MODIFYING SOME FOLIATED DYNAMICAL SYSTEMS TO GUIDE THEIR TRAJECTORIES TO SPECIFIED SUB-MANIFOLDS

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Abstract. We show that dynamical systems in inverse problems are sometimes foliated if the embedding dimension is greater than the dimension of the manifold on which the system resides. Under this condition, we end up reaching different leaves of the foliation if we start from different initial conditions. For some of these cases we have found a method by which we can asymptotically guide the system to a specific leaf even if we start from an initial condition which corresponds to some other leaf. We demonstrate the method by two examples. In the chosen cases of the harmonic oscillator and Duffing's oscillator we find an alternative set of equations which represent a collapsed foliation, such that no matter what initial conditions we choose, the system would asymptotically reach the same desired sub-manifold of the original system. This process can lead to cases for which a system begins in a chaotic region, but is guided to a periodic region and vice versa. It may also happen that we could move from an orbit of one period to an orbit of another period.

Keywords: manifold, foliation, duffing oscillator

MSC 2010: 34D35, 34A55

1. Introduction

The main purpose of this paper is to modify some foliated dynamical systems so that their trajectories eventually reach certain pre-specified sub-manifold. Formal definition of foliation is given in [1]. An n dimensional foliated manifold is composed of disjoint d dimensional ($n > d$) sub-manifolds. Each of these sub-manifolds is called a leaf of the foliation. Each leaf is a connected and invariant manifold. A common example of a topological foliation is a fiber bundle. Reference [2] gives an interesting example which has some relevance to dynamical systems. If G is a Lie group and H is a subgroup obtained by exponentiating a closed subalgebra of the Lie algebra of G, then G is foliated by cosets of H [2].

We define a dynamical system as foliated when its state phase portrait (plot of multiple trajectories corresponding to different initial conditions) is foliated. We can illustrate this by considering the example of a harmonic oscillator with a differential equation

$$\frac{\mathrm{d}x_0}{\mathrm{d}t} = x_1, \quad \frac{\mathrm{d}x_1}{\mathrm{d}t} = -x_0.$$

Let us consider solutions of this equation for all possible initial conditions. Thus, for example, if we choose $x_0 = 3$ and $x_1 = 4$ as the initial condition our solution will be a circle of radius 5. It is clear that any sets of initial conditions, sum of whose squares add up to 25 will lead to solutions which lie on the same circle. However, if the sum of the squares adds up to, say 30, we will get a circle which is concentric to the first one but of a larger radius. Now if we consider all possible solutions, it is clear that we have a set of concentric circles. These circles are one of the simplest examples of foliation.

In this paper we look at some n dimensional dynamical systems that are foliated by the choice of initial conditions. In these cases each leaf of the foliation is d dimensional ($n > d$). For each leaf there is an equivalence class of initial conditions such that if we start from any point of that equivalence class, the resulting trajectory will remain confined to a d dimensional sub-manifold shared by all other initial conditions in that class. Now if we start from a point of some different equivalence class of initial conditions, we would be confined to a different leaf. Our goal is to modify the dynamical system so that from wherever we start we always reach or converge to the same pre-specified leaf. An important constraint is that the original system and the modified system should remain identical on the target leaf. This is because the data that we wish to model in many cases, originates only on that leaf and the model must achieve stability while remaining faithful to the data.

This kind of problem arises when we embed a system in a higher dimension. For getting equations from time series data it is very common to embed the system in a higher dimension [3]. One of the most promising methods in this respect is Taken's embedding [4]. We have shown in reference [5] how embedding might create a foliated dynamical system. Foliation can also arise if the differential equation of a dynamical system is extended as, for example, in the case of applications to cryptography [6]. A related situation arises in reference [7]. In reference [7] some general conditions are found that lead to spatially localized periodic oscillations in networks of coupled oscillators. For conservative systems periodic solutions typically form a one-parameter family. To specify a single periodic solution, one extra equation is required which yields an overdetermined system. Reference [8] shows an algorithm by which this problem can be solved. In this paper we propose a method by which for some dynamical systems we can asymptotically guide the trajectory to a pre-specified sub-manifold.

