# ALMOST PERIODIC SOLUTIONS WITH A PRESCRIBED SPECTRUM OF LINEAR AND QUASILINEAR DIFFERENTIAL EQUATIONS WITH ALMOST PERIODIC COEFFICIENTS AND CONSTANT TIME LAG (NEUTRAL DIFFERENTIAL EQUATIONS)

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(Received October 6, 2010)

*Abstract.* The paper is the extension of the author's previous papers and solves more complicated problems. Almost periodic solutions of a certain type of almost periodic linear or quasilinear systems of neutral differential equations are dealt with.

*Keywords*: almost periodic function, Fourier coefficient, Fourier exponent, spectrum of almost periodic function, almost periodic system of differential equations, formal almost periodic solution, almost periodic solution, distance of two spectra, time lag, neutral differential equation

*MSC 2010*: 34C27, 34K40, 42A75

## 1. Introduction

**1.1. Preliminaries.** This article generalizes the method developed in [6] to some almost periodic cases of neutral differential equations (i.e. differential equations with time lag containing beside the highest derivative without the time lag also the highest derivative with the time lag). For analogous results, procedures and proofs we will often refer to [6]. In what follows all criteria of existence and uniqueness as well as estimates concern complex matrix (Bohr's uniformly) almost periodic functions.

**1.2. Notation and definitions.** The symbol $\mathbb{N}$ denotes the set of all positive integers, $\mathbb{N}_0$ the set of all non-negative integers, $\mathbb{R}$ the set of all real numbers (real axis), $\mathbb{C}$ the set of all complex numbers (complex plane).

If $\mathbb{E}$ is a non-void set and $m$, $n$ are from $\mathbb{N}$, then $\mathbb{E}^m$ denotes the Cartesian product $\mathbb{E} \times \ldots \times \mathbb{E}$ of $m$ factors and $\mathbb{E}^{m\times n}$ is the set of all matrices of $m$ rows and $n$ columns, the elements of which belong to $\mathbb{E}$; $\mathbb{E}^{1\times 1} = \mathbb{E}^1 = \mathbb{E}$. Analogously we could denote more-dimensional matrices.

If $n \in \mathbb{N}$ and $\overline{m} = (m_1, \dots, m_n) \in \mathbb{N}_0^{1\times n}$, $\overline{m}' = (m_1', \dots, m_n') \in \mathbb{N}_0^{1\times n}$, then the inequality $\overline{m} \leqslant \overline{m}'$ means the system of inequalities $m_j \leqslant m_j'$, $j = 1, \dots, n$.

If $\mathcal{M}$, $\mathcal{N}$ are non-void subsets of $\mathbb{C}$ or $\mathbb{R}$ and if $\omega$, $\xi$ are complex numbers, then $\omega\mathcal{M} = \{\omega\lambda\colon\ \lambda \in \mathcal{M}\}$, $\xi + \mathcal{N} = \{\xi + \mu\colon\ \mu \in \mathcal{N}\}$, $\mathcal{M} + \mathcal{N} = \{\lambda + \mu\colon\ \lambda \in \mathcal{M}, \mu \in \mathcal{N}\}$, $\emptyset + \mathcal{N} = \mathcal{M} + \emptyset = \emptyset$ and $S(\mathcal{M})$ stands for the smallest additive semigroup containing $\mathcal{M}$ and such that $S(\emptyset) = \emptyset$.

The distance of two non-void sets $\mathcal{M}$, $\mathcal{N}$, of a point $z$ and a non-void set $\mathcal{N}$ and of two points $z$, $w$ in $\mathbb{C}$ or $\mathbb{R}$, is denoted by $\operatorname{dist}[\mathcal{M},\mathcal{N}]$, $\operatorname{dist}[z,\mathcal{N}]$ and $\operatorname{dist}[z,w]$, respectively. The boundary of a set $\mathcal{M}$ is denoted by $\partial M$.

If $\alpha$ is a positive number then by a strip or an $\alpha$-strip in the complex plane we mean the set $\pi(\alpha) = \{z \in \mathbb{C}\colon\ |\Re z| \leqslant \alpha\}$. If $z_0 \in \mathbb{C}$ and $R \in (0, +\infty)$ then $\varkappa(z_0, R)$, $\overline{\varkappa}(z_0, R)$ and $K(z_0, R)$ denote an open disc, a closed disc and a circle centered at $z_0$ with its radius $R$ in the complex plane, respectively.

For number vectors or matrices, even more-dimensional, we use the norm $|\cdot|$ which is equal to the sum of absolute values of all coordinates of the vector or all elements of the matrix.

In addition to the usual symbol $\prod\limits_{j=1}^{k} a_j = a_1 \dots a_k$ for a product we will use the symbol $\prod\limits_{j=k}^{1} a_j = a_k \dots a_1$ for the product with a reversed order of factors.

For a vector $\overline{m} = (m_1, \dots, m_M) \in \mathbb{N}_0^{1\times M}$, $M \in \mathbb{N}$, we introduce the combinatorial number

$$\binom{|\overline{m}|}{\overline{m}} = \frac{|\overline{m}|!}{(m_1!)\dots(m_M!)} \quad \text{where} \quad |\overline{m}| = m_1 + \dots + m_M.$$

**1.3. Spaces and the starting problem.** We will deal with functions $f\colon\ \mathbb{R} \to \mathbb{X}$ where $\mathbb{X}$ is one of the spaces $\mathbb{E}$, $\mathbb{E}^m$, $\mathbb{E}^{m\times n}$ and $\mathbb{E} = \mathbb{R}$ or $\mathbb{E} = \mathbb{C}$. If all elements of an almost periodic matrix function $f\colon\ \mathbb{R} \to \mathbb{X}$ are trigonometric polynomials then $f$ is called a trigonometric or an $\mathbb{X}$-trigonometric polynomial. Here and hereafter the symbol 0 denotes both the zero number and the zero element in $\mathbb{X}$. The meaning of the symbol 0 is always clear from the context.

We denote by $C(\mathbb{X})$, $BC(\mathbb{X})$ and $AP(\mathbb{X})$ the space of all functions $f\colon\ \mathbb{R} \to \mathbb{X}$ continuous on $\mathbb{R}$, the space of all functions from $C(\mathbb{X})$ bounded on $\mathbb{R}$ and the space of all functions from $BC(\mathbb{X})$ (equivalently from $C(\mathbb{X})$) that are almost or $\mathbb{X}$-almost periodic, respectively. ($\mathbb{X}$-almost periodic functions are functions that can be approximated by $\mathbb{X}$-trigonometric polynomials with arbitrary accuracy on $\mathbb{R}$, i.e., for any given $f \in AP(\mathbb{X})$ and $\varepsilon > 0$ there exists an $\mathbb{X}$-trigonometric polynomial $\mathcal{T}_\varepsilon$ such that $|f(t) - \mathcal{T}_\varepsilon(t)|_{\mathbb{X}} \leqslant \varepsilon$ holds for every $t \in \mathbb{R}$.)

The spaces $BC(\mathbb{X})$ and $AP(\mathbb{X})$ are made Banach spaces (B-spaces) with the norm defined by

$$|f| = \sup\{|f(t)|_{\mathbb{X}}\colon\ t \in \mathbb{R}\}. \tag{1.1}$$

For any given positive integer $k$ we will denote by $C^k(\mathbb{X})$, $BC^k(\mathbb{X})$ and $AP^k(\mathbb{X})$ the space of all functions from $C(\mathbb{X})$ with derivatives up to the order $k$ on $\mathbb{R}$ which belong to $C(\mathbb{X})$, the space of all functions from $BC(\mathbb{X})$ with derivatives up to the order $k$ on $\mathbb{R}$ which belong to $BC(\mathbb{X})$, and the space of all functions from $AP(\mathbb{X})$ with derivatives up to the order $k$ on $\mathbb{R}$ which belong to $AP(\mathbb{X})$, respectively.

