ON HOLOMORPHIC CONTINUATION OF FUNCTIONS ALONG BOUNDARY SECTIONS

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Abstract. Let $D' \subset \mathbb{C}^{n-1}$ be a bounded domain of Lyapunov and $f(z', z_n)$ a holomorphic function in the cylinder $D = D' \times U_n$ and continuous on $\overline{D}$. If for each fixed a' in some set $E \subset \partial D'$, with positive Lebesgue measure $\operatorname{mes} E > 0$, the function $f(a', z_n)$ of z_n can be continued to a function holomorphic on the whole plane with the exception of some finite number (polar set) of singularities, then $f(z', z_n)$ can be holomorphically continued to $(D' \times \mathbb{C}) \setminus S$, where S is some analytic (closed pluripolar) subset of $D' \times \mathbb{C}$.

Keywords: holomorphic function, holomorphic continuation, pluripolar set, pseudoconcave set, Jacobi-Hartogs series

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The first result in this direction was obtained by Hartogs in [3], (also see [11]): let $f(z', z_n)$ be defined in the cylinder

$$U' \times U_n = \left\{z' = (z_1, \ldots, z_{n-1}) \in \mathbb{C}^{n-1} :\ |z'| < 1\right\} \times \{z_n \in \mathbb{C}\colon\ |z_n| < R\}$$

and holomorphic in the cylinder $U' \times \{z_n \in \mathbb{C}\colon\ |z_n| < r\}$, $0 < r < R$. If for each fixed $z'^0 \in U'$, $f(z'^0, z_n)$ is a holomorphic function of z_n in the disk $|z_n| < R$ then f is holomorphic in the cylinder $U' \times \{|z_n| < R\}$.

This result, which is called the Hartogs lemma, has several generalizations of distinct character and relates directly to the subject connected with holomorphic continuation along fixed direction. Subsequent results in this topic are contained in the papers of Rothstein [7], M. V. Kazaryan [4], A. S. Sadullaev and E. M. Chirka [10], T. T. Tuychiev [13].

More final result under the minimal conditions on sets of sections, along which such continuation exists is obtained in the paper [10] by A. S. Sadullaev and E. M. Chirka:

let $f(z', z_n)$ be holomorphic in the cylinder $U = U' \times U_n$ in $\mathbb{C}^n$, and assume that for each fixed z' in some nonpluripolar set $E \subset U'$ the function $f(z', z_n)$ of z_n can be continued to a function holomorphic on the whole plane with the exception of some polar set of singularities.

Then $f(z', z_n)$ can be continued holomorphically to $(U' \times \mathbb{C}) \setminus S$, where S is a closed pluripolar subset of $U' \times \mathbb{C}$.

The main results of the present paper are the following theorems.

Theorem 1. *Let $D' \subset \mathbb{C}^{n-1}$ be a bounded domain of Lyapunov and $f(z', z_n)$ a holomorphic function in the cylinder $D = D' \times U_n$ and continuous on $\overline{D}$. If for each fixed a' in some set $E \subset \partial D'$, with positive Lebesgue measure* $\operatorname{mes} E > 0$, *the function $f(a', z_n)$ of z_n can be continued to a function holomorphic on the whole plane with the exception of some finite number of singularities, then $f(z', z_n)$ can be holomorphically continued to $(D' \times \mathbb{C}) \setminus S$, where S is some analytic subset of $D' \times \mathbb{C}$.*

Theorem 2. *Let $D' \subset \mathbb{C}^{n-1}$ be a bounded domain of Lyapunov and a function $f(z', z_n)$ be holomorphic in the cylinder $D = D' \times U_n$ and continuous on $\overline{D}$. If for each fixed a' in some set $E \subset \partial D'$, with positive Lebesgue measure* $\operatorname{mes} E > 0$, *the function $f(a', z_n)$ of z_n can be continued to a function holomorphic on the whole plane with exception of some polar set of singularities, then the function $f(z', z_n)$ can be holomorphically continued to $(D' \times \mathbb{C}) \setminus S$, where S is a closed pluripolar subset of $D' \times \mathbb{C}$.*

Throughout the paper we suppose that $D' \subset \mathbb{C}^{n-1}$ is a bounded Lyapunov domain. Lemmas in Sect. 3 and in Sect. 4 are also proved for a such domains, though it is not excepted that they hold for domains with smooth boundary.

Proofs of these theorems are based on the Jacobi-Hartogs series in the variable z_n with coefficients holomorphic on D'. The difficult part of the proof of the theorems is to show that the set of singularities of the function f is a pseudoconcave set. That difficulty is overcome by applying the Jacobi-Hartogs series. Properties of convergence domains of such series, which we use, are described in the paper [10]. In Sections 1, 2 we discuss boundary behavior of plurisubharmonic functions, N-sets and some notices of Jacobi-Hartogs, which on the one hand are one of the main methods for proving the main results, on the other hand independent meanings are presented. In Sect. 3 and Sect. 4 the class R^0 and boundary behavior of pseudoconcave sets are studied, on which at the end the proofs of the theorem are based.

