ON THE LAGRANGE-SOURIAU FORM IN CLASSICAL FIELD THEORY

D. R. Grigore, O. T. Popp, Bucharest

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Abstract. The Euler-Lagrange equations are given in a geometrized framework using a differential form related to the Poincaré-Cartan form. This new differential form is intrinsically characterized; the present approach does not suppose a distinction between the field and the space-time variables (i.e. a fibration). In connection with this problem we give another proof describing the most general Lagrangian leading to identically vanishing Euler-Lagrange equations. This gives the possibility to have a geometric point of view of the usual Noetherian symmetries for classical field theories and strongly supports the usefulness of the above mentioned differential form.

Keywords: Lagrangian formalism, classical field theory, Noetherian symmetries

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1. Introduction

Since variational principles were accepted as the foundation of many important physical laws, it became apparent that one needs a coordinate independent formulation of the Euler-Lagrange equations. This was pioneered by Poincaré and Cartan for finite number of degrees of freedom using a differential 1-form instead of the Lagrangian [1]–[2].

It was later realized that it is more convenient to work with the exterior differential of the above Poincaré-Cartan form (see for instance [3]). This 2-form is generically presymplectic and was used by Souriau [4] (see also [5]) to obtain the phase space as a symplectic manifold in many physically interesting examples.

It is strongly advocated by Souriau that the fundamental mathematical object for a Lagrangian system must be this 2-form and not the Lagrangian function or the Poincaré-Cartan 1-form. His point of view keeps in the same framework the main interesting features of both the Lagrangian and the Hamiltonian formalism. Also it allows a natural definition for the usual Noetherian symmetries. We will call this 2-form the Lagrange-Souriau form.

The purpose of this paper is to investigate the generalization of the same point of view to the classical field theory. To our knowledge in the literature the main concern seems to be the identification of a suitable generalization of the Poincaré-Cartan form to classical field theory; there is a number of such generalizations [6]–[17]. In these references a fibration structure over the space-time manifold is used in an essential way. Without this fibration hypothesis the Poincaré-Cartan form cannot be intrinsically defined.

We will base our analysis on a generalization of the Lagrange-Souriau 2-form to classical field theory. This object will be intrinsically and globally defined without using the fibration hypothesis mentioned above. Moreover, this Lagrange-Souriau form can be locally written as the exterior differential of a Poincaré-Cartan form related to some chosen fibration. This Poincaré-Cartan form is the same as that given by Krupka [6], Betounes [7]–[8] and Rund [9]. The main feature of our globally defined Lagrange-Souriau form is that it determines (locally) a Lagrangian but only up to a variationally trivial Lagrangian, i.e. a Lagrangian giving trivial Euler-Lagrange equations. This is related to the fact that one can define in a geometrically nice way the Noetherian symmetries using the Lagrange-Souriau form. This is a very strong reason for considering as the main object the Lagrange-Souriau form and not the Poincaré-Cartan form.

In Part 2 of this paper we study the appropriate analog of the Lagrange-Souriau form in this nonfibrating case. For this reason we will need an auxiliary operator which seems to be new in literature. This is the main tool for an intrinsic definition of the Lagrange-Souriau form. We also discuss the Euler-Lagrange equations in this geometrical framework.

In Part 3 we prove that any Lagrangian determined by the Lagrange-Souriau form is given up to a variationally trivial Lagrangian. To this purpose we give an alternative proof for the most general expression of such a Lagrangian using the homogeneous formalism (other proofs can be found in [18]–[21]). We close Part 3 relating the above remarks to the Noetherian symmetries.

2. The geometry of the Euler-Lagrange equations

We present here the Lagrangian formalism in field theory from a purely geometric point of view following the ideas of Souriau [4], as was told in the Introduction.

Our point of view treats the space-time and field variables on the same footing. This is done in the same spirit as in the relativistic theories, where space and time do not have an intrinsic meaning.

2.1. Let S be a differentiable manifold of dimension $n+N$. Usually S is considered to be a fibre bundle over the space-time manifold M of dimension n, the N dimensional fibres describing the field variables. Also, in most cases, the above assumed bundle is a linear one. We emphasize once again that we do not make this hypothesis in this paper.

Remark 1. When S is a fibration over the manifold M, the space-time meaning for M is given by an additional structure on M related to some hyperbolic nature of the Euler-Lagrange equations. When this structure does not exists, M is usually interpreted as the space. The case $n=1$ is associated with dynamical systems having finite numbers of degrees of freedom and M is then interpreted as the axis of the absolute time.