The spaces $BC^1(\mathbb{X})$ and $AP^1(\mathbb{X})$ endowed with the norm

$$\|f\| = \max\{|f|, |\dot{f}|\} \tag{1.2}$$

become B-spaces.

If $f \in AP(\mathbb{X})$ then

$$\lim_{T\to+\infty} \frac{1}{T}\int_0^T f(t)\,\mathrm{d}t = \lim_{T\to+\infty} \frac{1}{T}\int_s^{s+T} f(t)\,\mathrm{d}t \quad \text{for any } s \in \mathbb{R}$$

exists uniformly with respect to the parameter $s \in \mathbb{R}$. It is called the mean value of the function $f$ and denoted by $M(f)$ or $M_t\{f(t)\}$.

For any $f \in AP(\mathbb{X})$ the function

$$a(\lambda) = a(\lambda, f) = M_t\{f(t)\exp(-\mathrm{i}\lambda t)\}, \quad \lambda \in \mathbb{R}$$

is called the Bohr transform of the function $f$. If $a(\lambda) \neq 0$ then $\lambda$ is called the Fourier exponent and $a(\lambda)$ is called the Fourier coefficient of the function $f$. The set of all Fourier exponents of the function $f$ will be denoted by $\Lambda_f$ and the set $\mathrm{i}\Lambda_f$ will be called the spectrum of $f$. These sets are at most countable (finite or can be arranged into a sequence). For any $\mathbb{X}$-almost periodic function $f$ the trigonometric series

$$\sum_{\lambda\in\Lambda_f} a(\lambda)\exp(\mathrm{i}\lambda t)$$

is called the Fourier series of the function $f$. It is uniquely determined up to the order of summation. For any $\lambda \in \Lambda_f$ the inequality

$$|a(\lambda)|_{\mathbb{X}} = |a(\lambda, f)|_{\mathbb{X}} \leqslant |f|$$

is true. Further, we introduce the quantity

$$\sum(f) = \sum_{\lambda\in\Lambda_f} |a(\lambda)|_{\mathbb{X}}.$$

The condition $\sum(f) < +\infty$ ensures the absolute and uniform convergence on $\mathbb{R}$ of the Fourier series of the function $f$.

For any function $f$ from $AP(\mathbb{X})$ there exists a sequence of the so-called Bochner-Fejér approximation (trigonometric) polynomials $B_m$, $m = 1, 2, \ldots$, of the function $f$ with their spectra contained in $\mathrm{i}\Lambda_f$ and uniformly convergent to $f$ on $\mathbb{R}$ and, moreover, $\sum(B_m) \leqslant \sum(f)$, $m = 1, 2, \ldots$ (see [1], [4], [7]).

The starting problem solved in [5] (256–268) is to find an almost periodic solution with a certain prescribed spectrum of the almost periodic neutral differential equation with constant coefficients

$$\dot{x}(t) = a_0 x(t) + b_0 x(t-\tau) + c_0 \dot{x}(t-\tau) + f(t), \quad t \in \mathbb{R}, \tag{1.3}$$

where $\tau$ is a positive constant, the so-called time lag, and $a_0$, $b_0$, $c_0$ belong to $\mathbb{C}^{n_0 \times n_0}$, $n_0 \in \mathbb{N}$, $f$ is a function from $AP(\mathbb{C}^{n_0 \times 1})$ and $x$ is a sought function (solution) from $C^1(\mathbb{C}^{n_0 \times 1})$. An important role is played by the properties of the matrix functions

$$\Omega = \Omega(z) = E - c_0 \exp(-z\tau), \quad z \in \mathbb{C}, \tag{1.4}$$

$$\Phi = \Phi(z) = z\Omega(z) - a_0 - b_0 \exp(-z\tau), \quad z \in \mathbb{C}, \tag{1.5}$$

where $E = E_{n_0}$ is the unit matrix from $\mathbb{C}^{n_0 \times n_0}$, by the properties of their determinants

$$\omega(z) = \det \Omega(z) \quad \text{and} \quad \Delta(z) = \det \Phi(z), \quad z \in \mathbb{C},$$

and by the equation $\Delta(z) = 0$, the so-called characteristic equation of (1.3). Under $\sigma(\Delta(z))$ we understand the set of all roots of the characteristic quasipolynomial $\Delta(z)$ in $\mathbb{C}$, which is a transcendent entire function (in general) of the complex variable $z$. Consequently, the quasipolynomial $\Delta(z)$ has an infinite number of (different) roots without any finite limit point for $z \in \mathbb{C}$. Suppose

$$\sigma(c_0) \cap K(0,1) = \emptyset, \tag{1.6}$$

where $\sigma(c_0)$ is the spectrum of the matrix $c_0$. Under this assumption there exists such a positive number $\delta_0 < 1/2$ that any characteristic number $\mu$ of the matrix $c_0$ satisfies

$$|\mu| \in [0,\, 1 - 2\delta_0) \cup (1 + 2\delta_0, +\infty) \tag{1.7}$$

(i.e. no $\mu$ satisfies the inequalities $1 - 2\delta_0 \leqslant |\mu| \leqslant 1 + 2\delta_0$).

If $\mu_1, \ldots, \mu_q$ are all mutually different characteristic numbers of the matrix $c_0$ and if $k_1, \ldots, k_q$ are their multiplicities, then

$$\omega(z) = \prod_{j=1}^{q} (1 - \mu_j \exp(-z\tau))^{k_j}, \quad z \in \mathbb{C}. \tag{1.8}$$

Consequently, we may choose a positive number $\alpha_0$ small enough such that for all $z$ from $\pi(\alpha_0)$ the inequalities

$$1 - \delta_0 \leqslant |\exp(\pm z\tau)| \leqslant 1 + \delta_0 \tag{1.9}$$

and at the same time

$$|\exp(\pm z\tau) - \mu_j| \geqslant \delta_0, \quad j = 1, \ldots, q, \tag{1.10}$$

are true. It is easy to prove that

$$|\omega(z)| \geqslant (1 - \delta_0)^k \delta_0^k > 0 \tag{1.11}$$

for $k = \sum\limits_{j=1}^{q} k_j$ and

$$|\Omega(z)| \leqslant |E| + |c_0|(1 + \delta_0) \tag{1.12}$$

in the strip $\pi(\alpha_0)$. Hence it follows that there exists a uniformly bounded $\Omega^{-1}(z)$ for every $z$ from $\pi(\alpha_0)$. For $z \in \pi(\alpha_0) \setminus \{0\}$ we obtain

$$\Phi(z)z^{-1} = \Omega(z) - (a_0 + b_0 \exp(-z\tau))z^{-1},$$

which means that for all sufficiently large (in absolute value) $z \in \pi(\alpha_0)$ the matrix $\Phi(z)z^{-1}$ is arbitrarily close to the regular matrix $\Omega(z)$ (for an arbitrarily given positive number $\varepsilon$ as an accuracy it suffices to choose $z \in \pi(\alpha_0) \setminus \varkappa(0, R_\varepsilon)$ with $R_\varepsilon \geqslant (|a_0| + |b_0|\delta_0)\varepsilon^{-1}$) and thus there exists a uniformly bounded inverse matrix $z\Phi^{-1}(z)$ for the matrix $\Phi(z)z^{-1}$ which is analytic and regular and uniformly bounded for such $z \in \pi(\alpha_0)$. Therefore the positive number $\alpha < \alpha_0/2$ can be chosen such that the set $\pi(2\alpha) \cap \sigma(\Delta(z))$ is finite and lies on the imaginary axis of the complex plane $\mathbb{C}$. Now we define the set

$$\theta = \begin{cases} \{\xi_j - \xi_k \colon\ j, k = 1, \ldots, j_0\} & \text{for } \pi(2\alpha) \cap \sigma(\Delta(z)) = \{\mathrm{i}\xi_1, \ldots, \mathrm{i}\xi_{j_0}\},\ j_0 \in \mathbb{N}, \\ \emptyset & \text{for } \pi(2\alpha) \cap \sigma(\Delta(z)) = \emptyset, \end{cases} \tag{1.13}$$

and the positive constant

$$d_\xi = \begin{cases} \min\{|\xi_j - \xi_k| \colon\ j \neq k;\ j, k = 1, \ldots, j_0\} & \text{for } j_0 > 1, \\ 2 & \text{for } j_0 = 1 \text{ or } \theta = \emptyset, \end{cases} \tag{1.14}$$

which will be needed in the sequel.