Perhaps the most important application of this method is to the problems of Mathematical Modeling. In this field it is often required to arrive at a set of differential equations or maps that fit a given observational data. In many problems the data points lie in a low dimensional manifold, but the model is of necessity of a higher dimension in which that low dimensional manifold is embedded. There are two problems with this. First is that the data can validate only the behavior which corresponds to the sub-manifold. Therefore these models can be seen as not being faithful to the data. The second problem, which is actually a possible consequence of the first, is that the predicted behavior of these equations can often be unstable. Under this circumstance we need to stabilize the system and confine it to a proper sub-manifold. Another important application is demonstrated at the end of this paper. This is to the problem of controlling chaos. In the next section we discuss how we can modify a foliated system so that the foliation would be collapsed. In the subsequent sections, we demonstrate practical application of these ideas by choosing, at first, a very simple example of the harmonic oscillator, followed by a case of a dynamical system generated by Duffing's oscillator. In each case we find a modification of the system such that no matter what initial conditions we choose, the system would asymptotically reach a specific leaf of the foliation. Our choice of the 4 dimensional form of Duffing's oscillator is an unusual one. It contains some leaves which are periodic and some which are chaotic. We show that we can control the system to go from periodic to chaotic or from chaotic to periodic.

2. Modifying dynamical system when foliation is created by embedding

It has been shown that if we embed a dynamical system into higher dimension it may create a foliation [5]. Let us consider the 2D system

$$\frac{\mathrm{d}x_0}{\mathrm{d}t} = x_1, \ \frac{\mathrm{d}x_1}{\mathrm{d}t} = f(x_0, x_1) \tag{2.1}$$

We want to embed it in 3D

$$\frac{\mathrm{d}x_0}{\mathrm{d}t} = x_1, \ \frac{\mathrm{d}x_1}{\mathrm{d}t} = f(x_0, x_1), \ \frac{\mathrm{d}x_2}{\mathrm{d}t} = \frac{\mathrm{d}f}{\mathrm{d}t}. \tag{2.2}$$

If we integrate the last equation of (2.2) we get $x_2 = f(x_0, x_1) + a_2 - f(a_0, a_1)$, where (a_0, a_1, a_2) are the initial conditions. Now for a given $(a_0, a_1), a_2$ may not be the same as $f(a_0, a_1)$. So we get a foliation as a_2 varies. For each a_2, we get a leaf of the foliation. Now we want to modify system (2.2) in such a way that this foliation gets collapsed. Let us consider the set of equations

$$\frac{\mathrm{d}x_0}{\mathrm{d}t} = x_1, \quad \frac{\mathrm{d}x_1}{\mathrm{d}t} = f(x_0, x_1), \quad \frac{\mathrm{d}x_2}{\mathrm{d}t} = \frac{\mathrm{d}f(x_0, x_1)}{\mathrm{d}t} + \lambda(x_2 - f(x_0, x_1)), \tag{2.3}$$

where λ is less than 0.

Let $g(x_0, x_1, x_2) = x_2 - f(x_0, x_1)$. So,

$$\frac{\mathrm{d}g}{\mathrm{d}t} = \frac{\mathrm{d}x_2}{\mathrm{d}t} - \frac{\mathrm{d}f}{\mathrm{d}t} = \lambda g.$$

Hence, $g(t) = g(0)\mathrm{e}^{\lambda t}$. Now λ is less than 0, so g tends to 0 as t tends to infinity. Therefore, if we obtain a numerical solution for this system and observe its long term behavior, we find that we eventually reach the leaf $x_2 - f(x_0, x_1) = 0$, whatever our initial conditions are. On the other hand, in case of the system (2.2) initial conditions decide which leaf we are going to reach.