Because we want to introduce the first derivatives of the fields in an intrinsic way, we proceed as follows: Let $T_p(S)$ be the tangent space to S at the point $p \in S$, and denote by $J^1_n(S)_p$ the manifold of all n-dimensional linear subspaces of $T_p(S)$. Then $J^1_n(S) \equiv \bigcup_{p \in S} J^1_n(S)_p$ has a natural fibre bundle structure over S. The canonical projection is denoted by $\pi\colon J^1_n(S) \to S$. This is the so called *bundle of* 1*-jets of* n*-dimensional submanifolds of* S.

To arrive at the usual form of the Euler-Lagrange equations we shall need a system of standard charts on $J^1_n(S)$. Let (x^μ, ψ^A) (where $\mu = 1, \dots, n$ and $A = 1, \dots, N$) be a coordinate system on the open set $U \subset S$. Then, on the open set $V \subset \pi^{-1}(U)$ we choose the coordinate system $(x^\mu, \psi^A, \chi^A{}_\mu)$. The n-plane in $T_{p_0}(S)$ corresponding to a given set of numbers $((x^\mu)_0, (\psi^A)_0, (\chi^A{}_\mu)_0)$ is spanned by the tangent vectors

$$\frac{\delta}{\delta x^\mu} \equiv \frac{\partial}{\partial x^\mu} + (\chi^A{}_\mu)_0 \frac{\partial}{\partial \psi^A}. \tag{2.1}$$

(p_0 corresponds to the set of numbers $((x^\mu)_0, (\psi^A)_0)$.

The purpose of the Lagrangian formalism (in this geometrical setting) is to describe n-dimensional immersed submanifolds, usually given in a parameterized form: $\Psi\colon M \to S$ (Ψ is the immersion). When S is a fibration over the space-time M, Ψ is considered to be a cross section and is called a *field evolution.*

We will need later the natural lift of the n-dimensional immersed submanifold $\Psi\colon M \to S$ to an n-dimensional immersed submanifold $\dot{\Psi}\colon M \to J^1_n(S)$. We describe it using coordinates: if the immersed n-dimensional submanifold (cleverly parameterized) is given by $\Psi\colon (x^\mu) \mapsto (x^\mu, \psi^A(x))$, then the lifted submanifold is: $\dot{\Psi}\colon (x^\mu) \mapsto \left(x^\mu, \psi^A(x), \frac{\partial \psi^A}{\partial x^\mu}(x)\right)$. If $L\colon U \to \mathbb{R}$ is a Lagrangian function (density), then the Euler-Lagrange equations for Ψ are

$$E_A(\Psi) \equiv -\frac{\partial}{\partial x^\mu}\left(\frac{\partial L}{\partial \chi^A{}_\mu} \circ \dot{\Psi}\right) + \frac{\partial L}{\partial \psi^A} \circ \dot{\Psi} = 0. \tag{2.2}$$

2.2. We review here the case $n = 1$. For simplicity it is presented in terms of local coordinates. In this case one usually denotes the x^μ variables by t, the ψ^A variables by q^A and the velocity variables $\chi^A{}_\mu$ by v^A.

Suppose L is a function of (t, q^A, v^A) named the Lagrangian. Then one defines the Poincaré-Cartan 1-form:

$$\theta_L \equiv \frac{\partial L}{\partial v^A}(\,\mathrm{d}q^A - v^A\,\mathrm{d}t) + L\,\mathrm{d}t, \tag{2.3}$$

and the Lagrange-Souriau 2-form:

$$\sigma_L \equiv \mathrm{d}\theta_L. \tag{2.4}$$

A curve $\gamma\colon \mathbb{R} \mapsto S$ is a solution of the Euler-Lagrange equations *iff* it is an integral curve of the foliation $\ker(\sigma_L)$ i.e.

$$(\dot{\gamma})^* i_X(\sigma_L) = 0 \tag{2.5}$$

for every vector field X on the evolution space $E \subset PT(S) \equiv J^1_1(S)$ (here $\dot{\gamma}$ is the natural lift of γ in $PT(S)$).

R e m a r k 2. One of the most important properties of the differential forms θ_L and σ_L is the following one: θ_L is exact (or $\sigma_L = 0$) *iff* the Euler-Lagrange equations for L are vanishing identically. In this case L is called a *variationally trivial Lagrangian.*

2.3. Here we generalize to fields the above frame, valid for $n = 1$.

First we give some notation: using the framework from Section 2.1 we denote by $D^k_q(U)$ the differential k-forms on the open set $U \subset J^1_n(S)$ vanishing when contracted with $q + 1$ vertical vector fields (i.e. vector fields tangent to the fibers of the fiber bundle $\pi\colon J^1_n(S) \to S$).