Theorems 1 and 2 are proved in Sect. 5.

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If $u_1, u_2, \dots, u_N$ are plurisubharmonic functions in $D \subset \mathbb{C}^n$, where N is a finite number, then $\sup\{u_1, u_2, \dots, u_N\}$ is also plurisubharmonic in D. The situation is different when we consider $u(z) = \sup u_\alpha(z)$ a supremum of an infinite number of plurisubharmonic locally uniformly upper-bounded functions $u_\alpha(z)$, $\alpha \in \Lambda$ (where α is an infinite cardinality). In this case $u(z)$ is not necessarily semi-continuous. However, if we consider the regularization $u^*(z) = \overline{\lim\limits_{\xi \to z}}\, u(\xi)$, then u^* is a plurisubharmonic function in D.

The situation is similar for the upper limit

$$u(z) = \overline{\lim_{j\to\infty}}\, u_j(z)$$

of locally uniformly upper-bounded sequence $\{u_j\}$: the regularization u^* also will be a plurisubharmonic function.

It is known that the set $N = \{z \in D\colon\ u(z) < u^*(z)\}$ is pluripolar (see [1], [8]).

The following assertion makes transition to boundary properties:

Lemma 1 (I. I. Privalov [6]). *Let D be a bounded domain of Lyapunov and a function $u(z) \not\equiv -\infty$ be subharmonic in D and upper-bounded. Then the function $u(z)$ almost everywhere has normal limit values on ∂D*

$$u(\xi) = \lim_{\varepsilon\to 0} u(\xi - \varepsilon\nu_\xi),$$

and $u(\xi)$ is a summable function on ∂D.

A bounded domain D is called a domain of Lyapunov if there exists an external normal ν_ξ for each boundary point ξ which is a continuous vector function satisfying Hölder's condition. This property implies a fairly good boundary behavior of the Green function $G(\xi, z)$: for each fixed $z \in D$ the function $G(\xi, z)$ is continuously differentiable in $\overline{D}$ and all its first partial derivatives satisfy Hölder's condition in $\overline{D}$. Therefore, in this case, every integrable function $\vartheta(\xi)$ on ∂D can be harmonically continued into D and this continuation is obviously given by the integral of Poisson:

$$\vartheta(z) = \int_{\partial D} P(\xi, z)\vartheta(\xi)\, \mathrm{d}\sigma(\xi),$$

where $P(\xi, z) = c_n \cdot \partial G(\xi, z)/\partial \nu_\xi$ is Poisson's kernel and c_n is a constant depending only on n.

The function $u(z)$ from lemma 1, even when it is harmonic in D, generally does not coincide with the Poisson integral of the function $u(\xi)$, it can differ by a singular part:

$$u(z)=\int_{\partial D}P(\xi,z)u(\xi)\,\mathrm{d}\sigma+\int_{\partial D}P(\xi,z)\,\mathrm{d}\lambda(\xi).$$

Here λ is a measure singular with respect to the Lebesgue measure. Therefore, for studying the boundary behavior of subharmonic functions it is more natural to consider the notion of a boundary measure, which is defined as follows: a subharmonic and upper-bounded function $u(z)$ in D has the least harmonic majorant in following form:

$$\vartheta(z)=\int_{\partial D}P(\xi,z)\,\mathrm{d}\mu(\xi).$$

The bounded measure μ is concentrated on ∂D and uniquely defined by the function $u(z)$; it is called boundary measure of the function $u(z)$ and almost everywhere the following equality holds:

$$\lim_{\varepsilon\to 0}u(\xi-\varepsilon\nu_\xi)=\lim_{\varepsilon\to 0}\vartheta(\xi-\varepsilon\nu_\xi)=\frac{\mathrm{d}\mu(\xi)}{\mathrm{d}\sigma}\tag{1}$$

(see [6]), where $\mathrm{d}\mu(\xi)/\,\mathrm{d}\sigma$ is the density of the measure μ with respect to the Lebesgue measure $\mathrm{d}\sigma$.

Let $D\subset\mathbb{C}^n$ be a bounded domain of Lyapunov and $\{u_j(z)\}$ a sequence of uniformly upper-bounded and subharmonic functions $u_j(z)$ in D. We extend the functions $u_j(z)$ to the boundary of the domain D, via

$$u_j(\xi)=\overline{\lim_{\varepsilon\to 0}}\,u_j(\xi-\varepsilon\nu_\xi),\ \ \xi\in\partial D.$$

Then the following lemma is true.

