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**Generalized vertex algebra
of intertwiners and correlation functions
on self-sewn Riemann surfaces**

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GENERALIZED VERTEX ALGEBRA OF INTERTWINERS AND CORRELATION FUNCTIONS ON SELF-SEWN RIEMANN SURFACES

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ABSTRACT. We review our recent results on computation of the partition and n -point "intertwined" functions for modules of vertex operator superalgebras with formal parameter associated to local parameters on Riemann surfaces obtained by self-sewing of a lower genus Riemann surface. We introduce the torus intertwined n -point functions containing two intertwining operators in the supertrace. Then we define the partition and n -point correlation functions for a vertex operator superalgebra on a genus two Riemann surface formed by self-sewing of the torus. For the free fermion vertex operator superalgebra we present a closed formula for the genus two continuous orbifold partition function in terms of an infinite dimensional determinant with entries arising from the original torus Szegő kernel. This partition function is holomorphic in the sewing parameters on a given suitable domain and possess natural modular properties. We describe modularity of the generating function for all n -point correlation functions in terms of a genus two Szegő kernel determinant.

1. INTRODUCTION

In these notes we review our recent results on construction and computation of correlation functions of vertex operator superalgebras with a formal parameter associated to local coordinates on a self-sewn Riemann surface of genus g which forms a genus $g + 1$ surface. In particular, we review result presented in the papers [TZ1]–[TZ5] accomplished in collaboration with M. P. Tuite (National University of Ireland, Galway). The ideas reflected in these notes were also presented in two seminars given in 02-03/2014 at Mathematical Department, Charles University, Prague.

1.1. Vertex operator super algebras. A Vertex Operator Superalgebra (VOSA) [B,DL,Ka,FHL,FLM] is a quadruple $(V, Y, \mathbf{1}, \omega)$: $V = V_0 \oplus V_1 = \bigoplus_{r \in \frac{1}{2}\mathbb{Z}} V_r$, $\dim V_r < \infty$, is a superspace, Y is a linear map $Y : V \rightarrow (\text{End}V)[[z, z^{-1}]]$: so that for any vector (state) $u \in V$ we have $u(k)\mathbf{1} = \delta_{k,-1}u$, $k \geq -1$,

$$Y(u, z) = \sum_{n \in \mathbb{Z}} u(n)z^{-n-1},$$

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$u(n)V_\alpha \subset V_{\alpha+p(u)}$, $p(u)$ - parity. The linear operators (modes) $u(n) : V \rightarrow V$ satisfy creativity

$$Y(u, z)\mathbf{1} = u + O(z),$$

and lower truncation

$$u(n)v = 0,$$

conditions for $u, v \in V$ and $n \gg 0$.

These axioms identity imply locality, associativity, commutation and skew-symmetry:

$$(z_1 - z_2)^m Y(u, z_1)Y(v, z_2) = (-1)^{p(u,v)}(z_1 - z_2)^m Y(v, z_2)Y(u, z_1),$$

$$(z_0 + z_2)^n Y(u, z_0 + z_2)Y(v, z_2)w = (z_0 + z_2)^n Y(Y(u, z_0)v, z_2)w,$$

$$u(k)Y(v, z) - (-1)^{p(u,v)}Y(v, z)u(k) = \sum_{j \geq 0} \binom{k}{j} Y(u(j)v, z)z^{k-j},$$

$$Y(u, z)v = (-1)^{p(u,v)}e^{zL(-1)}Y(v, -z)u,$$

for $u, v, w \in V$ and integers $m, n \gg 0$, $p(u, v) = p(u)p(v)$.

The vacuum vector $\mathbf{1} \in V_{\bar{0},0}$ is such that, $Y(\mathbf{1}, z) = Id_V$, and $\omega \in V_{\bar{0},2}$ the conformal vector satisfies

$$Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n)z^{-n-2},$$

where $L(n)$ form a Virasoro algebra for a central charge C

$$[L(m), L(n)] = (m - n)L(m + n) + \frac{C}{12}(m^3 - m)\delta_{m,-n}.$$

$L(-1)$ satisfies the translation property

$$Y(L(-1)u, z) = \partial_z Y(u, z).$$

$L(0)$ describes a grading with $L(0)u = wt(u)u$, and $V_r = \{u \in V | wt(u) = r\}$.

1.2. VOSA modules.

Definition 1.1. A V -module for a VOSA V is a pair (W, Y_W) , W is a \mathbb{C} -graded vector space $W = \bigoplus_{r \in \mathbb{C}} W_r$, $\dim W_r < \infty$, $W_{r+n} = 0$ for all r and $n \ll 0$. $Y_W : V \rightarrow \text{End}(W)[[z, z^{-1}]$

$$Y_W(u, z) = \sum_{n \in \mathbb{Z}} u_W(n)z^{-n-1},$$

for each $u \in V$ $u_W : W \rightarrow W$. $Y_W(\mathbf{1}, z) = Id_W$, and for the conformal vector

$$Y_W(\omega, z) = \sum_{n \in \mathbb{Z}} L_W(n)z^{-n-2},$$

where $L_W(0)w = rw$, $w \in W_r$. The module vertex operators satisfy the Jacobi identity:

$$\begin{aligned} & z_0^{-1} \delta \left(\frac{z_1 - z_2}{z_0} \right) Y_W(u, z_1) Y_W(v, z_2) \\ & - (-1)^{p(u,v)} \delta \left(\frac{z_2 - z_1}{-z_0} \right) Y_W(v, z_2) Y_W(u, z_1) \\ & = z_2^{-1} \delta \left(\frac{z_1 - z_0}{z_2} \right) Y_W(Y(u, z_0)v, z_2). \end{aligned}$$

Recall that

$$\delta(z) = \sum_{n \in \mathbb{Z}} z^n.$$

The above axioms imply that $L_W(n)$ satisfies the Virasoro algebra for the same central charge C and that the translation property

$$Y_W(L(-1)u, z) = \partial_z Y_W(u, z).$$

1.3. Twisted modules. We next define the notion of a twisted V -module [FHL, DLM2]. Let g be a V -automorphism g , i.e., a linear map preserving $\mathbf{1}$ and ω such that

$$gY(v, z)g^{-1} = Y(gv, z),$$

for all $v \in V$. We assume that V can be decomposed into g -eigenspaces

$$V = \bigoplus_{\rho \in \mathbb{C}} V^\rho,$$

where V^ρ denotes the eigenspace of g with eigenvalue $e^{2\pi i \rho}$.