For an intrinsic meaning of the Lagrange-Souriau form in the field case we define an auxiliary operator $K\colon\ D^k_1(J^1_n(S)) \to D^{k-1}_0(J^1_n(S))$ in

Lemma 1. *The operator* $K\colon\ D^k_1(U) \to D^{k-1}_0(U)$ *defined with respect to a local coordinate system* (x,ψ,χ) *by*

$$K(\omega) \equiv i_{\frac{\delta}{\delta x^\mu}} i_{\frac{\partial}{\partial \chi^A{}_\mu}} (\delta\psi^A \wedge \omega) \tag{2.6}$$

is independent of the chosen coordinates. Here

$$\delta\psi^A \equiv \mathrm{d}\psi^A - \chi^A{}_\mu \,\mathrm{d}x^\mu. \tag{2.7}$$

P r o o f. Let $(x^\mu,\psi^A) \mapsto (y^\mu,\varphi^A)$ be a change of coordinates on S. Here

$$y^\mu = y^\mu(x,\psi), \quad \varphi^A = \varphi^A(x,\psi).$$

The induced change of coordinates on $J^1_n(S)$ is

$$(x^\mu,\psi^A,\chi^A{}_\mu) \mapsto (y^\mu,\varphi^A,\zeta^A{}_\mu)$$

with y^μ and φ^A as above and

$$\zeta^A{}_\mu = \frac{\delta\varphi^A}{\delta y^\mu} = \frac{\delta x^\nu}{\delta y^\mu}\frac{\delta\varphi^A}{\delta x^\nu}. \tag{2.8}$$

From the last relation one can get the explicit dependence of ζ^A_μ on the variables (x,ψ,χ) using the following useful relation:

$$\frac{\delta}{\delta x^\mu} = \frac{\delta y^\nu}{\delta x^\mu}\frac{\delta}{\delta y^\nu} + \left(\frac{\delta\zeta^A{}_\nu}{\delta x^\mu}\right)\frac{\partial}{\partial\zeta^A{}_\nu}. \tag{2.9}$$

An easy consequence of (2.9) is

$$\frac{\delta x^\mu}{\delta y^\nu}\frac{\delta y^\nu}{\delta x^\varrho} = \delta^\mu_\varrho. \tag{2.10}$$

Now, one can get also

$$\delta\varphi^B = \left(\frac{\partial\varphi^B}{\partial\psi^A} - \zeta^B{}_\nu\frac{\partial y^\nu}{\partial\psi^A}\right)\delta\psi^A \tag{2.11}$$

and

$$\frac{\delta y^\nu}{\delta x^\mu}\frac{\partial}{\partial\chi^A{}_\mu} = \left(\frac{\partial\varphi^B}{\partial\psi^A} - \zeta^B{}_\nu\frac{\partial y^\nu}{\partial\psi^A}\right)\frac{\partial}{\partial\zeta^B{}_\nu}. \tag{2.12}$$

Hence it follows that

$$i_{\frac{\delta}{\delta x^\mu}} i_{\frac{\partial}{\partial\chi^A{}_\mu}}(\delta\psi^A\wedge\omega) = i_{\frac{\delta}{\delta y^\nu}} i_{\frac{\partial}{\partial\zeta^B{}_\nu}}(\delta\psi^B\wedge\omega) + \frac{\delta x^\mu}{\delta y^\varrho}\left(\frac{\delta\zeta^B{}_\nu}{\delta x^\mu}\right) i_{\frac{\partial}{\partial\zeta^B{}_\nu}} i_{\frac{\partial}{\partial\zeta^A{}_\varrho}}(\delta\varphi^A\wedge\omega).$$

But the second term above vanishes because $\omega \in D^k_1(U)$. □

R e m a r k 3. From (2.11) and (2.12) it follows that

$$\frac{\delta y^\nu}{\delta x^\mu}\left(\delta\psi^A\otimes\frac{\partial}{\partial\chi^A{}_\mu}\right)=\delta\varphi^B\otimes\frac{\partial}{\partial\zeta^B{}_\nu}. \tag{2.13}$$

Definition 1. A *Lagrange-Souriau* form on the open set $V\subset\pi^{-1}(U)\subset J^1_n(S)$ is a closed form $\sigma\in D^{n+1}_1(U)$ with $K\sigma=0$.