Lemma 2. *Let $u(z)=\overline{\lim\limits_{j\to\infty}}\,u_j(z)$, $z\in\overline{D}$, and let $u^*(z)=\overline{\lim\limits_{w\to z}}\,u(w)$, $z\in D$ be its regularization. Then the normal limit of the function $u^*(z)$ on the boundary ∂D is less than or equal to $u(\xi)$ almost everywhere with respect to the Lebesgue measure.*

Proof. First we shall prove the lemma for a monotone sequence, i.e. when

$$u_j(z)\leqslant u_{j+1}(z)\leqslant M,\quad j=1,2,\ldots.$$

Let $\vartheta_j(z)$ be the least harmonic majorant of the function $u_j(z)$. Then $\vartheta_j(z)\leqslant\vartheta_{j+1}(z)\leqslant M$ for each j and according to (1) the identity

$$u_j(\xi)=\vartheta_j(\xi).\tag{2}$$

is true almost everywhere on ∂D.

It is easy to check that the function $\vartheta(z) = \lim_{j\to\infty} \vartheta_j(z)$, $z \in \overline{D}$ is the least harmonic majorant for the function $u(z) = \lim_{j\to\infty} u_j(z)$, $z \in \overline{D}$, consequently $\vartheta(z)$ is the least harmonic majorant for the function $u^*(z)$ too. Hence according to (1) and (2) we obtain that $u^*(\xi) = \vartheta(\xi) = u(\xi)$ for almost all $\xi \in \partial D$.

In the general case, we consider the following sequence of subharmonic functions:

$$W_{j,k}(z) = \sup_{j\leqslant m\leqslant k} u_m(z).$$

It is obvious that

$$u(z) = \overline{\lim_{j\to\infty}}\, u_j(z) = \lim_{j\to\infty} \lim_{k\to\infty} W_{j,k}(z).$$

Since the sequence $W_{j,k}(z)$ is monotonously growing and the sequence

$$W_j(z) = \lim_{k\to\infty} W_{j,k}(z)$$

is monotonously decreasing, according to the first part of the proof we have

$$W_j^*(\xi) = W_j(\xi)$$

almost everywhere and therefore

$$u^*(\xi) = \lim_{\varepsilon\to\infty} u^*(\xi - \varepsilon v_\xi) \leqslant \lim_{j\to\infty} W_j^*(\xi) = \lim_{j\to\infty} W_j(\xi) = u(\xi)$$

for almost all $\xi \in \partial D$. Lemma 2 is proved. □

2. Jacobi-Hartogs series

In this paragraph we shall formulate some results from [10], which the proofs of the main theorems of this paper are based on.

We consider the rational lemniscate V_r in the plane $\mathbb{C}$, determined as the union of some connected components of the set $|g(z)| < r$, where g is a fixed rational function.

If f is holomorphic in a neighborhood of $\overline{V}_r$, then function

$$F(z,w) = \frac{1}{2\pi \mathrm{i}} \int_{\partial V_r} \frac{f(\xi)}{g(\xi) - w} \cdot \frac{g(\xi) - g(z)}{\xi - z}\, \mathrm{d}\xi$$

is holomorphic in the domain $V_r \times \{|w| < r\}$, and according to the Cauchy integral formula $F(z, g(z)) \equiv f(z)$ in V_r. We expand the function $F(z,w)$ into a Hartogs series with respect to w: $F(z,w)=\sum\limits_{k=0}^{\infty} c_k(z)w^k$. Substituting $w=g(z)$, we obtain a decomposition of the function f into a Jacobi series

$$f(z)=\sum_{k=0}^{+\infty} c_k(z)g^k(z), \tag{3}$$

with coefficients

$$c_k(z)=\frac{1}{2\pi \mathrm{i}}\int_{\partial V_r} f(\xi)\cdot\frac{g(\xi)-g(z)}{g^{k+1}(\xi)(\xi-z)}\,\mathrm{d}\xi. \tag{4}$$

From this formula it is easy to see that the functions $c_k(z)$ are rational functions with poles at the poles of g, and $\deg c_k \leqslant \deg g \leqslant m$.

Lemma 3 ([10]). *The domain of convergence of the series* (3) *is the interior of the lemniscate* $|g(z)|<R^{(g)}$, *where the radius of convergence* $R^{(g)}$ *is determined from the formula*

$$\varlimsup_{k\to\infty}\|c_k\|_K^{1/k}=\frac{1}{R^{(g)}}. \tag{5}$$

Here K *is an arbitrary nonpolar compact set which does not contain the poles of* g *and the limit on the left-hand side of the equality does not depend on the choice of such a compact set.*

Let's return to the function $f(z',z_n)$, which is holomorphic in the domain $D'\times U_n$.

