MONOTONE MODAL OPERATORS ON BOUNDED INTEGRAL RESIDUATED LATTICES

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Abstract. Bounded integral residuated lattices form a large class of algebras containing some classes of commutative and noncommutative algebras behind many-valued and fuzzy logics. In the paper, monotone modal operators (special cases of closure operators) are introduced and studied.

Keywords: residuated lattice, bounded integral residuated lattice, modal operator, closure operator

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Bounded integral residuated lattices form a large class of algebras containing some classes of algebras behind many-valued and fuzzy logics, such as pseudo MV-algebras [15] (or equivalently GMV-algebras [23]), pseudo BL-algebras [5], pseudo MTL-algebras [12] and Rl-monoids [10], and consequently, the classes of their commutative cases, i.e. MV-algebras [3], BL-algebras [16], MTL-algebras [11] and commutative Rl-monoids [9]. Moreover, Heyting algebras [2] which are algebras of the intuitionistic logic can be also viewed as residuated lattices.

Modal operators (special cases of closure operators) were introduced and investigated on Heyting algebras in [22], on MV-algebras in [17], on commutative Rl-monoids in [24] and on (non-commutative) Rl-monoids in [26]. Moreover, monotone modal operators on commutative bounded residuated lattices were studied in [19].

In the paper we define and study monotone modal operators on general (not necessarily commutative) residuated lattices.

A *bounded integral residuated lattice* is an algebra $M = (M; \odot, \vee, \wedge, \to, \rightsquigarrow, 0, 1)$ of type $(2, 2, 2, 2, 2, 0, 0)$ satisfying the following conditions:

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(i) $(M; \odot, 1)$ is a monoid,
(ii) $(M; \vee, \wedge, 0, 1)$ is a bounded lattice,
(iii) $x \odot y \leqslant z$ iff $x \leqslant y \rightarrow z$ iff $y \leqslant x \rightsquigarrow z$ for any $x, y \in M$.

In what follows, by a *residuated lattice* we will mean a bounded integral residuated lattice. If the operation "$\odot$" on a residuated lattice M is commutative then M is called a *commutative residuated lattice.*

In a residuated lattice M we define two unary operations "$^-$" and "$^\sim$" on M such that $x^- := x \rightarrow 0$ and $x^\sim := x \rightsquigarrow 0$ for each $x \in M$.

Recall that the above mentioned algebras of many-valued and fuzzy logics are characterized in the class of residuated lattices as follows:

A residuated lattice M is
(a) a pseudo MTL-algebra if M satisfies the identities of pre-linearity
(iv) $(x \rightarrow y) \vee (y \rightarrow x) = 1 = (x \rightsquigarrow y) \vee (y \rightsquigarrow x)$;
(b) an Rl-monoid if M satisfies the identities of divisibility
(v) $(x \rightarrow y) \odot x = x \wedge y = y \odot (y \rightsquigarrow x)$;
(c) a pseudo BL-algebra if M satisfies both (iv) and (v);
(d) a GMV-algebra (or equivalently a pseudo MV-algebra) if M satisfies (iv), (v) and the identities
(vi) $x^{-\sim} = x = x^{\sim-}$;
(e) a Heyting algebra if the operations "$\odot$" and "$\wedge$" coincide.

A residuated lattice M is called *good*, if M satisfies the identity $x^{-\sim} = x^{\sim-}$. For example, every commutative residuated lattice, every GMV-algebra and every pseudo BL-algebra which is a subdirect product of linearly ordered pseudo BL-algebras [7] are good.

By [4], every good residuated lattice satisfies the identity $(x^- \odot y^-)^\sim = (x^\sim \odot y^\sim)^-$. If M is good, we define a binary operation "$\oplus$" on M as

$$x \oplus y = (y^- \odot x^-)^\sim.$$

In the following proposition we recall some necessary basic properties of residuated lattices.

Proposition 1 ([1], [4], [14], [18]). *Let M be a residuated lattice. For all $x, y, z \in M$ we have*
(1) $x \odot y \leqslant x \wedge y$,
(2) $x \leqslant y \Longrightarrow x \odot z \leqslant y \odot z,\ z \odot x \leqslant z \odot y$,
(3) $x \leqslant y \Longrightarrow z \rightarrow x \leqslant z \rightarrow y,\ z \rightsquigarrow x \leqslant z \rightsquigarrow y$,
(4) $x \leqslant y \Longrightarrow x \rightarrow z \geqslant y \rightarrow z,\ x \rightsquigarrow z \geqslant y \rightsquigarrow z$,
(5) $(x \odot y) \rightarrow z = x \rightarrow (y \rightarrow z),\ (y \odot x) \rightsquigarrow z = x \rightsquigarrow (y \rightsquigarrow z)$,

