# OSCILLATION OF A HIGHER ORDER NEUTRAL DIFFERENTIAL EQUATION WITH A SUB-LINEAR DELAY TERM AND POSITIVE AND NEGATIVE COEFFICIENTS

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*Abstract.* We obtain sufficient conditions for every solution of the differential equation

$$[y(t) - p(t)y(r(t))]^{(n)} + v(t)G(y(g(t))) - u(t)H(y(h(t))) = f(t)$$

to oscillate or to tend to zero as $t$ approaches infinity. In particular, we extend the results of Karpuz, Rath and Padhy (2008) to the case when $G$ has sub-linear growth at infinity. Our results also apply to the neutral equation

$$[y(t) - p(t)y(r(t))]^{(n)} + q(t)G(y(g(t))) = f(t)$$

when $q(t)$ has sign changes. Both bounded and unbounded solutions are consideted here; thus some known results are expanded.

*Keywords*: oscillatory solution, neutral differential equation, asymptotic behaviour

*MSC 2000*: 34C10, 34C15, 34K40

## 1. Introduction

This article concerns the oscillation of solutions to the neutral functional differential equation

$$(y(t) - p(t)y(r(t)))^{(n)} + v(t)G(y(g(t))) - u(t)H(y(h(t))) = f(t), \tag{1.1}$$

where $n$ is an integer greater than 1, the functions $f$, $g$, $h$, $p$, $r$ are in $C([0,\infty),\mathbb{R})$, the functions $u$, $v$ are in $C([0,\infty),[0,\infty))$, the functions $G$ and $H$ are in $C(\mathbb{R},\mathbb{R})$, and $p$ is a function with $n$ continuous derivatives. The delay functions $g$, $h$, $r$ are non-decreasing with $g(t) \leqslant t$, $h(t) \leqslant t$, $r(t) \leqslant t$ and these functions approach $\infty$ as $t \to \infty$.

Let $t_0 \geqslant 0$ and $t_{-1} := \min\{r(t_0), g(t_0), h(t_0)\}$. By a *solution* of (1.1) we mean a function $y \in C([t_{-1}, \infty), \mathbb{R})$ such that $y(t) - p(t)y(r(t))$ is $n$ times continuously differentiable on $[t_0, \infty)$ and the neutral equation (1.1) is satisfied for $t \geqslant t_0$.

In this work we assume the existence of solutions and study only their qualitative behaviour. For results on existence and uniqueness of solutions, we refer the reader to [4].

Given an initial function $\varphi \in C([t_{-1}, t_0], \mathbb{R})$, we assume that there exists a unique solution to (1.1) satisfying $y(t) = \varphi(t)$ for $t \in [t_{-1}, t_0]$. Such a solution is said to be *non-oscillatory* if it is eventually positive or eventually negative; otherwise it is called *oscillatory*.

The function $G$ in (1.1) is said to have linear growth (or to be linear) at infinity, if $\lim\limits_{x\to\infty} |G(x)|/x$ is a positive constant. $G$ is super-linear if $\lim\limits_{x\to\infty} |G(x)|/x = \infty$, and $G$ is sub-linear if $\lim\limits_{x\to\infty} |G(x)|/x = 0$.

Oscillation and non-oscillation of neutral differential equations have been studied by many authors during the previous two decades [6], [10], [11], [13], [14], [21], [9], [24]. However, there are only a few publications on the oscillatory behaviour of higher order ($n \geqslant 2$) neutral differential equations with positive and negative coefficients. In a recent paper Karpuz et al [6] obtained results assuming that $G$ is linear or super-linear; see Theorem 2.1 below. In this article, we omit the superlinear growth condition on $G$. By doing this, our results also apply to the neutral differential equation

$$(1.2) \qquad (y(t) - p(t)y(r(t)))^{(n)} + q(t)G(y(g(t))) = f(t),$$

where $q$ is allowed to change sign; see Section 3. The majority of the existing publications have results for $q$ positive, and very few for $q$ having sign changes; see [1], [2], [12], [16], [17], [18], [19], [21], [23]. The main difficulty is that when $q(t)$ is oscillatory, then for a non-oscillatory solution $y(t)$ of (1.2), the function $(y(t) - p(t)y(r(t))^{(n)} - f(t)$ may be oscillatory. To the best of our knowledge, when $q(t)$ changes sign, (1.2) has been studied only for $n = 1$; see [3]. Thus, our results extend and generalize some results from [3], [6], [21].

## 2. Main results

In the sequel, unless otherwise specified, when we write a functional inequality, it will be assumed to hold for all $t$ sufficiently large. Our results will use the following hypotheses:

(H0) $\liminf\limits_{x\to\infty} G(x) > 0$, $\limsup\limits_{x\to-\infty} G(x) < 0$;
(H1) $xG(x) > 0$ for $x \neq 0$,
(H2) $H$ is bounded;
(H3) $\int_{t_0}^{\infty} v(t)\,\mathrm{d}t = \infty$;
(H4) $\int_{t_0}^{\infty} t^{n-1}u(t)\,\mathrm{d}t < \infty$;
(H5) there exists a bounded function $F$ such that $F^{(n)}(t) = f(t)$;
(H6) the function $F$ in (H5) satisfies $\lim\limits_{t\to\infty} F(t) = 0$.