Let $g(z_n)$ be a rational function of z_n with $g(0)=0$. Then for sufficiently small r there exists a connected component V_r of the set $\{z_n\colon |g(z_n)|<r\}$ such that $0\in V_r\subset U_n$. Since $f(z',z_n)$ is a holomorphic function in $D'\times V_r$, for each fixed $z'\in D'$ it can be decomposed into the Jacobi series (3):

$$f(z',z_n)=\sum_{k=0}^{\infty} c_k(z',z_n)g^k(z_n), \tag{6}$$

where

$$c_k(z',z_n)=\frac{1}{2\pi \mathrm{i}}\int_{\partial V_r} f(z',\xi_n)\cdot\frac{g(\xi_n)-g(z_n)}{g^{k+1}(\xi_n)(\xi_n-z_n)}\,\mathrm{d}\xi_n.$$

Consequently, the $c_k(z',z_n)$ are rational functions of z_n with coefficients holomorphic in D'.

Lemma 4 ([10]). *The Jacobi-Hartogs series* (6) *converges uniformly in the interior of the open set* $G_g = \{(z', z_n)\colon z' \in D',\ |g(z_n)| < R_*^{(g)}(z')\}$ *in* $D' \times \mathbb{C}$, *where* $R_*^{(g)}(z') = \varliminf_{\xi' \to z'} R^{(g)}(\xi')$ *is the lower regularization of* $R^{(g)}(z')$. *The function* $-\log R_*^{(g)}(z')$ *is plurisubharmonic in* D', $R_*^{(g)}(z') \leqslant R^{(g)}(z')$ *in* D' *and the set* $\{z' \in D'\colon R_*^{(g)}(z') < R^{(g)}(z')\}$ *is pluripolar.*

We denote by $\Re = \{g(z_n)\}$ the family of rational functions with coefficients from $Q + \mathrm{i}Q$ (here Q is the set of all rational numbers), such that every function $g(z_n) \in \Re$ has a unique zero on $z_n = 0$. To investigate the domain of convergence of the corresponding Jacobi-Hartogs series the following lemma on approximation of planar sets by rational lemniscates will be useful.

Lemma 5. *Let* $\sum$ *be a closed polar subset of* $\mathbb{C} \setminus \{0\}$ *and let* K *be a compact in* $\mathbb{C} \setminus \Sigma$. *Then there exists a rational function* $g \in \Re$, *such that the lemniscate* $\{w\colon |g(w)| < 1\}$ *is connected, belongs to* $\mathbb{C} \setminus \Sigma$ *and contains* K.

This lemma is given in a different formulation in [10].

P r o o f. Choose $r > 0$ such that $K \subset \{w\colon |w| < r\}$ and the circle $\{w\colon |w| = r\}$ does not intersect $\sum$. Then there is $\delta \in (0; r)$ such that the distances from $\sum$ to 0, K and $\{w\colon |w| = r\}$ as well as from K to $\{w\colon |w| = r\}$ are all greater than δ. Since $\sum$ is polar, for every $\varepsilon > 0$ there exist $a_1, a_2, \ldots, a_k \in \Sigma_r = \Sigma \cap \{w\colon |w| < r\}$ such that Σ_r belongs to the lemniscate $|P_k(w)| < \varepsilon^k$, where $P_k(w) = \prod_{j=1}^{k} (w - a_j)$. It is clear that $|P_k(w)| > \delta^k$ everywhere on K and for $|w| \geqslant r + \delta$. Put

$$g(w) = \Big(\frac{w}{r}\Big)^m \frac{1}{P_k(w)},$$

then, obviously,

1. $|g(w)| > \big((r+\delta)/r\big)^m 1/(3r)^k$ for $|w| = r + \delta$,
2. $|g(w)| > \big(\delta/r\big)^m 1/\varepsilon^k$ on Σ_r, and
3. $|g(w)| < \big((r-\delta)/r\big)^m 1/\delta^k$ on K.

The right-hand side of 1 is greater than 1 if $m > k \ln 3/\ln(1 + \delta/r)$, while the right-hand side of 3 is less than 1, if $m > k \ln\delta/\ln(1 - \delta/r)$. Therefore, there exists a constant $c > 1$ depending only on r and δ such that for $m = c \cdot k$ we have $|g(w)| > 1$ for $|w| \geqslant r + \delta$ and $|g(w)| < 1$ on K. If we choose then $\varepsilon < (\delta/r)^c$, we get from 2 that $|g(w)| > 1$ also on Σ_r. Thus, the lemniscate $\{w\colon |g(w)| < 1\}$ contains K, and its closure belongs to $\mathbb{C} \setminus \Sigma$. On the boundary of each component of this lemniscate the function g is constant in modulus, therefore such a component must contain zeros of g. Since $w = 0$ is the unique zero of g in $\mathbb{C}$, this implies that the lemniscate $\{w\colon |g(w)| < 1\}$ is connected. $\square$

Since the set of rational functions with rational coefficients is dense everywhere in the space of all rational functions, the function $g(w) = (w/r)^m 1/P_k(w)$ can be replaced by a function from the class $\Re$. Lemma 5 is proved.