3. Guiding trajectories of harmonic oscillator to a specific leaf

We have discussed in Section 1 that the harmonic oscillator

$$\frac{\mathrm{d}x_0}{\mathrm{d}t} = x_1, \quad \frac{\mathrm{d}x_1}{\mathrm{d}t} = -x_0 \tag{3.1}$$

is a foliated dynamical system. The choice of the initial condition gives rise to a leaf S1. For example, if we choose $x_0 = 3$ and $x_1 = 4$ as initial condition we would get a circle of radius 5. Now, if we choose some other initial condition we once again begin tracing the same circle, only if the sum of squares of the initial conditions add up to 25. In general, however, we generate many different concentric circles, as we keep choosing a variety of initial conditions and keep finding the trajectories generated by them. Our goal is to modify the system in such a way that we always reach a pre-specified circle, say, $g(x_0, x_1) = x_0^2 + x_1^2 - 25 = 0$ even if our initial condition lies outside of the circle. Let us consider the system

$$\frac{\mathrm{d}x_0}{\mathrm{d}t} = x_1 + \lambda_1 g x_0, \quad \frac{\mathrm{d}x_1}{\mathrm{d}t} = -x_0 + \lambda_2 g x_1 \tag{3.2}$$

with $g = x_0^2 + x_1^2 - 25$ and with both λ_1 and λ_2 negative. Multiplying the first equation of (3.2) by x_0 and the second equation of (3.2) by x_1 we get $(\mathrm{d}/\mathrm{d}t)(x_0^2 + x_1^2 - 25) = 2g(\lambda_1 x_0^2 + \lambda_2 x_1^2)$, i.e. $\mathrm{d}g/\mathrm{d}t = \lambda g$, where $\lambda = 2(\lambda_1 x_0^2 + \lambda_2 x_1^2)$.

So λ is a function of x_0 and x_1 and hence it is a function of time t. Integrating the above equation we get $g = g(0)\exp(\int_0^t \lambda \, \mathrm{d}t)$.

The Jacobian matrix of this system at the origin is

$$\begin{pmatrix} -25\lambda_1 & 1 \\ -1 & -25\lambda_2 \end{pmatrix}.$$

The eigenvalues are the solutions of the equation $x^2+25(\lambda_1+\lambda_2)x+(625\lambda_1\lambda_2+1) = 0$. The sum of the two eigenvalues is $-25(\lambda_1 + \lambda_2)$ which is positive and the product of the two eigenvalues is $(625\lambda_1\lambda_2+1)$ which is also positive. Hence both the eigenvalues of the Jacobian matrix of this system at the origin are greater than 0. So the origin is a repeller of this system. Since λ_1 and λ_2 are negative, x_0 and x_1 are not simultaneously 0 and the origin is a repeller, the value of λ will be strictly less than 0. Hence $\int_0^t \lambda \,\mathrm{d}t$ tends to $-\infty$ as t tends to ∞. So g tends to 0 as t tends to infinity. As a consequence we eventually reach the leaf $g = 0$, even when we start outside the leaf.

Empirically in the Runge Kutta method we choose $x_0 = 2$ and $x_1 = 3$ as the initial condition and the values of λ_1 and λ_2 are -0.2 and -0.3 respectively. In the case of the harmonic oscillator (3.1) we reach a circle of radius 3.6, but in system (3.2) when we start from the same initial condition we reach a circle of radius 5, i.e. $g = 0$ (Fig. 1). In fact, in system (3.2) we always converge to $g = 0$ whatever our initial conditions are.

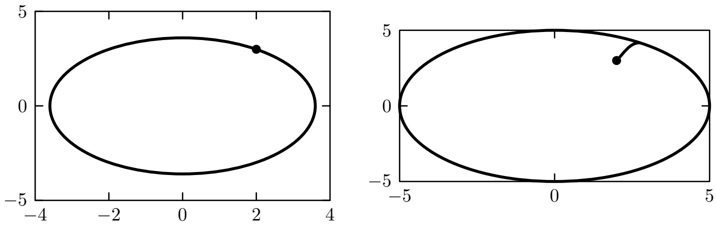

Figure 1. Left: trajectory of harmonic oscillator when initial condition is $(2,3)$, right: trajectory of modified harmonic oscillator when initial condition is $(2,3)$.

4. Guiding trajectories of Duffing system to a specific leaf

Duffing equation is a 2 dimensional non-autonomous system given by the equation

$$\frac{\mathrm{d}y_0}{\mathrm{d}t} = y_1, \quad \frac{\mathrm{d}y_1}{\mathrm{d}t} = -cy_1 - ky_0 - \delta y_0^3 + F\cos(\omega t + \alpha). \tag{4.1}$$

It is known to us that this equation shows chaos for the following values of the parameters: $c = 0.044964$, $k = 0$, $\delta = 1$, $\omega = 0.44964$, $\alpha = 0$ and $F = 1.02$ [7]. It is well known that keeping the other parameters fixed, if we change F we see that the system is periodic when F is 0.2 while if F is 1.02 this system becomes chaotic (as empirically we have found in Figure 2).