(6) $(y \to z) \odot (x \to y) \leqslant x \to z$, $(x \rightsquigarrow y) \odot (y \rightsquigarrow z) \leqslant x \rightsquigarrow z$,

(7) $x \leqslant x^{-\sim}$, $x \leqslant x^{\sim -}$,

(8) $x^{-\sim -} = x^{-}$, $x^{\sim - \sim} = x^{\sim}$,

(9) $x \leqslant y \Longrightarrow y^{-} \leqslant x^{-}$, $y^{\sim} \leqslant x^{\sim}$,

(10) $x \odot (x \rightsquigarrow y) \leqslant y$, $(x \to y) \odot x \leqslant y$,

(11) $y \leqslant x \to y$, $y \leqslant x \rightsquigarrow y$,

(12) $x \to y \leqslant y^{-} \to x^{-}$, $x \to y \leqslant y^{\sim} \rightsquigarrow x^{\sim}$.

Moreover, if M is good, then

(13) $(x \odot y)^{-} = x \to y^{-}$.

(14) $x^{-\sim} \oplus y^{-\sim} = x^{-\sim} \oplus y = x \oplus y^{-\sim} = x \oplus y$,

(15) $x \oplus 0 = x^{-\sim} = 0 \oplus x$,

(16) $x \oplus y = x^{-} \rightsquigarrow y^{-\sim} = y^{\sim} \to x^{-\sim}$,

(17) $y \oplus x^{-} = x \to y^{-\sim}$, $x^{\sim} \oplus y = x \rightsquigarrow y^{-\sim}$,

(18) $(x \oplus y) \oplus 0 = x \oplus y$,

(19) $x \leqslant y \Longrightarrow z \oplus x \leqslant z \oplus y$, $x \oplus z \leqslant y \oplus z$,

(20) $\oplus$ *is associative.*

Definition. Let M be a residuated lattice. A mapping $f\colon M \longrightarrow M$ is called a *modal operator* on M if for any $x, y \in M$

(M1) $x \leqslant f(x)$,

(M2) $f(f(x)) = f(x)$,

(M3) $f(x \odot y) = f(x) \odot f(y)$.

A modal operator f is called *monotone*, if for any $x, y \in M$

(M4) $x \leqslant y \Longrightarrow f(x) \leqslant f(y)$.

If M is a good residuated lattice and for any $x, y \in M$

(M5) $f(x \oplus y) = f(x \oplus f(y)) = f(f(x) \oplus y)$,

then f is called *strong*.

In all cases of Rl-monoids every modal operator is already monotone. However, in general residuated lattices the converse need not hold. The example below was given in [19].

Example 1. Let $X = (\{x/10 | 0 \leqslant x \leqslant 10, x \in Z\}, \wedge, \vee, 0, 1)$ be a bounded lattice where $x \wedge y = \min\{x, y\}$ and $x \vee y = \max\{x, y\}$. If we define operators $\odot$ and $\to$ on X as

$$x \odot y = \begin{cases} x & \text{if } y = 1, \\ y & \text{if } x = 1, \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad x \to y = \begin{cases} 1 & \text{if } x \leqslant y, \\ y & \text{if } x = 1, \\ 0.9 & \text{otherwise} \end{cases}$$

then it is easy to show that the structure $(X, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is a bounded commutative integral residuated lattice. We define an operator $f\colon X \to X$ by

$$f(x) = \begin{cases} 0 & \text{if } x = 0, \\ 1 - x & \text{if } 0 < x \leqslant 0.5, \\ x & \text{if } x > 0.5. \end{cases}$$

Although f is a modal operator it is not monotone, because we have $0.2 < 0.4$ but $f(0.2) = 0.8 \nleqslant 0.6 = f(0.4)$.

Now we will show examples of monotone modal operators.

Example 2. Let $M_1 = \{0, a, b, c, 1\}$. We define the operations $\odot$ and $\rightarrow$ on M_1 as follows:

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	a	a	a
b	0	a	b	a	b
c	0	a	a	c	c
1	0	a	b	c	1

$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	1	1	1
b	0	c	1	c	1
c	0	b	b	1	1
1	0	a	b	c	1

Then $M_1 = (M_1; \odot, \vee, \wedge, \rightarrow, 0, 1)$ is a commutative Rl-monoid which is both a BL-algebra and a Heyting algebra (i.e. a Gödel algebra). Since M_1 is commutative, we can also consider the operation $\oplus$.

Let now $f_1\colon M_1 \to M_1$ be the mapping such that $f_1(0) = 0, f_1(a) = f_1(b) = b$ and $f_1(c) = f_1(1) = 1$. Then f_1 is a strong monotone modal operator on M_1.

