NON-OSCILLATION OF SECOND ORDER LINEAR SELF-ADJOINT NONHOMOGENEOUS DIFFERENCE EQUATIONS

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Abstract. In the paper, conditions are obtained, in terms of coefficient functions, which are necessary as well as sufficient for non-oscillation/oscillation of all solutions of self-adjoint linear homogeneous equations of the form

$$\Delta(p_{n-1}\Delta y_{n-1}) + qy_n = 0, \quad n \geqslant 1,$$

where q is a constant. Sufficient conditions, in terms of coefficient functions, are obtained for non-oscillation of all solutions of nonlinear non-homogeneous equations of the type

$$\Delta(p_{n-1}\Delta y_{n-1}) + q_n g(y_n) = f_{n-1}, \quad n \geqslant 1,$$

where, unlike earlier works, $f_n \geqslant 0$ or $\leqslant 0$ (but $\not\equiv 0$) for large n. Further, these results are used to obtain sufficient conditions for non-oscillation of all solutions of forced linear third order difference equations of the form

$$y_{n+2} + a_n y_{n+1} + b_n y_n + c_n y_{n-1} = g_{n-1}, \quad n \geqslant 1.$$

Keywords: oscillation, non-oscillation, second order difference equation, third order difference equation, generalized zero

MSC 2010: 39A10, 39A12

1. Introduction

Oscillation theory of self-adjoint second order linear homogeneous difference equations of the form

$$\Delta(p_{n-1}\Delta y_{n-1}) + q_n y_n = 0, \quad n \geqslant 1 \tag{1}$$

is well-developed (see [2] and [5]), provided Δ denotes the forward difference operator defined by $\Delta y_n = y_{n+1} - y_n$, $\{p_n\}$, $n \geqslant 0$, and $\{q_n\}$, $n \geqslant 1$, are sequences of real numbers such that $p_n > 0$ for $n \geqslant 0$. After expansion, Eq. (1) takes the form

$$p_n y_{n+1} + (q_n - p_n - p_{n-1}) y_n + p_{n-1} y_{n-1} = 0, \quad n \geqslant 1 \tag{2}$$

By a solution of (1) we mean a sequence $\{y_n\}$, $n \geqslant 0$, of real numbers which satisfies the recurrence relation (2). A solution $\{y_n\}$ of (1) is said to be nontrivial if for every integer $N > 0$ there exists an integer $n > N$ such that $y_n \neq 0$. By a solution of (1) we always mean a non-trivial solution. A solution $\{y_n\}$ of (1) is said to be oscillatory if for every integer $N > 0$ we can find an integer $m > N$ such that $y_{m-1} y_m \leqslant 0$; otherwise, $\{y_n\}$ is called non-oscillatory. Equation (1) is called non-oscillatory if all its solutions are non-oscillatory.

An equation of the form

$$\alpha_n y_{n+1} + \beta_n y_n + \gamma_n y_{n-1} = 0, \quad n \geqslant 1,$$

can always be put in the self-adjoint form if $\alpha_n \gamma_n > 0$, $n \geqslant 1$ (p. 252, [5]). The Fibonacci equation

$$y_{n+1} - y_n - y_{n-1} = 0$$

cannot be put in the self-adjoint form. From the Sturm Separation Theorem (p. 261, [5] and p. 321, [2]) it follows that either all solutions of (1) are oscillatory or all are non-oscillatory. However, non-selfadjoint difference equations could admit both oscillatory and non-oscillatory solutions (see [6]). Indeed, the Fibonacci equation admits a positive solution $\{(1+\sqrt{5})^n/2^n\}$ and an oscillatory solution $\{(1-\sqrt{5})^n/2^n\}$.

Although many results concerning oscillation/nonoscillation of (1) are known, not many necessary and sufficient conditions for oscillation/nonoscillation of (1) are available in literature. For the three-term difference equations of order $(k+1)$ of the form

$$y_{n+1} - y_n + p_n y_{n-k} = 0, \tag{3}$$

where $k > 0$ is an integer and $\{p_n\}$ is a sequence of real numbers, the necessary and sufficient condition for oscillation of all solutions of (3) is that

$$p_n = p > \frac{k^k}{(k+1)^{k+1}},$$

where p is a constant (see p. 317, [2]). Such a result is not known for (1.1) even when $q_n = q$, a constant. In [1], S. Chen has obtained necessary and sufficient conditions, in terms of coefficient functions, for oscillation of (1). However, these conditions are not easy to verify. In this paper an attempt is made to obtain necessary and sufficient conditions for oscillation/non-oscillation of (1) with $q_n = q$, a constant.

In a series of papers (see [3], [4], [13], [14]), Hooker, Patula and Wong obtained sufficient conditions for oscillation/non-oscillation of all solutions of a class of second order linear homogeneous difference equations. Although the equations considered by them could be put in the self-adjoint form, their assumptions are such that the results could not be applied to (1). Moreover, there are no necessary and sufficient conditions for oscillation/non-oscillation of solutions of the equations they considered.

It seems that not many results concerning oscillation/non-oscillation of non-homogeneous equations of the form

$$\Delta(p_{n-1}\Delta y_{n-1}) + q_n y_n = f_{n-1}, \quad n \geqslant 1 \tag{4}$$

are known, provided $\{p_n\}$, $\{q_n\}$ and $\{f_n\}$ are sequences of real numbers such that $p_n > 0$ for $n \geqslant 0$. The definitions of oscillation/non-oscillation of a solution of (4) are the same as those of (1). In [7], [8], [10], the authors obtained sufficient conditions for oscillation of all solutions or all bounded solutions of self-adjoint forced non-linear second order difference equations of the form

$$\Delta(p_{n-1}\Delta y_{n-1}) + q_n g(y_{n-m}) = f_n, \quad n \geqslant 1,$$

where f_n changes sign. In this paper, sufficient conditions are obtained for non-oscillation of (4) and associated non-linear equations, where either $f_n \geqslant 0$ or $\leqslant 0$.

The results obtained for homogeneous equations (Section 2) and non-homogeneous equations (Section 3) of second order difference equations are applied to third order difference equations in Section 4.

2. Non-oscillation of homogeneous equations

In this section, sufficient conditions are obtained for non-oscillation of (1). If $q_n = q$, a constant, then necessary and sufficient conditions are obtained for non-oscillation/oscillation of (1).