Note that (H6) is satisfied for example if $\int_0^{\infty} t^{n-1}|f(t)|\,\mathrm{d}t < \infty$.

**2.1. Results for unbounded solutions.** First we state a theorem that assumes $G$ being linear or superlinear.

**Theorem 2.1** [6, Theorem 2.4]**.** *Assume that $G$ is nondecreasing and that there exist positive constants $p_1, p_2$ such that*

$$-1 < p_1 \leqslant p(t) \leqslant 0 \quad \forall t \quad \text{or} \quad 0 \leqslant p(t) \leqslant p_2 < 1 \quad \forall t. \tag{2.1}$$

*Also assume* (H1)–(H2), (H4)–(H5),

$$\int_{t_0}^{\infty} t^{n-2}v(t)\,\mathrm{d}t = \infty \quad (n \geqslant 2), \tag{2.2}$$

$$\liminf_{t\to\infty} g(t)/t > 0, \tag{2.3}$$

$$\liminf_{x\to\infty} G(x)/x > 0, \quad \liminf_{x\to-\infty} G(x)/x > 0. \tag{2.4}$$

*Then every unbounded solution of* (1.1) *is oscillatory.*

The following example shows that (2.4) is necessary in the setting of the above theorem.

Example 2.2. Consider the neutral equation

$$(y(t) - cy(t-1))''' + v(t)y^{1/3}(t-1) - u(t)H(y(t-2)) = 0 \tag{2.5}$$

for $t \geqslant 4$, where $v(t) = 3(t^{-6} + t^{-3/2} - c(t-1)^{-3/2})/(8\sqrt{t-1})$, $H(y) = y/(y^2+1)$, $u(t) = 3(1 + (t-2)^3)/(8t^6(t-2)^{3/2})$, and $0 < p(t) \equiv c < 1/2$ or $-1 < p(t) \equiv c < 0$. Note that $G(y) = y^{1/3}$ which is sub-linear. Clearly, the conditions for Theorem 2.1 are satisfied except for (2.4), and the solution $y(t) = t^{3/2}$ is unbounded and non-oscillatory.

Now we state our main result, without assuming that $G$ is non-decreasing and without conditions (2.2)–(2.4). In fact, we replace (2.2) by (H3), which is more restrictive than (2.2) when $n \geqslant 3$. However, these two conditions are the same when $n = 2$.

**Theorem 2.3.** *Assume* (H0)–(H5). *For $p(t)$ satisfying* (2.1), *every unbounded solution of* (1.1) *is oscillatory.*

P r o o f. For the sake of contradiction, assume that there exists an unbounded solution $y$ which is eventually positive. Because $g, h, r$ approach $\infty$ as $t \to \infty$, there exists $t_0$ such that $y(t) > 0$, $y(h(t)) > 0$, $y(g(t)) > 0$, $y(r(t)) > 0$ for $t \geqslant t_0$. Define

$$k(t) = \frac{(-1)^{n-1}}{(n-1)!} \int_t^{\infty} (s-t)^{n-1} u(s) H(y(h(s)))\,\mathrm{d}s \quad \text{for } t \geqslant t_0. \tag{2.6}$$

By assumptions (H2) and (H4) the above integral converges, thus $k(t)$ is a well defined real-valued function, and

$$\lim_{t\to\infty} k(t) = 0. \tag{2.7}$$

Note that the $n$th derivative of $k$ is $k^{(n)}(t) = -u(t)H(y(h(t)))$. For simplicity of notation, we define

$$w(t) = y(t) - p(t)y(r(t)) + k(t) - F(t), \tag{2.8}$$

where $F^{(n)}(t) = f(t)$. Then from (1.1),

$$w^{(n)}(t) = -v(t)G(y(g(t))). \tag{2.9}$$

Then by (H1) we have $w^{(n)}(t) \leqslant 0$. By (H3), $w^{(n)}(t)$ is not identically zero in any interval $[t_1, \infty)$. As in [8, Lemma 5.2.1], we can show that there exists $t_1 \geqslant t_0$ such that $w, w', \ldots, w^{(n-1)}$ are monotonic and of constant sign on $[t_1, \infty)$. However, we do not know yet that $w > 0$.

Since $y$ is positive and unbounded, there exists an increasing sequence $\{a_j\}$ such that

$$\lim_{j\to\infty} a_j = \infty, \quad \lim_{j\to\infty} y(a_j) = \infty, \quad \text{with } y(a_j) = \max_{t_1 \leqslant s \leqslant a_j} y(s).$$

By (2.7), for each $\varepsilon > 0$ there exists $N_0$ such that

$$k(a_j) < \varepsilon \quad \text{for } j \geqslant N_0.$$

Since $g(t), h(t), r(t)$ approach $\infty$ as $t \to \infty$, there exists $N_1 \geqslant N_0$ such that: $a_j, g(a_j), h(a_j), r(a_j) > t_1$ for $j \geqslant N_1$.