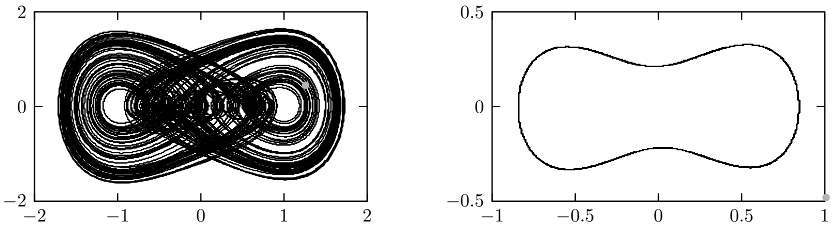

Figure 2. Left: Duffing oscillator when F is 1.02, right: Duffing oscillator when F is 0.2.

As F varies we get different trajectories in the state space picture of Duffing. So we can treat F as a parameter of foliation. We will show that we can modify this system in such a way that we can eventually reach a specific periodic orbit even if we start from an orbit which has chaotic behavior. It has been shown that the 2 dimensional non-autonomous Duffing equation can be extended to a 4 dimensional autonomous equation [6]. The equation is

$$\begin{aligned} &\frac{\mathrm{d}y_0}{\mathrm{d}t} = y_1, \quad \frac{\mathrm{d}y_1}{\mathrm{d}t} = y_2, \quad \frac{\mathrm{d}y_2}{\mathrm{d}t} = y_3, \\ &\frac{\mathrm{d}y_3}{\mathrm{d}t} = -cy_3 - ky_2 - 3\delta y_0^2 y_2 - 6\delta y_0 y_1^2 - \omega^2 y_2 - \omega^2 c y_1 - \omega^2 k y_0 - \omega^2 \delta y_0^3 \end{aligned} \tag{4.2}$$

Since $\mathrm{d}y_1/\mathrm{d}t = y_2$, from (4.1) we get

$$F\cos(\omega t + \alpha) = y_2 + cy_1 + ky_0 + \delta y_0^3. \tag{4.3}$$

By differentiating (4.3) we get

$$F\omega\sin(\omega t + \alpha) = -cy_2 - ky_1 - 3\delta y_0^2 y_1 - y_3 \tag{4.4}$$

We can have an arbitrary choice of initial condition for y_0 and y_1 in system (4.2). But we have to select the initial condition for y_2 and y_3 by using the relations (4.3) and (4.4). If Y_0, Y_1, Y_2 and Y_3 are the values of y_0, y_1, y_2 and y_3 at $t = 0$, these relations yield

$$Y_2 = F\cos(\alpha) - (cY_1 + kY_0 + \delta Y_0^3), \; Y_3 = -F\omega\sin(\alpha) + (-cY_2 - kY_1 - 3\delta Y_0^2 Y_1).$$

In this paper we want to propose an alternative form of the 4 dimensional Duffing equation. From (4.1) we get

$$F\cos(\omega t + \alpha) = \frac{\mathrm{d}w_1}{\mathrm{d}t} + cw_1 + kw_0 + \delta w_0^3.$$

We define w_2 as $w_2 = F\cos(\omega t+\alpha)$. Then $\mathrm{d}w_1/\mathrm{d}t = w_2 - (cw_1 + kw_0 + \delta w_0^3)$, and if we further let $w_3 = \mathrm{d}w_2/\mathrm{d}t = -\omega F\sin(\omega t+\alpha)$, then $\mathrm{d}w_3/\mathrm{d}t = -\omega^2 w_2$.