Example 3. Let $M_2 = \{0, a, b, c, 1\}$ and let the operations $\odot, \rightarrow, \rightsquigarrow$ on M_2 be defined as follows:

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	0	0	a	a
b	0	a	b	a	b
c	0	0	0	c	c
1	0	a	b	c	1

$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	c	1	1	1	1
b	c	c	1	c	1
c	0	b	b	1	1
1	0	a	b	c	1

$\rightsquigarrow$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	1	1	1
b	0	c	1	c	1
c	b	b	b	1	1
1	0	a	b	c	1

Then $M_2 = (M_2; \odot, \vee, \wedge, \rightarrow, \rightsquigarrow, 0, 1)$ is a non-commutative residuated lattice which is a pseudo MTL-algebra but not an Rl-monoid beause $(b \rightarrow a) \odot b = c \odot b = 0 \neq a = a \wedge b$. (Notice that the lattices $(M_1; \vee, \wedge)$ and $(M_2; \vee, \wedge)$ are isomorphic.)

Let us consider the mapping $f_2\colon M_2 \to M_2$ such that $f_2(0) = f_2(a) = f_2(b) = b$ and $f_2(c) = f_2(1) = 1$. Then f_2 is a monotone modal operator on M_2.

Since $a^{-\sim} = b \neq c = a^{\sim -}$, the residuated lattice M_2 is not good, hence the addition on M_2 does not exist.

E x a m p l e 4. Let $M_3 = \{0, a, b, c, 1\}$. We define operations $\odot, \to, \rightsquigarrow$ as follows:

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	a	a	a
b	0	a	a	b	b
c	0	a	a	c	c
1	0	a	b	c	1

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	1	1	1
b	0	c	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

$\rightsquigarrow$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	1	1	1
b	0	b	1	1	1
c	0	b	b	1	1
1	0	a	b	c	1

Then $\mathbf{M}_3 = (M_3; \odot, \vee, \wedge, \to, \rightsquigarrow, 0, 1)$ is a linearly ordered (non-commutative) residuated lattice, which is a pseudo MTL-algebra. Since $c \odot (c \rightsquigarrow b) = c \odot 1 = c \neq b = b \wedge c$, $\mathbf{M}_3$ is not an Rl-monoid.

Let $f_3\colon M_3 \to M_3$ be the mapping such that $f_3(0) = f_3(a) = a$, $f_3(b) = b$, $f_3(c) = c$ and $f_3(1) = 1$. Then f_3 is a monotone modal operator on $\mathbf{M}_3$. Moreover, the residuated lattice $\mathbf{M}_3$ is good, hence the operation $\oplus$ exists and one can easily see that the operator f_3 is strong.

R e m a r k. Recall [22] that the notion of a modal operator has its main source in the theory of topoi and sheafification (see [13], [20], [21], [28]). Moreover, modal operators have come also from the theory of frames, where frame maps can be recognized as modal operators on a complete Heyting algebra (see [6]). Therefore the modal operators do not have direct and explicit connections to modal logics. Moreover, modal operators have some diferent properties than e.g. the logic operator "necessarily". Among other, we show that for every modal operator f on any good residuated lattice satisfying the identity $x^{-\sim} = x$, $f(0) = 0$ if and only if f is the identity.

Proposition 2. *Let $\mathbf{M}$ be a residuated lattice. If f is a monotone modal operator on $\mathbf{M}$ and $x, y \in M$, then*

(i) $f(x \to y) \leqslant f(x) \to f(y) = f(f(x) \to f(y)) = x \to f(y) = f(x \to f(y))$, $f(x \rightsquigarrow y) \leqslant f(x) \rightsquigarrow f(y) = f(f(x) \rightsquigarrow f(y)) = x \rightsquigarrow f(y) = f(x \rightsquigarrow f(y))$,

(ii) $f(x) \leqslant (x \rightsquigarrow f(0)) \to f(0)$, $f(x) \leqslant (x \to f(0)) \rightsquigarrow f(0)$,

(iii) $x^{-} \odot f(x) \leqslant f(0)$, $f(x) \odot x^{\sim} \leqslant f(0)$,

(iv) $f(x \vee y) = f(x \vee f(y)) = f(f(x) \vee f(y))$.

Moreover, if $\mathbf{M}$ is good, then for any $x \in M$

(v) $x \oplus f(0) \geqslant f(x^{-\sim}) \geqslant f(x)$, $f(0) \oplus x \geqslant f(x^{-\sim}) \geqslant f(x)$.

P r o o f. (i) By Proposition 1 (10), $(x \to y) \odot x \leqslant y$. It follows immediately that $f((x \to y) \odot x) = f(x \to y) \odot f(x) \leqslant f(y)$. Thus we have $f(x \to y) \leqslant f(x) \to f(y)$. By Proposition 1, $f(x) \to f(y) \leqslant x \to f(y) \leqslant f(x \to f(y)) \leqslant f(x) \to f(f(y)) = f(x) \to f(y)$, therefore $f(x) \to f(y) = x \to f(y) = f(x \to f(y))$.