Theorem 1. *If $q_n \leqslant 0$ for large n, then all solutions of (1) are non-oscillatory.*

Proof. Let $q_n \leqslant 0$ for $n \geqslant n_0 > 0$. In view of the Sturm Separation Theorem, it is enough to show that (1) admits a non-oscillatory solution. Let $\{y_n\}$ be a solution of (1) with $y_{n_0} = 0$ and $y_{n_0+1} > 0$. Writing (2) as

$$p_n(y_{n+1} - y_n) = (p_{n-1} - q_n)(y_n - y_{n-1}) - q_n y_{n-1},$$

we obtain

$$
\begin{aligned}
p_{n_0+1}(y_{n_0+2}-y_{n_0+1}) &= (p_{n_0}-q_{n_0+1})(y_{n_0+1}-y_{n_0})-q_{n_0+1}y_{n_0}\\
&= (p_{n_0}-q_{n_0+1})y_{n_0+1} > 0,
\end{aligned}
$$

which implies that $y_{n_0+2} > y_{n_0+1} > 0$. Taking $n = n_0 + 2$, we get

$$p_{n_0+2}(y_{n_0+3}-y_{n_0+2}) = (p_{n_0+1}-q_{n_0+2})(y_{n_0+2}-y_{n_0+1})-q_{n_0+2}y_{n_0+1} > 0.$$

Hence $y_{n_0+3} > y_{n_0+2} > 0$. Proceeding as above we get $y_n > 0$ for $n \geqslant n_0 + 1$. This completes the proof of the theorem.

Remark. In [5], there are no sufficient conditions for non-oscillation of (1). However, there are sufficient conditions for non-oscillation of (1) in [2] but the verification of the conditions requires a lot of effort (see Theorem 7.16 and Example 7.17 on p. 325, [2]). On the other hand, a simple observation of the sign of q_n in (1) yields the non-oscillation of (1) by Theorem 1. In Example 7.17 ([2]), $q_n = -1/n < 0$, $n \geqslant 1$ and hence the equation is non-oscillatory.

Example. Let $p_n = p > 0$, $n \geqslant 0$ and $q_n = q \leqslant 0$, $n \geqslant 1$ in (1). From Theorem 1 it follows that all solutions of

$$py_{n+1} + (q-2p)y_n + py_{n-1} = 0 \tag{5}$$

are non-oscillatory. The characteristic equation of (5) is given by $p\lambda^2-(2p-q)\lambda+p = 0$. Hence

$$\lambda = \frac{(2p-q)\pm\sqrt{(2p-q)^2-4p^2}}{2p}.$$

Further, $p > 0$ and $q \leqslant 0$ imply that $2p - q > 0$ and hence

$$\lambda_1 = \frac{(2p-q)+\sqrt{(2p-q)^2-4p^2}}{2p} > 0 \quad\text{and}\quad \lambda_2 = \frac{(2p-q)-\sqrt{(2p-q)^2-4p^2}}{2p} > 0.$$

A basis of the solution space of (5) is given by $\{\{\lambda_1^n\},\{\lambda_2^n\}\}$. Thus all solutions of (5) are non-oscillatory.

Corollary 2. *If all solutions of* (1) *are oscillatory, then there exists a sub-sequence* $\{n_k\}$ *of* $\{n\}$ *with* $n_k \to \infty$ *as* $k \to \infty$ *such that* $q_{n_k} > 0$.

This follows from Theorem 1.

Remark. We cannot make this observation using the results in the existing literature.

E x a m p l e. Consider

$$\Delta^2 y_{n-1} + \tfrac{1}{2}(-1)^n y_n = 0, \quad n \geqslant 1.$$

All solutions of this equation are oscillatory (see Example 6.23, p. 296, [5]). Here $q_n = \frac{1}{2}(-1)^n > 0$ for n even and < 0 for n odd.

Theorem 3. *Let* $\lim_{n\to\infty} p_n = p > 0$ *and* $\liminf_{n\to\infty} q_n = q > 0$. *Then all solutions of* (1) *are oscillatory.*

P r o o f. Let us assume that $\{y_n\}$ is a non-oscillatory solution of (1). Hence there exists an integer $N_1 > 1$ such that either $y_n > 0$ or < 0 for all $n \geqslant N_1$. Without any loss of generality, we may assume that $y_n > 0$ for $n \geqslant N_1$. Since $q > 0$, there exists an integer $N_2 > 1$ such that $q_n > 0$ for $n \geqslant N_2$. Let $N > \max\{N_1, N_2\}$. For $n \geqslant N$, we set

$$z_n = \frac{y_n}{y_{n-1}} > 0$$

to obtain from (1)

$$p_n z_{n+1} + (q_n - p_n - p_{n-1}) + p_{n-1} z_n^{-1} = 0.$$

Hence $p_{n-1} z_n^{-1} = (p_n + p_{n-1} - q_n) - p_n z_{n+1} < p_n + p_{n-1}$ implies that $z_n^{-1} < p_n/p_{n-1} + 1$. Hence

$$\limsup_{n\to\infty} z_n^{-1} \leqslant \limsup_{n\to\infty} \Big(\frac{p_n}{p_{n-1}} + 1\Big) \leqslant \limsup_{n\to\infty} \frac{p_n}{p_{n-1}} + 1 = 2$$

implies that $\mu \geqslant \frac{1}{2}$, where $\mu = \liminf_{n\to\infty} z_n$. Further,

$$p_n z_{n+1} = (p_n + p_{n-1} - q_n) - p_{n-1} z_n^{-1} < p_n + p_{n-1}$$

implies that

$$\mu = \liminf_{n\to\infty} z_{n+1} \leqslant \limsup_{n\to\infty} z_{n+1} \leqslant \limsup_{n\to\infty} \Big(1 + \frac{p_{n-1}}{p_n}\Big) = 2.$$

Moreover, $p_{n-1} z_n^{-1} = p_n + p_{n-1} - q_n - p_n z_{n+1}$ implies that

$$\begin{aligned}\limsup_{n\to\infty} z_n^{-1} &= \limsup_{n\to\infty} \Big(\frac{p_n}{p_{n-1}} + 1 - \frac{q_n}{p_{n-1}} - \frac{p_n}{p_{n-1}} z_{n+1}\Big) \\ &\leqslant \limsup_{n\to\infty} \Big(\frac{p_n}{p_{n-1}} + 1\Big) - \liminf_{n\to\infty} \frac{q_n}{p_{n-1}} - \liminf_{n\to\infty} \Big(\frac{p_n}{p_{n-1}} z_{n+1}\Big),\end{aligned}$$

that is, $1/\mu \leqslant 2 - q/p - \mu$, that is, $q \leqslant f(\mu)$, where

$$f(\mu) = \frac{2\mu p - p - p\mu^2}{\mu}.$$

We notice that $f(\mu)$ has its maximum at $\mu = 1$ and $f(1) = 0$. This implies that $q \leqslant 0$, a contradiction. Hence the theorem is proved.

E x a m p l e. Consider (5) with $p > 0$ and $q > 0$. All solutions of (5) are oscillatory by Theorem 3. If $q = 4p$, then $\lambda_1 = \lambda_2 = -1$. If $q < 4p$, then $\lambda_1 = a + ib$ and $\lambda_2 = a - ib$, where a and b are real. If $q > 4p$, then $\lambda_1 < 0$ and $\lambda_2 < 0$. In each case, all solutions of (5) are oscillatory.