By (H5), there is an upper bound $\eta$ for $|F|$. Using that $y(t) > 0$, the definition of $\{a_j\}$ and that $r(t) \leqslant t$, we have: for the case $0 \leqslant p(t) \leqslant p_2 < 1$,

$$w(a_j) = y(a_j) - p(a_j)y(r(a_j)) + k(a_j) - F(a_j) \geqslant (1 - p_2)y(a_j) - \varepsilon - \eta, \quad j \geqslant N_1;$$

and for the case $-1 < p_1 \leqslant p(t) \leqslant 0$,

$$w(a_j) = y(a_j) - p(a_j)y(r(a_j)) + k(a_j) - F(a_j) \geqslant y(a_j) - \varepsilon - \eta, \quad j \geqslant N_1.$$

Taking limits in both the inequalities above, we have $\lim\limits_{j\to\infty} w(a_j) = \infty$. Since $w, w', \ldots, w^{(n-1)}$ are monotonic and of constant sign, it follows that $w > 0$ and $w' > 0$. Now by [8, Lemma 5.2.1], $w^{(n)} \leqslant 0$ and $w > 0$ imply $w^{(n-1)}(t) > 0$ for $t \geqslant t_1$.

Next we show that $y$ is bounded below by a positive constant, which will be used for bounding the $G$ term from below.

Using that $w$ is positive and increasing and that $r(t) \leqslant t$, we have for the case $-1 < p_1 \leqslant p(t) \leqslant 0$,

$$\begin{aligned}(1 + p_1)w(t) &\leqslant w(t) + p_1 w(r(t)) \leqslant w(t) + p(t)w(r(t)) \\ &= y(t) + k(t) - F(t) + p(t)[-p(r(t))y(r(r(t))) + k(r(t)) - F(r(t))];\end{aligned}$$

and for the case $0 \leqslant p(t) \leqslant p_2 < 1$,

$$\begin{aligned}w(t) &\leqslant w(t) + p(t)w(r(t)) \\ &= y(t) + k(t) - F(t) + p(t)[-p(r(t))y(r(r(t))) + k(r(t)) - F(r(t))].\end{aligned}$$

Since $p(t)$ and $p(r(t))$ have the same sign and $y > 0$ in each of the two inequalities above, and $1 - p_2 \leqslant 1$, we have

$$(1 - p_2)w(t) \leqslant y(t) + \varepsilon + \eta + p_2\varepsilon + p_2\eta, \quad \text{for } t \geqslant t_1.$$

Since $\lim\limits_{t\to\infty} w(t) = \infty$, it follows that $\lim\limits_{t\to\infty} y(t) = \infty$. Then there exists $t_2 \geqslant t_1$ such that for $t \geqslant t_2$, $y(t), y(g(t)), y(h(t)), y(r(t))$ are bounded below by a positive constant. By (H0)–(H1), for $s \geqslant t_2$, $G(y(g(s)))$ is bounded below by a positive constant $\alpha$. Integrating (2.9) we obtain

$$w^{(n-1)}(t) = w^{(n-1)}(t_2) + \int_{t_2}^{t} -v(s)G(y(g(s)))\,\mathrm{d}s \leqslant w^{(n-1)}(t_2) - \alpha \int_{t_2}^{t} v(s)\,\mathrm{d}s.$$

Note that by (H3) the right-hand side approaches $-\infty$, while the left-hand side is positive. This contradiction implies that the solution cannot be unbounded and eventually positive.

Now assume that the solution $y$ is unbounded and eventually negative. Because $g, h, r$ approach $\infty$ as $t \to \infty$, there exists $t_0$ such that $y(t) < 0$, $y(h(t)) < 0$, $y(g(t)) < 0$, $y(r(t)) < 0$ for $t \geqslant t_0$. Define $w(t)$ as in (2.8). Then $w^{(n)}(t) \geqslant 0$ and it is not identically zero in any interval $[t_1, \infty)$. As in [8, Lemma 5.2.1], we can show that there exists $t_1 \geqslant t_0$ such that $w, w', \ldots, w^{(n-1)}$ are monotonic and of constant sign on $[t_1, \infty)$. However, we do not know yet that $w < 0$.

Since $y$ is negative and unbounded, there exists an increasing sequence $\{a_j\}$ such that

$$\lim_{j\to\infty} a_j = \infty, \quad \lim_{j\to\infty} y(a_j) = -\infty, \quad \text{with } y(a_j) = \min_{t_1 \leqslant s \leqslant a_j} y(s).$$

Since $g(t), h(t), r(t)$ approach $\infty$ as $t \to \infty$, there exists $N_1 \geqslant N_0$ such that: $a_j, g(a_j), h(a_j), r(a_j) > t_1$ for $j \geqslant N_1$.