## 2. Equation with almost periodic coefficients

**2.1. Basic equations.** We shall study the differential equations

$$\dot{x}(t) = a_0 x(t) + b_0 x(t-\tau) + c_0 \dot{x}(t-\tau) + a(t)x(t) + b(t)x(t-\tau) + f(t), \quad t \in \mathbb{R}, \tag{2.1}$$

where $a_0$, $b_0$, $c_0$, $f$, $x$ have the same meaning as in (1.3) and further, $\{a, b\} \subset AP(\mathbb{C}^{n_0 \times n_0})$ with $\sum(a) < +\infty$, $\sum(b) < +\infty$. Our aim is to prove the existence and uniqueness of an almost periodic solution of Equation (2.1) the spectrum of which is contained in a certain à priori given set $\mathrm{i}\Lambda$, $\Lambda \subset \mathbb{R}$. Such a solution will be called an almost periodic $\Lambda$-solution.

**2.2. Formal solutions.** First, we solve Equation (2.1) in a formal manner. This means that we are looking for the so-called formal almost periodic solution $x_f$ represented by a trigonometric series

$$x_f = x_f(t) \sim \sum_{\sigma} c(\sigma) \exp(\mathrm{i}\sigma t), \quad \sigma \in \Lambda(\subset \mathbb{R}),\ t \in \mathbb{R}, \tag{2.2}$$

with coefficients $c(\sigma)$ from $\mathbb{C}^{n_0 \times 1}$ which formally satisfies the considered equation, where $\Lambda$ is an at most countable set of real numbers or in the case of an uncountable set $\Lambda$ the coefficients $c(\sigma)$ are non-zero only for $\sigma$ from an at most countable subset $\Lambda'$ of $\Lambda$. Such formal solution $x_f$ we gain by the substitution of the representing trigonometric series in Equation (2.1) and by formal execution of the denoted arithmetic, differential and integral operations, the shift and the mean value. The formality of these operations consists in applying such operations to every separate member of the proper trigonometric series without any regard to a convergence and any justification (as concerns the convergence) of the operations performed. The formal mean value of a trigonometric series is defined to be its absolute term. (The more detailed explanation of a formal solution is in [6].)

R e m a r k 2.1. Let us note that under the assumption of the appropriate convergence of the trigonometric series entering into the formal operations these formal operations coincide with the non-formal ones.

The above constructed matrix function $\Phi = \Phi(z)$, $z \in \mathbb{C}$, is not identical with the matrix function $\Phi = \Phi(z)$, $z \in \mathbb{C}$, in [5], [6], but it has the same properties ($\Phi$ and its inverse matrix $\Phi^{-1}$ are analytic, regular and bounded on a certain closed region in $\mathbb{C}$) needed for the construction of the (formal or non-formal) almost periodic $\Lambda$-solution $x_f$ of Equation (2.1) in a way analogous to [5], [6].

R e m a r k 2.2. The assertions presented without proofs in the sequel can be proved analogously to the corresponding analogous assertions in [6].

**2.3. Construction of formal solution.** We begin with the case when $a$, $b$ and $f$ are trigonometric polynomials.

**Theorem 2.1.** *If in Equation* (2.1) *$a$, $b$ are nonconstant trigonometric polynomials $a = a(t) = \sum_{k=1}^{M} \alpha(\mu_k)\exp(\mathrm{i}\mu_k t)$, $M(a) = 0$, $b = b(t) = \sum_{k=1}^{N} \beta(\nu_k)\exp(\mathrm{i}\nu_k t)$, $M(b) = 0$, $t \in \mathbb{R}$, where $M$, $N$ are from $\mathbb{N}$, and if $f$ is a (non-zero) trigonometric polynomial $f = f(t) = \sum_{\lambda} \varphi(\lambda)\exp(\mathrm{i}\lambda t)$, for $\lambda \in \Lambda_f$, $t \in \mathbb{R}$, and if (for $\theta$ see* (1.13)*)*

$$\Delta = \inf(\Lambda_a \cup \Lambda_b) > 0, \tag{2.3}$$

$$d_\theta = \begin{cases} \operatorname{dist}[\theta, S(\Lambda_a \cup \Lambda_b)] > 0 & \text{for } \theta \neq \emptyset, \\ 4 & \text{for } \theta = \emptyset, \end{cases} \tag{2.4}$$

$$d = \operatorname{dist}[\mathrm{i}\Lambda, \sigma(\Delta(z))] > 0, \tag{2.5}$$

$$\sigma(c_0) \cap K(0,1) = \emptyset, \tag{2.6}$$

*where $\Lambda = \Lambda_f + S(\Lambda_a \cup \Lambda_b \cup \{0\})$, then there exists a unique formal almost $\Lambda$-solution $x_f$ of Equation* (2.1).

P r o o f. Here for the reader's convenience we show the proof of the analogous Theorem 2.1 in [6] at least in the reduced form. We gain the unique formal almost periodic $\Lambda$-solution of Equation (2.1) in the form

$$x_f = x_f(t) = \sum_{\lambda} x_\lambda \sim \sum_{\lambda} \sum_{\overline{s} \geqslant \overline{0}} \sum_{P} \Phi_P(\mathrm{i}\lambda)\varphi(\lambda)\exp(\mathrm{i}(\lambda + \overline{s}\,\overline{\omega})t), \tag{2.7}$$

$$x_\lambda = x_\lambda(t) \sim \sum_{\overline{s} \geqslant \overline{0}} \sum_{P} \Phi_P(\mathrm{i}\lambda)\varphi(\lambda)\exp(\mathrm{i}(\lambda + \overline{s}\,\overline{\omega})t) \tag{2.8}$$

for $\lambda \in \Lambda_f$, $t \in \mathbb{R}$. Here for given $\lambda \in \Lambda_f$ and $\overline{s} \in \mathbb{N}_0^{1\times(M+N)}$, $P = P(\overline{s})$ denotes an increasing sequence $\overline{0} = \overline{P}_0 \leqslant \overline{P}_1 \leqslant \ldots \leqslant \overline{P}_{|\overline{s}|} = \overline{s}$ of vectors from $\mathbb{N}_0^{1\times(M+N)}$ which satisfies $|\overline{P}_j - \overline{P}_{j-1}| = 1$, $j = 1, \ldots, |\overline{s}|$. With every such sequence $P = P(\overline{s})$ for a fixed $\lambda \in \Lambda_f$ we can associate in a unique manner a sequence $p = p(\overline{s})$ of vectors $\overline{p}_0, \overline{p}_1, \ldots, \overline{p}_{|\overline{s}|}$ satisfying $\overline{p}_0 = \overline{0}$, $|\overline{p}_j| = 1$, $j = 1, \ldots, |\overline{s}|$, and $\overline{P}_k = \sum_{j=0}^{k} \overline{p}_j$, $k = 0, 1, \ldots, |\overline{s}|$, while $\overline{p}_j = (\overline{q}_j, \overline{r}_j)$, $\overline{q}_j \in \mathbb{N}_0^{1\times M}$ and $\overline{r}_j \in \mathbb{N}_0^{1\times N}$, $j = 0, 1, \ldots, |\overline{s}|$. Define the function

$$\Phi_P(z) = \prod_{j=|\overline{s}|}^{0} \Phi^{-1}(z + \mathrm{i}\overline{P}_j\overline{\omega})\gamma(\overline{p}_j\overline{\omega}) \tag{2.9}$$

with $\gamma(0) = 1$, $\gamma(\overline{p}_j\overline{\omega}) = \alpha(\overline{q}_j\overline{\mu}) + \beta(\overline{r}_j\overline{\nu})\exp(-\mathrm{i}\overline{P}_{j-1}\overline{\omega}\tau)$, $j = 1, \ldots, |\overline{s}|$, while $\alpha(\mu) = 0$ for $\mu \notin \Lambda_a$, $\beta(\nu) = 0$ for $\nu \notin \Lambda_b$, $(\alpha(0) = \beta(0) = 0)$.