Corollary 4. *Let* $\lim\limits_{n\to\infty} p_n = p > 0$. *If all solutions of* (1) *are nonoscillatory, then* $q \leqslant 0$, *where* $q = \liminf\limits_{n\to\infty} q_n$.

It follows from Theorem 3, because $q > 0$ implies that all solutions of (1) are oscillatory.

R e m a r k. $q_n \leqslant 0$ for large n implies that $\liminf\limits_{n\to\infty} q_n \leqslant 0$. But the converse is not necessarily true. Indeed, if $q_n = \frac{1}{2} - \sin n$, then $\liminf\limits_{n\to\infty} q_n = -\frac{1}{2} < 0$. However, $\limsup\limits_{n\to\infty} q_n = \frac{3}{2}$ implies that $q_{n_k} > 0$, where $\{n_k\}$ is a subsequence of $\{n\}$ with $n_k \to \infty$ as $k \to \infty$.

Theorem 5. *Let* $p_n > 0$ *for* $n \geqslant 0$ *with* $\lim\limits_{n\to\infty} p_n = p > 0$. *Then all solutions of*

$$\Delta(p_{n-1}\Delta y_{n-1}) + qy_n = 0,\ \ n \geqslant 0,$$

are non-oscillatory (oscillatory) if and only if $q \leqslant 0$ $(q > 0)$.

This follows from Theorem 1 and Corollary 4 (Corollary 2 and Theorem 3).

Theorem 6 (see Lemma 7.10, [2]). *If* $q_{n_k} \geqslant p_{n_k} + p_{n_k-1}$ *for a subsequence* $\{n_k\}$ *of* $\{n\}$, *then all solutions of* (1) *are oscillatory.*

E x a m p l e. All solutions of

$$y_{n+1} + y_n + y_{n-1} = 0,\ \ n \geqslant 1,$$

are oscillatory by Theorem 6. In particular, $u_n = \cos(2\pi n/3)$ and $v_n = \sin(2\pi n/3)$ are linearly independent oscillatory solutions of the equation.

R e m a r k. $q_{n_k} \geqslant p_{n_k} + p_{n_k-1}$ for a subsequence $\{n_k\}$ of $\{n\}$ implies that $q_{n_k} > 0$. We shall get necessary and sufficient conditions for non-oscillation of (1) if the condition in Theorem 6 is weakened to $q_{n_k} > 0$ for a subsequence $\{n_k\}$ of $\{n\}$ due to Corollary 2.

3. Non-oscillation of non-homogeneous equations

This section deals with non-oscillation of non-homogeneous equations of the form (4). They may be written as

$$p_{n+1}y_{n+2} + (q_{n+1} - p_{n+1} - p_n)y_{n+1} + p_n y_n = f_n, \ n \geqslant 0. \tag{6}$$

On some occasions we use the following two forms of (6):

$$p_{n+1}(y_{n+2} - y_{n+1}) = (p_n - q_{n+1})(y_{n+1} - y_n) - q_{n+1}y_n + f_n, \ n \geqslant 0, \tag{7}$$

and

$$p_{n+1}(y_{n+2} - y_{n+1}) = (p_n - q_{n+1})y_{n+1} - p_n y_n + f_n, \ n \geqslant 0. \tag{8}$$

A solution $\{y_n\}$ of (6) is said to have a simple zero at $n_0 \geqslant 0$ if $y_{n_0} = 0$. It is said to have a sign-changing zero at $n_0 \geqslant 1$ if $y_{n_0-1}y_{n_0} < 0$. A solution $\{y_n\}$ of (6) is said to have a generalized zero at n_0 if $n_0 \geqslant 0$ is a simple zero or $n_0 \geqslant 1$ is a sign-changing zero of the solution.

Remark. We note that in the definition of oscillation of a solution $\{y_n\}$ of (1), $y_{m-1}y_m \leqslant 0$ for $m > N > 0$ implies that $y_{m-1} = 0$ or $y_m = 0$ or $y_{m-1}y_m < 0$, that is, $m-1$ or m is a simple zero or m is a sign-changing zero. Thus $\{y_n\}$ is oscillatory if it has arbitrarily large generalized zeros.

Theorem 7. *If $q_n \leqslant 0$ and $f_n \geqslant 0$ or $\leqslant 0$ but $\not\equiv 0$ for large n, then all solutions of (6) are non-oscillatory.*

Proof. Let $q_n \leqslant 0$ and $f_n \geqslant 0$ for $n \geqslant N > 0$. The case $f_n \leqslant 0$ for $n \geqslant N$ can be dealt with similarly. Let us assume that $\{y_n\}$ is an oscillatory solution of (6). Hence it has arbitrarily large generalized zeros. Let n_0 and m_0 ($m_0 > n_0 > N$) be two consecutive generalized zeros of $\{y_n\}$. We consider four possible cases and arrive at a contradiction in each case to complete the proof of the theorem.

Case (i): Both n_0 and m_0 are simple zeros of $\{y_n\}$, that is, $y_{n_0} = 0$ and $y_{m_0} = 0$, where $m_0 \geqslant n_0 + 1$. Suppose $m_0 = n_0 + 1$. Then $y_{n_0+2} \geqslant 0$ by (6). We note that $y_{n_0+2} = 0$ if and only if $f_{n_0} = 0$. From (6) we obtain $y_{n_0+3} > 0$ if $f_{n_0+1} > 0$ or $f_{n_0} > 0$ and $y_{n_0+3} = 0$ if $f_{n_0+1} = 0$ and $f_{n_0} = 0$. Further, from (6) we get $y_{n_0+4} > 0$ if $f_{n_0+2} > 0$ or $f_{n_0+1} > 0$ or $f_{n_0} > 0$ and $y_{n_0+4} = 0$ if $f_{n_0+2} = 0$, $f_{n_0+1} = 0$ and $f_{n_0} = 0$. Since $f_n \not\equiv 0$ for large n, we can find $n^* > N$ such that $f_{n^*} > 0$ and $f_n = 0$ for $n < n^*$. In this case, our consecutive zeros are $y_{n^*} = 0$ and $y_{n^*+1} = 0$. From (6) we obtain $y_{n^*+2} > 0$. Hence, without any loss of generality, we may take $f_{n_0} > 0$ and hence $y_{n_0+2} > 0$. From (8) we have

$$p_{n_0+2}(y_{n_0+3} - y_{n_0+2}) = (p_{n_0+1} - q_{n_0+2})y_{n_0+2} + f_{n_0+1} > 0$$

and hence $y_{n_0+3} > y_{n_0+2} > 0$. We note that we can have $f_{n_0+1} \geqslant 0$. From (7) we get

$$p_{n_0+3}(y_{n_0+4} - y_{n_0+3}) = (p_{n_0+2} - q_{n_0+3})(y_{n_0+3} - y_{n_0+2}) - q_{n_0+3}y_{n_0+2} + f_{n_0+2} > 0,$$