By (H5) there is an upper bound $\eta$ for $|F|$. Using that $y(t) < 0$, the definition of $\{a_j\}$, and that $r(t) \leqslant t$, we have: For the case $0 \leqslant p(t) \leqslant p_2 < 1$,

$$w(a_j) = y(a_j) - p(a_j)y(r(a_j)) + k(a_j) - F(a_j) \leqslant (1 - p_2)y(a_j) + \varepsilon + \eta, \quad j \geqslant N_1;$$

and for the case $-1 < p_1 \leqslant p(t) \leqslant 0$,

$$w(a_j) = y(a_j) - p(a_j)y(r(a_j)) + k(a_j) - F(a_j) \leqslant y(a_j) + \varepsilon + \eta, \quad j \geqslant N_1.$$

Taking limits in both the inequalities above, we have $\lim_{j\to\infty} w(a_j) = -\infty$. Since $w, w', \ldots, w^{(n-1)}$ are monotonic and of constant sign, it follows that $w < 0$ and $w' < 0$. Now by [8, Lemma 5.2.1], $w^{(n)} \geqslant 0$ and $w < 0$ imply $w^{(n-1)}(t) < 0$ for $t \geqslant t_1$.

Using that $w$ is negative and increasing, and that $r(t) \leqslant t$, we have: For the case $-1 < p_1 \leqslant p(t) \leqslant 0$,

$$\begin{aligned}(1 + p_1)w(t) &\geqslant w(t) + p_1 w(r(t)) \geqslant w(t) + p(t)w(r(t)) \\ &= y(t) + k(t) - F(t) + p(t)[-p(r(t))y(r(r(t))) + k(r(t)) - F(r(t))];\end{aligned}$$

and for the case $0 \leqslant p(t) \leqslant p_2 < 1$,

$$\begin{aligned}w(t) &\geqslant w(t) + p(t)w(r(t)) \\ &= y(t) + k(t) - F(t) + p(t)[-p(r(t))y(r(r(t))) + k(r(t)) - F(r(t))].\end{aligned}$$

Since $p(t)$ and $p(r(t))$ have the same sign and $y < 0$ in each of the two inequalities above, and $0 < 1 + p_1 \leqslant 1$, we have

$$w(t) \geqslant y(t) - \varepsilon - \eta + p_1\varepsilon + p_1\eta, \quad \text{for } t \geqslant t_1.$$

Since $\lim_{t\to\infty} w(t) = -\infty$, it follows that $\lim_{t\to\infty} y(t) = -\infty$. Then there exists $t_2 \geqslant t_1$ such that for $t \geqslant t_2$, $y(t), y(g(t)), y(h(t)), y(r(t))$ are bounded above by a negative constant. By (H0)–(H1), for $s \geqslant t_2$, $G(y(g(s)))$ is bounded above by a negative constant $\delta$. Integrating (2.9), we obtain

$$w^{(n-1)}(t) = w^{(n-1)}(t_2) + \int_{t_2}^{t} -v(s)G(y(g(s)))\,\mathrm{d}s \geqslant w^{(n-1)}(t_2) - \delta \int_{t_2}^{t} v(s)\,\mathrm{d}s.$$

Note that by (H3) the right-hand side approaches $+\infty$, while the left-hand side is negative. This contradiction implies that the solution cannot be unbounded and eventually negative. This completes the proof. $\square$

The necessity of (H3) in the above theorem can be shown using Example 2.2. Note that by setting $p(t) = 0$, Theorems 2.1, 2.3 apply to the equation

$$y^{(n)}(t) + v(t)G(y(g(t))) - u(t)H(y(h(t))) = f(t).$$

Also note that Theorems 2.1, 2.3 answer the open problem posed in [23, p. 195]; i.e., to study oscillatory behaviour of unbounded solutions of (1.1) when $p(t)$ satisfies (2.1).

**2.2. Results for bounded solutions.** The superlinearity conditions (2.4) and (H0) are not needed in this subsection because the solution $y$ does not approach $\pm\infty$. The next result is a modification of [6, Theorem 2.10], we omit the assumption that $G$ is non-decreasing.

**Theorem 2.4.** *Assume* (H1)–(H2), (H4)–(H6), *that the delayed argument* $r(t)$ *is strictly increasing, and that*

$$\int_{t_0}^{\infty} t^{n-1}v(t)\,\mathrm{d}t = \infty. \tag{2.10}$$

*Then every bounded solution of* (1.1) *is oscillatory or tends to zero as* $t \to \infty$, *for each of the following four cases:*

$$\begin{gathered} p_4 \leqslant p(t) \leqslant p_3 < -1 \ \ \forall t; \quad -1 < p_1 \leqslant p(t) \leqslant 0 \ \ \forall t; \\ 0 \leqslant p(t) \leqslant p_2 < 1 \ \ \forall t; \quad 1 < p_5 \leqslant p(t) \leqslant p_6 \ \ \forall t, \end{gathered} \tag{2.11}$$