3. Some results connected with the class R^0

The class R^0 was introduced by A. A. Gonchar to investigate the rapid convergence of sequences of rational functions and the main results about the properties of the class R^0 also belong to him (see [2]).

Let f be a holomorphic function on some neighborhood U of $0 \in \mathbb{C}^n$ such that there exists a sequence of rational functions $r_k(z)$ with $\deg r_k \leqslant k$, which rapidly converges to the function f with respect to the Lebesgue measure: $\lim\limits_{k\to\infty} \operatorname{mes}\{z \in U : |f - r_k|^{1/k} \geqslant \varepsilon\} = 0$ for every $\varepsilon > 0$. Then f is globally single-valued and the sequence r_k rapidly converges to f in measure everywhere on the natural domain of existence $W_f \subset \mathbb{C}^n$ of the function f. The class of all such functions f is denoted by R^0.

Besides being entire and meromorphic in $\mathbb{C}^n$ the functions of the class R^0 also contain every function f having a pluripolar set of singularities. Moreover, the class R^0 is much wider than above mentioned classes of functions. In the case of $n = 1$ a criterion for the function f to belong to the class R^0 in the terms of numerical sequence expressed through the Taylor coefficients of the function f at the origin was obtained by A. Sadullaev [9]. Recall this criterion: let $f(z) = \sum\limits_{k=0}^{\infty} a_k \cdot z^k$ be holomorphic in a neighborhood of the unit ball $\overline{U}\colon\ |z| \leqslant 1$ and

$$V_k = \sup_{j_1,\ldots,j_k} \operatorname{mod} \begin{vmatrix} a_{j_1} a_{j_1+1} \ldots a_{j_1+k-1} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ a_{j_k} a_{j_k+1} \ldots a_{j_k+k-1} \end{vmatrix}, \ k = 1, 2, \ldots$$

where $\operatorname{mod}|\cdot|$ is modulus of the appropriate determinant.

Then

$$f \in R^0 \Longleftrightarrow \lim_{k\to\infty} V_k^{1/k^2} = 0. \tag{7}$$

With the help of this criterion we shall prove the following lemma, which will supply single-valuedness of the analytical continuation of the function f in one variable with discrete singularities.

Lemma 6. *If*

$$f(z', z_n) \in O(D' \times U_n) \cap C\overline{(D' \times U_n)}$$

and $f(a', z_n) \in R^0$ *for all* $a' \in E \subset \partial D'$, $\operatorname{mes} E > 0$, *then* $f(z', z_n) \in R^0$ *for all* $z' \in D'$.

In the case of $E \subset D'$ a nonpluripolar set in D' this lemma is proved in [9].

P r o o f. We expand the function $f(z', z_n)$ into the Hartogs series

$$f(z', z_n) = \sum_{j=0}^{\infty} a_j(z') z_n^j,$$

where

$$a_j(z') = \frac{1}{2\pi \mathrm{i}} \int_{\partial U_n} \frac{f(z', \xi)}{\xi^{j+1}} \,\mathrm{d}\xi.$$

It is clear that $a_j(z') \in O(D') \cap C(\overline{D'})$.

Now we define $V_k(z')$ as the following

$$V_k(z') = \sup_{j_1, \dots, j_k} \operatorname{mod} \begin{vmatrix} a_{j_1}(z') \; a_{j_1+1}(z') \; \dots \; a_{j_1+k-1}(z') \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{j_k}(z') \; a_{j_k+1}(z') \; \dots \; a_{j_k+k-1}(z') \end{vmatrix}, \; k = 1, 2, \dots.$$

Obviously (see [9]), that functions $\varphi_k(z') = \frac{1}{k^2} \ln V_k(z')$ are plurisubharmonic and upper bounded in $\overline{D'}$ uniformly in k.