that is, $y_{n_0+4} > y_{n_0+3} > 0$. Proceeding as above we obtain $y_n > 0$ for $n \geqslant n_0 + 2$, a contradiction to the fact that $\{y_n\}$ is oscillatory. Suppose that $m_0 = n_0 + 2$. We have two possibilities, viz. $y_{n_0+1} > 0$ or $y_{n_0+1} < 0$, that is, $y_{n_0} = 0$, $y_{n_0+1} > 0$ and $y_{n_0+2} = 0$ or $y_{n_0} = 0$, $y_{n_0+1} < 0$ and $y_{n_0+2} = 0$. Let $y_{n_0+1} > 0$. From (6) it follows that

$$0 = p_{n_0+1}y_{n_0+2} = (p_{n_0+1} + p_{n_0} - q_{n_0+1})y_{n_0+1} - p_{n_0}y_{n_0} + f_{n_0} > 0,$$

a contradiction. Let $y_{n_0+1} < 0$. From (6) we get

$$p_{n_0+2}y_{n_0+3} = (p_{n_0+2} + p_{n_0+1} - q_{n_0+2})y_{n_0+2} - p_{n_0+1}y_{n_0+1} + f_{n_0+1} > 0.$$

Hence $y_{n_0+3} > 0$. Equation (8) yields

$$p_{n_0+3}(y_{n_0+4} - y_{n_0+3}) = (p_{n_0+2} - q_{n_0+3})y_{n_0+3} - p_{n_0+2}y_{n_0+2} + f_{n_0+2} > 0,$$

that is, $y_{n_0+4} > y_{n_0+3} > 0$. Using (7) we obtain

$$p_{n_0+4}(y_{n_0+5} - y_{n_0+4}) = (p_{n_0+3} - q_{n_0+4})(y_{n_0+4} - y_{n_0+3}) - q_{n_0+4}y_{n_0+3} + f_{n_0+3} > 0,$$

which implies that $y_{n_0+5} > y_{n_0+4} > 0$. Repeated use of (7) yields $y_n > 0$ for $n \geqslant n_0 + 6$. Thus $y_n > 0$ for $n \geqslant n_0 + 3$, a contradiction. If $m_0 = n_0 + 3$, then we consider two cases, viz. $y_{n_0+1} > 0$ and $y_{n_0+2} > 0$ or $y_{n_0+1} < 0$ and $y_{n_0+2} < 0$. As we are considering two consecutive simple zeros at $n = n_0$ and $n = m_0$, neither the case $y_{n_0+1} > 0$ and $y_{n_0+2} < 0$ nor the case $y_{n_0+1} < 0$ and $y_{n_0+2} > 0$ arises. Suppose $y_{n_0} = 0$, $y_{n_0+1} > 0$, $y_{n_0+2} > 0$ and $y_{n_0+3} = 0$. From (8) we get

$$p_{n_0+1}(y_{n_0+2} - y_{n_0+1}) = (p_{n_0} - q_{n_0+1})y_{n_0+1} - p_{n_0}y_{n_0} + f_{n_0} > 0,$$

that is, $y_{n_0+2} > y_{n_0+1} > 0$. Equation (7) yields

$$0 > p_{n_0+2}(y_{n_0+3} - y_{n_0+2}) = (p_{n_0+1} - q_{n_0+2})(y_{n_0+2} - y_{n_0+1}) - q_{n_0+2}y_{n_0+1} + f_{n_0+1} > 0,$$

a contradiction. Let $y_{n_0} = 0, y_{n_0+1} < 0, y_{n_0+2} < 0$ and $y_{n_0+3} = 0$. Equation (6) yields

$$p_{n_0+3}y_{n_0+4} + (q_{n_0+3} - p_{n_0+3} - p_{n_0+2})y_{n_0+3} = -p_{n_0+2}y_{n_0+2} + f_{n_0+2} > 0,$$

that is, $y_{n_0+4} > 0$. From (7) we obtain

$$p_{n_0+4}(y_{n_0+5} - y_{n_0+4}) = (p_{n_0+3} - q_{n_0+4})(y_{n_0+4} - y_{n_0+3}) - q_{n_0+4}y_{n_0+3} + f_{n_0+3} > 0.$$

Hence $y_{n_0+5} > y_{n_0+4} > 0$. Repeated use of (7) yields $y_{n_0+6} > y_{n_0+5} > 0$, $y_{n_0+7} > y_{n_0+6} > 0$ and so on. Hence $y_n > 0$ for $n \geqslant n_0 + 4$, a contradiction. If $m_0 = n_0 + 4$, then either $y_{n_0} = 0$, $y_{n_0+1} > 0$, $y_{n_0+2} > 0$, $y_{n_0+3} > 0$ and $y_{n_0+4} = 0$ or $y_{n_0} = 0$, $y_{n_0+1} < 0$, $y_{n_0+2} < 0$, $y_{n_0+3} < 0$ and $y_{n_0+4} = 0$. Consider the former case. Equation (8) yields

$$p_{n_0+1}(y_{n_0+2} - y_{n_0+1}) = (p_{n_0} - q_{n_0+1})y_{n_0+1} - p_{n_0}y_{n_0} + f_{n_0} > 0,$$

that is, $y_{n_0+2} > y_{n_0+1} > 0$. From (7) we get

$$p_{n_0+2}(y_{n_0+3} - y_{n_0+2}) = (p_{n_0+1} - q_{n_0+2})(y_{n_0+2} - y_{n_0+1}) - q_{n_0+2}y_{n_0+1} + f_{n_0+1} > 0,$$

that is, $y_{n_0+3} > y_{n_0+2} > 0$. Further use of (7) yields

$$0 > p_{n_0+3}(y_{n_0+4} - y_{n_0+3}) = (p_{n_0+2} - q_{n_0+3})(y_{n_0+3} - y_{n_0+2}) - q_{n_0+3}y_{n_0+2} + f_{n_0+2} > 0,$$

a contradiction. Next consider the latter case. Equation (6) yields $y_{n_0+5} > 0$ by virtue of the facts that $y_{n_0+4} = 0$ and $y_{n_0+3} < 0$. Repeated use of (7) gives $y_{n_0+6} > y_{n_0+5} > 0$, $y_{n_0+7} > y_{n_0+6} > 0$ and so on. Hence $y_n > 0$ for $n \geqslant n_0 + 5$, a contradiction. In a similar way, we arrive at a contradiction for $m_0 \geqslant n_0 + 5$. Thus the solution $\{y_n\}$ cannot have two consecutive simple zeros.