The linear space of these forms is denoted by $S(U)$. Lemma 1 implies that σ is intrinsically defined.

R e m a r k 4. This definition is justified by considering the case $n=1$ and proving that σ can be written as in (2.3) and (2.4). We will prove below that such a (local) expression also exists in the general case $n\geqslant 1$.

Namely, we have

Theorem 1. a) *If (x,ψ,χ) is a system of coordinates on U and $\sigma\in S(U)$, then there exists a smooth function $L\colon U\mapsto R$ such that $\sigma=\mathrm{d}\theta_L$ where*

$$\theta_L\equiv\varepsilon_{\mu_1,\ldots,\mu_n}\sum_{k=0}^{n}\frac{1}{k!}C^k_n\frac{\partial^k L}{\partial\chi^{A_1}{}_{\mu_1}\ldots\partial\chi^{A_k}{}_{\mu_k}}\delta\psi^{A_1}\wedge\ldots\wedge\delta\psi^{A_k}\wedge\mathrm{d}x^{\mu_{k+1}}\wedge\ldots\wedge\mathrm{d}x^{\mu_n}. \tag{2.14}$$

(Here $\varepsilon_{\mu_1,\ldots,\mu_n}$ is the signature of the permutation $(1,\ldots,n)\mapsto(\mu_1,\ldots,\mu_n)$.)

b) *The n-dimensional immersed submanifold $\Psi\colon M\mapsto S$ satisfies the Euler-Lagrange equations for the Lagrangian L (above) iff*

$$\dot\Psi^* i_Z\sigma=0 \tag{2.15}$$

for any vector field Z on $J^1_n(S)$.

P r o o f. a) We begin with a general result:

Lemma 2. *If $E\to S$ is a differentiable bundle and $D^k_q(U)$ is defined as at the beginning of Section 2.3, then for a sufficiently small U we have*

$$\mathrm{d}D^k_p(U)\cap D^{k+1}_q\subset\mathrm{d}D^k_{q-1}(U).$$

P r o o f of Lemma 2. The case $q>p$ is trivial. Suppose now that $q\leqslant p$, and let $\omega\in\mathrm{d}D^k_p(U)\cap D^{k+1}_q(U)$. It follows that $\omega=\mathrm{d}\lambda$ with $\lambda\in D^k_p(U)$. We choose now (x^α,y^i) a set of local coordinates on U such that the projection $E\mapsto S$ is $(x^\alpha,y^i)\mapsto(x^\alpha)$. Because $\lambda\in D^k_p(U)$ there exists $\lambda'\in D^k_{p-1}(U)$ and λ'' of the form

$$\lambda''=\lambda_{i_1,\ldots,i_p}\wedge\mathrm{d}y^{i_1}\wedge\ldots\wedge\mathrm{d}y^{i_p}$$

with $\lambda_{i_1,\dots,i_p} \in D_0^{k-p}(U)$ such that $\lambda = \lambda' + \lambda''$. From $\mathrm{d}\lambda \in D_q^{k+1}(U)$ it follows that

$$\sum_{r=0}^{p} (-1)^r \frac{\partial}{\partial y^{i_r}} (\lambda_{i_0,\dots,\hat{i_r},\dots,i_p}) = 0.$$

Using the Poincaré lemma we can find $\varphi_{i_1,\dots,i_{p-1}} \in D_0^{k-p}(U)$ such that

$$\lambda_{i_1,\dots,i_p} = \sum_{r=1}^{p} (-1)^{r-1} \frac{\partial}{\partial y^{i r}} (\varphi_{i_1,\dots,\hat{i_r},\dots,i_p}).$$

Taking $\hat{\lambda}'' = \lambda'' - \mathrm{d}\varphi$ with $\varphi \equiv p\varphi_{i_1,\dots,i_{p-1}} \wedge \mathrm{d}y^{i_1} \wedge \dots \wedge \mathrm{d}y^{i_{p-1}}$ we have $\hat{\lambda}'' \in D_{p-1}^{k}(U)$ and $\mathrm{d}\hat{\lambda}'' = \mathrm{d}\lambda''$. Now, if $\hat{\lambda} \equiv \lambda' + \lambda''$ then $\mathrm{d}\hat{\lambda} = \mathrm{d}\lambda = \omega$ and $\hat{\lambda} \in D_{p-1}^{k}(U)$. Iterating the above argument we prove the result. □