For Equation (2.1) with the given assumptions and for $f = f(t)$, $t \in \mathbb{R}$, equal to some "harmonic" $\varphi(\lambda)\exp(\mathrm{i}\lambda t)$, $t \in \mathbb{R}$, we get a formal almost periodic $\Lambda$-solution $x_f = x_\lambda$ from (2.8) with $\lambda \in \Lambda_f$. Every $\sigma \in \Lambda$ can be represented in the form $\sigma = \lambda + \overline{s\omega} = \lambda + \overline{m\mu} + \overline{n\nu}$, where $\lambda \in \Lambda$ and

$$\overline{\mu} = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_M \end{pmatrix}, \quad \overline{\nu} = \begin{pmatrix} \nu_1 \\ \vdots \\ \nu_N \end{pmatrix}, \quad \overline{\omega} = \begin{pmatrix} \overline{\mu} \\ \overline{\nu} \end{pmatrix},$$

$\overline{m} = (m_1, \dots, m_M) \in \mathbb{N}_0^{1\times M}$, $\overline{n} = (n_1, \dots, n_N) \in \mathbb{N}_0^{1\times N}$, $\overline{s} = (\overline{m}, \overline{n})$. The number of all possible different "descents" from $\lambda + \overline{s\omega}$ to $\lambda$ represented by all sequences $P = P(\overline{s})$ is given by the combinatorial number

$$\binom{|\overline{s}|}{\overline{s}} = \frac{|\overline{s}|!}{(m_1!)\dots(m_M!)(n_1!)\dots(n_N!)}.$$

$\square$

R e m a r k 2.3. Every almost periodic $\Lambda$-solution $x_f$ of Equation (2.1) is simultaneously a formal almost periodic $\Lambda$-solution $x_f$ of Equation (2.1). The contrary is not true in general.

## 3. Almost periodic solutions

**3.1. Closed regions.** In the sequel we will take up the case $\theta \neq \emptyset$ but the case $\theta = \emptyset$ when $\Delta(z)$ has no purely imaginary roots would be even easier (for $\theta$ see (1.13)). Hence, let $\mathrm{i}\xi_1, \dots, \mathrm{i}\xi_{j_0}$, $j_0 \in \mathbb{N}$, be all mutually different purely imaginary roots in $\mathbb{C}$ of the quasipolynomial $\Delta(z)$ and let $\varrho_1, \dots, \varrho_{j_0}$ be their multiplicities. We pick the positive constant $\delta = \frac{1}{4}\min\{\alpha, \delta_0, \Delta, d_\theta, d_\xi, d, \tau, 4\}$ (where $d_\xi, d_\theta$ are from (1.14), (2.4)), and where a positive number $\alpha$ is chosen so that $\pi(2\alpha) \cap \sigma(\Delta(z))$ is equal to $\{\mathrm{i}\xi_1, \dots, \mathrm{i}\xi_{j_0}\}$ for $\theta \neq \emptyset$ and $\emptyset$ for $\theta = \emptyset$. Further, unless stated otherwise, we assume that we are given a fixed vector $\overline{s}$ and a fixed sequence of vectors $P = P(\overline{s})$. Recall that $\varkappa(z,\delta)$ and $\overline{\varkappa}(z,\delta)$ are the open disc and the closed disc centred at the point $z$ with the radius $\delta$ in the complex plane $\mathbb{C}$. In $\mathbb{C}$ we construct closed regions

$$G_k = \pi(\alpha) \setminus \bigcup_{j=1}^{j_0} \varkappa(\mathrm{i}\xi_j - \mathrm{i}\overline{P}_k\overline{\omega}, \delta), \quad k = 0, 1, \dots, |\overline{s}|, \tag{3.1}$$

and we denote by $G_P$ their intersection, so that

$$G_P = \bigcap_{k=0}^{|\overline{s}|} G_k = \pi(\alpha) \setminus \bigcup_{k=0}^{|\overline{s}|} \bigcup_{j=1}^{j_0} \varkappa(\mathrm{i}\xi_j - \mathrm{i}\overline{P}_k\overline{\omega}, \delta), \tag{3.2}$$

what evidently is a closed region. Each of the closed regions $G_k$ is a shift of the region $G_0$ in the complex plane by $\overline{P}_k\overline{\omega}$ units downward, $k = 0, 1, \ldots, |\overline{s}|$.

Since the matrix function $\Phi(z)$ introduced in (1.5) is analytic and regular on $G_0$, the matrix function $\Phi(z + \mathrm{i}\overline{P}_k\overline{\omega})$ is analytic and regular on $G_k$ and the same holds also for $\Phi^{-1}(z + \mathrm{i}\overline{P}_k\overline{\omega})$, $k = 0, 1, \ldots, |\overline{s}|$. It follows that the matrix function $\Phi_P$ is analytic and regular on the closed region $G_P$.

In the case $\theta = \emptyset$ the boundary $L_P = \partial G_P$ of the closed region $G_P$ is formed by two lines $|\Re z| = \alpha$ which form the boundary of the strip $\pi(\alpha)$. For $\theta \neq \emptyset$ the boundary $L_P = \partial G_P$ is formed by two lines $|\Re z| = \alpha$ and by circles $K_{j,k} = K(\mathrm{i}\xi_j - \mathrm{i}\overline{P}_k\overline{\omega}, \delta)$, $j = 1, \ldots, j_0$, $k = 0, 1, \ldots, |\overline{s}|$. In virtue of the assumptions of Theorem 2.1 and of the choice of the positive number $\delta$ it is ensured for $\theta \neq \emptyset$ that no point $z \in K_{j,k}$ belongs to any disc $\varkappa_{l,m}$, $l = 1, \ldots, j_0$, $m = 0, 1, \ldots, |\overline{s}|$ (the more detailed explanation is in [6]).

**3.2. Integral representation.** For a given vector $\overline{s}$ we can choose a sufficiently large positive number $R$ such that all circles $K_{j,l}$, $j = 1, \ldots, j_0$, $l = 0, 1, \ldots, |\overline{s}|$ belong to the interior of the closed region $\pi(2\alpha) \cap \overline{\varkappa}(0, R)$ the boundary of which we denote by $L_R$. We refer to the proof in [6] which proves a certain (type of) absolute and uniform convergence on $\mathbb{R}$ of the trigonometric series

$$\sum_{\overline{s} \geqslant \overline{0}} \Big[ \sum_P \sum_\lambda \Phi_P(\mathrm{i}\lambda)\varphi(\lambda)\exp(\mathrm{i}\lambda t) \Big] \exp(\mathrm{i}\overline{s}\overline{\omega}t), \quad t \in \mathbb{R}, \tag{3.3}$$

which arises by a rearrangement of the trigonometric series $x_f$. Namely, the convergence of the series

$$\sum_{\overline{s} \geqslant \overline{0}} \sum_P \Big| \sum_\lambda \Phi_P(\mathrm{i}\lambda)\varphi(\lambda)\exp(\mathrm{i}\lambda t) \Big|, \quad t \in \mathbb{R}, \tag{3.4}$$

is considered. In the case of one-point spectrum for $f(t) = \varphi(\lambda)\exp(\mathrm{i}\lambda t)$, $t \in \mathbb{R}$, when $x_f$ and $x_\lambda$ coincide and $x_\lambda$ coincides with (3.3), the convergence of the series (3.4) ensures the absolute and uniform convergence of $x_\lambda$. This implies the absolute and uniform convergence of the trigonometric series $x_f = \sum\limits_\lambda x_\lambda$ for trigonometric polynomials $a$, $b$, $f$. Eventually, with the use of passing to limits we proceed to the case when $a$, $b$, $f$ are not trigonometric polynomials.