Let

$$\varphi(z') = \varlimsup_{k \to \infty} \varphi_k(z'), \; z' \in \overline{D'}.$$

According to the criterion (7) $\varphi(\xi') \equiv -\infty$, for each $\xi' \in E$, and according to Lemma 2 $\varphi^*(\xi') = \varlimsup_{\varepsilon \to 0} \varphi^*(\xi' - \varepsilon \cdot \nu_{\xi'}) \leqslant \varphi(\xi')$ for almost all $\xi' \in \partial D'$, i.e. the boundary function $\varphi^*(\xi')$ is not integrable. Hence we have that plurisubharmonic function $\varphi^*(z') \equiv -\infty$ in D'. Indeed, if we suppose that $\varphi^*(z') \neq -\infty$, then according to Lemma 1 the boundary function $\varphi^*(\xi')$ must be integrable. Since $\varphi(z') \leqslant \varphi^*(z')$, then $\varphi(z') = -\infty$ for each $z' \in D'$. Consequently,

$$\varlimsup_{k \to \infty} V_k^{1/k^2}(z') = 0, \quad z' \in D'.$$

Hence, using the criterion (7) we find that $f(z', z_n) \in R^0$ for every $z' \in D'$. Lemma 6 is proved. □

4. Boundary properties of pseudoconcave sets

In the papers of Oka [5], Slodkowski [12] and Sadullaev [8], [9] some properties of pseudoconcave sets have been established. Let S be a pseudoconcave subset of the domain $U' \times U_n$ and let

$$S_{a'} = S \cap \{z' = a'\}.$$

Assume that the closure of S does not intersect $U' \times \partial U_n$. Then

1. (Slodkowski). The function $\log(\operatorname{cap} S_{z'})$, where "cap" denotes the capacity (transfinite diameter) of a planar set, is plurisubharmonic in U'.
2. (Oka). If $S_{z'}$ are finite for all z' in some nonpluripolar set $E \subset U'$, then S is an analytic set.
3. (Sadullaev). If the sets $S_{z'}$ are polar for all z' in some nonpluripolar set $E \subset U'$, then S is pluripolar set.

By $l_{a'}$ and $S^*_{a'}$, $a' \in \partial D'$, we denote respectively a line in the space $\mathbb{C}^{n-1}$ passing through a point a' on the direction of the normal $\nu_{a'}$, and the normal boundary fiber of a pseudoconcave set $S \subset D' \times \mathbb{C}$ at a point $a' \in \partial D'$, which is defined as follows:

$$S^*_{a'} = \overline{(l_{a'} \times \mathbb{C}) \cap S} \cap \{z' = a'\}.$$

It is easy to check that in general $S^*_{a'} \neq \overline{S} \cap \{z' = a'\}$.

Lemma 7. *Let S be a pseudoconcave and bounded subset of the domain $D' \times \mathbb{C} \subset \mathbb{C}^n$. If for every a' from a set $E \subset \partial D'$, with positive Lebesgue measure $\operatorname{mes} E > 0$, the normal boundary fiber of $S^*_{a'}$ consists of a finite number of points, then S is an analytic set.*

P r o o f. Consider the plurisubharmonic function

$$\ln \prod |w_i - w_j| = \sum \ln |w_i - w_j|, \quad w = (w_1, w_2, \ldots, w_k) \in \mathbb{C}^k,$$

where the product and the sum are taken over all $1 \leqslant i < j \leqslant k$. According to Slodkowski's Lemma (see [11, p. 460]) the function

$$\delta_k(z') = \frac{2}{k(k-1)} \max_{w \in S^k_{z'}} \Sigma \ln |w_i - w_j|$$

is plurisubharmonic in D' for arbitrary k, where $S^k_{z'} = S_{z'} \times S_{z'} \times \ldots \times S_{z'}$ (k-times).

Denote $E_j = \{a' \in E, \operatorname{card} S^*_{a'} \leqslant j\}$, $j = 1, 2, \ldots$, where "card" stands for the number of points. Then E_j is a growing sequence of sets such that $E = \bigcup\limits_{j=1}^{\infty} E_j$.

Since E has positive Lebesgue measure, at least one of these sets (let it be E_k) also has positive Lebesgue measure. It is clear that the set $S^*_{a'}$, $a' \in \partial D'$, is the set of limit points of the set S, thus for any sequence $z'^p \in D' \cap l_{a'}$, converging to the point a', the following inclusion is true:

$$\bigcap_{m=1}^{\infty} \overline{\bigcup_{p=m}^{\infty}} S_{z'^p} \subseteq S^*_{a'}.$$

Thus the plurisubharmonic function $\delta_{k+1}(z')$ is upper bounded and $\lim\limits_{\varepsilon\to 0} \delta_{k+1}(a' - \varepsilon\nu_{a'}) = -\infty$ for each $a' \in E_k$. Hence, we see that the boundary function $\delta_{k+1}(\xi') = \overline{\lim\limits_{\varepsilon\to 0}}\, \delta_{k+1}(\xi' - \varepsilon \cdot \nu_{\xi'})\xi' \in \partial D'$, is not integrable. Thus by Lemma 1 the plurisubharmonic function $\delta_{k+1}(z')$ has a finite value in no point, i.e. $\delta_{k+1}(z') \equiv -\infty$ in D'. This means that card $S_{z'} \leqslant k$, for all $z' \in D'$. By applying here the theorem of Oka [5] we obtain the analyticity of the set S. Lemma 7 is proved. □