Now we apply the above lemma for σ satisfying $\mathrm{d}\sigma = 0$ and $\sigma \in D_1^{n+1}(U)$. Hence, there exists $\theta \in D_0^n(U)$ such that $\sigma = \mathrm{d}\theta$. The general form of such a θ is

$$\theta \equiv \varepsilon_{\mu_1,\dots,\mu_n} \sum_{k=0}^{n} \frac{1}{k!} C_n^k L_{A_1,\dots,A_k}^{\mu_1,\dots,\mu_k} \delta\psi^{A_1} \wedge \dots \wedge \delta\psi^{A_k} \wedge \mathrm{d}x^{\mu_{k+1}} \wedge \dots \wedge \mathrm{d}x^{\mu_n}. \tag{2.16}$$

Hereafter one has

$$\begin{aligned} \sigma = \mathrm{d}\theta = \varepsilon_{\mu_1,\dots,\mu_n} \sum_{k=0}^{n} \frac{1}{k!} C_n^k \left(\mathrm{d}L_{A_1,\dots,A_k}^{\mu_1,\dots,\mu_k} - L_{A_0,\dots,A_k}^{\mu_0,\dots,\mu_k} \, \mathrm{d}\chi^{A_0}{}_{\mu_0} \right) \wedge \\ \delta\psi^{A_1} \wedge \dots \wedge \delta\psi^{A_k} \wedge \mathrm{d}x^{\mu_{k+1}} \wedge \dots \wedge \mathrm{d}x^{\mu_n}. \end{aligned} \tag{2.17}$$

Next, from $K\sigma = 0$ we obtain equivalently

$$L_{A_1,\dots,A_k}^{\mu_1,\dots,\mu_k} = \frac{1}{k^2} \sum_{i,j=1}^{k} (-1)^{i+j} \frac{\partial}{\partial \chi^{A_j}{}_{\mu_i}} \left(L_{A_1,\dots,\hat{A}_j,\dots,A_k}^{\mu_1,\dots\hat{\mu}_i,\dots,\mu_k} \right).$$

Iterating from $k = 0$ we obtain

$$L_{A_1,\dots,A_k}^{\mu_1,\dots,\mu_k} = \frac{1}{k!} \sum_{\sigma \in P_k} (-1)^{|\sigma|} \frac{\partial^k L}{\partial \chi^{A_1}{}_{\mu_{\sigma(1)}} \dots \partial \chi^{A_k}{}_{\mu_{\sigma(k)}}}, \tag{2.18}$$

which proves (2.14). □

b) Let $(x^\mu) \mapsto (x^\mu, \Phi^A(x))$ be an n-dimensional submanifold and $Z = Z^\mu \frac{\delta}{\delta x^\mu} + Z^A \frac{\partial}{\partial \psi^A} + Z_\mu^A \frac{\partial}{\partial \chi^A{}_\mu}$ a vector field on $U \subset J_n^1(S)$. Making the left hand side of (2.15) explicit by using

$$\dot{\Psi}_* \frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial x^\mu} + \frac{\partial \Psi^A}{\partial x^\mu} \frac{\partial}{\partial \psi^A} + \frac{\partial^2 \Psi^A}{\partial x^\mu \partial x^\nu} \frac{\partial}{\partial \chi^A{}_\nu} \tag{2.19}$$

we get

$$
\left(\dot{\Psi}^* i_Z \sigma\right)\left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right) = \text{const.} Z^A E_A(\Psi). \tag{2.20}
$$

(For notation, see (2.2).) □

2.4. The above proof was based on an explicit computation worked out in a local system of coordinates. The formula (2.15) suggests that the proof of part b) of the above theorem can be provided in an intrinsic way. For this we use the extremal property of the Euler-Lagrange equations.

If L is a Lagrangian function and $\theta \in D_0^n(U)$ such that for some θ_A we have

$$
\theta = L\, \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n + \theta_A \wedge \delta\psi^A \tag{2.21}
$$

then the action functional is

$$
A(\Psi) \equiv \int_M \dot{\Psi}^* \theta. \tag{2.22}
$$

(Here $\Psi(M) \subset U$.)

The variation of A along a vector field Y on S is given by

$$
\delta_Y A(\Psi) = \int_M \dot{\Psi}^* i_{\dot{Y}}\, \mathrm{d}\theta + \int_{\partial M} \dot{\Psi}^* i_{\dot{Y}} \theta
$$

(where $\dot{Y}$ is the natural lift of Y to $J_1^n(U)$). Then, for fixed boundary variations, i.e. $\dot{Y}|_{\dot{\Psi}(\partial M)} = 0$, the second term on the right hand side vanishes. Moreover, suppose that θ in (2.21) has the property

$$
\dot{\Psi}^* i_V\, \mathrm{d}\theta = 0 \tag{2.23}
$$

for every vertical vector field V and every immersed n-dimensional submanifold Ψ. Y being arbitrary in the equation

$$
\int_M \dot{\Psi}^* i_{\dot{Y}}\, \mathrm{d}\theta = 0,
$$

(2.15) follows. Note that (2.21) and (2.23) are fulfilled by θ_L by virtue of (2.14).