Now, we use the Cauchy integral for the expression inside the norm in the series (3.4). If we denote by $L_R(P)$ the boundary of the closed region $G_P \cap \overline{\varkappa}(0,R)$ then

$$\sum_{\lambda} \Phi_P(\mathrm{i}\lambda)\varphi(\lambda)\exp(\mathrm{i}\lambda t) = \frac{1}{2\pi\mathrm{i}} \oint_{L_R(P)} \Phi_P(z)F(t,z)\,\mathrm{d}z$$
$$= \frac{1}{2\pi\mathrm{i}} \oint_{L_R} \Phi_P(z)F(t,z)\,\mathrm{d}z - \sum_{j=1}^{j_0}\sum_{k=0}^{|\xi|} \frac{1}{2\pi\mathrm{i}} \oint_{K_{j,k}} \Phi_P(z)F(t,z)\,\mathrm{d}z, \quad \lambda \in \Lambda, \tag{3.5}$$

where

$$F(t,z) = \sum_{\lambda} \frac{\exp(\mathrm{i}\lambda t)}{z-\mathrm{i}\lambda}\varphi(\lambda), \quad \lambda \in \Lambda,\ t \in \mathbb{R}. \tag{3.6}$$

The function $F$ has the following property:

$$\lim_{|z|\to+\infty} |F(t,z)| = 0 \tag{3.7}$$

uniformly with respect to $t \in \mathbb{R}$. This implies the existence of a positive constant $R'$ such that the inequality

$$|F(t,z)| \leqslant 1 \tag{3.8}$$

holds uniformly with respect to $t \in \mathbb{R}$ for all $z \in \mathbb{C}$, $|z| \geqslant \mathbb{R}'$.

With the notation $\|f\| = \max\{|f|, |\dot{f}|\}$ (introduced in (1.2)) the estimate

$$|F(t,z)| \leqslant \frac{1+\alpha}{\alpha|z|}\|f\| \tag{3.9}$$

holds uniformly with respect to $t \in \mathbb{R}$ for all $z \in \mathbb{C}$ for which $|\Re z| = \alpha$.

Indeed, for $|\Re z| \neq 0$ the equalities

$$\frac{\exp(\mathrm{i}\lambda t)}{z-\mathrm{i}\lambda} = \exp(zt)\int_t^{+\infty\Re z} \exp\left((\mathrm{i}\lambda - z)s\right)\,\mathrm{d}s \qquad \text{and}$$
$$F(t,z) = \int_0^{+\infty\Re z} f(t+s)\exp(-zs)\,\mathrm{d}s = \frac{1}{z}\left[f(t) + \int_0^{+\infty\Re z} \dot{f}(t+s)\exp(-zs)\,\mathrm{d}s\right]$$

are valid.

**3.3. Estimates.** Assume that $\theta \neq \emptyset$. Owing to the choice of positive numbers $\alpha$, $\delta$ and to the properties of the matrix function $\Phi = \Phi(z)$ and of the quasipolynomial $\Delta(z)$ there exists a positive number $C_1$ such that the inequalities

$$\begin{cases} |\Phi^{-1}(z)| \leqslant C_1 & \text{for } z \in G_0, \\ |\Phi^{-1}(z)| \leqslant C_1|z|^{-1} & \text{for } z \in G_0 \setminus \{0\} \end{cases} \tag{3.10}$$

are valid. If we pass to the limit for $R \to +\infty$ on the right-hand side of the equality (3.5) we get the equality

$$\begin{aligned}(3.11)\qquad \sum_{\lambda} \Phi_P(\mathrm{i}\lambda)\varphi(\lambda)\exp(\mathrm{i}\lambda t) &= \Big(\int_{-\alpha-\mathrm{i}\infty}^{-\alpha+\mathrm{i}\infty} + \int_{\alpha-\mathrm{i}\infty}^{\alpha+\mathrm{i}\infty}\Big)\Phi_P(z)F(t,z)\,\mathrm{d}z \\ &\quad - \sum_{j=1}^{j_0}\sum_{k=0}^{|\overline{\xi}|}\frac{1}{2\pi\mathrm{i}}\oint_{K_{j,k}} \Phi_P(z)F(t,z)\,\mathrm{d}z,\ \ \lambda\in\Lambda_f, t\in\mathbb{R}.\end{aligned}$$

This can be seen by taking into account the estimates (3.9) and (3.10) which imply the absolute convergence of the improper integrals on the right-hand side in (3.11) and the convergence to zero uniformly with respect to $t \in \mathbb{R}$ for $R \to +\infty$ of the integrals over arcs of the circle $K(0,R)$ lying in the $\alpha$-strip.

The quasipolynomial $\Delta(z)$ may be expressed in the forms $\Delta(z) = (z - \mathrm{i}\xi_j)^{\varrho_j}\Delta_j(z)$ where $\Delta_j(z) \neq 0$ for $z \in \overline{\varkappa}(\mathrm{i}\xi_j,\delta)$, $j = 1,\dots,j_0$. Hence, the inverse matrix function $\Phi^{-1}$ may be expressed in the form $\Phi^{-1}(z) = (z - \mathrm{i}\xi_j)^{-\varrho_j}\Gamma_j(z)$ where $\Gamma_j(z) = \Delta_j^{-1}(z)\widetilde{\Phi}(z)$, $j = 1,\dots,j_0$, and $\widetilde{\Phi}(z)$ is the matrix function whose elements with subscripts $k$, $l$ are equal to the algebraic complements of $\Phi(z)$ with subscripts $l$, $k$: $k,l = 1,\dots,n_0$. The matrix function $\Gamma_j$ is analytic in the closed disc $\overline{\varkappa}(\mathrm{i}\xi_j,\delta)$, $j = 1,\dots,j_0$.

According to these decompositions and in view of $|\Omega(z)| \leqslant |E| + |c_0|\exp(\alpha\tau) = c_{00}$ and $|\Omega^{(h)}(z)| = |\tau^h c_0\exp(z\tau)| \leqslant \tau^h|c_0|\exp(\alpha\tau) = c_{0h}$ for $z \in \pi(\alpha)$, $h = 1,2,\dots$, and for $z \in G_0$ we have $|\Phi^{(h)}(z)| = |z\Omega^{(h)}(z) + h\Omega^{(h-1)}(z) - (-\tau)^h b_0\exp(-z\tau)| \leqslant |z|c_{0h} + c_{1h} \leqslant |z|(c_{0h}+c_{1h})$, where $c_{10} = |a_0| + |b_0|\exp(\alpha\tau)$ and $c_{1h} = \tau^h|b_0|\exp(\alpha\tau)$ for $|z| \geqslant R_0$ and $|\Phi^{(h)}(z)| \leqslant R_0c_{0h} + c_{1h} \leqslant R_0(c_{0h}+c_{1h})$ for $|z| \leqslant R_0$ where $R_0 \geqslant 1$ is a large enough positive number, $h = 0,1,\dots$. It is possible to choose the already defined positive constant $C_1$ large enough so that besides the estimates (3.10) also the following ones are true:

$$(3.12)\qquad \begin{cases} |\Phi^{-1}(z)| \leqslant C_1 & \text{for } z\in G_0,\\ |(\Phi^{-1}(z))^{(h)}| \leqslant C_1|z|^{-1} & \text{for } z\in G_0\setminus\{0\},\\ |\Gamma_j^{(h)}(\mathrm{i}\xi_j)| \leqslant C_1 & \end{cases}$$

for $j = 1,\dots,j_0$, $h = 0,1,\dots,\varrho$ where $\varrho = \max\{\varrho_1,\dots,\varrho_{j_0}\}$. This implies that our matrix function $\Phi$ and the matrix function $\Phi$ from [6], although they are essentially different, have the same properties needed for analogous constructions of the formal almost periodic and finally almost periodic solutions with a certain prescribed spectrum of the studied almost periodic neutral differential equations (with constant time lag). These solutions are in the formal identical forms as in [6] (with respect to the identical notation of these two matrix functions $\Phi$). Thus, we can refer to the validity of analogous assertions and remarks in [6] but here with additional assumptions: $\sigma(c_0) \cap K(0,1) = \emptyset$, $\alpha_0 > 0$, $\delta > 0$, $0 < \alpha \leqslant \alpha_0/2$.