Definition 1.2. A g -twisted V -module for a VOVA V is a pair (W^g, Y_g) $W^g = \bigoplus_{r \in \mathbb{C}} W_r^g$, $\dim W_r^g < \infty$, $W_{r+n}^g = 0$ for all r and $n \ll 0$. $Y_g : V \rightarrow \text{End } W^g\{z\}$, the vector space of $\text{End } W^g$ -valued formal series in z with arbitrary complex powers of z . For $v \in V^\rho$

$$Y_g(v, z) = \sum_{n \in \rho + \mathbb{Z}} v_g(n) z^{-n-1},$$

with $v_g(\rho + l)w = 0$, $w \in W^g$, $l \in \mathbb{Z}$ sufficiently large. $Y_g(\mathbf{1}, z) = \text{Id}_{W^g}$,

$$Y_g(\omega, z) = \sum_{n \in \mathbb{Z}} L_g(n) z^{-n-2},$$

where $L_g(0)w = rw$, $w \in W_r^g$. The g -twisted vertex operators satisfy the twisted Jacobi identity:

$$\begin{aligned} & z_0^{-1} \delta \left(\frac{z_1 - z_2}{z_0} \right) Y_g(u, z_1) Y_g(v, z_2) \\ & - (-1)^{p(u,v)} z_0^{-1} \delta \left(\frac{z_2 - z_1}{-z_0} \right) Y_g(v, z_2) Y_g(u, z_1) \\ & = z_2^{-1} \left(\frac{z_1 - z_0}{-z_2} \right)^{-\rho} \delta \left(\frac{z_1 - z_0}{-z_2} \right) Y_g(Y(u, z_0)v, z_2), \end{aligned}$$

for $u \in V^\rho$.

1.4. **Creative intertwining operators.** We define the notion of creative intertwining operators in [TZ3]. Suppose we have a VOA V with a V -module (W, Y_W) .

Definition 1.3. A *Creative Intertwining Vertex Operator* Y for a VOA V -module (W, Y_W) is defined by a linear map

$$Y(w, z) = \sum_{n \in \mathbb{Z}} w(n) z^{-n-1},$$

for $w \in W$ with modes $w(n) : V \rightarrow W$; satisfies creativity

$$Y(w, z)\mathbf{1} = w + O(z),$$

for $w \in W$ and lower truncation

$$w(n)v = 0,$$

for $v \in V$, $w \in W$ and $n \gg 0$. The intertwining vertex operators satisfy the Jacobi identity:

$$\begin{aligned} & z_0^{-1} \delta \left(\frac{z_1 - z_2}{z_0} \right) Y_W(u, z_1) Y(w, z_2) \\ & - z_0^{-1} \delta \left(\frac{z_2 - z_1}{-z_0} \right) Y(w, z_2) Y(u, z_1) \\ & = z_2^{-1} \delta \left(\frac{z_1 - z_0}{z_2} \right) Y(Y_W(u, z_0)w, z_2), \end{aligned}$$

for all $u \in V$ and $w \in W$.

These axioms imply that the intertwining vertex operators satisfy translation, locality, associativity, commutativity and skew-symmetry:

$$Y(L_W(-1)w, z) = \partial_z Y(w, z),$$

$$(z_1 - z_2)^m Y_W(u, z_1) Y(w, z_2) = (z_1 - z_2)^m Y(w, z_2) Y(u, z_1),$$

$$(z_0 + z_2)^n Y_W(u, z_0 + z_2) Y(w, z_2)v = (z_0 + z_2)^n Y(Y_W(u, z_0)w, z_2)v,$$

$$u_W(k)Y(w, z) - Y(w, z)u(k) = \sum_{j \geq 0} \binom{k}{j} Y(u_W(j)w, z) z^{k-j},$$

$$Y(w, z)v = e^{zL_W(-1)} Y_W(v, -z)w,$$

for $u, v \in V$, $w \in W$ and integers $m, n \gg 0$.

1.5. **Example: Heisenberg intertwiners.** Consider the Heisenberg vertex operator algebra M , [Ka] generated by weight one normalized Heisenberg vector a with modes obeying

$$[a(n), a(m)] = n\delta_{n, -m},$$

$n, m \in \mathbb{Z}$. In [TZ3] we consider an extension $M = \cup_{\alpha \in \mathbb{C}} M_\alpha$ of M by its irreducible modules M_α generated by a \mathbb{C} -valued continuous parameter α automorphism $g = e^{2\pi i \alpha a(0)}$.

We introduce an extra operator q which is canonically conjugate to the zero mode $a(0)$, i.e.,

$$[a(n), q] = \delta_{n,0}.$$

The state $\mathbf{1} \otimes e^\alpha \in M$ is created by the action of $e^{\alpha q}$ on the state $\mathbf{1} \otimes e^0$. Using q -conjugation and associativity properties, we explicitly construct in [TZ3] the creative intertwining operators $Y(u, z) : M \rightarrow M_\alpha$. We then prove

Theorem 1.4 (Tuite-Z). *The creative intertwining operators Y for M are generated by q -conjugation of vertex operators of M . For a Heisenberg state u ,*

$$\begin{aligned} Y(u \otimes e^\alpha, z) &= e^{\alpha q} Y_-(e^\alpha, z) Y(u \otimes e^0) Y_+(e^\alpha, z) z^{\alpha a(0)}, \\ Y_\pm(e^\alpha, z) &\equiv \exp\left(\mp \alpha \sum_{n>0} a(\pm n) \frac{z^{\mp n}}{n}\right). \end{aligned}$$

The operators Y with some extra cocycle structure satisfy a natural extension from rational to complex parameters of the notion of a *Generalized VOA* as described by Dong and Lepowsky [DL, DLM3]. We then prove in [TZ3]:

Theorem 1.5 (Tuite-Z). *$Y(u \otimes e^\alpha, z)$ satisfy the generalized Jacobi identity*

$$\begin{aligned} & z_0^{-1} \left(\frac{z_1 - z_2}{z_0}\right)^{-\alpha\beta} \delta\left(\frac{z_1 - z_2}{z_0}\right) Y(u \otimes e^\alpha, z_1) Y(v \otimes e^\beta, z_2) \\ & - C(\alpha, \beta) z_0^{-1} \left(\frac{z_2 - z_1}{z_0}\right)^{-\alpha\beta} \delta\left(\frac{z_2 - z_1}{-z_0}\right) \\ & \quad Y(v \otimes e^\beta, z_2) Y(u \otimes e^\alpha, z_1) \\ & = z_2^{-1} \delta\left(\frac{z_1 - z_0}{z_2}\right) Y(Y(u \otimes e^\alpha, z_0)(v \otimes e^\beta), z_2) \left(\frac{z_1 - z_0}{z_2}\right)^{\alpha a(0)}, \end{aligned}$$

for all $u \otimes e^\alpha, v \otimes e^\beta \in M$.