*where* $p_1, p_2, p_3, p_4, p_5, p_6$ *are constants.*

P r o o f. We follow the steps in [6, Theorem 2.10]. By contradiction assume $y$ is an eventually positive solution of (1.1) which does not tend to zero as $t \to \infty$. Then there exists a $t_0$ such that for $t \geqslant t_0$, $y(t)$, $y(h(t))$, $y(g(t))$, $y(r(t))$ are positive. Define $w(t)$ by (2.8). Then, as above, $w^{(n)} \leqslant 0$ and $w, w', \ldots, w^{(n-1)}$ are monotonic and of constant sign on some interval $[t_1, \infty)$. Let $\lambda := \lim_{t\to\infty} w(t)$ which exists as a finite number because $w$ is monotonic and bounded. Integrating (2.9) $n$ times, we obtain

$$w(t) - \lambda = \frac{(1)^{n-1}}{(n-1)!} \int_t^\infty (s-t)^{n-1} v(s) G(y(g(s)))\,\mathrm{d}s.$$

Since $w$ is bounded, the above integral is convergent. This in turn, by (2.10), implies $\liminf_{s\to\infty} G(y(g(s))) = 0$. As $G(x) \neq 0$ for $x \neq 0$, we have $\liminf_{s\to\infty} y(g(s)) = 0$ and because $\lim_{t\to\infty} g(t) = \infty$, we have $\liminf_{t\to\infty} y(t) = 0$.

Since $\lim_{t\to\infty} w(t)$ exists, $k(t)$, $F(t)$ approach zero and $p(t)$ is bounded, it follows that $\lim_{t\to\infty} y(t) - p(t)y(r(t))$ exists as a finite number. A result in [23, Lemma 1] shows that $\liminf_{s\to\infty} y(s) = 0$ implies $\lim_{t\to\infty} y(t) - p(t)y(r(t)) = 0$ under each of the assumptions in (2.11), with $r(t)$ strictly increasing. Therefore $\lambda = \lim_{t\to\infty} w(t) = 0$.

For $p(t) \leqslant 0$ (including $p(t) \leqslant p_3 < -1$) we have $w(t) \geqslant y(t) + k(t) - F(t)$ and

$$0 = \lim_{t\to\infty} w(t) \geqslant \limsup_{t\to\infty} y(t) \geqslant 0.$$

For $0 \leqslant p(t) \leqslant p_2 < 1$ we have $w(t) \geqslant y(t) - p_2 y(r(t)) + k(t) - F(t)$ and

$$\begin{aligned} 0 = \lim_{t\to\infty} w(t) &\geqslant \limsup_{t\to\infty} [y(t) - p_2 y(r(t))] \\ &\geqslant \limsup_{t\to\infty} y(t) + \liminf_{t\to\infty} [-p_2 y(r(t))] \\ &= (1-p_2) \limsup_{t\to\infty} y(t) \geqslant 0. \end{aligned}$$

For $1 < p_5 \leqslant p(t) \leqslant p_6$ we have $w(t) \leqslant y(t) - p_5 y(r(t)) + k(t) - F(t)$ and

$$\begin{aligned} 0 = \lim_{t\to\infty} w(t) &\leqslant \liminf_{t\to\infty} [y(t) - p_5 y(r(t))] \\ &\leqslant \limsup_{t\to\infty} y(t) + \liminf_{t\to\infty} [-p_5 y(r(t))] \\ &= (1-p_5) \limsup_{t\to\infty} y(t) \leqslant 0. \end{aligned}$$

Therefore, under each of the four hypotheses in (2.11), $\limsup_{t\to\infty} y(t) = 0$. Therefore, bounded eventually positive solutions must approach zero.

The proof for bounded eventually negative solutions is similar to the proof above; so we omit it. The proof is complete. $\square$

**2.3. Results for bounded or unbounded solutions.**

Theorems 2.3 and 2.4 are combined to give a general result as follows.

**Theorem 2.5.** *Assume* (H0)–(H6) *and that* $r(t)$ *is strictly increasing. Then every solution of* (1.1) *is oscillatory or tends to zero as* $t \to \infty$, *for* $p(t)$ *satisfying* (2.1).

The following oscillation result does not assume that $r(t)$ is increasing, but restricts $p(t)$ furthermore than the theorem above.

**Theorem 2.6.** *Assume* (H0)–(H6) *and that* $0 \leqslant p(t) \leqslant p_2 < 1$. *Then every solution of* (1.1) *is oscillatory or tends to zero as* $t \to \infty$.