**Lemma 3.1.** *The magnitudes of the integrals*

$$I_{j,l}(P) = -\frac{1}{2\pi\mathrm{i}} \oint_{K_{j,l}} \Phi_P(z) F(t,z)\,\mathrm{d}z$$

*are estimated by*

$$|I_{j,l}(P)| \leqslant \frac{C_1}{|\overline{s}|!} \binom{|\overline{s}|}{\overline{s}} \widetilde{M}\widetilde{N} \sum_{k=1}^{\varrho} (2M_d)^k \|f\|,$$

*where*

$$\widetilde{M} = \prod_{k=1}^{M} \frac{2^{\varrho}(L+1)|\alpha(\mu_k)|}{2\delta}, \quad \widetilde{N} = \prod_{k=1}^{N} \frac{2^{\varrho}(L+1)|\beta(\nu_k)|}{2\delta},$$

$j = 1, \ldots, j_0$, $l = 0, 1, \ldots, |\overline{s}|$, *and the positive constants* $L$, $M_d$ *are independent of both* $\overline{s}$ *and* $P(\overline{s})$.

**Lemma 3.2.** *The improper integrals*

$$I_-(P) + I_+(P) = \frac{1}{2\pi\mathrm{i}} \biggl( -\int_{-\alpha-\mathrm{i}\infty}^{-\alpha+\mathrm{i}\infty} + \int_{\alpha-\mathrm{i}\infty}^{\alpha+\mathrm{i}\infty} \biggr) \Phi_P(z) F(t,z)\,\mathrm{d}z$$

*converge absolutely and the following estimate is valid:*

$$|I_-(P)| + |I_+(P)| \leqslant \frac{1+\alpha}{\alpha^2} C_1 \biggl[ \prod_{k=1}^{M} \Bigl( \frac{\sqrt{2}C_1|\alpha_k|}{\delta} \Bigr)^{m_k} \biggr] \biggl[ \prod_{k=1}^{N} \Bigl( \frac{\sqrt{2}C_1|\beta_k|}{\delta} \Bigr)^{n_k} \biggr] \frac{\|f\|}{|\overline{s}|!},$$

$|\overline{s}| = 0, 1, \ldots$, $\alpha_k = \alpha(\mu_k)$, $k = 1, \ldots, M$, $\beta_k = \beta(\nu_k)$, $k = 1, \ldots, N$.

**3.4. Almost periodic solutions.** (See Remark 2.2.)

**Theorem 3.1** (see T. 3.3 in [6])**.** *The formal almost periodic* $\Lambda$*-solution* $x_f$ *from Theorem* 2.1 *is an almost periodic* $\Lambda$*-solution of Equation* (2.1). *Moreover, it is unique and satisfies the estimate*

$$\|x_f\| \leqslant A\|f\| \tag{3.13}$$

*where the positive constant* $A$ *depends only on* $a_0$, $b_0$, $c_0$, $\Delta$, $d_\theta$, $d_\xi$, $d$, $\tau$, $S$, $T$ *where* $S = \sum|\alpha(\mu)| = \sum(a)$, $\mu \in \Lambda_a$, $T = \sum|\beta(\nu)| = \sum(b)$, $\nu \in \Lambda_b$.

As a corollary we obtain

**Corollary 3.1** (see C. 3.4 in [6]). *Let $\Lambda_1$, $\Lambda_2$ be two non-void sets of real numbers and let $T$, $S$ be positive constants. If $a$, $b$, $f$ from Equation* (2.1) *are trigonometric polynomials with $\Lambda_f \subset \Lambda_1$, $\Lambda_a \subset \Lambda_2$, $\Lambda_b \subset \Lambda_2$ and $\sum(a) \leqslant S$, $\sum(b) \leqslant T$ and if (for $\theta$ and $d_\xi$ see* (1.13) *and* (1.14))

$$\sigma(c_0) \cap K(0,1) = \emptyset, \tag{3.14}$$

$$\Delta' = \inf \Lambda_2 > 0, \tag{3.15}$$

$$d'_\theta = \begin{cases} \operatorname{dist}[\theta, S(\Lambda_2)] > 0 & \text{for } \theta \neq \emptyset, \\ 4 & \text{for } \theta = \emptyset, \end{cases} \tag{3.16}$$

$$d' = \operatorname{dist}[\mathrm{i}\Lambda', \sigma(\Delta(z))] > 0, \tag{3.17}$$

*where $\Lambda' = \Lambda_f + S(\Lambda_2 \cup \{0\})$, then there exists exactly one almost periodic $\Lambda'$-solution $x_f$ of Equation* (2.1)*. This solution satisfies estimate* (3.13) *where the positive constant $A$ depends on $a_0$, $b_0$, $c_0$, $\Delta'$, $d'_\theta$, $d_\xi$, $d'$, $\tau$, $S$, $T$ only.*

Remark 3.1. Corollary 3.1 ensures the validity of the estimate (3.13) with a constant $A$ common for all almost periodic $\Lambda'$-solutions $x_f$ of Equation (2.1) for the whole class of trigonometric polynomials $a$, $b$, $f$ from Corollary 3.1.

**3.5. Limit passages.** The conclusions obtained under the assumptions that $a$, $b$, $f$ are trigonometric polynomials remain valid even under more general assumptions. They follow by means of Bochner-Fejér approximation polynomials and limit passages.

**Theorem 3.2** (see T. 3.6 in [6]). *If in Equation* (2.1) *$a$, $b$ are trigonometric polynomials and $f$ is an almost periodic function with an almost periodic derivative $\dot f$ and if* (2.3), (2.4), (2.5), (2.6) *from Theorem* 2.1 *are fulfilled then Equation* (2.1) *has exactly one almost periodic $\Lambda$-solution $x_f$ and this solution satisfies estimate* (3.13).

Remark 3.2. Equation (2.1) may admit infinitely many almost periodic solutions but only one of them has its spectrum contained in $\mathrm{i}\Lambda$ (hence it is an almost periodic $\Lambda$-solution).

**Corollary 3.2.** *Let $\Lambda_1$, $\Lambda_2$ be two non-void sets of real numbers and let $S$, $T$ be two positive constants. If assumptions* (3.14), (3.15), (3.16), (3.17) *are satisfied and if $f$ is an almost periodic function with its spectrum contained in $\mathrm{i}\Lambda_1$ having the almost periodic derivative $\dot f$ and if $a$, $b$ are trigonometric polynomials with their spectra contained in $\mathrm{i}\Lambda_2$ for which $\sum(a) \leqslant S$, $\sum(b) \leqslant T$, then Equation* (2.1) *has exactly one almost periodic $\Lambda'$-solution $x_f$ where $\Lambda' = \Lambda_1 + S(\Lambda_2 \cup \{0\})$ and this* *solution satisfies estimate* (3.13) *where the positive constant $A$ depends on $a_0$, $b_0$, $c_0$, $\Delta'$, $d'_\theta$, $d_\xi$, $d'$, $\tau$, $S$, $T$ only.*

R e m a r k 3.3. Corollary 3.2 ensures the validity of the estimate (3.13) with a constant $A$ common for all almost periodic $\Lambda'$-solutions $x_f$ of Equation (2.1) for the whole class of trigonometric polynomials $a$, $b$ and an almost periodic function $f$ from Corollary 3.2.