1.6. Invariant form for extended Heisenberg algebra. The definitions of invariant forms [FHL, L] for a VOSA and its g -twisted modules were given by Scheithauer [S] and in [TZ2] correspondingly. A bilinear form $\langle \cdot, \cdot \rangle$ on M is said to be invariant if for all $u \otimes e^\alpha, v \otimes e^\beta, w \otimes e^\gamma \in M$ we have

$$\langle Y(u \otimes e^\alpha, z)v \otimes e^\beta, w \otimes e^\gamma \rangle = e^{i\pi\alpha\beta} \langle v \otimes e^\beta, Y^\dagger(u \otimes e^\alpha, z)w \otimes e^\gamma \rangle,$$

$$Y^\dagger(u \otimes e^\alpha, z) = Y\left(e^{-z\lambda^{-2}L(1)}\left(-\frac{\lambda}{z}\right)^{2L(0)}(u \otimes e^\alpha), -\frac{\lambda^2}{z}\right).$$

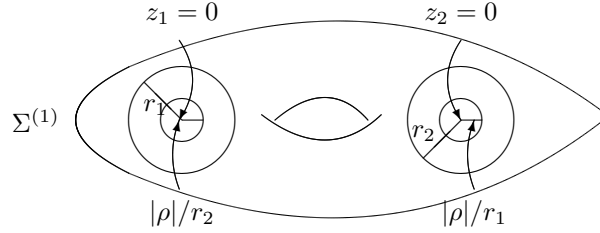
We are interested in the Möbius map $z \mapsto w = \frac{\rho}{z}$ associated with the sewing condition so that $\lambda = -\xi \rho^{\frac{1}{2}}$, with $\xi \in \{\pm\sqrt{-1}\}$. We prove in [TZ3]

Theorem 1.6 (Tuite-Z). *The invariant form $\langle \cdot, \cdot \rangle$ on M is symmetric, unique and invertible with*

$$\langle v \otimes e^\alpha, w \otimes e^\beta \rangle = \lambda^{-\alpha^2} \delta_{\alpha, -\beta} \langle v \otimes e^0, w \otimes e^0 \rangle.$$

2. THE SZEGŐ KERNEL

2.1. Torus self-sewing to form a genus two Riemann surface. In [TZ1] we describe procedures of sewing Riemann surfaces [G, FK]. Consider a self-sewing of the oriented torus $\Sigma^{(1)} = \mathbb{C}/\Lambda$, $\Lambda = 2\pi i(\mathbb{Z}\tau \oplus \mathbb{Z})$, $\tau \in \mathbb{H}_1$.



Define annuli \mathcal{A}_a , $a = 1, 2$ centered at $z = 0$ and $z = w$ of $\Sigma^{(1)}$ with local coordinates $z_1 = z$ and $z_2 = z - w$ respectively. We use the convention $\bar{1} = 2$, $\bar{2} = 1$. Take the outer radius of \mathcal{A}_a to be $r_a < \frac{1}{2}D(q) = \min_{\lambda \in \Lambda, \lambda \neq 0} |\lambda|$. Introduce a complex parameter ρ , $|\rho| \leq r_1 r_2$. Take inner radius to be $|\rho|/r_{\bar{a}}$, with $|\rho| \leq r_1 r_2$. r_1, r_2 must be sufficiently small to ensure that the disks do not intersect. Excise the disks

$$\{z_a, |z_a| < |\rho| r_{\bar{a}}^{-1}\} \subset \Sigma^{(1)},$$

to form a twice-punctured surface

$$\widehat{\Sigma}^{(1)} = \Sigma^{(1)} \setminus \bigcup_{a=1,2} \{z_a, |z_a| < |\rho| r_{\bar{a}}^{-1}\}.$$

Identify annular regions $\mathcal{A}_a \subset \widehat{\Sigma}^{(1)}$, $\mathcal{A}_a = \{z_a, |\rho| r_{\bar{a}}^{-1} \leq |z_a| \leq r_a\}$ as a single region $\mathcal{A} = \mathcal{A}_1 \simeq \mathcal{A}_2$ via the sewing relation

$$z_1 z_2 = \rho,$$

to form a compact genus two Riemann surface $\Sigma^{(2)} = \widehat{\Sigma}^{(1)} \setminus \{\mathcal{A}_1 \cup \mathcal{A}_2\} \cup \mathcal{A}$, parameterized by

$$\mathcal{D}^\rho = \{(\tau, w, \rho) \in \mathbb{H}_1 \times \mathbb{C} \times \mathbb{C}, |w - \lambda| > 2|\rho|^{\frac{1}{2}} > 0, \lambda \in \Lambda\}.$$

2.2. The Prime form. Recall the prime form $E^{(g)}(z, z')$ [M, F1, F2]

$$E^{(g)}(z, z') = \frac{\vartheta \left[\begin{smallmatrix} \gamma \\ \delta \end{smallmatrix} \right] \left(\int_{z'}^z \nu | \Omega^{(g)} \right)}{\zeta(z)^{\frac{1}{2}} \zeta(z')^{\frac{1}{2}}} \sim (z - z') dz^{-\frac{1}{2}} dz'^{-\frac{1}{2}} \quad \text{for } z \sim z',$$

is a holomorphic differential form of weight $(-\frac{1}{2}, -\frac{1}{2})$ on $\widetilde{\Sigma}^{(g)} \times \widetilde{\Sigma}^{(g)}$,

$$E^{(g)}(z, z') = -E^{(g)}(z', z),$$

and has multipliers 1 and $e^{-i\pi\Omega_{jj}^{(g)} - \int_{z'}^z \nu_j}$ along the a_i and b_j cycles in z . Here

$$\zeta(z) = \sum_{i=1}^g \partial_{z_i} \vartheta \begin{bmatrix} \gamma \\ \delta \end{bmatrix} (0|\Omega^{(g)}) \nu_i(z),$$

(a holomorphic 1-form, and let $\zeta(z)^{\frac{1}{2}}$ denote the form of weight $\frac{1}{2}$ on the double cover $\tilde{\Sigma}^{(g)}$ of $\Sigma^{(g)}$).