So we get a new system

$$\begin{aligned}\frac{\mathrm{d}w_0}{\mathrm{d}t} &= w_1, \quad \frac{\mathrm{d}w_1}{\mathrm{d}t} = w_2 - (cw_1 + kw_0 + \delta w_0^3),\\ \frac{\mathrm{d}w_2}{\mathrm{d}t} &= w_3, \quad \frac{\mathrm{d}w_3}{\mathrm{d}t} = -\omega^2 w_2.\end{aligned} \tag{4.5}$$

It will be equivalent to (4.1) if at $t=0$ we take $w_2 = F\cos(\alpha)$, $w_3 = -\omega F\sin(\alpha)$. Hence we can conclude that the system (4.5) with the initial condition (W_0, W_1, W_2, W_3) is equivalent to the system (4.1) with the initial condition (Y_0, Y_1), where $W_0 = Y_0$, $W_1 = Y_1$, $W_2 = F\cos(\alpha)$, $W_3 = -\omega F\sin(\alpha)$, and F is the same as in (4.1). From the definition of the initial conditions W_2 and W_3 we get $W_2^2 + W_3^2/\omega^2 = F^2$. System 4.5 will show chaotic behavior if we choose F, and hence $(W_2^2 + W_3^2/\omega^2)$ as 1.02, and it will show periodic behavior when F is 0.2. So the choice of our initial conditions (W_2 and W_3) decides which leaf we will reach. Now let us consider the system

$$\begin{aligned}\frac{\mathrm{d}w_0}{\mathrm{d}t} &= w_1, \quad \frac{\mathrm{d}w_1}{\mathrm{d}t} = w_2 - (cw_1 + kw_0 + \delta w_0^3),\\ \frac{\mathrm{d}w_2}{\mathrm{d}t} &= w_3 + \lambda_1 w_2\Big(w_2^2 + \frac{w_3^2}{\omega^2} - G^2\Big),\\ \frac{\mathrm{d}w_3}{\mathrm{d}t} &= -\omega^2 w_2 + \lambda_2 w_3\Big(w_2^2 + \frac{w_3^2}{\omega^2} - G^2\Big),\end{aligned} \tag{4.6}$$

with $G = 0.2$ and both λ_1 and λ_2 negative.

Let $g = w_2^2 + w_3^2/\omega^2 - G^2$. So by virtue of (4.6) we can write

$$\frac{\mathrm{d}w_2}{\mathrm{d}t} = w_3 + \lambda_1 w_2 g, \quad \frac{\mathrm{d}w_3}{\mathrm{d}t} = -\omega^2 w_2 + \lambda_2 w_3 g.$$

Multiplying the first equation by w_2 and the second equation by w_3/ω^2 we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(w_2^2 + \frac{w_3^2}{\omega^2} - G^2\Big) = 2\Big(\lambda_1 w_2^2 + \frac{\lambda_2 w_3^2}{\omega^2}\Big)g,$$

i.e. $\mathrm{d}g/\mathrm{d}t = 2\lambda g$, where $\lambda = \lambda_1 w_2^2 + \lambda_2 w_3^2/\omega^2$. So λ is a function of w_2 and w_3 and hence it is a function of time t. Integrating the above equation we get $g = g(0)\exp(\int_0^t \lambda\,\mathrm{d}t)$. The Jacobian matrix of this system at the origin is

$$\begin{pmatrix} -G^2\lambda_1 & 1 \\ -1 & -G^2\lambda_2 \end{pmatrix}.$$

The eigenvalues are the solutions of the equation $x^2+G^2(\lambda_1+\lambda_2)x+(G^4\lambda_1\lambda_2+1)=0$. The sum of the two eigenvalues is $-G^2(\lambda_1+\lambda_2)$ which is positive and the product of the eigenvalues is $(G^4\lambda_1\lambda_2+1)$ which is also positive. Hence both the eigenvalues of the Jacobian matrix of this system at the origin are greater than 0. So the origin is a repeller of this system. Since λ_1 and λ_2 are negative, x_0 and x_1 are not simultaneously 0 and the origin is a repeller, the value of λ will be strictly less than 0. Hence $\int_0^t \lambda \, dt$ tends to $-\infty$ as t tends to ∞.

So g tends to 0 as t tends to infinity. As a consequence we eventually reach $g=0$, i.e. $w_2^2+w_3^2/\omega^2-G^2=0$, even if the initial conditions W_2 and W_3 were chosen in such a way that $w_2^2+w_3^2/\omega^2=F^2$, where F is not equal to G.