Case (ii): Each of n_0 and m_0 is a sign-changing zero, that is, $y_{n_0-1}y_{n_0} < 0$ and $y_{m_0-1}y_{m_0} < 0$, where $m_0 \geqslant n_0 + 1$. Let $m_0 = n_0 + 1$. We have to consider two cases, viz. $y_{n_0-1} > 0$, $y_{n_0} < 0$ and $y_{n_0+1} > 0$ or $y_{n_0-1} < 0$, $y_{n_0} > 0$ and $y_{n_0+1} < 0$. Consider the first case. From (8) we get

$$p_{n_0+1}(y_{n_0+2} - y_{n_0+1}) = (p_{n_0} - q_{n_0+1})y_{n_0+1} - p_{n_0}y_{n_0} + f_{n_0} > 0,$$

that is, $y_{n_0+2} > y_{n_0+1} > 0$. Repeated use of (7) yields $y_{n_0+3} > y_{n_0+2} > 0$, $y_{n_0+4} > y_{n_0+3} > 0$ and so on. Thus $y_n > 0$ for $n \geqslant n_0 + 1$, a contradiction. For the second case, we use (6) to conclude that

$$0 > p_{n_0}y_{n_0+1} = (p_{n_0} + p_{n_0-1} - q_{n_0})y_{n_0} - p_{n_0-1}y_{n_0-1} + f_{n_0-1} > 0,$$

a contradiction. If $m_0 = n_0 + 2$, then we consider two cases, viz. $y_{n_0-1} > 0$, $y_{n_0} < 0$, $y_{n_0+1} < 0$ and $y_{n_0+2} > 0$ or $y_{n_0-1} < 0, y_{n_0} > 0$, $y_{n_0+1} > 0$ and $y_{n_0+2} < 0$. Consider the former case. From (8) we have

$$p_{n_0+2}(y_{n_0+3} - y_{n_0+2}) = (p_{n_0+1} - q_{n_0+2})y_{n_0+2} - p_{n_0+1}y_{n_0+1} + f_{n_0+1} > 0,$$

which implies that $y_{n_0+3} > y_{n_0+2} > 0$. Repeated use of (7) yields $y_{n_0+4} > y_{n_0+3} > 0$, $y_{n_0+5} > y_{n_0+4} > 0$ and so on. Thus $y_n > 0$ for $n \geqslant n_0 + 2$, a contradiction. Considering the latter case, we obtain from (8) that

$$p_{n_0}(y_{n_0+1} - y_{n_0}) = (p_{n_0-1} - q_{n_0})y_{n_0} - p_{n_0-1}y_{n_0-1} + f_{n_0-1} > 0,$$

that is, $y_{n_0+1} > y_{n_0} > 0$. From (7) we have

$$0 > p_{n_0+1}(y_{n_0+2} - y_{n_0+1}) = (p_{n_0} - q_{n_0+1})(y_{n_0+1} - y_{n_0}) - q_{n_0+1}y_{n_0} + f_{n_0} > 0,$$

a contradiction, because $y_{n_0+1} > 0$ and $y_{n_0+2} < 0$. Suppose that $m_0 = n_0 + 3$. We consider two cases, viz. $y_{n_0-1} > 0$, $y_{n_0} < 0$, $y_{n_0+1} < 0$, $y_{n_0+2} < 0$ and $y_{n_0+3} > 0$ or $y_{n_0-1} < 0$, $y_{n_0} > 0$, $y_{n_0+1} > 0$, $y_{n_0+2} > 0$ and $y_{n_0+3} < 0$. The use of (8) and then the repeated use of (7) in the former case yield $y_n > 0$ for $n \geqslant n_0+3$, a contradiction. In the latter case, the use of (8) yields $y_{n_0+1} > y_{n_0}$. The use of (7) gives $y_{n_0+2} > y_{n_0+1}$ and a further use of (7) yields $0 > p_{n_0+2}(y_{n_0+3} - y_{n_0+2}) > 0$, a contradiction. The above procedure is adopted for $m_0 \geqslant n_0 + 4$ to obtain a contradiction in each case.

Case (iii): Let n_0 be a sign-changing zero and m_0 a simple zero, that is, $y_{n_0-1}y_{n_0} < 0$ and $y_{m_0} = 0$, where $m_0 \geqslant n_0 + 1$. Let $m_0 = n_0 + 1$. As usual we consider two cases, viz. $y_{n_0-1} > 0$, $y_{n_0} < 0$ and $y_{n_0+1} = 0$ or $y_{n_0-1} < 0$, $y_{n_0} > 0$ and $y_{n_0+1} = 0$. For the former case, we use (6) to obtain $y_{n_0+2} > 0$. Successive use of (7) yields $y_{n_0+3} > y_{n_0+2} > 0$, $y_{n_0+4} > y_{n_0+3} > 0$ and so on. Thus $y_n > 0$ for $n \geqslant n_0 + 2$, a contradiction. For the latter case, (6) yields $0 > (q_{n_0} - p_{n_0} - p_{n_0-1})y_{n_0} > 0$, a contradiction. Let $m_0 = n_0 + 2$. Then $y_{n_0-1} > 0$, $y_{n_0} < 0$, $y_{n_0+1} < 0$ and $y_{n_0+2} = 0$ or $y_{n_0-1} < 0$, $y_{n_0} > 0$, $y_{n_0+1} > 0$ and $y_{n_0+2} = 0$. For the former case, (6) yields $y_{n_0+3} > 0$.Then using (7) repeatedly, we obtain $y_{n_0+4} > y_{n_0+3} > 0$, $y_{n_0+5} > y_{n_0+4} > 0$ and so on. Hence $y_n > 0$ for $n \geqslant n_0 + 3$, a contradiction. Next consider the latter case. Equation (8) yields $y_{n_0+1} > y_{n_0}$. From (7) we obtain $0 > p_{n_0+1}(y_{n_0+2} - y_{n_0+1}) > 0$, a contradiction. If $m_0 = n_0 + 3$, then we consider two cases, viz. $y_{n_0-1} > 0$, $y_{n_0} < 0$, $y_{n_0+1} < 0$, $y_{n_0+2} < 0$ and $y_{n_0+3} = 0$ or $y_{n_0-1} < 0$, $y_{n_0} > 0$, $y_{n_0+1} > 0$, $y_{n_0+2} > 0$ and $y_{n_0+3} = 0$. One may proceed as in the case $m_0 = n_0 + 2$ to arrive at a contradiction in each case. For $m_0 \geqslant n_0 + 4$, the same procedure is used to get a contradiction.