In particular, the prime form on the torus is [M]

$$E^{(1)}(z, z') = K^{(1)}(z - z', \tau) dz^{-\frac{1}{2}} dz'^{-\frac{1}{2}},$$

$$K^{(1)}(z, \tau) = \frac{\vartheta_1(z, \tau)}{\partial_z \vartheta_1(0, \tau)},$$

for $z \in \mathbb{C}$ and $\tau \in \mathbb{H}_1$ and where $\vartheta_1(z, \tau) = \vartheta \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} (z, \tau)$.

2.3. The Szegő kernel. The Szegő Kernel [M, F1, F2] is defined by

$$S^{(g)} \begin{bmatrix} \theta \\ \phi \end{bmatrix} (z, z'|\Omega) = \frac{\vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (\int_{z'}^z \nu)}{\vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (0) E^{(g)}(z, z')} \sim \frac{dz^{\frac{1}{2}} dz'^{\frac{1}{2}}}{z - z'} \quad \text{for } z \sim z',$$

with $\vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (0) \neq 0$,

$$\theta_j = -e^{-2\pi i \beta_j}, \quad \phi_j = -e^{2\pi i \alpha_j}, \quad j = 1, \dots, g,$$

where $E^{(g)}(z_1, z_2)$ is the genus g prime form. The Szegő kernel has multipliers along the a_i and b_j cycles in z given by $-\phi_i$ and $-\theta_j$ respectively and is a meromorphic $(\frac{1}{2}, \frac{1}{2})$ -form on $\tilde{\Sigma}^{(g)} \times \tilde{\Sigma}^{(g)}$.

$$S^{(g)} \begin{bmatrix} \theta \\ \phi \end{bmatrix} (z, z') = -S^{(g)} \begin{bmatrix} \theta^{-1} \\ \phi^{-1} \end{bmatrix} (z', z),$$

where $\theta^{-1} = (\theta_i^{-1})$ and $\phi^{-1} = (\phi_i^{-1})$.

Finally, we describe the modular invariance of the Szegő kernel under the symplectic group $Sp(2g, \mathbb{Z})$ where we find [Fay]

$$S^{(g)} \begin{bmatrix} \tilde{\theta} \\ \tilde{\phi} \end{bmatrix} (z, z'|\tilde{\Omega}^{(g)}) = S^{(g)} \begin{bmatrix} \theta \\ \phi \end{bmatrix} (z, z'|\Omega^{(g)}),$$

with $\tilde{\theta}_j = -e^{-2\pi i \tilde{\beta}_j}$, $\tilde{\phi}_j = -e^{2\pi i \tilde{\alpha}_j}$,

$$\begin{pmatrix} -\tilde{\beta} \\ \tilde{\alpha} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} -\beta \\ \alpha \end{pmatrix} + \frac{1}{2} \begin{pmatrix} -\text{diag}(AB^T) \\ \text{diag}(CD^T) \end{pmatrix},$$

$$\tilde{\Omega} = (A\Omega + B)(C\Omega + D)^{-1},$$

where $\text{diag}(M)$ denotes the diagonal elements of a matrix M .

On the torus $\Sigma^{(1)}$ the Szegő kernel for $(\theta, \phi) \neq (1, 1)$ is

$$S^{(1)} \begin{bmatrix} \theta \\ \phi \end{bmatrix} (z, z'|\tau) = P_1 \begin{bmatrix} \theta \\ \phi \end{bmatrix} (z - z', \tau) dz^{\frac{1}{2}} dz'^{\frac{1}{2}},$$

where

$$\begin{aligned} P_1 \begin{bmatrix} \theta \\ \phi \end{bmatrix} (z, \tau) &= \frac{\vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (z, \tau)}{\vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (0, \tau)} \frac{\partial_z \vartheta_1(0, \tau)}{\vartheta_1(z, \tau)} \\ &= - \sum_{k \in \mathbb{Z}} \frac{q_z^{k+\lambda}}{1 - \theta^{-1} q^{k+\lambda}}, \end{aligned}$$

for $\vartheta_1(z, \tau) = \vartheta \left[\frac{1}{2} \right] (z, \tau)$, $q_z = e^z$, and $\phi = \exp(2\pi i \lambda)$ for $0 \leq \lambda < 1$.

2.4. Genus two Szegő kernel in the ρ -formalism. It is convenient to define $\kappa \in \left[-\frac{1}{2}, \frac{1}{2}\right)$ by $\phi_2 = -e^{2\pi i \kappa}$. Then we prove [TZ1] the following

Theorem 2.1 (Tuite-Z). *$S^{(2)}$ is holomorphic in ρ for $|\rho| < r_1 r_2$ with*

$$S^{(2)}(x, y) = S_\kappa^{(1)}(x, y) + O(\rho),$$

for $x, y \in \widehat{\Sigma}^{(1)}$ where $S_\kappa^{(1)}(x, y)$ is defined for $\kappa \neq -\frac{1}{2}$, by

$$\begin{aligned} S_\kappa^{(1)} \begin{bmatrix} \theta_1 \\ \phi_1 \end{bmatrix} (x, y | \tau, w) &= \left(\frac{\vartheta_1(x-w, \tau) \vartheta_1(y, \tau)}{\vartheta_1(x, \tau) \vartheta_1(y-w, \tau)} \right)^\kappa \\ &\cdot \frac{\vartheta^{(1)} \begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix} (x-y + \kappa w, \tau)}{\vartheta^{(1)} \begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix} (\kappa w, \tau) K^{(1)}(x-y, \tau)} dx^{\frac{1}{2}} dy^{\frac{1}{2}}, \end{aligned}$$

with similar expression for $S_{-\frac{1}{2}}^{(1)}(x, y)$ for $\kappa = -\frac{1}{2}$.