R e m a r k 5. The Euler-Lagrange equations can be written as (2.15) if $\theta \in D_0^n(U)$ satisfies (2.21) and (2.23). These two conditions do not determine θ uniquely (except for the case $n = 1$). This explains why for $n > 1$ there are other expressions for θ (also named Poincaré-Cartan forms) in literature. For example one can consider only the terms corresponding to $k = 0, 1$ in the sum appearing in (2.14) [11]. The Poincaré-Cartan form (2.14) was also found starting from a different point of view in [6], [7].

3. Variationally trivial Lagrangians

In this part of the paper, we describe the most general Lagrangian function L for which the Euler-Lagrange equations (2.2) are identically satisfied. In other words, using Theorem 1, we shall describe the Poincaré-Cartan forms θ_L such that:

$$\dot{\Psi}^* i_Z \, \mathrm{d}\theta_L = 0$$

for every n-dimensional submanifold Ψ and every vector field Z. This will give an important property of the Lagrange-Souriau form σ from Def. 1.

As was mentioned in Introduction, this problem was already solved in literature [18]–[21].

We give here an alternative approach to this problem. For this we will generalize the homogeneous formalism in the Lagrangian description of particles (i.e. $n = 1$). In this formalism one uses as Lagrangian a homogeneous function of degree one on the tangent space to S.

3.1. Thus we need a suitable substitute for $T(S)$ (used for $n = 1$) in the case $n > 1$. So, we introduce the principal bundle of n-frames of $T(S)$, denoted by $R_n(S)$. The fibre over $p \in S$ is, by definition, the set $R_n(S)_p$ of linear injective maps from $\mathbb{R}^n$ into $T_p(S)$. The structural group is $GL(n, \mathbb{R})$ (it acts freely on $R_n(S)$) and we have

$$R_n(S)/GL(n, \mathbb{R}) \cong J^1_n(S),$$

i.e. the base manifold of the principal bundle $R_n(S)$ is $J^1_n(S)$. Let $\xi^a (a = 1, \ldots, n+N)$ be a local system of coordinates on the open set $U \in S$. Then on the open set $V \subset R_n(S)$, projecting into U, we give the coordinates (ξ^a, ϱ^a_i) such that the corresponding linear injective map in $R_n(S)$ is

$$\mathbb{R}^n \ni \tau \mapsto \tau^i \varrho^a_i \frac{\partial}{\partial \xi^a} \in T_p(S). \tag{3.1}$$

From now on it is convenient to parameterize an immersed n-submanifold in S with parameters in $I \subset \mathbb{R}^n$, i.e. $\Psi \colon I \mapsto S$. Then there exists a natural lift $\widetilde{\Psi} \colon I \mapsto R_n(S)$ given by $\widetilde{\Psi}(\tau) \equiv \Psi_{*,\tau}$; using the above local coordinates system it becomes

$$\widetilde{\Psi}(\tau) = \Big(\Psi^a(\tau), \frac{\partial \Psi^a}{\partial \tau^i}(\tau)\Big).$$

Definition 2. A *homogeneous Lagrangian* is a smooth function L defined on an open $GL(n, \mathbb{R})$-invariant set $E \subset R_n(S)$ such that

$$\widetilde{L}(\varrho\alpha) = \det(\alpha)\widetilde{L}(\varrho) \tag{3.2}$$

for any $\alpha \in GL(n, \mathbb{R})$ and $\varrho \in E$.

3.2. Now we give the connection with the Lagrangian formalism from Part 2. If (x, ψ, χ) is a local system of coordinates on $J_n^1(S)$, we define ξ^a by $\xi^\mu = x^\mu \quad (\mu = 1, \dots, n)$ and $\xi^{n+A} = \psi^A \quad (A = 1, \dots, N)$. If (ξ^a, ϱ_i^a) is a point in $R_n(S)$, projecting on $J_n^1(S)$ at the point (x, ψ, χ), we have

$$x^\mu = \xi^\mu \quad (\mu = 1, \dots, n) \tag{3.3}$$

$$\psi^A = \xi^{n+A} \quad (A = 1, \dots, N) \tag{3.4}$$

$$\chi^A{}_\mu = \varrho_i^{n+A} (\varrho^{-1})_\mu^i \tag{3.5}$$

where $(\varrho^{-1})_\mu^i$ is the inverse matrix of the submatrix ϱ_i^μ of ϱ_i^a.