Proof. By contradiction assume $y$ is an eventually positive solution of (1.1) which does not tend to zero as $t \to \infty$. Then there exists a $t_0$ such that for $t \geqslant t_0$, $y(t)$, $y(h(t))$, $y(g(t))$, $y(r(t))$ are positive and $\limsup\limits_{t\to\infty} y(t) > 0$. Define $w(t)$ by (2.8). Then, as above, $w^{(n)} \leqslant 0$ and $w, w', \ldots, w^{(n-1)}$ are monotonic and of constant sign on some interval $[t_1, \infty)$. We do not know that $w > 0$ yet. Since $0 \leqslant p(t) \leqslant p_2 < 1$ and $y > 0$, we have

$$w(t) \geqslant y(t) - p_2 y(r(t)) + k(t) - F(t).$$

Taking the limit superior, using that $w$ is monotonic and that $k(t)$ and $F(t)$ converge to zero, we obtain

$$\lambda = \lim_{t\to\infty} w(t) \geqslant (1 - p_2) \limsup_{t\to\infty} y(t) > 0.$$

Then $w(t)$ is positive for $t$ large enough. By [8, Lemma 5.2.1], $w^{(n)} \leqslant 0$ and $w > 0$ imply the existence of $t_1$ such that $w^{(n-1)}(t) > 0$ for $t \geqslant t_1$. Next we show that $\liminf\limits_{t\to\infty} y(t) > 0$, which will be used for bounding $G(y(g(s)))$ from below by a positive constant. Using that $0 \leqslant p(t)$ and $y > 0$, we have

$$w(t) \leqslant y(t) + k(t) - F(t).$$

Taking the limit inferior, using that $w$ is monotonic and that $k(t)$ and $F(t)$ approach zero, we have

$$0 < \lambda = \lim_{t\to\infty} w(t) \leqslant \liminf_{t\to\infty} y(t).$$

Then there exists a $t_2 \geqslant t_1$ such that for $t \geqslant t_2$, $y(t)$, $y(h(t))$, $y(g(t))$, $y(r(t))$ are bounded below by a positive constant. By (H0)–(H1), for $s \geqslant t_2$, $G(y(g(s)))$ is bounded below by a positive constant $\alpha$. Integrating (2.9), we obtain

$$w^{(n-1)}(t) = w^{(n-1)}(t_2) + \int_{t_2}^{t} -v(s) G(y(g(s)))\,\mathrm{d}s \leqslant w^{(n-1)}(t_2) - \alpha \int_{t_2}^{t} v(s)\,\mathrm{d}s.$$

Note that by (H3) the right-hand side approaches $-\infty$, while the left-hand side is positive. This contradiction implies that the solution cannot be eventually positive without approaching zero.

Now assume that $y$ is eventually negative and does not tend to zero as $t \to \infty$. Then there exists a $t_0$ such that for $t \geqslant t_0$, $y(t)$, $y(h(t))$, $y(g(t))$, $y(r(t))$ are negative and $\liminf_{t\to\infty} y(t) < 0$. Define $w(t)$ by (2.8). Then $w^{(n)} \geqslant 0$. As above, $w, w', \dots, w^{(n-1)}$ are monotonic and of constant sign on some interval $[t_1, \infty)$. Since $0 \leqslant p(t) \leqslant p_2 < 1$ and $y < 0$, we have

$$w(t) \leqslant y(t) - p_2 y(r(t)) + k(t) - F(t).$$

Taking the limit inferior, using that $w$ is monotonic and that $k(t)$ and $F(t)$ approach zero, we have

$$\lambda = \lim_{t\to\infty} w(t) \leqslant (1 - p_2) \liminf_{t\to\infty} y(t) < 0.$$

Then $w(t) < 0$ for $t$ large enough. Now by [8, Lemma 5.2.1], $w^{(n)} \geqslant 0$ and $w < 0$ imply the existence of $t_1$ such that $w^{(n-1)}(t) < 0$ for $t \geqslant t_1$. Next we show that $\limsup_{t\to\infty} y(t) < 0$, which will be used for bounding $G(y(g(s)))$ from above by a negative constant. Using that $0 \leqslant p(t) \leqslant p_2 < 1$ and $y < 0$, we obtain

$$w(t) \geqslant y(t) + k(t) - F(t).$$

Taking the limit superior, using that $w$ is monotonic and that $k(t)$ and $F(t)$ approach zero, we have

$$0 > \lambda = \lim_{t\to\infty} w(t) \geqslant \limsup_{t\to\infty} y(t).$$

Then there exists a $t_2 \geqslant t_1$ such that for $t \geqslant t_2$, $y(t)$, $y(h(t))$, $y(g(t))$, $y(r(t))$ are bounded above by a negative constant. By (H0)–(H1), for $s \geqslant t_2$, $G(y(g(s)))$ is bounded above by a negative constant $\delta$. Integrating (2.9), we arrive at

$$w^{(n-1)}(t) = w^{(n-1)}(t_1) + \int_{t_1}^{t} -v(s) G(y(g(s)))\,\mathrm{d}s \geqslant w^{(n-1)}(t_1) - \delta \int_{t_1}^{t} v(s)\,\mathrm{d}s.$$

Note that by (H3) the right-hand side approaches $\infty$, while the left-hand side is negative. This contradiction implies that an eventually negative solution must approach zero as $t \to \infty$. This completes the proof. $\square$

The following example illustrates some of our main results.