Let $k_a = k + (-1)^{\bar{a}} \kappa$, for $a = 1, 2$ and integer $k \geq 1$. We introduce the moments for $S_\kappa^{(1)}(x, y)$:

$$\begin{aligned} G_{ab}(k, l) &= G_{ab} \begin{bmatrix} \theta^{(1)} \\ \phi^{(1)} \end{bmatrix} (\kappa; k, l) \\ &= \frac{\rho^{\frac{1}{2}(k_a + l_b - 1)}}{(2\pi i)^2} \oint_{\mathcal{C}_{\bar{a}}(x_{\bar{a}})} \oint_{\mathcal{C}_b(y_b)} (x_{\bar{a}})^{-k_a} (y_b)^{-l_b} S_\kappa^{(1)}(x_{\bar{a}}, y_b) dx_{\bar{a}}^{\frac{1}{2}} dy_b^{\frac{1}{2}}, \end{aligned}$$

with associated infinite matrix $G = (G_{ab}(k, l))$. We define also half-order differentials

$$\begin{aligned} h_a(k, x) &= h_a \begin{bmatrix} \theta^{(1)} \\ \phi^{(1)} \end{bmatrix} (\kappa; k, x) = \frac{\rho^{\frac{1}{2}(k_a - \frac{1}{2})}}{2\pi i} \oint_{\mathcal{C}_a(y_a)} y_a^{-k_a} S_\kappa^{(1)}(x, y_a) dy_a^{\frac{1}{2}}, \\ \bar{h}_a(k, y) &= \bar{h}_a \begin{bmatrix} \theta^{(1)} \\ \phi^{(1)} \end{bmatrix} (\kappa; k, y) = \frac{\rho^{\frac{1}{2}(k_a - \frac{1}{2})}}{2\pi i} \oint_{\mathcal{C}_{\bar{a}}(x_{\bar{a}})} x_{\bar{a}}^{-k_a} S_\kappa^{(1)}(x_{\bar{a}}, y) dx_{\bar{a}}^{\frac{1}{2}}, \end{aligned}$$

and let $h(x) = (h_a(k, x))$ and $\bar{h}(y) = (\bar{h}_a(k, y))$ denote the infinite row vectors indexed by a, k . From the sewing relation $z_1 z_2 = \rho$ we have

$$dz_a^{\frac{1}{2}} = (-1)^{\bar{a}} \xi \rho^{\frac{1}{2}} \frac{dz_{\bar{a}}^{\frac{1}{2}}}{z_{\bar{a}}},$$

for $\xi \in \{\pm\sqrt{-1}\}$, depending on the branch of the double cover of $\Sigma^{(1)}$ chosen. It is convenient to define

$$T = \xi G D^\theta,$$

with an infinite diagonal matrix

$$D^\theta(k, l) = \begin{bmatrix} \theta^{-1} & 0 \\ 0 & -\theta \end{bmatrix} \delta(k, l).$$

Defining $\det(I - T)$ by the formal power series in ρ

$$\log \det(I - T) = \text{Tr} \log(I - T) = - \sum_{n \geq 1} \frac{1}{n} \text{Tr}(T^n),$$

we prove in [TZ1]

Theorem 2.2 (Tuite-Z).

a.) $(I - T)^{-1} = \sum_{n \geq 0} T^n$ is convergent for $|\rho| < r_1 r_2$,

b.) $\det(I - T)$ is non-vanishing and holomorphic in ρ on D^ρ .

Theorem 2.3 (Tuite-Z). $S^{(2)}(x, y)$ is given by

$$S^{(2)}(x, y) = S_\kappa^{(1)}(x, y) + \xi h(x) D^\theta (I - T)^{-1} \bar{h}^T(y).$$

3. INTERTWINED n -POINT FUNCTIONS

As in ordinary (non-intertwined) case [DLM1, H, MN, MT1, MT3, MT4, MTZ, TZ2, Z1] we construct in [TZ4] the partition and n -point functions [DVFHLS, EO, FS, GKV, GV, KNTY, Pe, R, TUY, U] for vertex operator algebra modules.

3.1. Torus intertwined n -point functions. Let $g_i, f_i, i = 1, 2$ be VOSA V automorphisms commuting with $\sigma v = (-1)^{p(v)} v$. For $u \in V_{\sigma g_2}$ and the states $v_1, \dots, v_n \in V$ we define the *intertwined n -point function* [TZ4] on the torus by

$$\begin{aligned} Z^{(1)} \begin{bmatrix} f_1 \\ g_1 \end{bmatrix} (u, z_2; v_1, x_1; \dots; v_n, x_n; \bar{u}, z_1; \tau) \\ \equiv \text{STr}_{V_{\sigma g_1}} \left(f_1 Y \left(q_{z_2}^{L_{\sigma g_2}(0)} u, q_{z_2} \right) Y(q_1^{L(0)} v_1, q_1) \right. \\ \left. \dots Y(q_n^{L(0)} v_n, q_n) Y \left(q_{z_1}^{L_{\sigma g_2^{-1}}(0)} \bar{u}, q_{z_1} \right) q^{L_{\sigma g_1}(0) - c/24} \right), \end{aligned}$$

where $q = \exp(2\pi i \tau)$, $q_k = \exp(x_k)$, $q_{z_j} = \exp(z_j)$, $j = 1, 2$; $1 \leq k \leq n$, for variables x_1, \dots, x_n associated to the local coordinates on the torus, and \bar{u} is dual for u with respect to the invariant form on $V_{\sigma g_2}$. The supertrace over a V -module N is defined by

$$\text{STr}_N(X) = \text{Tr}_N(\sigma X).$$

For an element $u \in V_{\sigma_{g_2}}$ of a VOSA g -twisted V -module we introduce also the differential form

$$\begin{aligned} \mathcal{F}^{(1)} \begin{bmatrix} f_1 \\ g_1 \end{bmatrix} (u, z_2; v_1, x_1; \dots; v_n, x_n; \bar{u}, z_1; \tau) \\ \equiv Z^{(1)} \begin{bmatrix} f_1 \\ g_1 \end{bmatrix} (u, z_2; v_1, x_1; \dots; v_n, x_n; \bar{u}, z_1; \tau) \\ \cdot dz_2^{wt[u]} dz_1^{wt[\bar{u}]} \prod_{i=1}^n dx_i^{wt[v_i]}, \end{aligned}$$

associated to the torus intertwined n -point function.