If L is a Lagrangian function in the variables (x, ψ, χ), then the corresponding homogeneous Lagrangian is

$$\widetilde{L}(\xi^a, \varrho_i^a) = \det(\varrho_i^\mu) L(x^\mu, \psi^A, \chi^A{}_i) \tag{3.6}$$

where the variables on the right hand side are given by the above expressions. Indeed, it is easy to see that (3.2) is verified. Moreover, for every $\widetilde{L}$ verifying (3.2) there exists an L such that we have (3.6). The formula (3.6) has an important consequence:

$$A(\psi) = \int \widetilde{L} \circ \widetilde{\psi} \, \mathrm{d}\tau$$

so that if L gives trivial Euler-Lagrange equations, the same is true for $\widetilde{L}$ and conversely.

Because $\widetilde{L}$ above does not depend on τ we can use a result from [21] obtaining

Theorem 2. *The Euler-Lagrange equations for the homogeneous Lagrangian $\widetilde{L}$ are identically satisfied iff there exists a set of smooth functions $\widetilde{\Omega}_{a_1, \dots, a_n}$ depending only on (ξ^a) and completely antisymmetric in $a_1, \dots, a_n$, such that*

$$\widetilde{L}(\xi^a, \varrho_i^a) = \frac{1}{n!} \varrho_1^{a_1} \dots \varrho_n^{a_n} \widetilde{\Omega}_{a_1, \dots, a_n}(\xi) \tag{3.7}$$

and the differential form

$$\widetilde{\Omega} \equiv \widetilde{\Omega}_{a_1, \dots, a_n}(\xi) \, \mathrm{d}\xi^{a_1} \wedge \dots \wedge \mathrm{d}\xi^{a_n} \tag{3.8}$$

is exact.

Proof. From [21] it follows that

$$\widetilde{L}(\xi^a, \varrho_i^a) = \sum_{k=1}^{n} \varrho_{i_1}^{a_1} \dots \varrho_{i_k}^{a_k} \widetilde{\Omega}_{a_1, \dots, a_k}^{i_1, \dots, i_k}(\xi) \tag{3.9}$$

where $\widetilde{\Omega}^{\cdots}_{\cdots}$ depends only on (ξ^a), is completely antisymmetric in the upper and lower indices (separately), and the differential forms on S

$$\widetilde{\Omega}^{i_1,\ldots,i_k} = \widetilde{\Omega}^{i_1,\ldots,i_k}_{a_1,\ldots,a_k}\, \mathrm{d}\xi^{a_1} \wedge \ldots \wedge \mathrm{d}\xi^{a_k}$$

are closed. This is a consequence of the triviality of the Euler-Lagrange equations. Now, using the condition (3.2), only the term corresponding to $k = n$ in (3.9) can be nonzero. □

Corollary 1. *The Euler-Lagrange equations for the homogeneous Lagrangian* $\widetilde{L}$ *are identically satisfied iff there exists a set of functions* $\widetilde{\Lambda}_{a_1,\ldots,a_{n-1}}$ *depending only on* (ξ^a) *which are completely antisymmetric in a's and such that*

$$\widetilde{L}(\xi, \varrho) = \varrho^a_i \frac{\partial}{\partial \xi^a} \widetilde{\Lambda}^i \tag{3.10}$$

where

$$\widetilde{\Lambda}^i(\xi, \varrho) = \frac{n}{(n!)^2} \varrho^{a_1}_{i_1} \ldots \varrho^{a_{n-1}}_{i_{n-1}} \varepsilon^{i,i_1,\ldots,i_{n-1}} \widetilde{\Lambda}_{a_1,\ldots,a_{n-1}}(\xi).$$

P r o o f. From $d\widetilde{\Omega} = 0$ and the Poincaré lemma, there exists an $n-1$ form $\widetilde{\Lambda}$ such that $\widetilde{\Omega} = \mathrm{d}\widetilde{\Lambda}$, with:

$$\widetilde{\Lambda} = \widetilde{\Lambda}_{a_1,\ldots,a_{n-1}}(\xi)\, \mathrm{d}\xi^{a_1} \wedge \ldots \wedge \mathrm{d}\xi^{a_{n-1}}.$$