Moreover, $f(x) \to f(y) \leqslant f(f(x) \to f(y)) \leqslant f(f(x)) \to f(f(y)) = f(x) \to f(y)$, which implies that $f(f(x) \to f(y)) = f(x) \to f(y)$. The proof can be done similarly for "$\rightsquigarrow$".

(ii) By (i), $f(x) \rightsquigarrow f(0) = x \rightsquigarrow f(0)$ and by Proposition 1 (10), $f(x) \odot (f(x) \rightsquigarrow f(0)) \leqslant f(0)$. Thus we have $f(x) \leqslant (f(x) \rightsquigarrow f(0)) \to f(0) = (x \rightsquigarrow f(0)) \to f(0)$.

(iii) Since $0 \leqslant f(0)$, it follows that $x^- = x \to 0 \leqslant x \to f(0) = f(x) \to f(0)$. Therefore $x^- \odot f(x) \leqslant f(0)$. In a similar way we get $f(x) \odot x^\sim \leqslant f(0)$.

(iv) By the monotony of f we get $f(x \vee y) \leqslant f(x \vee f(y)) \leqslant f(f(x) \vee f(y)) \leqslant f(f(x \vee y)) = f(x \vee y)$.

(v) By Proposition 1 and by (i), $x \oplus f(0) = x^- \rightsquigarrow f(0)^{-\sim} \geqslant x^- \rightsquigarrow f(0) = f(x^- \rightsquigarrow f(0)) \geqslant f(x^- \rightsquigarrow 0) = f(x^{-\sim}) \geqslant f(x)$.

Analogously we prove the remaining inequalities. □

Proposition 3. *If M is a good residuated lattice and f is a strong monotone modal operator on M, then for any $x, y \in M$*

(i) $f(x \oplus y) = f(f(x) \oplus f(y))$,

(ii) $x \oplus f(0) = f(x^{-\sim}) = f(0) \oplus x$.

P r o o f. (i) Obvious.

(ii) Since f is strong, we have $f(x \oplus f(0)) = f(x \oplus 0) = f(x^{-\sim})$. This means that by Proposition 2 (v), $f(x^{-\sim}) = f(x \oplus f(0)) \geqslant x \oplus f(0) \geqslant f(x^{-\sim})$. The proof of $f(x^{-\sim}) = f(0) \oplus x$ follows in the same manner. □

Proposition 4. *Let M be a good residuated lattice and f a monotone modal operator on M.*

(1) *If for any $x \in M$ we have $x \oplus f(0) = f(x \oplus 0)$, then*

  a) $f(x) \oplus f(0) = x \oplus f(0)$,

  b) $f(0) \oplus f(x) = f(0) \oplus x$.

(2) *If for any $x \in M$ we have $f(0) \oplus x = f(0 \oplus x)$, then*

  a) $f(x) \oplus f(0) = f(0) \oplus x$,

  b) $f(x) \oplus f(0) = x \oplus f(0)$.

P r o o f. Let f be a monotone modal operator on a good residuated lattice M.

(1) It follows from Proposition 2 (v) that $f(x) \leqslant x \oplus f(0)$. Thus $f(x) \oplus f(0) \leqslant x \oplus f(0) \oplus f(0)$. By the assumption, we have $f(0) \oplus f(0) = f(f(0) \oplus 0) = f(0 \oplus f(0)) = f(f(0 \oplus 0)) = f(0 \oplus 0) = f(0)$. Therefore $f(x) \oplus f(0) \leqslant x \oplus f(0)$. Conversely, it is obvious that $x \oplus f(0) \leqslant f(x) \oplus f(0)$. Thus we get $f(x) \oplus f(0) = x \oplus f(0)$. It can be shown in a similar manner that $f(0) \oplus f(x) = f(0) \oplus x$.

(2) Analogously. □

From the above proposition we get a characterization of strong modal operators.

Proposition 5. *Let f be a monotone modal operator on a good residuated lattice M. Then it is strong if and only if for any $x \in M$*

$$x \oplus f(0) = f(x^{-\sim}) = f(0) \oplus x.$$

P r o o f. If f is strong, then by Proposition 3 (ii) $x \oplus f(0) = f(x^{-\sim}) = f(0) \oplus x$.