3.2. Genus two partition and n -point functions in ρ -formalism. Let $f_i, i = 1, 2$ be automorphisms, and $V_{\sigma_{g_j}}$ be twisted V -modules of a vertex operator superalgebra V . For $x_1, \dots, x_n \in \Sigma^{(1)}$ with $|x_k| \geq |\rho|/r_2$ and $|x_k - w| \geq |\rho|/r_1, k = 1, \dots, n$, we define the genus two n -point function [TZ4] in the ρ -formalism by

$$\begin{aligned} Z^{(2)} \begin{bmatrix} f \\ g \end{bmatrix} (v_1, x_1; \dots; v_n, x_n; \tau, w, \rho) \\ = \sum_{k \geq 0} \sum_{u \in V_{\sigma_{g_2}}[k]} \rho^k Z^{(1)} \begin{bmatrix} f_1 \\ g_1 \end{bmatrix} (u, w + z_2; v_1, x_1; \dots; v_n, x_n; f_2 \bar{u}, z_1; \tau), \end{aligned}$$

where $(f, g) = ((f_i), (g_i))$, where f (respectively g) denotes the pair f_1, f_2 (respectively g_1, g_2). The sum is taken over any $V_{\sigma_{g_2}}$ -basis.

In particular, introduce the genus two partition function

$$Z^{(2)} \begin{bmatrix} f \\ g \end{bmatrix} (\tau, w, \rho) = \sum_{u \in V_{\sigma_{g_2}}} Z^{(1)} \begin{bmatrix} f_1 \\ g_1 \end{bmatrix} (u, w; f_2 \bar{u}, 0; \tau),$$

where $Z^{(1)} \begin{bmatrix} f_1 \\ g_1 \end{bmatrix} (u, w; f_2 \bar{u}, 0; \tau)$ is the genus one intertwined two point function.

Remark 3.1. *We can generalize the genus two n -point function by introducing and computing the differential form associated to the torus n -point function containing several intertwining operators in the supertrace as well as corresponding genus two n -point functions.*

Similar to the ordinary genus two case [TZ2], we define the differential form [TZ4] associated to the n -point function on a sewn genus two Riemann surface for $v_i \in V$ and $x_i \in \Sigma^{(2)}, i = 1, \dots, n$ with $|x_i| \geq |\rho|/r_2, |x_i - w| \geq |\rho|/r_1$,

$$\begin{aligned} \mathcal{F}^{(2)} \begin{bmatrix} f \\ g \end{bmatrix} (v_1, \dots, v_n; \tau, w, \rho) \\ \equiv Z^{(2)} \begin{bmatrix} f \\ g \end{bmatrix} (v_1, x_1; \dots; v_n, x_n; \tau, w, \rho) \prod_{i=1}^n dx_i^{wt[v_i]}. \end{aligned}$$

4. FREE FERMION VOSA

4.1. Torus intertwined two-point function. The rank two free fermionic VOSA $V(H, \mathbb{Z} + \frac{1}{2})^{\otimes 2}$, [Ka] is generated by ψ^\pm with

$$[\psi^+(m), \psi^-(n)] = \delta_{m, -n-1}, [\psi^+(m), \psi^+(n)] = 0, [\psi^-(m), \psi^-(n)] = 0,$$

The rank two free fermion VOSA intertwined torus n -point function is parameterized by $\theta_1 = -e^{-2\pi i \beta_1}$, $\phi_1 = -e^{2\pi i \alpha_1}$, and $\phi_2 = -e^{-2\pi i \kappa}$, [TZ2, TZ4] where

$$\sigma f_1 = e^{2\pi i \beta_1 a(0)}, \quad \sigma g_1 = e^{-2\pi i \alpha_1 a(0)}, \quad \sigma g_2 = e^{2\pi i \kappa a(0)},$$

for real valued $\alpha_1, \beta_1, \kappa$, $(\theta_1, \phi_1) \neq (1, 1)$.

For $u = \mathbf{1} \otimes e^\kappa \equiv e^\kappa \in V_{\sigma g_2}$ and $v_i = \mathbf{1}$, $i = 1, \dots, n$ we obtain [TZ4] the basic intertwined two-point function on the torus

$$\begin{aligned} Z^{(1)} \begin{bmatrix} f_1 \\ g_1 \end{bmatrix} (e^\kappa, z_2; e^{-\kappa}, z_1; \tau) \\ \equiv \text{STr}_{V_{\sigma g_1}} \left(f_1 Y \left(q_{z_2}^{L(0)} e^\kappa, q_{z_2} \right) Y \left(q_{z_1}^{L(0)} e^{-\kappa}, q_{z_1} \right) q^{L_{\sigma g_1(0)} - c/24} \right). \end{aligned}$$

We then consider the differential form

$$\begin{aligned} \mathcal{G}_n^{(1)} \begin{bmatrix} f_1 \\ g_1 \end{bmatrix} (x_1, y_1, \dots, x_n, y_n) \\ \equiv F^{(1)} \begin{bmatrix} f_1 \\ g_1 \end{bmatrix} (e^\kappa, w; \psi^+, x_1; \psi^-, y_1; \dots; \psi^+, x_n; \psi^-, y_n; e^{-\kappa}, 0; \tau), \end{aligned}$$

associated to the torus intertwined $2n$ -point function

$$Z^{(1)} \begin{bmatrix} f_1 \\ g_1 \end{bmatrix} (e^\kappa, w; \psi^+, x_1; \psi^-, y_1; \dots; \psi^+, x_n; \psi^-, y_n; e^{-\kappa}, 0; \tau),$$

with alternatively inserted n states ψ^+ and n states ψ^- distributed on the resulting genus two Riemann surface $\Sigma^{(2)}$ at points $x_i, y_i \in \Sigma^{(2)}$, $i = 1, \dots, n$.