We run the Runge Kutta method for both systems (4.5) and (4.6). We take $F=1.02$ for both (4.5) and (4.6) for choosing the initial conditions W_2 and W_3. We choose the rest of the parameters the same as before for both the systems. For system (4.6) we take the value of λ_1 to be -0.002 and the value of λ_2 to be -0.003. We take 10^7 points in the method. If we plot the first 2800 points we get rather similar pictures for system (4.5) and system (4.6) as we can see in Figure 3. Then we plot the last 60000 points and we can see in Figure 4 that in the case of (4.5) we get a chaotic orbit, whereas in the case of (4.6) we get a periodic orbit.

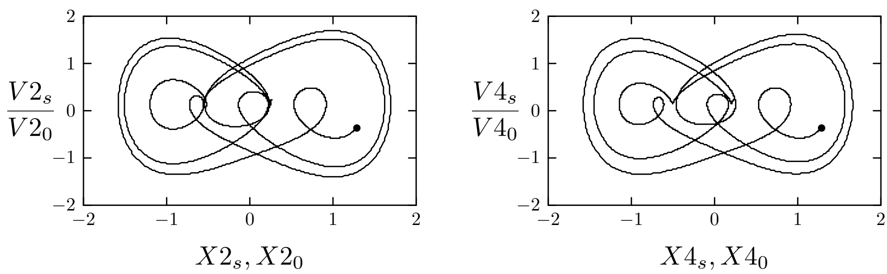

Figure 3. Left: initial behavior of the trajectory of system (4.5), right: initial behavior of the trajectory of system (4.6).

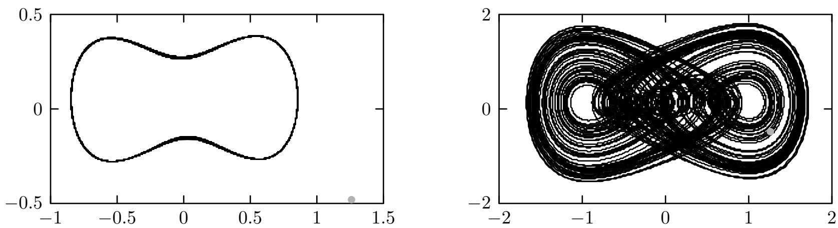

Figure 4. Left: in the end, system (4.6) shows periodic behavior, right: in the end, system (4.5) still acts chaotically.

In system (4.5) we take $F=1.02$ and it shows chaos. In system (4.6) we also take $F=1.02$, so initially it shows a trajectory similar to that of system (4.5). But as we have taken both λ_1 and λ_2 negative, $w_2^2 + w_3^2/\omega^2 - G^2$ tends to 0 eventually. As a consequence, the long term behavior of the system is governed not by the value of F, but by the value of G and since we have chosen in this example $G = 0.2$, the long term behavior is similar to that generated by the system (4.5) when F for that system is 0.2. At $F = 0.2$ the conventional Duffing system (given by equation (4.5)) is periodic, so system (4.6) ends up showing periodic behavior. Thus, the main interesting characteristic of system (4.6) is that it initially shows similarity with a system which is chaotic, but in the end it converges to a periodic orbit.

5. Conclusion

In this paper we have discussed what is a foliation and how it appears in practical problems. We have shown how we can guide a trajectory of a foliated dynamical system to a specific leaf of the foliation. We have demonstrated this method by taking the examples of the harmonic oscillator and Duffing's oscillator. In each of these cases we find an alternative set of equations which represent a collapsed foliation, such that no matter what initial conditions we choose, the system would asymptotically reach the same desired sub-manifold. We have shown that this can be achieved while making sure that the original system and the modified system remain identical on the target leaf. This is because the data that we wish to model, in many cases originates only on that leaf and the model must achieve stability while remaining faithful to the data.

In the case of the Duffing system, as the parameter F varies we get different trajectories. These different trajectories show different qualitative behavior. Some of them are periodic whereas some of them are chaotic. We have shown how we can create a trajectory which acts chaotically at the beginning but eventually shows periodic behavior. So one of the important aspects of this method is that it shows control of chaos. Thus it is very easy to synthesize cases for which a system begins in a chaotic region, but is guided to a periodic region and vice versa. It may also happen that we could move from an orbit of one period to an orbit of another period.

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