Lemma 8. *Let S be a pseudoconcave, bounded subset of the domain $D' \times \mathbb{C} \subset \mathbb{C}^n$. If for every a' from a set $E \subset \partial D'$, with positive Lebesgue measure,* $\operatorname{mes} E > 0$, *the normal boundary fibers $S^*_{a'}$ are polar, then S is a pluripolar set.*

Proof. First note that the cardinality

$$\operatorname{cap} S_{z'} = \lim_{k\to\infty} \Bigl(\max \prod_{1\leqslant i\leqslant j\leqslant k} |w_i - w_j|\Bigr)^{2/k(k-1)}$$

is called the capacity of the set $S_{z'} \subset \mathbb{C}$, where the maximum is taken over all possible arrangements of points $w_1, w_2, \ldots, w_k \in S_{z'}$. Here the limit exists, because the sequence in question decreases. Since the sequence

$$\delta_k(z') = \frac{2}{k(k-1)} \max_{w\in S^k_{z'}} \sum_{1\leqslant i\leqslant j\leqslant k} \ln |w_i - w_j|$$

of plurisubharmonic functions in D' is decreasing, it converges to a function $\psi(z') = \ln \operatorname{cap} S_{z'}$, where $\psi(z')$ is an upper-bounded plurisubharmonic function in D' and $\psi(a) = \ln \operatorname{cap} S^*_{a'} = -\infty$, for any $a' \in E$, because $\operatorname{cap} S^*_{a'} = 0$. Hence we have $\psi(z') = \ln \operatorname{cap} S_{z'} \equiv -\infty$ in D'. Consequently $\operatorname{cap} S_{z'} \equiv 0$, i.e. the layers of $S_{z'}$ are polar sets for all $z' \in D'$. Then by the theorem of Sadullaev (see [9], Proposition 1) it follows that S is pluripolar in $D' \times \mathbb{C}$. Lemma 8 is proved.

5. Proof of the theorems

1. Let $f(z', z_n)$ satisfy the conditions of Theorem 1 (Theorem 2). Then according to Lemma 6 for every fixed $z' \in D' \cup E$ the function as a function of z_n belongs to the class R^0.

We expand $f(z', z_n)$ into the Jacobi-Hartogs series of powers of a rational function $g(z_n) \in \Re$:

$$f(z', z_n) = \sum_{k=0}^{\infty} c_k(z', z_n) g^k(z_n),$$

where $c_k(z', z_n) \in O(D' \times U_n) \cap C(\overline{D'} \times U_n)$. This can be done since $f(z', z_n)$ is holomorphic in $D = D' \times U_n$, continuous on $\overline{D' \times U_n}$, and for small $\delta > 0$ the lemniscate $\{z_n\colon\ |g(z_n)| < \delta\}$ belongs to U_n. By Lemma 4 the series converges uniformly in the interior of the open set $G_g = \{(z', z_n)\colon\ z' \in D',\, |g(z_n)| < R_*^{(g)}(z')\}$ and hence its sum is holomorphic there. (We recall that

$$R^{(g)}(z') = \frac{1}{\overline{\lim\limits_{k\to\infty}} \sqrt[k]{\|c_k(z', z_n)\|_{|z_n| \leqslant \delta}}},$$

for all $z' \in \overline{D'}$ and $R_*^{(g)}(z') = \underline{\lim\limits_{\zeta' \to z'}}\ R^{(g)}(\zeta')$, $z' \in D'$). According to the definition of the family $\Re$ of rational functions (see Sect. 2) the lemniscate

$$\{z_n\colon\ |g(z_n)| < R_*^{(g)}(z')\}$$

is connected and contains some neighborhood of $z_n = 0$. Therefore the set G_g is a domain, which contains $D' \times \{0\}$. The sum of the constructed series coincides with $f(z', z_n)$ in neighborhood $D' \times \{0\}$ and, thus, this sum is a holomorphic continuation of $f(z', z_n)$ in G_g.