**Theorem 3.3** (see T. 3.10 in [6]). *If $a$ and $b$ are almost periodic functions with absolutely convergent Fourier series having almost periodic the first derivatives and $f$ is the function from Theorem* 3.2 *and if assumptions* (2.3), (2.4), (2.5), (2.6) *are satisfied, then Equation* (2.1) *has exactly one almost periodic $\Lambda$-solution $x_f$ where $\Lambda = \Lambda_f + S(\Lambda_a \cup \Lambda_b \cup \{0\})$, and this solution satisfies estimate* (3.13) *in which the positive constant $A$ depends only on $a_0$, $b_0$, $c_0$, $\Delta$, $d_\theta$, $d_\xi$, $d$, $\tau$, $S$, $T$ where $S = \sum(a)$, $T = \sum(b)$.*

**Corollary 3.3** (see C. 3.11 in [6]). *Let $\Lambda_1$, $\Lambda_2$ be two non-void sets of real numbers and let $S$, $T$ be positive constants. If assumptions* (3.14), (3.15), (3.16), (3.17) *are satisfied and if $f$ is an almost periodic function with its spectrum contained in $\mathrm{i}\Lambda_1$ having the almost periodic derivative $\dot{f}$ and if $a$, $b$ are almost periodic functions with their spectra contained in $\mathrm{i}\Lambda_2$ satisfying $\sum(a) \leqslant S$, $\sum(b) \leqslant T$, then Equation* (2.1) *has exactly one almost periodic $\Lambda'$-solution $x_f$ where $\Lambda' = \Lambda_1 + S(\Lambda_2 \cup \{0\})$ and this solutions satisfies estimate* (3.13) *in which the positive constant $A$ depends only on $a_0$, $b_0$, $c_0$, $\Delta'$, $d'_\theta$, $d_\xi$, $d'$, $\tau$, $S$, $T$.*

R e m a r k 3.4. Corollary 3.3 ensures the validity of the estimate (3.13) with a constant $A$ common for all almost periodic $\Lambda'$-solutions $x_f$ of Equation (2.1) for the whole class of almost periodic functions $a$, $b$, $f$ from Corollary 3.3.

## 4. Quasilinear equations

**4.1. Functions of several variables.** We recall that a set $\mathcal{M}$ of real numbers is said to be relatively dense in $\mathbb{R}$ if there exists a positive constant $l$ such that for any real number $\alpha$ the intersection $\mathcal{M} \cap [\alpha, \alpha + l]$ is non-void. This positive constant $l$ is called the inclusion length (of the relative density of $\mathcal{M}$ in $\mathbb{R}$).

Let $g = g(t, x)$ be a continuous function $g\colon \mathbb{R} \times D \to \mathbb{C}^{p\times q}$ where $D \subset \mathbb{C}^{m\times n}$ is a non-void set. The function $g$ is said to be

- ▷ almost periodic in the variable $t$ on $\mathbb{R} \times D$ if $g(t, x)$ is almost periodic as a function of $t \in \mathbb{R}$ for any fixed $x \in D$;

- ▷ uniformly almost periodic in $t \in \mathbb{R}$ on $\mathbb{R} \times D$ if $g(t,x)$ is almost periodic in $t \in \mathbb{R}$ on $\mathbb{R} \times D$ and for any $\varepsilon > 0$ there exists a set $\{\zeta\}_\varepsilon \subset \mathbb{R}$ relatively dense in $\mathbb{R}$ such that $|g(t+\zeta,x) - g(t,x)| \leqslant \varepsilon$ for every $\zeta \in \{\zeta\}_\varepsilon$, $t \in \mathbb{R}$ and $x \in D$;
- ▷ locally uniformly almost periodic in the variable $t \in \mathbb{R}$ on $\mathbb{R} \times D$ if for any non-void compact set $K \subset D$ the restriction $g_K$ of the function $g$ on $\mathbb{R} \times K$ is uniformly almost periodic in the variable $t$ on $\mathbb{R} \times K$.

**Lemma 4.1.** *Let a continuous function $g\colon \mathbb{R} \times D \to \mathbb{C}^{p\times q}$, $D \subset \mathbb{C}^{m\times n}$ be almost periodic in $t$ on $\mathbb{R} \times D$. A necessary and sufficient condition for $g$ to be locally uniformly almost periodic in $t$ on $\mathbb{R} \times D$ is that $g$ be continuous in $x$ uniformly with respect to the variable $t \in \mathbb{R}$ on $\mathbb{R} \times D$.*

Proof. It is sufficient to take $p = q = 1$. To prove the sufficiency, let $K \subset D$ be a compact set and $\varepsilon > 0$. The restriction $g_K$ is uniformly continuous in $x$ uniformly with respect to $t \in \mathbb{R}$ on $\mathbb{R} \times K$. Hence, there exists $\delta = \delta(\varepsilon/3) > 0$ such that for any $x, y \in K$ and $t \in \mathbb{R}$ we have $|g_K(t,x) - g_K(t,y)| < \varepsilon/3$ provided $|x-y| < \delta$. Further, there exists a finite $\delta$-net for $K$, namely, $x_1, \ldots, x_N \in K$ such that $\min\{|x-x_j|\colon j = 1,\ldots,N\} < \delta$ for any $x \in K$. Since the functions $h_j(t) = g_K(t,x_j)$, $t \in \mathbb{R}$, $j = 1,\ldots,N$, are almost periodic, there exists a set $\{\tau\} \subset \mathbb{R}$ of $\varepsilon/3$-almost periods common for the functions $h_1,\ldots,h_N$ which is relatively dense in $\mathbb{R}$, i.e. $|h_j(t+\tau) - h_j(t)| < \varepsilon/3$ for any $t \in \mathbb{R}$, $\tau \in \{\tau\}$ and $j = 1,\ldots,N$. Now, let $\tau \in \{\tau\}$, $t \in \mathbb{R}$ and $x \in K$. Choose $j$ such that $|x - x_j| < \delta$. Then

$$\begin{aligned}|g_K(t+\tau,x) - g_K(t,x)| &\leqslant |g_K(t+\tau,x) - g_K(t+\tau,x_j)| \\ &\quad + |g_K(t+\tau,x_j) - g_K(t,x_j)| + |g_K(t,x_j) - g_K(t,x)| < \varepsilon.\end{aligned}$$

Thus, $g$ is locally uniformly almost periodic in $t$ on $\mathbb{R} \times D$ and the sufficiency is proved. Let us remark that, in the same way, the function $g_K$ may be shown to be uniformly continuous on $\mathbb{R} \times K$ for any non-void compact $K \subset D$.

On the other hand, to prove the necessity we take an arbitrary non-void compact set $K \subset D$ and $\varepsilon > 0$ and use the uniform continuity of $g_K$ on $[0,\,l] \times K$ where $l = l(\varepsilon/3)$ is the inclusion length of the relative density of the set $\{\tau\}$ of $\varepsilon/3$-almost periods. For any $x, y \in K$ and $t, s \in [0,\,l]$ we have $|g_K(t,x) - g_K(s,y)| < \varepsilon/3$ provided $|t-s| + |x-y| < \delta$. For any $t \in \mathbb{R}$ there exists $\tau = \tau(t) \in \{\tau\}$ such that $t + \tau \in [0,\,l]$. Consequently, for any $x, y \in K$, $|x-y| < \delta$ and $t \in \mathbb{R}$ we get

$$\begin{aligned}|g_K(t,x) - g_K(t,y)| &\leqslant |g_K(t,x) - g_K(t+\tau,x)| \\ &\quad + |g_K(t+\tau,x) - g_K(t+\tau,y)| + |g_K(t+\tau,y) - g_K(t,y)| < \varepsilon\end{aligned}$$

and the assertion follows. Let us remark that it is easy to prove that the local uniform almost periodicity of $g$ on $\mathbb{R} \times D$ implies the local uniform continuity of $g$ on $\mathbb{R} \times D$ (i.e. the uniform continuity of $g$ on $\mathbb{R} \times K$ for any non-void compact set $K \subset D$). $\square$

In the sequel we deal with the cases in which the conditions for the locally uniform almost periodicity of the introduced functions are fulfilled.