Conversely, suppose that $x \oplus f(0) = f(x^{-\sim}) = f(x \oplus 0)$. By Proposition 1 (18), $x \oplus y = x \oplus y \oplus 0$ holds for all $x, y \in M$, and by Proposition 4 we have

$$\begin{aligned} f(x \oplus f(y)) &= f((x \oplus f(y)) \oplus 0) \\ &= x \oplus f(y) \oplus f(0) \\ &= x \oplus y \oplus f(0) \\ &= f(x \oplus y \oplus 0) \\ &= f(x \oplus y). \end{aligned}$$

By Proposition 4 we can find in the same manner that $f(f(x) \oplus y) = f(x \oplus y)$. Therefore f is a strong modal operator. $\square$

Theorem 6. *Let M be a residuated lattice and $f\colon M \longrightarrow M$ a mapping. Then f is a monotone modal operator on M if and only if we have for any $x, y \in M$:*

(i) $x \to f(y) = f(x) \to f(y)$,

(ii) $x \rightsquigarrow f(y) = f(x) \rightsquigarrow f(y)$,

(iii) $f(x) \odot f(y) \geqslant f(x \odot y)$.

P r o o f. Suppose a mapping f satisfies (i)–(iii). We will show that f also satisfies the conditions (M1)–(M4) from the definition of a monotone modal operator.

(M1) By (i), $x \to f(x) = f(x) \to f(x) = 1$, which implies that $x \leqslant f(x)$.

(M2) Since $1 = f(x) \to f(x) = f(f(x)) \to f(x)$, it follows that $f(f(x)) \leqslant f(x)$, thus by (1) we have $f(f(x)) = f(x)$.

(M3) By (M1) , $x \odot y \leqslant f(x \odot y)$, and it follows that $y \leqslant x \rightsquigarrow f(x \odot y) = f(x) \rightsquigarrow f(x \odot y)$ and $f(x) \odot y \leqslant f(x \odot y)$. Thus we get $f(x) \leqslant y \to f(x \odot y) = f(y) \to f(x \odot y)$ and $f(x) \odot f(y) \leqslant f(x \odot y)$. Therefore $f(x) \odot f(y) = f(x \odot y)$.

(M4) Note that if $x \leqslant y$, then $x \leqslant f(y)$. From the fact that $1 = x \to f(y) = f(x) \to f(y)$ we obtain $f(x) \leqslant f(y)$. $\square$

In general, if f is a monotone modal operator, the equation $f(0) = 0$ need not hold. An example is shown in [19]. Thus we will investigate under which condition this equality holds.

Proposition 7. *Let M be a residuated lattice and f a monotone modal operator. Then the following conditions are equivalent.*

(i) $f(0) = 0$,

(ii) $f(x^{\sim}) = x^{\sim}$, *for all* $x \in M$,

(iii) $f(x^{-}) = x^{-}$, *for all* $x \in M$.

Proof. (i) $\Longrightarrow$ (ii): Suppose that $f(0) = 0$. It follows from Proposition 2 (ii) that $f(x) \leqslant (x \to f(0)) \rightsquigarrow f(0) = (x \to 0) \rightsquigarrow 0 = x^{-\sim}$. Therefore $f(x) \leqslant x^{-\sim}$ and $f(x^{\sim}) \leqslant (x^{\sim})^{-\sim} = x^{\sim}$. Since $x^{\sim} \leqslant f(x^{\sim})$, we have that $f(x^{\sim}) = x^{\sim}$ for all $x \in M$.

(ii) $\Longrightarrow$ (i): Suppose that $f(x^{\sim}) = x^{\sim}$ for all $x \in M$. Then we get $f(0) = f(1^{\sim}) = 1^{\sim} = 0$.

It can be proved in a similar manner that (i) $\Longrightarrow$ (iii) and (iii) $\Longrightarrow$ (i). $\square$

Corollary 8. *Let M be a good residuated lattice satisfying $x^{-\sim} = x$ for all $x \in M$. Let f be a monotone modal operator on M such that $f(0) = 0$. Then f is the identity on M.*

A residuated lattice M is called *normal* if it satisfies the identities

$$(x \odot y)^{-\sim} = x^{-\sim} \odot y^{-\sim},$$
$$(x \odot y)^{\sim -} = x^{\sim -} \odot y^{\sim -}.$$

For example, every Heyting algebra and every good pseudo BL-algebra is normal [27], [8].

Proposition 9 ([25]). *Let M be a good and normal residuated lattice. Then for any $x, y \in M$*

(i) $(x \oplus y)^{-} = y^{-} \odot x^{-}$, $(x \oplus y)^{\sim} = y^{\sim} \odot x^{\sim}$,

(ii) $x^{-} \oplus y^{-} = (y \odot x)^{-}$, $x^{\sim} \oplus y^{\sim} = (y \odot x)^{\sim}$.

Denote by

$$I(M) = \{a \in M; a \odot a = a\}$$

the set of all multiplicative idempotents in a residuated lattice M. Clearly $0, 1 \in M$.

Proposition 10. *Let M be a good and normal residuated lattice. Then the following conditions are equivalent.*

(i) $a^{-} \in I(M)$,

(ii) $a^{\sim} \in I(M)$,

(iii) $a \oplus a = a^{-\sim}$.