E x a m p l e 2.7. Consider the neutral functional differential equation with positive and negative coefficients, for $t \geqslant 1$,

$$[y(t)-\mathrm{e}^{-5t/3}y(t/3)]''' + (\mathrm{e}^{-t}+1)G(y(t/2)) - 8\mathrm{e}^{-t}(\mathrm{e}^{-t}+1)H(y(t/4)) = 0, \tag{2.12}$$

where $G(x) = x^2\operatorname{sgn}(x)/(1+x^2)$ and $H(x) = x^4\operatorname{sgn}(x)/(1+x^4)$. Here $\operatorname{sgn}(x)$ assumes values $-1, 0, 1$ for $x<0$, $x=0$, $x>0$, respectively. Clearly, all the conditions of Theorems 2.3, 2.6 and 2.4 are satisfied by the neutral equation (2.12). As such, by these theorems, (2.12) has a solution $y(t) = \mathrm{e}^{-t}$, which tends to zero as $t\to\infty$. Note that in (2.12), $G$ is sublinear; as such, none of the results published before this article can be applied to this neutral equation.

In the next result we remove the barrier at $-1$ for $p(t)$. However, we introduce additional hypotheses.

**Theorem 2.8.** *Assume* (H0)–(H2), (H4)–(H6), $p_4 \leqslant p(t) \leqslant 0$, *and let the delay functions satisfy* $g(r(t)) = r(g(t)$. *Also assume that*

$$\int_{t_0}^{\infty} \min\{v(t), v(r(t))\}\,\mathrm{d}t = \infty; \tag{2.13}$$

*that there exists a positive constant* $\delta$ *such that for* $x, y, z > 0$,

$$G(x+y) \leqslant \delta(G(x)+G(y)), \quad G(zx) \leqslant G(z)G(x); \tag{2.14}$$

*and that for* $x, y < 0$ *and* $z > 0$,

$$G(x+y) \geqslant \delta(G(x)+G(y)), \quad G(zx) \geqslant G(z)G(x). \tag{2.15}$$

*Then every solution of* (1.1) *is oscillatory or tends to zero as* $t\to\infty$.

P r o o f. By contradiction assume $y$ is an eventually positive solution of (1.1) which does not tend to zero as $t\to\infty$. Then there exists a $t_0$ such that for $t \geqslant t_0$, $y(t)$, $y(h(t))$, $y(g(t))$, $y(r(t))$ are positive and $\limsup\limits_{t\to\infty} y(t) > 0$. Define $w(t)$ by (2.8). Then, as above, $w^{(n)} \leqslant 0$ and $w, w', \dots, w^{(n-1)}$ are monotonic and of constant sign on some interval $[t_1,\infty)$. From $p(t)\leqslant 0$ and $y>0$ it follows that $w(t) \geqslant y(t) + k(t) - F(t)$. Taking the limit we have

$$\lambda = \lim_{t\to\infty} w(t) \geqslant \limsup_{t\to\infty} y(t) > 0.$$

Since $k(t)$ and $F(t)$ approach zero, $y(t) - p(t)y(r(t))$ is bounded below by a positive constant for all $t$ large enough. Using $y(t) - p_4 y(r(t)) \geqslant y(t) - p(t)y(r(t))$, $\lim\limits_{t\to\infty} g(t) =$

$\infty$, and $g(r(t)) = r(g(t))$, it follows that $y(g(t)) - p_4 y(g(r(t)))$ is also bounded below by a positive constant on some interval $[t_2, \infty)$. Then by (H0)–(H1) there exists a positive constant $\alpha$ such that $\alpha \leqslant G(y(g(t)) - p_4 y(g(r(t))))$. Using (2.14) we obtain

$$\begin{aligned}\alpha &\leqslant G(y(g(t)) - p_4 y(g(r(t)))) \\ &\leqslant \delta[G(y(g(t))) + G(-p_4) y(g(r(t)))] \\ &\leqslant \delta[G(y(g(t))) + G(-p_4) G(y(g(r(t))))].\end{aligned}$$

From (2.9) we have

$$\begin{aligned}&w^{(n)}(t) + G(-p_4) w^{(n)}(r(t)) \\ &\qquad \leqslant -\min\{v(t), v(r(t))\}[G(y(g(t))) + G(-p_4) G(y(g(r(t))))] \\ &\qquad \leqslant -\min\{v(t), v(r(t))\}\alpha/\delta\end{aligned}$$

Integrating, we arrive at

$$\begin{aligned}&w^{(n-1)}(t) + G(-p_4) w^{(n-1)}(r(t)) \\ &\qquad \leqslant w^{(n-1)}(t_2) + G(-p_4) w^{(n-1)}(r(t_2)) - (\alpha/\delta) \int_{t_2}^{t} \min\{v(s), v(r(s))\} \,\mathrm{d}s.\end{aligned}$$

Taking the limit as $t \to \infty$, by (H3) and (2.12) we obtain that the right-hand side approaches $-\infty$ while the left-hand side is positive. This contradiction proves that eventually positive solutions must converge to zero. For eventually negative solutions, we proceed as above. Thus the proof is complete. □