2. Let g_1 and g_2 be arbitrary rational functions from the class $\Re$ and let $f_1(z', z_n)$ and $f_2(z', z_n)$ be analytic continuations of the function $f(z', z_n)$ in the domains G_{g_1} and G_{g_2}respectively. Since for arbitrary $z'^0 \in D'$ the intersections

$$G_{g_1} \cap \{z' = z'^0\} = \{(z'^0, z_n)\colon\ |g_1(z_n)| < R_*^{(g_1)}(z'^0)\}$$

and

$$G_{g_2} \cap \{z' = z'^0\} = \{(z'^0, z_n)\colon\ |g_2(z_n)| < R_*^{(g_2)}(z'^0)\}$$

are connected and $f_1(z'^0, z_n) = f(z'^0, z_n)$, $f_2(z'^0, z_n) = f(z'^0, z_n)$ (we recall that $f(z'^0, z_n) \in R^0$ for every fixed $z'^0 \in D'$) in the corresponding intersections, we have

$$f_1(z'^0, z_n) = f_2(z'^0, z_n)$$

for any $(z'^0, z_n) \in G_{g_1} \cap G_{g_2}$. It follows that $f(z', z_n)$ has a single-valued continuation to $G_{g_1} \cup G_{g_2}$. Hence the function $f(z', z_n)$ has a single-valued continuation to the domain $G = \bigcup G_g$, where the union is over all rational functions in $\Re$.

3. Let $\widehat{G} \subset D' \times \mathbb{C}$ be the natural domain of existence of the analytic function $f(z', z_n)$ (in the sense of Weierstrass) relatively from $D' \times \mathbb{C}$. Since for every fixed $z'^0 \in D'$ the function $f(z'^0, z_n)$ belongs to the class R^0, it follows that the function $f(z', z_n)$ is single-valued in the domain $\widehat{G}$. Hence, the domain $\widehat{G}$ is one-sheeted and holomorphically nonexpandable at every boundary point $(z', z_n) \in S = (D' \times \mathbb{C}) \setminus \widehat{G}$ (i.e. for every point $(z', z_n) \in S$ there exists its neighborhood U and a function $\varphi \in O(\widehat{G} \cap U)$ holomorphically non continuable at the point (z', z_n)). Hence we obtain that for every ball $B' \subset D'$ the complement $(B' \times \mathbb{C}) \setminus S$ is pseudoconvex, i.e. S-pseudoconcave.

By Lemma 2 the Lebesgue measure of

$$E_g = \{a' \in E\colon \lim_{\varepsilon \to 0} R_*^{(g)}(a' - \varepsilon \cdot \nu_{a'}) < R^{(g)}(a')\}$$

is equal to zero. This implies that the Lebesgue measure of $\bigcup E_g$ is also equal to zero, where the union is taken over $\Re$, and, consequently, for any $a' \in E_0 = E \setminus \left(\bigcup E_g\right)$ and $g \in \Re$ the following inequality holds

$$R_*^{(g)}(a') = \lim_{\varepsilon \to 0} R_*^{(g)}(a' - \varepsilon \cdot \nu_{a'}) \geqslant R^{(g)}(a').$$

Using Lemma 5 once again we obtain that for any $a' \in E_0$ the normal boundary fiber $S^*_{a'}$ of pseudoconcave set S coincides with the set of singularities $\Lambda_{a'}$ of the function $f(a', z_n)$: Indeed, since by the hypothesis of Theorem 1 (Theorem 2) the singular set $\Lambda_{a'}$ of the function $f(z', z_n)$ consists of a finite number of points (a polar set), by Lemma 5 for each point (a', z_n^0), $a' \in E_0, z_n^0 \in \mathbb{C} \setminus \Lambda_{a'}$, there exists a rational function $g \in \Re$ whose lemniscate $\{(a', z_n)\colon |g(z_n)| < R_*^{(g)}(a')\}$ contains the point (a', z_n^0). It follows that for each $a' \in E_0$

$$\bigcap_{g \in \Re} \overline{(\ell_{a'} \times \mathbb{C}) \setminus G_g} \cap \{z' = a'\} = \bigcap_{g \in \Re} \{(a', z_n)\colon |g(z_n)| \geqslant R_*^{(g)}(a')\} = \Lambda_{a'}.$$

From these equalities it follows that $S^*_{a'} = \Lambda_{a'}$. In particular, the fibers $S^*_{a'}$ consist of a finite number (a polar set) of points.

5. Let Ω be the image of the domain $\widehat{G} = (D' \times \mathbb{C}) \setminus S$ under the mapping $(z', z_n) \to (z', 1/z_n)$. The set $\Sigma = (D' \times \mathbb{C}) \setminus \Omega$ is also pseudoconcave. Since $\overline{S}$ does not intersect the set $\overline{D'} \times \{0\}$, it follows that Σ is bounded and for every $a' \in E_0$ the normal boundary fiber of Σ consists of a finite number (a polar set) of points, i.e. the set Σ satisfies all the conditions of Lemma 7 (Lemma 8), consequently, Σ is an analytic (pluripolar) subset of $D' \times \mathbb{C}$. Thus, S is also analytic (pluripolar). The theorems are proved.

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