P r o o f. (ii) $\Longleftrightarrow$ (iii): If $a^{\sim} \in I(M)$, then $a \oplus a = (a^{\sim} \odot a^{\sim})^{-} = (a^{\sim})^{-} = a^{-\sim}$. Conversely, suppose that $a \oplus a = a^{-\sim}$. By Proposition 9 (i), we have $a^{\sim} = (a^{-\sim})^{\sim} = (a \oplus a)^{\sim} = a^{\sim} \odot a^{\sim}$. Therefore $a^{\sim} \in \mathrm{I}(M)$.

(i) $\Longleftrightarrow$ (iii): Analogously. $\square$

Let M be a good residuated lattice and $a \in M$. We denote by $\varphi_a \colon M \to M$ the mapping such that $\varphi_a(x) = a \oplus x$ for all $x \in M$.

Proposition 11. *Let M be a good and normal residuated lattice and let $a \in M$. If φ_a is a strong monotone modal operator on M, then $a^{-}, a^{\sim}, a^{-\sim} \in I(M)$.*

P r o o f. Since $\varphi_a(x \odot y) = \varphi_a(x) \odot \varphi_a(y)$, we have $a \oplus (x \odot y) = (a \oplus x) \odot (a \oplus y)$ for any $x, y \in M$. By setting $x = y = 0$, we obtain $a \oplus 0 = (a \oplus 0) \odot (a \oplus 0)$, thus $a^{-\sim} = a^{-\sim} \odot a^{-\sim}$, which implies that $a^{-\sim} \in I(M)$.

Further, $a \oplus (x \oplus y) = \varphi_a(x \oplus y) = \varphi_a(x \oplus \varphi_a(y)) = a \oplus (x \oplus (a \oplus y))$ for any $x, y \in M$. For $x = y = 0$ we have $a^{-\sim} = a \oplus 0 = a \oplus (0 \oplus 0) = a \oplus (0 \oplus (a \oplus 0)) = (a \oplus 0) \oplus a^{-\sim} = a^{-\sim} \oplus a^{-\sim}$, thus $a^{-\sim} = (a^{-} \odot a^{-})^{\sim}$. This implies that $a^{-} = (a^{-} \odot a^{-})^{\sim -} = a^{-\sim -} \odot a^{-\sim -} = a^{-} \odot a^{-}$ and so $a^{-} \in I(M)$.

Moreover, by Proposition 10, $a^{\sim} \in I(M)$. $\square$

Proposition 12. *If M is a good and normal residuated lattice and $a \in M$ is such that $a^{-}, a^{-\sim} \in I(M)$, then φ_a satisfies conditions* (M1), (M2), (M4) *from the definition of a strong monotone modal operator, and*

(M5′) $f(x \oplus y) = f(f(x) \oplus y)$.

Moreover, if a commutes with every $x \in M$, then φ_a satisfies (M5).

P r o o f. (M1) For any we have $x \in M$ $\varphi_a(x) = a \oplus x = (x^{-} \odot a^{-})^{\sim} \geqslant x^{-\sim} \geqslant x$.

(M2) Since $a^{-} \in I(M)$, we get $\varphi_a(\varphi_a(x)) = a \oplus (a \oplus x) = a \oplus x = \varphi_a(x)$.

(M4) If $x \leqslant y$, then $\varphi_a(x) = a \oplus x \leqslant a \oplus y = \varphi_a(y)$.

(M5′) Let $x, y \in M$. We have $\varphi_a(\varphi_a(x) \oplus y) = \varphi_a(a \oplus x \oplus y) = a \oplus a \oplus x \oplus y = a \oplus x \oplus y = \varphi_a(x \oplus y)$.

Now suppose that a commutes with every $x \in M$. For any $x, y \in M$ we get $\varphi_a(x \oplus \varphi_a(y)) = a \oplus (x \oplus (a \oplus y)) = ((a \oplus a) \oplus x) \oplus y = (a^{-\sim} \oplus x) \oplus y = a \oplus (x \oplus y) = \varphi_a(x \oplus y)$. $\square$

Proposition 13. *Let M be a good and normal residuated lattice and f a monotone modal operator on M such that $f(x) = f(x^{-\sim})$ for all $x \in M$. Then f is strong if and only if $f = \varphi_{f(0)}$ and $f(0)^{-} \in I(M)$.*

P r o o f. Let f be a monotone modal operator on M satisfying the identity $f(x) = f(x^{-\sim})$.

If f is strong then by Proposition 5, $f(x) = f(x^{-\sim}) = x \oplus f(0)$ for any $x \in M$, hence $f = \varphi_{f(0)}$ and therefore, by Proposition 11, $f(0)^{-}, f(0)^{-\sim} \in I(M)$.