Inserting this in (3.7) we get (3.10). □

R e m a r k 6. Using coordinates (x, ψ, χ) instead of (ξ, ϱ) we get from (3.6) the corresponding statement of (3.10):

$$L(x, \psi, \chi) = \frac{\delta \Lambda^\mu}{\delta x^\mu} \tag{3.11}$$

with

$$\Lambda^\mu(x, \psi, \chi) = \sum_{k=0}^{n} \chi^{A_1}{}_{\mu_1} \ldots \chi^{A_k}{}_{\mu_k} \Lambda^{\mu,\mu_1,\ldots,\mu_k}_{A_1,\ldots,A_k}(x, \psi)$$

where $\Lambda^{\mu,\mu_1,\ldots,\mu_k}_{A_1,\ldots,A_k}$ is completely antisymmetric in the upper and lower indices.

Formulæ (3.10) and (3.11) suggest

Definition 3. The (homogeneous) Lagrangians of the form (3.10) or (3.11) are called *variationally trivial Lagrangians.*

Theorem 3. *Let σ be a Lagrange-Souriau form. Then the (local) Lagrangian corresponding to it according to Theorem* 1 *is determined up to addition of a variationally trivial Lagrangian.*

P r o o f. Let L_1, L_2 be two Lagrangians such that $\mathrm{d}\theta_{L_1} = \mathrm{d}\theta_{L_2} = \sigma$. Hereafter $\mathrm{d}\theta_{L_1-L_2} = 0$. Using Theorem 1, part b, it follows that the Euler-Lagrange equations for $L_1 - L_2$ are identically satisfied. Applying Theorem 2 we get the result. □

R e m a r k 7. Looking back we see that the definition of the operator K and implicitly of the Lagrange-Souriau form is rather hard to be justified. A possible way to find out the explicit form of σ, given in Theorem 1 proposed in [6], [7] is to use the fact that $\mathrm{d}\theta_L = 0$ *iff* L is a total divergence Lagrangian.

3.3. We close with some remarks concerning Noetherian symmetries. Let $\Phi\colon S \to S$ be a diffeomorphism such that the natural lift $\widetilde{\Phi}\colon R_n(S) \to R_n(S)$ leaves $\widetilde{E} \subset R_n(S)$ invariant (here $\widetilde{E} = \widetilde{\pi}^{-1}(E)$ with $\widetilde{\pi}\colon R_n(S) \to J^1_n(S)$ the bundle projection). Such a Φ is called a *kinematical symmetry*. If moreover, for a given homogeneous Lagrangian $\widetilde{L}$, there exists a variationally trivial Lagrangian $\widetilde{L}_\Phi$ such that

$$\widetilde{L} \circ \widetilde{\Phi} - \widetilde{L} = \widetilde{L}_\Phi \tag{3.12}$$

then Φ is called a *restricted Noetherian symmetry for* L. From (3.12) we get

$$\dot{\Phi}^* \sigma = \sigma. \tag{3.13}$$

The above property of Φ, related to σ, can be taken as the definition of a restricted Noetherian symmetry (equivalent to that in (3.12)). This suggests to define, more generally, a *Noetherian symmetry* as any diffeomorphism $\varphi\colon E \to E$ (not necessary a lift from S) verifying:

$$\varphi^* \sigma = \sigma; \tag{3.14}$$

(this kind of symmetries are sometimes called *dynamical*).

4. Final remarks

The necessity of having a formalism for the general case (without a fibre bundle structure) becomes apparent, for instance, when studying the relativistic particle. Then S is the Minkowski space which does not have a fibre bundle structure over some absolute time [22].

Using the Lagrange-Souriau form for studying Lagrangian systems with groups of Noetherian symmetries seems to be fruitful. This was already used to find all possible Lagrangians having a given group of Noetherian symmetries. Remarkably enough, this can be done also for infinite-dimensional (gauge-like) groups [23]–[25].

It would be interesting to rephrase some results from this paper using the language of sheaf theory in the spirit of the analysis for a finite number of degrees of freedom [26]. This might be profitable in the classification of Lagrangians admitting a given group of Noetherian symmetries analogous to [26], [27].

Secondly, one could try to generalize the main results of this paper to Lagrangians depending on derivatives of orders higher than one [12].

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Author's address: *D. R. Grigore*, *O. T. Popp*, Dept. of Theor. Phys., Inst. Atomic Phys., Bucharest-Măgurele, P. O. Box MG 6, România, e-mail: `grigore@theor1.ifa.ro`, `grigore@roifa.ifa.ro`.