We then prove in [TZ4]

Theorem 4.1 (Tuite-Z). *For the rank two free fermion vertex operator superalgebra V and for $(\theta, \phi) \neq (1, 1)$ the generating form is given by*

$$\begin{aligned} G_n^{(1)} \begin{bmatrix} f_1 \\ g_1 \end{bmatrix} (x_1, y_1, \dots, x_n, y_n) \\ = Z^{(1)} \begin{bmatrix} f_1 \\ g_1 \end{bmatrix} (e^\kappa, w; e^{-\kappa}, 0; \tau) \det S_\kappa^{(1)}, \end{aligned}$$

$$Z^{(1)} \begin{bmatrix} f_1 \\ g_1 \end{bmatrix} (e^\kappa, w; e^{-\kappa}, 0; \tau) = \frac{1}{\eta(\tau)} \frac{\vartheta^{(1)} \begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix} (\kappa w, \tau)}{K^{(1)}(w, \tau)^{\kappa^2}},$$

is the basic intertwined two-point function on the torus, and $n \times n$ -matrix $S_\kappa^{(1)} = \left[S_\kappa^{(1)} \begin{bmatrix} \theta_1 \\ \phi_1 \end{bmatrix} (x_i, y_j \mid \tau, w) \right]$, $i, j = 1, \dots, n$, with elements given by parts of the Szegő kernel.

4.2. Genus two partition function. In [TZ4] we then prove:

Theorem 4.2 (Tuite-Z). *Let $V_{\sigma g_i}$, $i = 1, 2$ be σg_i -twisted V -modules for the rank two free fermion vertex operator superalgebra V . Let $(\theta, \phi) \neq (1, 1)$. Then the partition function on a genus two Riemann surface obtained in the ρ -self-sewing formalism of the torus is a non-vanishing holomorphic function on D^ρ given by*

$$Z^{(2)} \begin{bmatrix} f \\ g \end{bmatrix} (\tau, w, \rho) = Z^{(1)} \begin{bmatrix} f_1 \\ g_1 \end{bmatrix} (e^\kappa, w; e^{-\kappa}, 0; \tau) \det(1 - T),$$

where $Z^{(1)} \begin{bmatrix} f_1 \\ g_1 \end{bmatrix} (e^\kappa, w; e^{-\kappa}, 0; \tau)$ is the intertwined V -module $V_{\sigma g_1}$ torus basic two-point function.

We may similarly compute the genus two partition function in the ρ -formalism for the original rank one fermion VOSA $V(H, \mathbb{Z} + \frac{1}{2})$ in which case we can only construct a σ -twisted module. Then we have [TZ4] the following

Corollary 4.3 (Tuite-Z). *Let V be the rank one free fermion vertex operator superalgebra and $f_1, g_1 \in \{\sigma, 1\}$, $a = 1, 2$ be automorphisms. Then the partition function for V -module $V_{\sigma g_1}$ on a genus two Riemann surface obtained from ρ formalism of a self-sewn torus $\Sigma^{(1)}$ is given by*

$$Z_{\text{rank } 1}^{(2)} \begin{bmatrix} f \\ g \end{bmatrix} (\tau, w, \rho) = Z_{\text{rank } 1}^{(1)} \begin{bmatrix} f_1 \\ g_1 \end{bmatrix} (e^\kappa, w; e^{-\kappa}, 0; \tau) \det(I - T)^{1/2},$$

where $Z_{\text{rank } 1}^{(1)} \begin{bmatrix} f_1 \\ g_1 \end{bmatrix} (e^\kappa, w; e^{-\kappa}, 0; \tau)$ is the rank one fermion intertwined partition function on the original torus.

4.3. Genus two generating form. In [TZ4] we define matrices

$$S^{(2)} = (S^{(2)}(x_i, y_j)), \quad S_\kappa^{(1)} = (S_\kappa^{(1)}(x_i, y_j)), \\ H^+ = ((h(x_i))(k, a)), \quad H^- = ((\bar{h}(y_i))(l, b))^T.$$

$S^{(2)}$ and $S_\kappa^{(1)}$ are finite matrices indexed by x_i, y_j for $i, j = 1, \dots, n$; H^+ is semi-infinite with n rows indexed by x_i and columns indexed by $k \geq 1$ and $a = 1, 2$ and H^- is semi-infinite with rows indexed by $l \geq 1$ and $b = 1, 2$ and with n columns indexed by y_j . We then prove

Lemma 4.4 (Tuite-Z).

$$\det \begin{bmatrix} S_\kappa^{(1)} & \xi H^+ D^{\theta_2} \\ H^- & I - T \end{bmatrix} = \det S^{(2)} \det(I - T),$$

with T, D^{θ_2} .

Introduce the differential form

$$\mathcal{G}_n^{(2)} \begin{bmatrix} f \\ g \end{bmatrix} (x_1, y_1, \dots, x_n, y_n) \\ = F^{(2)} \begin{bmatrix} f \\ g \end{bmatrix} (\psi^+, \psi^-, \dots, \psi^+, \psi^-; \tau, w, \rho),$$

associated to the rank two free fermion VOSA genus two $2n$ -point function

$$Z^{(2)} \begin{bmatrix} f \\ g \end{bmatrix} (\psi^+, x_1; \psi^-, y_1; \dots; \psi^+, x_n; \psi^-, y_n; \tau, w, \rho),$$

with alternatively inserted n states ψ^+ and n states ψ^- . The states are distributed on the genus two Riemann surface $\Sigma^{(2)}$ at points $x_i, y_i \in \Sigma^{(2)}$, $i = 1, \dots, n$. Then we have

Theorem 4.5 (Tuite-Z). *All n -point functions for rank two free fermion VOSA twisted modules $V_{\sigma g}$ on self-sewn torus are generated by the differential form*

$$\mathcal{G}_n^{(2)} \begin{bmatrix} f \\ g \end{bmatrix} (x_1, y_1, \dots, x_n, y_n) = Z^{(2)} \begin{bmatrix} f \\ g \end{bmatrix} (\tau, w, \rho) \det S^{(2)},$$

where the elements of the matrix $S^{(2)} = \left[S^{(2)} \begin{bmatrix} \theta \\ \phi \end{bmatrix} (x_i, y_j \mid \tau, w) \right]$, $i, j = 1, \dots, n$ and $Z^{(2)} \begin{bmatrix} f \\ g \end{bmatrix} (\tau, w, \rho)$ is the genus two partition function.