Conversely, let f be any modal operator on M. Then $f(0)^{-\sim} = f(0 \odot 0)^{-\sim} = (f(0) \odot f(0))^{-\sim} = f(0)^{-\sim} \odot f(0)^{-\sim}$, thus $f(0)^{-\sim} \in I(M)$. Let now f be monotone, $f = \varphi_{f(0)}$ and $f(0)^{-} \in I(M)$. Then by Proposition 11 we get that f is strong. $\square$

Let M be a residuated lattice and $a \in I(M)$. Consider the mappings $\psi_a^1 \colon M \longrightarrow M$ and $\psi_a^2 \colon M \longrightarrow M$ such that $\psi_a^1(x) = a \to x$ and $\psi_a^2(x) = a \rightsquigarrow x$.

Proposition 14. *Let M be a good residuated lattice and $a \in I(M)$. Then for any $x, y \in M$*

(1) $\psi_a^1(x \oplus y) = \psi_a^1(x \oplus \psi_a^1(y))$,
(2) $\psi_a^1(x \oplus y) \leqslant \psi_a^1(\psi_a^1(x) \oplus y)$,
(3) $\psi_a^2(x \oplus y) = \psi_a^2(\psi_a^2(x) \oplus y)$,
(4) $\psi_a^2(x \oplus y) \leqslant \psi_a^2(x \oplus \psi_a^2(y))$.

Proof. (1) We have $y \leqslant a \to y = \psi_a^1(y)$, thus $\psi_a^1(x \oplus y) \leqslant \psi_a^1(x \oplus \psi_a^1(y))$.

To prove the converse inequality first note that since $(a \to x) \odot a \leqslant x$, we have $(a \to x) \odot (a \odot x^{\sim}) \leqslant x \odot x^{\sim} = 0$, hence $a \odot x^{\sim} \leqslant (a \to x)^{\sim}$. Thus we have $\psi_a^1(x \oplus \psi_a^1(y)) = \psi_a^1((\psi_a^1(y)^{\sim} \odot x^{\sim})^{-}) = a \to (\psi_a^1(y)^{\sim} \odot x^{\sim})^{-} = (a \odot \psi_a^1(y)^{\sim} \odot x^{\sim})^{-}$, hence $a \odot \psi_a^1(y)^{\sim} \odot x^{\sim} = a \odot (a \to y)^{\sim} \odot x^{\sim} \geqslant a \odot (a \odot y^{\sim}) \odot x^{\sim} = (a \odot a) \odot (y^{\sim} \odot x^{\sim}) = a \odot (y^{\sim} \odot x^{\sim})$, therefore $\psi_a^1(x \oplus \psi_a^1(y)) = (a \odot \psi_a^1(y)^{\sim} \odot x^{\sim})^{-} \leqslant (a \odot y^{\sim} \odot x^{\sim})^{-} = a \to (y^{\sim} \odot x^{\sim})^{-} = a \to (x \oplus y) = \psi_a^1(x \oplus y)$, i.e. $\psi_a^1(x \oplus \psi_a^1(y)) \leqslant \psi_a^1(x \oplus y)$.

(2) Since $x \leqslant a \to x = \psi_a^1(x)$, we get $x \oplus y \leqslant \psi_a^1(x) \oplus y$, thus $\psi_a^1(x \oplus y) \leqslant \psi_a^1(\psi_a^1(x) \oplus y)$.

(3) We have $x \leqslant a \rightsquigarrow x = \psi_a^2(x)$, hence $x \oplus y \leqslant \psi_a^2(x) \oplus y$, and so $\psi_a^2(x \oplus y) \leqslant \psi_a^2(\psi_a^2(x) \oplus y)$. Further, since $a \odot (a \rightsquigarrow y) \leqslant y$, we get $(y^{-} \odot a) \odot (a \rightsquigarrow y) \leqslant y^{-} \odot y = 0$, and so $y^{-} \odot a \leqslant (a \rightsquigarrow y)^{-}$.

We have $\psi_a^2(\psi_a^2(x) \oplus y) = \psi_a^2((y^{-} \odot \psi_a^2(x)^{-})^{\sim}) = a \rightsquigarrow (y^{-} \odot \psi_a^2(x)^{-})^{\sim} = ((y^{-} \odot \psi_a^2(x)^{-} \odot a)^{\sim}$, hence $y^{-} \odot \psi_a^2(x)^{-} \odot a = y^{-} \odot (a \rightsquigarrow x)^{-} \odot a \geqslant y^{-} \odot (x^{-} \odot a) \odot a = y^{-} \odot x^{-} \odot a$, thus $\psi_a^2(\psi_a^2(x) \oplus y) = (y^{-} \odot \psi_a^2(x)^{-} \odot a)^{\sim} \leqslant (y^{-} \odot x^{-} \odot a)^{\sim} = ((y^{-} \odot x^{-}) \odot a)^{\sim} = a \rightsquigarrow (x \oplus y) = \psi_a^2(x \oplus y)$. Therefore $\psi_a^2(x \oplus y) = \psi_a^2(\psi_a^2(x) \oplus y)$.