Case (iv): Let n_0 be a simple zero and m_0 a sign-changing zero, that is, $y_{n_0} = 0$ and $y_{m_0-1}y_{m_0} < 0$, where $m_0 \geqslant n_0 + 2$. Let $m_0 = n_0 + 2$. Then we have two possibilities, viz. $y_{n_0} = 0$, $y_{n_0+1} > 0$ and $y_{n_0+2} < 0$ or $y_{n_0} = 0$, $y_{n_0+1} < 0$ and $y_{n_0+2} > 0$. For the former case, the use of (6) yields

$$0 > p_{n_0+1}y_{n_0+2} = (p_{n_0+1} + p_{n_0} - q_{n_0+1})y_{n_0+1} - p_{n_0}y_{n_0} + f_{n_0} > 0,$$

a contradiction. Consider the latter case. From (8) we obtain

$$p_{n_0+2}(y_{n_0+3} - y_{n_0+2}) = (p_{n_0+1} - q_{n_0+2})y_{n_0+2} - p_{n_0+1}y_{n_0+1} + f_{n_0+1} > 0,$$

that is, $y_{n_0+3} > y_{n_0+2} > 0$. Repeated use of (7) yields $y_{n_0+4} > y_{n_0+3} > 0$, $y_{n_0+5} > y_{n_0+4} > 0$ and so on. Hence $y_n > 0$ for $n \geqslant n_0 + 2$, a contradiction. If $m_0 = n_0 + 3$, then we have two cases, viz. $y_{n_0} = 0$, $y_{n_0+1} > 0$, $y_{n_0+2} > 0$ and $y_{n_0+3} < 0$ or $y_{n_0} = 0$, $y_{n_0+1} < 0$, $y_{n_0+2} < 0$ and $y_{n_0+3} > 0$. For the former case, (7) yields $y_{n_0+2} > y_{n_0+1}$. Further use of (7) gives $0 > p_{n_0+2}(y_{n_0+3} - y_{n_0+2}) > 0$, a contradiction. For the latter case, we obtain from (8) that $y_{n_0+4} > y_{n_0+3} > 0$. Then the repeated use of (7) yields $y_{n_0+5} > y_{n_0+4} > 0$, $y_{n_0+6} > y_{n_0+5} > 0$ and so on. Thus $y_n > 0$ for $n \geqslant n_0 + 3$, a contradiction. If $m_0 \geqslant n_0 + 4$, then one may proceed as above to arrive at a contradiction in each case.

Thus (6) does not admit an oscillatory solution, that is, all solutions of (6) are non-oscillatory.

If $f_n \leqslant 0$ for $n \geqslant N$, then we set $z_n = -y_n$ in (6) to obtain

$$p_{n+1}z_{n+2} + (q_{n+1} - p_{n+1} - p_n)z_{n+1} + p_n z_n = g_n, \tag{9}$$

where $g_n = -f_n \geqslant 0$ for $n \geqslant N$. Proceeding as above, one can show that all solutions of (9) are non-oscillatory. Hence all solutions of (6) are non-oscillatory.

This completes the proof of the theorem.

Example. Consider

$$y_{n+2} - 3y_{n+1} + y_n = n + 1, \ \ n \geqslant 0. \tag{10}$$

Here $p_n = 1, q_n = -1$ and $f_n = n + 1$. From Theorem 7 it follows that all solutions of (10) are non-oscillatory. The homogeneous equation associated with (10) is given by

$$y_{n+2} - 3y_{n+1} + y_n = 0, \ \ n \geqslant 0. \tag{11}$$

Equation (11) has two linearly independent solutions $\{(\frac{1}{2}(3+\sqrt{5}))^n\}$ and $\{(\frac{1}{2}(3-\sqrt{5}))^n\}$ which form a basis of the solution space of the equation. A particular solution of (10) is given by $\{-n\}$. Hence the general solution of (10) is

$$y_n = c_1\left(\frac{3+\sqrt{5}}{2}\right)^n + c_2\left(\frac{3-\sqrt{5}}{2}\right)^n - n, \quad n \geqslant 0.$$

In the sequel, we show that $\{y_n\}$ is non-oscillatory. Writing $\{y_n\}$ as

$$y_n = \left(\frac{3+\sqrt{5}}{2}\right)^n \left[c_1 + c_2\left(\frac{3-\sqrt{5}}{3+\sqrt{5}}\right)^n - \frac{2^n n}{(3+\sqrt{5})^n}\right]$$

and observing that $\lim\limits_{n\to\infty}(n2^n/(3+\sqrt{5})^n) = 0$ and $\lim\limits_{n\to\infty}((3-\sqrt{5})/(3+\sqrt{5}))^n = 0$, we obtain $\lim\limits_{n\to\infty} y_n = +\infty$ or $-\infty$ as $c_1 > 0$ or < 0. If $c_1 = 0$ but $c_2 \neq 0$, then $\lim\limits_{n\to\infty} y_n = -\infty$ because $\lim\limits_{n\to\infty}((3-\sqrt{5})/2)^n = 0$. If $c_1 = 0$ and $c_2 = 0$, then clearly $\lim\limits_{n\to\infty} y_n = -\infty$. Hence $\{y_n\}$ is non-oscillatory.

Example. Consider

$$y_{n+2} - 3y_{n+1} + y_n = 1 + (-1)^n, \quad n \geqslant 0. \tag{12}$$

Hence $f_n \geqslant 0$ but $\not\equiv 0$ for $n \geqslant 0$. From Theorem 7 it follows that all solutions of (12) are non-oscillatory. Clearly, $u_n = -1 + \frac{1}{5}(-1)^n$ is a particular solution of (12) with $u_n = -\frac{4}{5}$ for n even and $u_n = -\frac{6}{5}$ for n odd. The general solution of (12) is

$$y_n = c_1\left(\frac{3+\sqrt{5}}{2}\right)^n + c_2\left(\frac{3-\sqrt{5}}{2}\right)^n + \left(\frac{1}{5}(-1)^n - 1\right), \quad n \geqslant 0.$$

Since $3+\sqrt{5} > 2$ and $3-\sqrt{5} < 3+\sqrt{5}$, we have $\lim\limits_{n\to\infty} y_n = +\infty$ or $-\infty$ as $c_1 > 0$ or < 0. Hence y_n is non-oscillatory. If $c_1 = 0$, then

$$y_n = c_2\left(\frac{3-\sqrt{5}}{2}\right)^n + \left(\frac{1}{5}(-1)^n - 1\right), \quad n \geqslant 0.$$

As $((3-\sqrt{5})/2)^n \to 0$ when $n \to \infty$, for $0 < \varepsilon < 4/5$ we can find $N_0 > 0$ such that $|c_2((3-\sqrt{5})/2)^n| < \varepsilon$ for $n \geqslant N_0$. Hence, for $n \geqslant N_0$ we have $y_n < \varepsilon - \frac{4}{5} < 0$. Thus y_n is non-oscillatory. If $c_1 = 0$ and $c_2 = 0$, then $y_n \leqslant -\frac{4}{5} < 0$, $n \geqslant 0$, and hence it is non-oscillatory.

Theorem 8. *Let $p_n > 0$ for $n \geqslant 0$ and $q_n \leqslant 0$ and $f_n \geqslant 0$ or $\leqslant 0$ but $\not\equiv 0$ for large n. Let $g\colon \mathbb{R} \to \mathbb{R}$ be continuous with $xg(x) > 0$ for $x \neq 0$. Then all solutions of*

$$\Delta(p_{n-1}\Delta y_{n-1}) + q_n g(y_n) = f_{n-1}, \quad n \geqslant 1 \tag{13}$$

are non-oscillatory.

