \{dx\}$$

the quantity $[\bar{C}] = [F][F]^T$ ... left Cauchy-Green def. t.

cm0107b

Different forms of Almansi strain tensor

$$[A] = \frac{1}{2}\left([I] - [F]^{-T}[F]^{-1}\right) = \frac{1}{2}\left([I] - [\bar{C}]^{-1}\right) =$$

using $[F]^{-1} = [I] - [\bar{Z}]$

$$= \frac{1}{2}\left([\bar{Z}] + [\bar{Z}]^T - [\bar{Z}]^T[\bar{Z}]\right)$$

$$A_{ij} = \frac{1}{2}\left(\bar{Z}_{ij} + \bar{Z}_{ji} - \bar{Z}_{ki}\bar{Z}_{kj}\right) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right) =$$

$$= \frac{1}{2}\left(\delta_{ij} - F_{ki}^{-1} F_{kj}^{-1}\right) = \frac{1}{2}\left(\delta_{ij} - \bar{C}_{ij}^{-1}\right).$$

$$[A] = \frac{1}{2}\left(\left(\{\nabla_t\}\{u\}^T\right)^T + \left(\{\nabla_t\}\{u\}^T\right) - \left(\{\nabla_t\}\{u\}^T\right)\left(\{\nabla_t\}\{u\}^T\right)^T\right)$$

where

$$\{\nabla_t\} = \left\{\frac{\partial}{\partial x_1} \quad \frac{\partial}{\partial x_2} \quad \frac{\partial}{\partial x_3}\right\}^T$$