**4.2. Harmonic analysis.** Let $g\colon \mathbb{R} \times D \to \mathbb{C}^{p\times q}$, $D \subset \mathbb{C}^{m\times n}$ be a function almost periodic in $t$ on $\mathbb{R} \times D$. For any given $x \in D$ the Bohr transformation

$$a(\lambda, x) = a(\lambda, x, g) = \lim_{T\to+\infty} \int_s^{s+T} g(t, x) \exp(-\mathrm{i}\lambda t)\,\mathrm{d}t$$

exists for each given $\lambda \in \mathbb{R}$ uniformly with respect to $s \in \mathbb{R}$. If $a(\lambda, x)$ is non-zero for a given $\lambda \in \mathbb{R}$ for at least one point $x \in D$, i.e. $a(\lambda, x) \not\equiv 0$ for $x \in D$, then $\lambda$ is called the Fourier exponent and (the function) $a(\lambda, x)$ is called the Fourier coefficient of the function $g$. The set of all Fourier exponents of the function $g$ is denoted by $\Lambda_g$. If $D$ is a compact set or a region (open connected non-void set) then the set $\Lambda_g$ is at most countable, which is not difficult to prove ([6]). If $g$ is locally uniformly almost periodic in the variable $t$ on $\mathbb{R} \times D$ and $D$ is a region, then the Fourier series

$$g(t, x) \sim \sum_\lambda a(\lambda, x) \exp(\mathrm{i}\lambda t), \quad \lambda \in \Lambda_g,\ t \in \mathbb{R},\ x \in D,$$

can be uniquely determined except for its order of summation. If the function $g$ is also analytic in the variable $x$ on a closed ball lying in $D$ and containing the set $R_f$ of all values of the almost periodic function $f$, then $\Lambda_F \subset \Lambda_g + S(\Lambda_f \cup \{0\})$ is valid for the function $F(t) = g(t, f(t))$, $t \in \mathbb{R}$.

**4.3. Derivatives.** Now we will deal with a function $g = g(t, u, v, \varepsilon)\colon \mathbb{R} \times D = \mathbb{R} \times \mathbb{C}^{n\times 1} \times \mathbb{C}^{n\times 1} \times \overline{\varkappa}_0 \to \mathbb{C}^{n\times 1}$ where $\overline{\varkappa}_0 = \overline{\varkappa}(0, \delta') \subset \mathbb{C}$, $\delta' > 0$. In order to avoid complicated expressions we will use the symbolic records $g_t$, $g_u$, $g_v$, $g_{tu}$, $g_{tv}$, $g_{uu}$, $g_{uv}$, $g_{vv}$ of Jacobi matrices which arise during the evaluation of the derivatives of the function $g$. The norm of a matrix is the sum of absolute values of all its elements (for example $|g_{uv}| = \sum\limits_j \sum\limits_k \sum\limits_l |\partial^2 g_j / \partial u_k \partial v_l|$).

**4.4. Quasilinear equations.** Using the Banach contraction principle we shall deal with the following quasilinear (weakly nonlinear) system

$$\begin{aligned}\dot{x}(t) = a_0 x(t) + b_0 x(t-\tau) + c_0 \dot{x}(t-\tau) + a(t)x(t) + b(t)x(t-\tau)\\ + f(t) + \varepsilon g(t, x(t), x(t-\tau), \varepsilon), \quad t \in \mathbb{R},\end{aligned} \tag{4.1}$$

where $\varepsilon$ is a small complex parameter. For $\varepsilon = 0$ we get the generating equation (2.1) with its conditions for $a_0$, $b_0$, $c_0$, $a$, $b$, $f$. Assume that the function $g = g(t, u, v, \varepsilon)$ (from Sec. 4.3) together with its derivative $g_t$ is locally uniformly almost periodic in the variable $t$ on $\mathbb{R} \times D$ and $g$ is analytic in the variables $u$, $v$, $\varepsilon$.

Put $\Lambda = S(\Lambda_f \cup \Lambda_g + S(\Lambda_a \cup \Lambda_b \cup \{0\}))$. If $\Lambda_\xi \subset \Lambda$ for a function $\xi \in AP(\mathbb{C}^{n\times 1})$, then the composite function $F = F(t) = g(t, \xi(t), \xi(t-\tau), \varepsilon)$, $t \in \mathbb{R}$, is an almost periodic function whose spectrum is contained in $\mathrm{i}\Lambda$ for each $\varepsilon \in \overline{\varkappa}_0$. This follows from the fact that $\Lambda_F \subset \Lambda_g + S(\Lambda_f \cup \{0\}) \subset \Lambda_f \cup \Lambda_g + S(\Lambda \cup \{0\}) \subset \Lambda$ due to the analyticity of the function $g$ in the variables $u$, $v$. Thus the "spectrum" $\mathrm{i}\Lambda$ is wide enough in order to allow the existence of an almost $\Lambda$-solution of Equation (4.1).

If a positive number $R$ is given then the norm $\|g\|_R$ is the maximum value among the least upper bounds of magnitudes of the function $g$ and its derivatives $g_t$, $g_u$, $g_v$, $g_{tu}$, $g_{tv}$, $g_{uu}$, $g_{uv}$, $g_{vv}$ on the (metric) space $\Omega_R = \mathbb{R} \times \mathbb{C}_R^{n\times 1} \times \mathbb{C}_R^{n\times 1} \times \overline{\varkappa}_0$ where $\mathbb{C}_R^{n\times 1} = \{w \in \mathbb{C}^{n\times 1} \colon |w| \leqslant R\}$. For any two points $U = [t, u, v, \varepsilon]$, $\widetilde{U} = [t, \tilde{u}, \tilde{v}, \varepsilon]$ from the space $\Omega_R$ the inequality

$$\|g(U) - g(\widetilde{U})\|_R \leqslant \|g\|_R |U - \widetilde{U}| = \|g\|_R (|u - \tilde{u}| + |v - \tilde{v}|)$$

holds. As stated in Remark 2.2 the complete proof of the following theorem is analogous to the proof of Theorem 4.1 in [6].

**Theorem 4.1** (see T. 4.1 in [6]). *If the conditions* (3.14), (3.15), (3.16), (3.17) *are fulfilled for*

$$\Lambda = S(\Lambda_f \cup \Lambda_g + S(\Lambda_a \cup \Lambda_b \cup \{0\})) \tag{4.2}$$

*then for each positive number $R > A\|f\|$, where $A$ is from estimate* (3.13), *there exists such a positive number $\varepsilon(R)$ that Equation* (4.1) *has a unique almost periodic $\Lambda$-solution $x_\varepsilon$ with the norm $\|x_\varepsilon\| \leqslant R$ for each $\varepsilon \in \overline{\varkappa}_0 \cap \varkappa_{\varepsilon(R)}$.*

Remark 4.1. Equation (4.1) may admit even infinitely many almost periodic $\Lambda$-solutions but only one of them has its norm $\|\cdot\| \leqslant R$.

Conclusion. The method developed in this paper for the construction of almost periodic solutions of the studied almost periodic neutral differential equations can be used also for finding an approximative solution of the problem.

*References*

[1] *L. Amerio, G. Prouse*: Almost Periodic Functions and Functional Equations. Van Nostrand Reihold Company, New York, 1971. zbl

[2] *S. Bochner*: Abstrakte fastperiodischen Funktionen. Acta Math. *61* (1933), 149–184. zbl

[3] *C. Corduneanu*: Almost Periodic Functions. Wiley Interscience, New York, 1968. zbl

[4] *A. M. Fink*: Almost Periodic Differential Equations. Lecture Notes in Mathematics, Springer, New York, 1978. zbl

[5] *A. Fischer*: Existence of almost periodic solution of systems of linear and quasilinear differential equations with time lag. Czech. Math. J. *106* (1981), 256–268.

[6] *A. Fischer*: Almost periodic solutions with a prescribed spectrum of systems of linear and quasilinear differential equations with almost periodic coefficients and constant lag (Cauchy integral). Math. Bohem. *124* (1999), 351–379. zbl

[7] *B. M. Levitan, V. V. Zhikov*: Almost Periodic Functions and Differential Equations. Cambridge Univ. Press, Cambridge, 1982. zbl

[8] *S. N. Shimanov*: Almost Periodic Functions and Differential Equations with Time Lag. VUZ SSSR, Matematika *4* (1959). (In Russian.) zbl



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