5. MODULAR INVARIANCE PROPERTIES

Following the ordinary case [DLM1, MT3, MT5] we would like to describe modular properties of genus two "intertwined" partition and n -point generating functions. As in [MT3], consider $\hat{H} \subset Sp(4, \mathbb{Z})$ with elements

$$\mu(a, b, c) = \begin{pmatrix} 1 & 0 & 0 & b \\ a & 1 & b & c \\ 0 & 0 & 1 & -a \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

\hat{H} is generated by $A = \mu(1, 0, 0)$, $B = \mu(0, 1, 0)$ and $C = \mu(0, 0, 1)$ with relations $[A, B]C^{-2} = [A, C] = [B, C] = 1$. We also define $\Gamma_1 \subset Sp(4, \mathbb{Z})$ where $\Gamma_1 \cong SL(2, \mathbb{Z})$ with elements

$$\gamma_1 = \begin{pmatrix} a_1 & 0 & b_1 & 0 \\ 0 & 1 & 0 & 0 \\ c_1 & 0 & d_1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad a_1 d_1 - b_1 c_1 = 1.$$

Together these groups generate $L = \hat{H} \rtimes \Gamma_1 \subset Sp(4, \mathbb{Z})$. From [MT3] we find that L acts on the domain \mathcal{D}^ρ of as follows:

$$\begin{aligned} \mu(a, b, c) \cdot (\tau, w, \rho) &= (\tau, w + 2\pi i a \tau + 2\pi i b, \rho), \\ \gamma_1 \cdot (\tau, w, \rho) &= \left(\frac{a_1 \tau + b_1}{c_1 \tau + d_1}, \frac{w}{c_1 \tau + d_1}, \frac{\rho}{(c_1 \tau + d_1)^2} \right). \end{aligned}$$

We then define [TZ4] a group action of $\gamma_1 \in SL(2, \mathbb{Z})$ on the torus intertwined two-point function $Z^{(1)} \begin{bmatrix} f_1 \\ g_1 \end{bmatrix} (u, w; v, 0; \tau)$ for $u, v \in V_{\sigma g}$:

$$Z^{(1)} \begin{bmatrix} f_1 \\ g_1 \end{bmatrix} \Big|_{\gamma_1} (u, w; v, 0; \tau) = Z^{(1)} \left(\gamma_1 \cdot \begin{bmatrix} f_1 \\ g_1 \end{bmatrix} \right) (u, \gamma_1 \cdot w; v, 0; \gamma_1 \cdot \tau),$$

with the standard action $\gamma_1 \cdot \tau$ and $\gamma_1 \cdot w$, and $\gamma_1 \cdot \begin{bmatrix} f_1 \\ g_1 \end{bmatrix} = \begin{bmatrix} f_1^{a_1} g_1^{b_1} \\ f_1^{c_1} g_1^{d_1} \end{bmatrix}$, and the torus multiplier $e_{\gamma_1}^{(1)} \begin{bmatrix} f_1 \\ g_1 \end{bmatrix} \in U(1)$, [MTZ], [TZ1]. Then we have [TZ4]

Theorem 5.1 (Tuite-Z). *The torus intertwined two-point function for the rank two free fermion VOSA is a modular form (up to multiplier) with respect to L*

$$\begin{aligned} & Z^{(1)} \begin{bmatrix} f_1 \\ g_1 \end{bmatrix} \Big|_{\gamma_1} (u, w; v, 0; \tau) \\ &= e_{\gamma_1}^{(1)} \begin{bmatrix} f_1 \\ g_1 \end{bmatrix} (c_1 \tau + d_1)^{wtu + wtv + \kappa^2} Z^{(1)} \begin{bmatrix} f_1 \\ g_1 \end{bmatrix} (u, w; v, 0; \tau), \end{aligned}$$

where $u, v \in V_{\sigma g}$.

The action of the generators A, B and C is given by [TZ1]

$$A \begin{bmatrix} f_1 \\ f_2 \\ g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_1 f_2 \sigma \\ g_1 g_2^{-1} \sigma \\ g_2 \end{bmatrix}, \quad B \begin{bmatrix} f_1 \\ f_2 \\ g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} f_1 g_2 \sigma \\ f_2 g_1 \sigma \\ g_1 \\ g_2 \end{bmatrix}, \quad C \begin{bmatrix} f_1 \\ f_2 \\ g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 g_2 \sigma \\ g_1 \\ g_2 \end{bmatrix}.$$

In a similar way we may introduce the action of $\gamma \in L$ on the genus two partition function [TZ4]

$$\begin{aligned} & Z^{(2)} \begin{bmatrix} f \\ g \end{bmatrix} \Big|_{\gamma} (\tau, w, \rho) = Z^{(2)} \left(\gamma \cdot \begin{bmatrix} f \\ g \end{bmatrix} \right) \gamma \cdot (\tau, w, \rho), \\ & \gamma_1 \cdot \begin{bmatrix} f_1 \\ f_2 \\ g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} f_1^{a_1} g_1^{b_1} \\ f_2 \\ f_1^{c_1} g_1^{d_1} \\ g_2 \end{bmatrix}. \end{aligned}$$

We may now describe the modular invariance of the genus two partition function for the rank two free fermion VOSA under the action of L . Define a genus two multiplier $e_{\gamma}^{(2)} \begin{bmatrix} f \\ g \end{bmatrix} \in U(1)$ for $\gamma \in L$ in terms of the genus one multiplier as follows

$$e_{\gamma}^{(2)} \begin{bmatrix} f \\ g \end{bmatrix} = e_{\gamma_1}^{(1)} \begin{bmatrix} f_1 \\ g_1 \end{bmatrix},$$

for the generator $\gamma_1 \in \Gamma_1$. We then find [TZ4]

Theorem 5.2 (Tuite-Z). *The genus two partition function for the rank two VOSA is modular invariant with respect to L with the multiplier system, i.e.,*

$$Z^{(2)} \begin{bmatrix} f \\ g \end{bmatrix} \Big|_{\gamma} (\tau, w, \rho) = e_{\gamma}^{(2)} \begin{bmatrix} f \\ g \end{bmatrix} Z^{(2)} \begin{bmatrix} f \\ g \end{bmatrix} (\tau, w, \rho).$$

Finally, we can also obtain modular invariance for the generating form

$$G_n^{(2)} \begin{bmatrix} f \\ g \end{bmatrix} (x_1, y_1, \dots, x_n, y_n),$$

for all genus two n -point functions [TZ4].

Theorem 5.3. $G_n^{(2)} \left[\begin{smallmatrix} f \\ g \end{smallmatrix} \right] (x_1, y_1, \dots, x_n, y_n)$ is modular invariant with respect to L with a multiplier.

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