(4) Similarly to (2). $\square$

Proposition 15. *If M and a are as in Proposition 14 and, moreover, a commutes with every element in M, then in (2) and (4) we have equalities.*

Proof. (2) We have $\psi_a^1(\psi_a^1(x) \oplus y) = \psi_a^1((y^{\sim} \odot \psi_a^1(x)^{\sim})^{-}) = a \to (y^{\sim} \odot \psi_a^1(x)^{\sim})^{-} = (a \odot y^{\sim} \odot \psi_a^1(x)^{\sim})^{-}$ by Proposition 1 (13), hence $a \odot y^{\sim} \odot \psi_a^1(x)^{\sim} =$

$a \odot y^{\sim} \odot (a \to x)^{\sim} \geqslant a \odot y^{\sim} \odot (a \odot x^{\sim}) = (a \odot a) \odot (y^{\sim} \odot x^{\sim}) = a \odot (y^{\sim} \odot x^{\sim})$, and similarly to the proof of (1) in Proposition 14 we get $\psi_a^1(\psi_a^1(x) \oplus y) \leqslant \psi_a^1(x \oplus y)$.

(4) Analogously as for (2). $\square$

Corollary 16. *If M is a commutative residuated lattice or M is a bounded* Rl-*monoid (not necessarily commutative), and $a \in I(M)$, then in* (2) *and* (4) *we have equalities.*

Proof. For bounded Rl-monoids see [26]. $\square$

Corollary 17. *If $a \in M$ satisfies the conditions from Proposition* 15 *or Corollary* 16, *and ψ_a^1 and ψ_a^2 are monotone modal operators on M, then they are strong.*

Let M be a residuated lattice and f a modal operator on M. We denote by

$$\mathrm{Fix}(f) = \{x \in M; f(x) = x\}$$

the set of all fixed elements of the operator f. By the definition of a modal operator it is obvious that $\mathrm{Fix}(f) = \mathrm{Im}(f)$.

Proposition 18. *If f is a monotone modal operator on a residuated lattice M, then* $\mathrm{Fix}(f) = (\mathrm{Fix}(f); \odot, \vee_{\mathrm{Fix}(f)}, \wedge, \to, \rightsquigarrow, f(0), 1)$, *where $x \vee_{\mathrm{Fix}(f)} y = f(x \vee y)$ for any $x, y \in \mathrm{Fix}(f)$, and $\wedge, \to, \rightsquigarrow$ are the restrictions of the binary operations from M to* $\mathrm{Fix}(f)$, *is a residuated lattice.*

Proof. Let M be a residuated lattice and f a monotone modal operator on M.

(i) If $x, y \in \mathrm{Fix}(f)$, then $f(x \odot y) = f(x) \odot f(y) = x \odot y$, thus $x \odot y \in \mathrm{Fix}(f)$. Therefore $(\mathrm{Fix}(f); \odot, 1)$ is a residuated lattice.

(ii) Since f is a closure operator on the lattice $(M; \vee, \wedge)$, it follows that $x \wedge y \in \mathrm{Fix}(f)$ for each $x, y \in \mathrm{Fix}(f)$ and $x \vee_{\mathrm{Fix}(f)} y = f(x \vee y)$. Therefore $(\mathrm{Fix}(f); \wedge, f(0), 1)$ is a bounded lattice.

(iii) Let $x, y \in \mathrm{Fix}(f)$. Then by Proposition 2, $x \to y = f(x) \to f(y) = f(f(x) \to f(y)) = f(x \to y)$, hence $x \to y \in \mathrm{Fix}(f)$. Analogously $x \rightsquigarrow y \in \mathrm{Fix}(f)$.

(iv) Now, let $x, y, z \in \mathrm{Fix}(f)$. Then $x \odot y, y \to z, x \rightsquigarrow z \in \mathrm{Fix}(f)$, hence $x \odot_{\mathrm{Fix}(f)} y \leqslant z$ iff $x \leqslant y \to_{\mathrm{Fix}(f)} z$ iff $y \leqslant x \rightsquigarrow_{\mathrm{Fix}(f)} z$. $\square$

Conclusions. In the paper we have investigated monotone modal operators, which are special cases of closure operators on bounded integral residuated lattices. The results are applicable to a wide class of algebras containing algebras of some algebras behind many-valued and fuzzy logics. One can expect that these results will also be useful for studying analogous operators on further classes of algebras, e.g. on algebras of several quantum logics.

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