Indeed, after expansion Eq. (13) *takes the form*

$$p_{n+1}y_{n+2} - (p_{n+1} + p_n)y_{n+1} + p_n y_n + q_{n+1}g(y_{n+1}) = f_n, \ n \geqslant 0. \tag{14}$$

Observing that $g(0) = 0$, writing (14) *in two different forms as follows and then proceeding as in the proof of Theorem* 7 *one can complete the proof of Theorem* 8*:*

$$p_{n+1}(y_{n+2} - y_{n+1}) = p_n(y_{n+1} - y_n) - q_{n+1}g(y_{n+1}) + f_n, \quad n \geqslant 0$$

and

$$p_{n+1}(y_{n+2} - y_{n+1}) = p_n y_{n+1} - p_n y_n - q_{n+1}g(y_{n+1}) + f_n, \quad n \geqslant 0.$$

Example. All solutions of

$$y_{n+2} - 2y_{n+1} + y_n - y_{n+1}^3 = (n+1)^3, \quad n \geqslant 0$$

are non-oscillatory by Theorem 8. In particular, $y_n = -n$ is a non-oscillatory solution of the equation.

Remark. The conclusion of Theorem 8 remains true if $f_n \equiv 0$, because $g(0) = 0$ implies that two consecutive simple zeroes of the form $y_{n_0} = 0$ and $y_{n_0+1} = 0$ of a non-trivial solution $\{y_n\}$ of the equation leads to $y_n \equiv 0$. Thus this possibility does not arise at all. The other cases are similar to those of Theorem 7.

Theorem 9. *Let f_n change sign for large n. If $q_n \geqslant p_n + p_{n+1}$, then all solutions of* (4), (6) *are oscillatory.*

Proof. Let f_n change sign for $n \geqslant n_1 > 0$. Let us assume that $\{y_n\}$ is a non-oscillatory solution of (6). Hence $y_n > 0$ or < 0 for $n \geqslant n_2 > 0$. Let $y_n > 0$ for $n \geqslant n_2$. The case $y_n < 0$ for $n \geqslant n_2$ can be dealt with similarly. Let $n_0 > \max\{n_1, n_2\}$. For $n \geqslant n_0$,

$$f_n = p_{n+1}y_{n+2} + (q_{n+1} - p_{n+1} - p_n)y_{n+1} + p_n y_n > 0,$$

a contradiction. If $y_n < 0$ for $n \geqslant n_2$, then

$$f_n = p_{n+1}y_{n+2} + (q_{n+1} - p_{n+1} - p_n)y_{n+1} + p_n y_n < 0$$

for $n \geqslant n_0$, a contradiction. Thus the theorem is proved. $\square$

E x a m p l e. All solutions of

$$y_{n+2} + y_{n+1} + y_n = (-1)^n, \quad n \geqslant 0$$

are oscillatory by Theorem 9. In particular, $y_n = (-1)^n$ is an oscillatory solution of the equation. Indeed, the general solution of the equation is given by

$$y_n = c_1 \cos(2\pi n/3) + c_2 \sin(2\pi n/3) + (-1)^n,$$

which is oscillatory.

4. Applications to third order difference equations

Oscillatory/non-oscillatory behaviour of solutions of linear homogeneous third order difference equations of the form

$$y_{n+2} + a_n y_{n+1} + b_n y_n + c_n y_{n-1} = 0, \quad n \geqslant 1, \tag{15}$$

is studied in [9], [11], [12]. However, sufficient conditions in terms of coefficient sequences are not yet available for non-oscillation of all solutions of (15). In this section, an attempt is made in this direction. Moreover, sufficient conditions, in terms of coefficient sequences and the forcing sequence, are obtained for non-oscillation of all solutions of nonhomogeneous third order difference equations of the form

$$y_{n+2} + a_n y_{n+1} + b_n y_n + c_n y_{n-1} = g_{n-1}, \quad n \geqslant 1. \tag{16}$$

In literature not many results on third order difference equations are available.

Theorem 10. *If $q_n \leqslant 0$ for large n, then all solutions of*

$$\Delta(p_{n-1}\Delta^2 y_{n-1}) + q_n \Delta y_n = 0, \quad n \geqslant 1 \tag{17}$$

are non-oscillatory, provided $p_n > 0$ for $n \geqslant 0$.

P r o o f. This follows from Theorem 1. Let $\{y_n\}$ be a solution of (17). Setting $z_n = \Delta y_n$, we obtain from (17) that

$$\Delta(p_{n-1}\Delta z_{n-1}) + q_n z_n = 0, \quad n \geqslant 1.$$

From Theorem 1 it follows that $z_n > 0$ or < 0 for large n. Hence $\Delta y_n > 0$ or < 0 for large n, that is, $\{y_n\}$ is either increasing or decreasing for large n. Thus $\{y_n\}$ is non-oscillatory. Since $\{y_n\}$ is an arbitrary solution of (17), all solutions of (17) are non-oscillatory. Hence the theorem is proved.

After expansion, (17) takes the form

$$p_n y_{n+2} + (q_n - 2p_n - p_{n-1}) y_{n+1} + (p_n + 2p_{n-1} - q_n) y_n - p_{n-1} y_{n-1} = 0. \tag{18}$$

Comparing (18) with (15) we obtain

$$\frac{q_n - 2p_n - p_{n-1}}{p_n} = a_n, \quad \frac{p_n + 2p_{n-1} - q_n}{p_n} = b_n \quad \text{and} \quad -\frac{p_{n-1}}{p_n} = c_n.$$

If $c_n \neq 0$ for $n \geqslant 1$, then

$$p_n = \frac{(-1)^n p_0}{\prod\limits_{i=1}^{n} c_i}, \quad q_n = p_n(a_n + 2 - c_n) = p_n(1 - b_n - 2c_n). \tag{19}$$

Hence $p_n > 0$ for $n \geqslant 0$ if and only if $p_0 > 0$, $\prod\limits_{i=1}^{2m} c_i > 0$ and $\prod\limits_{i=1}^{2m+1} c_i < 0$ and $a_n + b_n + c_n + 1 = 0$. Thus the following theorem follows from Theorem 10.

Theorem 11. *If* $p_0 > 0$, $c_n \neq 0$ *for* $n \geqslant 1$, $\prod\limits_{i=1}^{2m} c_i > 0$, $\prod\limits_{i=1}^{2m+1} c_i < 0$, $a_n+b_n+c_n+1 = 0$ *and* $a_n + 2 - c_n \leqslant 0$ *or* $1 - b_n - 2c_n \leqslant 0$ *for large* n, *then all solutions of* (15) *are non-oscillatory.*

Indeed, consider (17) *with* $\{p_n\}$ *and* $\{q_n\}$ *given by* (19). *Hence* $p_n > 0$ *for* $n \geqslant 0$ *and* $q_n \leqslant 0$ *for large* n. *If* $\{y_n\}$ *is a solution of* (15), *then it is a solution of* (17). *From Theorem* 10 *it follows that* $\{y_n\}$ *is non-oscillatory. Thus all solutions of* (15) *are non-oscillatory.*

E x a m p l e. Consider

$$y_{n+2} - 4y_{n+1} + 4y_n - y_{n-1} = 0, \quad n \geqslant 1. \tag{20}$$

If we choose $p_0 = 1$, then all conditions of Theorem 11 are satisfied and hence all solutions of (20) are non-oscillatory. The characteristic equation of (20) is $\lambda^3 - 4\lambda^2 + 4\lambda - 1 = 0$, that is, $(\lambda - 1)(\lambda^2 - 3\lambda + 1) = 0$. Hence a basis of the solution space of (20) is

$$\left\{\{1\}, \left\{\left(\frac{3+\sqrt{5}}{2}\right)^n\right\}, \left\{\left(\frac{3-\sqrt{5}}{2}\right)^n\right\}\right\}.$$

Thus all solutions of (20) are non-oscillatory.