As prototypes of functions $G$ satisfying the conditions (H0), (H1), (2.14)–(2.15), we have $G(x) = |x|^\lambda \operatorname{sgn}(x)$ and $G(x) = (\beta + |x|^\mu)|x|^\lambda \operatorname{sgn}(x)$ with $\lambda > 0$, $\mu > 0$, $\lambda + \mu \geqslant 1$, $\beta \geqslant 1$. For verifying these conditions, we may use the well known inequality [5, p. 292]

$$x^p + y^p \geqslant \begin{cases} (x+y)^p, & 0 \leqslant p < 1, \\ 2^{1-p}(x+y)^p, & 1 \leqslant p. \end{cases}$$

Clearly, (2.12) implies (H3), but not the other way around. For example $v(t) = \max\{0, \sin(t)\}$ and $r(t) = t - \pi$ yield $\int_0^\infty v(t) \,\mathrm{d}t = \infty$ and $\int_0^\infty v(r(t)) \,\mathrm{d}t = \infty$ but $\int_0^\infty \min\{v(t), v(r(t))\} \,\mathrm{d}t = 0$. However, when $v$ is monotonic, (2.12) is equivalent to (H3).

A result similar to Theorem 2.8 is shown in [6, Theorem 2.20]. There it is assumed that

$$\int_{t_0}^{\infty} t^{n-2} \min\{v(t), v(r(t))\} \,\mathrm{d}t = \infty$$

which is less restrictive than (2.12). This is a trade off for $G$ being non-decreasing and of superlinear growth there.

R e m a r k 2.9. Note that, even in the particular cases of our results for $u \equiv 0$ in (1.1), i.e., for the equation

$$[y(t) - p(t)y(r(t))]^{(n)} + v(t)G(y(g(t))) = f(t),$$

Theorems 2.3, 2.6, 2.4 and 2.8 generalize the results in [17], [18]. Due to this generalization, particularly, by relaxing the conditions that $G$ is non-decreasing and super linear, it is now possible to apply these results to the oscillatory and asymptotic behaviour of the higher-order neutral equation (1.2) with an oscillating coefficient $q(t)$ in our next section, which was not hitherto possible.

## 3. Application to neutral equations with oscillating coefficients

In this section we find sufficient conditions for every solution of the higher order ($n \geqslant 2$) neutral differential equation

$$[y(t) - p(t)y(r(t))]^{(n)} + q(t)G(y(g(t))) = f(t) \tag{3.1}$$

to oscillate or tend to zero as $t \to \infty$, where $q(t)$ is allowed to change sign. Let $q^+(t) = \max\{q(t), 0\}$ and $q^-(t) = \max\{-q(t), 0\}$. Then $q(t) = q^+(t) - q^-(t)$ and the above equation can be written as

$$[y(t) - p(t)y(r(t))]^{(n)} + q^+(t)G(y(g(t))) - q^-(t)G(y(g(t))) = f(t). \tag{3.2}$$

Now we proceed as in the previous section by setting $v(t) = q^+(t)$, $u(t) = q^-(t)$ and $H(x) = G(x)$. Assumptions (H3) and (H4) become

$$\int_{t_0}^{\infty} q^+(t)\,\mathrm{d}t = \infty, \quad \int_{t_0}^{\infty} t^{n-1}q^-(t)\,\mathrm{d}t < \infty,$$

which are feasible conditions. Therefore, the study of (3.1) reduces to the study of (1.1) in Theorems 2.3, 2.4, 2.6, 2.4. However, Theorem 2.1 cannot be applied because (H2) and (2.4) are incompatible conditions.

For the results in this section we need $G$ to be bounded, continuous, and to satisfy (H0) and (H1). The prototype of such a function $G(y)$ is $y^{2n}\operatorname{sgn}(y)/(1 + y^{2n})$. To emphasize the need for condition (H4) in the results in this section, we consider the equation

$$y''(t) + q(t)y(t - 2\pi) = 0,$$

where $q(t) = (\sin(t) - \cos^2(t))$, $n = 2$ and $p(t) = 0$. Then $q^+(t) = (\sin(t))^+$ and $q^-(t) = ((\sin(t))^- + \cos^2(t))$. Note that $\int_0^\infty tq^-(t)\,\mathrm{d}t = \infty$ and that the solution $y(t) = \exp(\sin(t))$ neither oscillates nor tends to zero as $t \to \infty$.

F i n a l  c o m m e n t s. Since (H2) and (2.4) are incompatible, it would be interesting to study the oscillation of solutions to (1.1) or to (3.1), by either relaxing the conditions or by considering the corresponding linear equations.

While studying (1.1) and (3.1), we assumed (H4). However, we do not know yet what would happen if these conditions are not met. Hence it would be very interesting to do research in this direction.

We observe that in the majority of the results for forced equations, non-oscillatory solutions tend to zero at $\infty$. Can we change this asymptotic behaviour of the non-oscillatory solutions by imposing additional conditions on the coefficient functions of (1.1) or (3.1)?

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