Equation (16) can be put in the form

$$\Delta(p_{n-1}\Delta^2 y_{n-1}) + q_n \Delta y_n = f_{n-1}, \quad n \geqslant 1, \tag{21}$$

where $\{p_n\}$ and $\{q_n\}$ are given by (19) and $f_{n-1} = p_n g_{n-1}$. As above, the application of Theorem 7 to (21) leads to the following theorem.

Theorem 12. *If* $p_0 > 0$, $c_n \neq 0$ *for* $n \geqslant 1$, $\prod_{i=1}^{2m} c_i > 0$, $\prod_{i=1}^{2m+1} c_i < 0$, $a_n + b_n + c_n + 1 = 0$, $a_n + 2 - c_n \leqslant 0$ *or* $1 - b_n - 2c_n \leqslant 0$ *and* $g_n \geqslant 0$ *or* $\leqslant 0$ *but* $\not\equiv 0$ *for large* n, *then all solutions of* (16) *are non-oscillatory.*

Example. Consider

$$y_{n+2} - 4y_{n+1} + 4y_n - y_{n-1} = -1 - 2n, \quad n \geqslant 1. \tag{22}$$

All solutions of (22) are non-oscillatory by Theorem 12 if we choose $p_0 = 1$. Indeed, the general solution of (22) is given by

$$y_n = c_1 + c_2\left(\frac{3+\sqrt{5}}{2}\right)^n + c_3\left(\frac{3-\sqrt{5}}{2}\right)^n + n^2, \quad n \geqslant 1,$$

because $u_n = n^2$ is a particular solution of (22). Writing

$$y_n = \left(\frac{3+\sqrt{5}}{2}\right)^n \left[c_1\left(\frac{2}{3+\sqrt{5}}\right)^n + c_2 + c_3\left(\frac{3-\sqrt{5}}{3+\sqrt{5}}\right)^n + \frac{n^2 2^n}{(3+\sqrt{5})^n}\right],$$

we observe that $y_n \to +\infty$ or $-\infty$ as $n \to \infty$ if $c_2 > 0$ or < 0, respectively. If $c_2 = 0$, then

$$y_n = c_1 + c_3\left(\frac{3-\sqrt{5}}{2}\right)^n + n^2, \quad n \geqslant 1,$$

implies that $y_n \to +\infty$ as $n \to \infty$ for all values of c_1 and c_3. Hence $\{y_n\}$ is non-oscillatory.

Theorem 13. *Let* $p_n > 0$ *for* $n \geqslant 0$. *Let* $q_n \leqslant 0$ *and* $r_n \geqslant 0$ *or* $\leqslant 0$ *but* $\not\equiv 0$ *for large* n. *If* $h\colon \mathbb{R} \to \mathbb{R}$ *is continuous with* $xh(x) > 0$ *for* $x \neq 0$, *then all solutions of*

$$\Delta(p_{n-1}\Delta^2 y_{n-1}) + q_n h(\Delta y_n) = r_{n-1}, \quad n \geqslant 1,$$

are non-oscillatory.

An application of Theorem 8 yields the proof.

E x a m p l e. All solutions of

$$\Delta^3 y_{n-1} - \frac{25}{2}(\Delta y_n)^3 = -6n - \frac{25}{2}, \quad n \geqslant 1,$$

are non-oscillatory by Theorem 13. In particular, $y_n = n$ is such a solution. As the equation is nonlinear, we cannot get explicitly all its non-oscillatory solutions. If we consider

$$\Delta^3 y_{n-1} - \frac{25}{2}\Delta y_n = -6n - \frac{25}{2}, \quad n \geqslant 1,$$

then its general solution

$$y_n = c_1\left(\frac{1}{2}\right)^n + c_2\left(\frac{15+\sqrt{217}}{2}\right)^n + c_3\left(\frac{15-\sqrt{217}}{2}\right)^n + n$$

is non-oscillatory.

5. Conclusions

Necessary and sufficient conditions for oscillation/non-oscillation of (1) in terms of coefficient functions are yet to be established. It seems that (1) is oscillatory if $\{q_n\}$ changes sign (see the example following Corollary 2). However, the converse is not necessarily true (consider $y_{n+1} + y_n + y_{n-1} = 0$). Suppose in (6), $f_n \geqslant 0$ or $\leqslant 0$ but $\not\equiv 0$ for large n. It seems that the sign of $\{q_n\}$ plays an important role. If $q_n \leqslant 0$ for large n, then (6) is non-oscillatory (Theorem 7). If $q_n > 2$ for large n, then (6) can be oscillatory. For example, the general solution of

$$y_{n+2} + y_{n+1} + y_n = \tfrac{3}{2} + (-1)^n, \quad n \geqslant 0,$$

is given by

$$y_n = c_1\cos(2\pi n/3) + c_2\sin(2\pi n/3) + \tfrac{1}{2} + (-1)^n$$

which is oscillatory. If $0 \leqslant q_n \leqslant 2$, then (6) can admit both oscillatory and non-oscillatory solutions. For example,

$$y_{n+2} - y_{n+1} + y_n = \frac{(n+1)^2+1}{n(n+1)(n+2)}, \quad n \geqslant 1,$$

admits an oscillatory solution $\{\sin(n\pi/3)+1/n\}$ and a non-oscillatory solution $\{1/n\}$. Similarly,

$$y_{n+2} + y_n = \frac{1}{n} + \frac{1}{n+2}, \quad n \geqslant 1,$$

admits an oscillatory solution $\{1/n + \sin n\pi\}$ and a non-oscillatory solution $\{1/n\}$. We notice that in the above examples $\{f_n\}$ is bounded. If $f_n \to +\infty$ or $-\infty$ as $n \to \infty$, then (6) can be non-oscillatory. For example, the general solution of

$$y_{n+2} - y_{n+1} + y_n = n + 1, \quad n \geqslant 0,$$

can be written as

$$y_n = c_1 \cos(n\pi/3) + c_2 \sin(n\pi/3) + (n + 1)$$

which tends to ∞ as $n \to \infty$. The problem becomes complex when $\{q_n\}$ changes sign. For third order equations, very few results concerning non-oscillation of solutions are known.

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